Some remarks on characters of symmetric groups, Schur functions, Littlewood-Richardson and Kronecker coefficients

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Content - in no particular order

- Schur functions
- Littlewood-Richardson coefficients
- The hive model
- Stretched Littlewood-Richardson coefficients
- Characters of the symmetric group
- Characters in reduced notation
- Kronecker coefficients
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- Stretched Kronecker coefficients
- Polynomial and quasi-polynomial behaviour

Schur functions

Symmetric functions over $\mathbf{x} = (x_1, x_2, \ldots, x_n)$

- Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ be a partition of length $\leq n$
- Schur functions: $s_\lambda(\mathbf{x}) = \dfrac{|\, x_i^{\lambda_j+n-j} \,|}{|\, x_i^{n-j} \,|}$
- Complete: $h_k(\mathbf{x}) = s_k(\mathbf{x})$, $h_\lambda(\mathbf{x}) = h_{\lambda_1}(\mathbf{x}) h_{\lambda_2}(\mathbf{x}) \cdots$
- Elementary: $e_k(\mathbf{x}) = s_{1^k}(\mathbf{x})$, $e_\lambda(\mathbf{x}) = e_{\lambda_1}(\mathbf{x}) e_{\lambda_2}(\mathbf{x}) \cdots$

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Transition matrices

- $h_\mu(\mathbf{x}) = \sum_\lambda K_{\lambda\mu}\, s_\lambda(\mathbf{x})$ coefficients $K_{\lambda\mu} \in \mathbb{Z}_{\geq 0}$
- $s_\lambda(\mathbf{x}) = \sum_\mu B_{\lambda\mu}\, h_\mu(\mathbf{x})$ where $B = K'^{-1}$ with $B_{\lambda\mu} \in \mathbb{Z}$
- Jacobi-Trudi: $s_\lambda(\mathbf{x}) = |\, h_{\lambda_i - i + j}(\mathbf{x}) \,|$

  with $h_0(\mathbf{x}) = 1$ and $h_k(\mathbf{x}) = 0$ if $k < 0$.

Schur function products and quotients

- Let Λ be the ring of symmetric functions of $x_1, x_2, \ldots$
- Basis of Λ: Schur functions s_λ for all partitions λ
- Stability: $s_\lambda(x_1, x_2, \ldots, x_n) = s_\lambda(x_1, x_2, \ldots, x_n, 0, 0, \ldots)$
- Product: $s_\mu\, s_\nu = \sum_\lambda c_{\mu\nu}^\lambda\, s_\lambda$ outer product

  Littlewood-Richardson coefficients $c_{\lambda\mu}^\nu \in \mathbb{Z}_{\geq 0}$
- Coproduct: $\Delta : s_\lambda = \sum_{\mu,\nu} c_{\mu\nu}^\lambda\, s_\mu \otimes s_\nu$
- Skew Schur functions: quotient

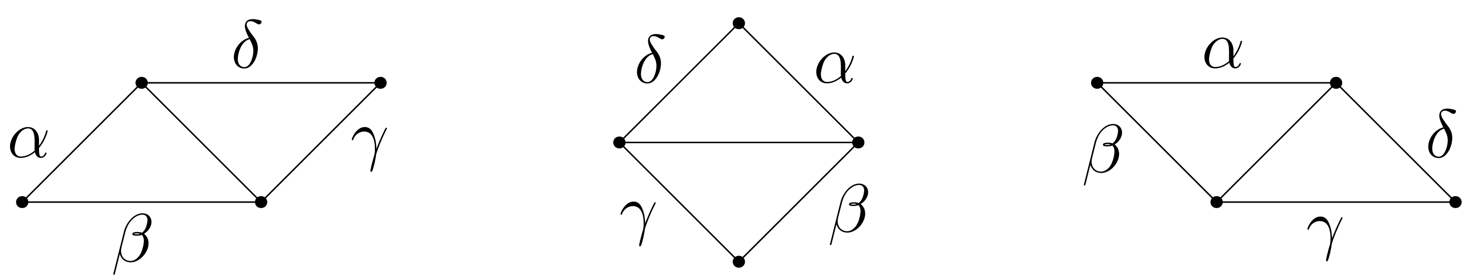
  $$D_{s_\mu} s_\lambda = s_{\lambda/\mu} = \sum_\mu c_{\mu\nu}^\lambda\, s_\nu = |\, h_{\lambda_i - \mu_j - i + j}\, |$$
- Scalar product: $(\cdot\,|\,\cdot)$ on Λ such that

$$(s_\lambda\,|\,s_\mu) = \delta_{\lambda\mu} \qquad \text{and} \qquad c_{\lambda\mu}^\nu = (s_\nu\,|\,s_\lambda\, s_\mu) = (s_{\nu/\lambda}\,|\,s_\mu)$$

Hive model for LR coefficients

- Knutson and Tao [1999], as described by Buch [2000], and adapted by Tollu, Toumazet and K [2006]
- An integer n-hive is a triangular graph with non-negative integer edge labels satisfying:
  - Elementary triangle condition: $\sigma + \tau = \rho$

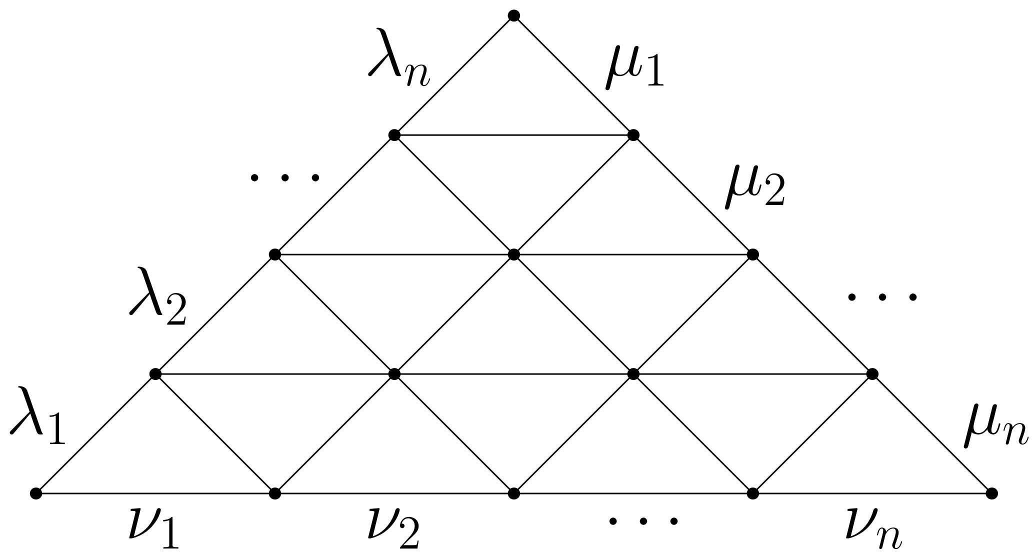
  - Elementary rhombi condition: $\alpha \geq \gamma$ and $\beta \geq \delta$



  - Note: The triangle condition implies $\alpha + \delta = \beta + \gamma$.

LR-hives edge labels

Definition An LR n-hive is an integer n-hive for which the boundary edge labels are determined by partitions λ, μ, ν of no more than n parts with $|\lambda| + |\mu| = |\nu|$



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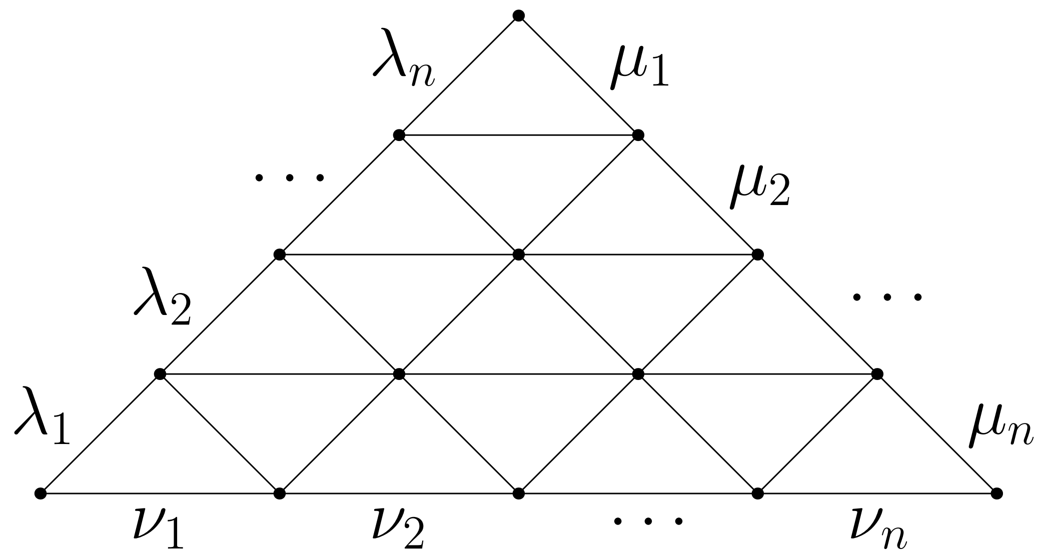
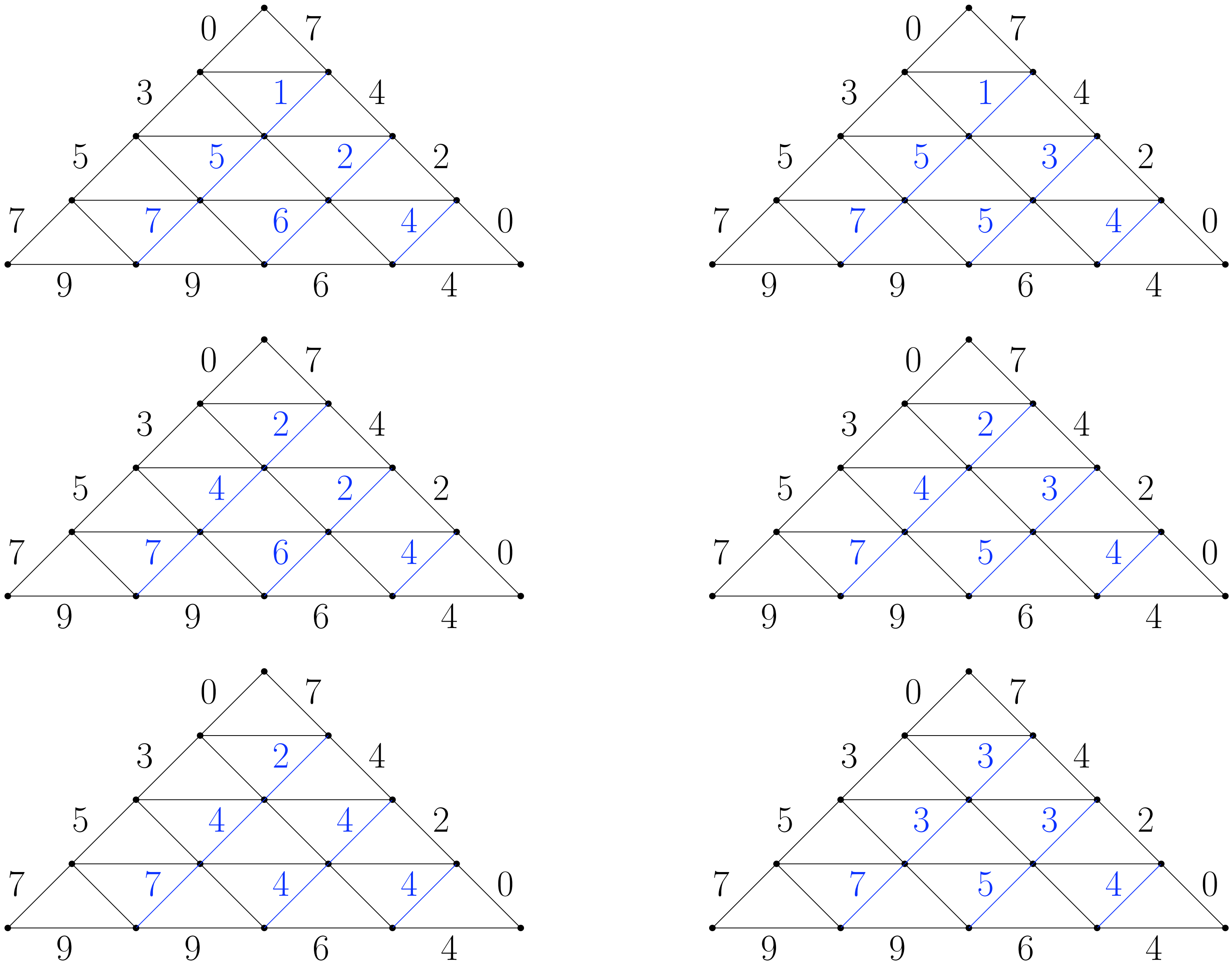


Theorem The LR-coefficient $c^{\nu}_{\lambda\mu}$ is the number of distinct LR n-hives with boundary labels determined by λ, μ and ν.

Ex: $\lambda = (753), \mu = (742), \nu = (9964), \quad c^{\nu}_{\lambda\mu} = 6$



Note The triangle condition fixes all other edge labels

Stretched LR coefficients

- Partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$
- Stretching parameter $t \in \mathbb{N}$. Let $t\lambda = (t\lambda_1, t\lambda_2, \ldots, t\lambda_n)$.
- Littlewood-Richardson coefficients $c_{\lambda\mu}^{\nu}$.
- Stretched Littlewood-Richardson coefficients $c_{t\lambda,t\mu}^{t\nu}$.
- Theorem [Rassart, 2004]

  $P_{\lambda,\mu}^{\nu}(t) = c_{t\lambda,t\mu}^{t\nu}$ is a polynomial in t.

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- Stretched Littlewood-Richardson coefficients $c^{t\nu}_{t\lambda,t\mu}$.
- Theorem [Rassart, 2004]

  $P^{\nu}_{\lambda,\mu}(t) = c^{t\nu}_{t\lambda,t\mu}$ is a polynomial in t.

Ex: $n = 4, \lambda = (7530), \mu = (7420), \nu = (9964), \quad c^{\nu}_{\lambda\mu} = 6$

$$P^{\nu}_{\lambda\mu}(t) = \frac{1}{2}(5t^2 + 5t + 2)$$

$$G^{\nu}_{\lambda\mu}(z) = \sum_{t=0}^{\infty} P^{\nu}_{\lambda\mu}(t)\, z^t = \frac{1 + 3z + z^2}{(1-z)^3}.$$

Ehrhart quasi-polynomials

- Let $\mathcal{P} \in \mathbb{R}^m$ be a rational convex polytope
- Let $\overline{\mathcal{P}}$ be the interior of $\mathcal{P}$
- For $t \in \mathbb{N}$ let $i(\mathcal{P}, t) = \#\{t\mathcal{P} \cap \mathbb{Z}^m\}$
- For $t \in \mathbb{N}$ let $\bar{i}(\mathcal{P}, t) = \#\{t\overline{\mathcal{P}} \cap \mathbb{Z}^m\}$

Theorem There exist polynomials $P_l(t)$ of degree d in t and quasi-period k such that

$$i(\mathcal{P}, t) = P_l(t) \text{ for } t \equiv l (\operatorname{mod} k)$$

ie. $i(\mathcal{P}, t)$ is a quasi-polynomial of degree d in t.

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ie. $i(\mathcal{P}, t)$ is a quasi-polynomial of degree d in t.

Theorem If $\mathcal{P}$ is an integer convex polytope then

$$i(\mathcal{P}, t) = P(t)$$

is a polynomial of degree d in t

Generating functions

Corollary For all rational convex polytopes $\mathcal{P}$ there exists

$$G(\mathcal{P}, z) = \sum_{t \geq 0} i(\mathcal{P}, t)\ z^t = \frac{N(z)}{\prod_{p|k}(1 - z^p)^{d_p}}$$

- with $N(z)$ a polynomial in z of degree $\leq d$
- $\sum_p d_p = d + 1$ and $N(0) = i(\mathcal{P}, 0) = G(\mathcal{P}, 0) = 1$

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Corollary For all integer convex polytopes $\mathcal{P}$ there exists

$$G(\mathcal{P}, z) = \sum_{t \geq 0} i(\mathcal{P}, t)\ z^t = \sum_{t \geq 0} P(t)\ z^t = \frac{N(z)}{(1 - z)^{d+1}}$$

- with $P(t)$ a polynomial in t of degree d
- with $N(z)$ a polynomial in z of degree $\leq d$
- and $N(0) = P(0) = i(\mathcal{P}, 0) = G(\mathcal{P}, 0) = 1$

Application to stretched LR-coefficients

- The LR-hive conditions define a rational convex polytope
- There exist λ, μ, ν such that this polytope is not integer
- Theorem $i(\mathcal{P}, t) = P^{\nu}_{\lambda\mu}(t)$ is polynomial in t
- Ehrhart reciprocity: $i(\mathcal{P}, -t) = (-1)^d \, \bar{i}(\mathcal{P}, t)$ for all $t \in \mathbb{N}$
- Corollary $P^{\nu}_{\lambda\mu}(t)$ contains $(t+1)(t+2)\cdots(t+m)$ as a factor if $m\mathcal{P}$ contains no interior integer points.

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Ex: $n = 7, \lambda = (433210), \mu = (432210), \nu = (7444321), c^{\nu}_{\lambda\mu} = 13$

$$
\begin{aligned}
P^{\nu}_{\lambda\mu}(t) &= \frac{1}{10080}(t+1)(t+2)(t+3)(t+4)(t+5) \\
&\qquad \cdot (5t+21)(t^2+2t+4) \\
G^{\nu}_{\lambda\mu}(z) &= \sum_{t=0}^{\infty} P^{\nu}_{\lambda\mu}(t)\, z^t = \frac{1+4z+12z^2+3z^3}{(1-z)^9}.
\end{aligned}
$$

Observations

Stretched LR polynomial:

- $P^{\nu}_{\lambda\mu}(t) \ = \ \sum_{i=0}^{d} a_i \, t^i$ is a polynomial of degree d
  - Problem predict d.
  - Conjecture $a_i \geq 0$ for all i
- $P^{\nu}_{\lambda\mu}(t)$ may contain factors $(t+1)(t+2)\cdots(t+m)$ for some $m \in \mathbb{N}$ Problem predict maximum value of m.

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Generating function:

- $G^{\nu}_{\lambda\mu}(z) \; = \; \sum_{t=0}^{\infty} P^{\nu}_{\lambda\mu}(t)\, z^t \; = \; N(z)\big/(1-z)^{d+1}$ with $N(z) = \sum_{i=0}^{n} b_i\, z^i$ a polynomial of degree $n \leq d$
  - Problem predict n
  - Conjecture b_i non-negative integer for all i

Characters of the symmetric group S_n

- Irreducible representations specified by partitions $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \vdash n$
- Conjugacy classes specified by partitions $\rho = (1^{\alpha_1} 2^{\alpha_2} \cdots n^{\alpha_n}) \vdash n$
- Characters χ^λ_ρ
- Dimensions $\chi^\lambda_{1^n} = f^\lambda_n = \dfrac{n!}{H(\lambda)}$

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Ex. $n = 8$, $\lambda = (4, 3, 1)$, $f^\lambda_8 = 8!/(6.4.3.1.4.2.1.1) = 70.$

$F^{431} =$

hook:

	*	*	*
	*		

hook lengths:

6	**4**	**3**	**1**
4	**2**	**1**	
1			

S_n character formulae

- Power sums: $p_k = x_1^k + x_2^k + \cdots$ for $k = 1, 2, \ldots$
- Let $\rho = (1^{\alpha_1}, 2^{\alpha_2}, \cdots, n^{\alpha_n}) \vdash n$ and $z_\rho = \prod_{k=1}^n k^{\alpha_k} \alpha_k!$
- Power sum functions: $p_\rho = p_1^{\alpha_1} \, p_2^{\alpha_2} \, \cdots \, p_n^{\alpha_n}$

Frobenius $\quad p_\rho = \sum_{\lambda \vdash n} \chi_\rho^\lambda \, s_\lambda$ and $s_\lambda = \sum_{\rho \vdash n} \frac{1}{z_\rho} \chi_\rho^\lambda \, p_\rho$

- Hence $\quad \chi_\rho^\lambda = (s_\lambda \,|\, p_\rho)$

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- Hence $\quad \chi^\lambda_\rho = (s_\lambda \,|\, p_\rho)$

Murnaghan-Nakayama $\quad \chi^\lambda_\rho = \sum_{\mu:\lambda/\mu=\xi} (-1)^{\mathrm{row}(\xi)-1}\, \chi^\mu_{\rho\backslash k}$

- $\rho = (\cdots, k^{\alpha_k}, \cdots) \implies \rho\backslash k = (\cdots, k^{\alpha_k - 1}, \cdots)$
- The sum is over those μ for which λ/μ is a continuous boundary strip ξ of k boxes occupying $\mathrm{row}(\xi)$ rows.

Reduced notation - Murnaghan, 1938

Scharf and Thibon [1994]

- For $\mu \vdash m$ let $\langle \mu \rangle = \sum_{n \in \mathbb{Z}} s_{(n-m,\mu)} = \sigma_1 \, D_{\lambda_{-1}} \, s_\mu$, then

$$\langle \mu \rangle = \sum_{n \in \mathbb{Z}} \begin{vmatrix} h_{n-m-1+j} \\ h_{\mu_{i-1}-i+j} \end{vmatrix} = \sum_{k \in \mathbb{Z}_{\geq 0}} h_k \sum_{j \geq 0} (-1)^j s_{\mu/1^j}$$

- Let $\chi_\rho^{\langle \mu \rangle} = (\langle \mu \rangle \,|\, p_\rho)$ with $\rho = (1^{\alpha_1}, 2^{\alpha_2}, \ldots) \vdash n$, then

$$\chi_\rho^{\langle \mu \rangle} = \chi_\rho^{(n-m,\mu)} = \left(\sigma_1 \, D_{\lambda_{-1}} \, s_\mu \,|\, p_\rho\right) = \left(D_{\lambda_{-1}} \, s_\mu \,|\, D_{\sigma_1} \, p_\rho\right)$$

with $$D_{\lambda_{-1}} \, s_\mu = \sum_{j \geq 0} (-1)^j \, s_{\mu/1^j}$$

and $$D_{\sigma_1} \, p_\rho = \prod_{i \geq 1} (p_i + 1)^{\alpha_i}$$

Character polynomial - Frobenius, 1904

Gupta [1952] Character expressed as a polynomial in the α_i

$$\chi_\rho^{\langle\mu\rangle} = \sum_{k\geq 0} \sum_{\xi\vdash m-k} (-1)^k \, \chi_\xi^{\mu/1^k} \prod_{i\geq 1} \binom{\alpha_i}{\beta_i}$$

with $\rho = (1^{\alpha_1}, 2^{\alpha_2}, \ldots)$ and $\xi = (1^{\beta_1}, 2^{\beta_2}, \ldots)$

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Butler and K. [1973]

- Let $f_n^{\langle\mu\rangle} = \chi_{1^n}^{\langle\mu\rangle} = \chi_{1^n}^{(n-m,\mu)} = f_n^{(n-m,\mu)}$. Then

$$\chi_\rho^{\langle\mu\rangle} = \sum_{k\geq 0} \sum_{\kappa\vdash k} f_{\alpha_1}^{\langle\kappa\rangle} \sum_{\eta\vdash m-k} \chi_\eta^{\mu/\kappa} \prod_{i\geq 2} \binom{\alpha_i}{\gamma_i}$$

with $\rho = (1^{\alpha_1}, 2^{\alpha_2}, \ldots)$ and $\eta = (2^{\gamma_2}, 3^{\gamma_3}, \ldots)$

Example

- Character in case $\langle\mu\rangle = \langle 3,2,1\rangle$ with $\rho = (1^{\alpha_1}, 2^{\alpha_2}, \ldots)$

$$\chi_\rho^{\langle 321\rangle} = \frac{1}{45}\alpha_1(\alpha_1-1)(\alpha_1-2)(\alpha_1-4)(\alpha_1-6)(\alpha_1-8)$$
$$-\ \frac{1}{6}\alpha_1(\alpha_1-1)(\alpha_1-5)\,\alpha_3 - \ \frac{1}{6}(\alpha_1-1)(\alpha_1-2)(\alpha_1-3)\,\alpha_3$$
$$-\ \alpha_3(\alpha_3-1)\ +\ (\alpha_1-1)\,\alpha_5$$

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- Note: $|\mu| = 6$, $F^{321} = \begin{array}{|c|c|c|}\hline \mathbf{5} & \mathbf{3} & \mathbf{1} \\ \hline \mathbf{3} & \mathbf{1} & \\ \cline{1-2} \mathbf{1} & & \\ \cline{1-1}\end{array}$, $H(\mu) = 5\cdot 3\cdot 1\cdot 3\cdot 1\cdot 1 = 45$

- $f_n^{\langle 321\rangle} = \dfrac{1}{45}n(n-1)(n-2)(n-4)(n-6)(n-8)$ with zeros at $n = 0,1,2,4,6,8$ determined by $s_{(n-6,3,2,1)} = 0$

- $\chi_\rho^{\langle 321\rangle}$ is a polynomial of degree 6 depending only on α_1, α_3, α_5 with sums of indices ≤ 6

Case $\langle\mu\rangle$ with $\mu \vdash m$ and $\rho = (1^{\alpha_1}, 2^{\alpha_2}, \ldots)$

- The character $\chi_\rho^{\langle\mu\rangle}$ is a polynomial of degree m
- Depends only on α_k with k a hook length of F^μ
- In each term the sum of the indices k is $\leq m$
- $s_{(n-m,\mu)} = \begin{vmatrix} h_{n-m-1+j} \\ h_{\mu_i-i-1+j} \end{vmatrix} = 0 \Leftrightarrow$ $n = m + \mu_i - i$ for $i = 1, 2, \ldots, m$
- Butler and K. [1973] $f_n^{\langle\mu\rangle} = \dfrac{1}{H(\mu)} \prod_{i=1}^{m} (n - m - \mu_i + i)$

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- Butler and K. [1973] $f_n^{\langle\mu\rangle} = \frac{1}{H(\mu)} \prod_{i=1}^{m} (n - m - \mu_i + i)$
- In fact setting $\chi_\rho^{\langle\mu\rangle} = 0$ for all $\rho \vdash m + \mu_i - i$ with $i = 1, 2, \ldots, m$ completely determines $\chi_\rho^{\langle\mu\rangle}$ as a polynomial in the α_k of degree m, with leading term $\alpha_1^m / H(\mu)$ [K. and Welsh, 2009]

Example

Nature of character polynomial

- Let $X^{21}(\alpha) = \chi_\rho^{(n-3,2,1)} = \chi_\rho^{\langle 2,1\rangle}$ with $\rho = (1^{\alpha_1}, 2^{\alpha_2}, \ldots)$
- Then $X^{21}(\alpha)$ is a polynomial in $\alpha_1, \alpha_2, \alpha_3$ of degree 3
- In each summand the sum of the indices k on the α_k is no greater than 3
- $X^{21}(\alpha) = c_0\alpha_1^3 + c_1\alpha_1^2 + c_2\alpha_1 + c_3 + c_4\alpha_1\alpha_2 + c_5\alpha_2 + c_6\alpha_3$

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Identification of zeros

- $s_{(n-3,2,1)} = \begin{vmatrix} h_{n-3} & h_{n-2} & h_{n-1} & h_n \\ h_1 & h_2 & h_3 & h_4 \\ 0 & h_0 & h_1 & h_2 \\ 0 & 0 & 0 & h_0 \end{vmatrix} = 0 \quad \Rightarrow \quad n \in \{0, 2, 4\}$

Dimension

Dimension $\rho = 1^{\alpha_1}$ with $\alpha_1 = n$ and $\alpha_k = 0$ for $k > 1$

- $X^{21}(n, 0, \ldots, 0) = \chi^{(n-3,2,1)}_{1^n} = f^{(n-3,2,1)}_n = \dfrac{n!}{H(n-3,2,1)}$

$$= \frac{n!}{\begin{array}{|c|c|c|c|c|c|}\hline n-1 & n-3 & n-5 & \cdots & 2 & 1 \\ \hline 3 & 1 \\ \cline{1-2} 1 \\ \cline{1-1}\end{array}} = \frac{n(n-2)(n-4)}{3}$$

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- $X^{21}(n, 0, \ldots, 0) = \chi_{1^n}^{(n-3,2,1)} = f_n^{(n-3,2,1)} = \dfrac{n!}{H(n-3,2,1)}$

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Alternatively

- $f_n^{(n-3,2,1)}$ is a polynomial in n of degree 3 with leading term $\dfrac{n^3}{H(21)} = \dfrac{n^3}{3}$ and $f_n^{(n-3,2,1)} = 0$ for $n = 0, 2, 4$
- Hence $f_n^{(n-3,2,1)} = \dfrac{1}{3}\, n(n-2)(n-4)$

Implication of zeros for complete polynomial

$$X^{21}(\alpha) = c_0\,\alpha_1^3 + c_1\,\alpha_1^2 + c_2\,\alpha_1 + c_3 + c_4\,\alpha_1\alpha_2 + c_5\,\alpha_2 + c_6\,\alpha_3$$

- $c_0 = \frac{1}{H(21)} = \frac{1}{3}$ and $X^{21}(\alpha) = 0$ for all $\rho \vdash n \in \{0, 2, 4\}$
  - $\rho = 0$: $c_3 = 0$
  - $\rho = 2$: $c_3 + c_5 = 0$
  - $\rho = 1^2$: $8c_0 + 4c_1 + 2c_2 + c_3 = 0$
  - $\rho = 4$: no information
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- Unique solution: $X^{21}(\alpha) = \dfrac{1}{3}\,\alpha_1(\alpha_1 - 2)(\alpha_1 - 4) - \alpha_3$

Kronecker coefficients

- Characters χ_ρ^λ form an orthogonal basis of the space of class functions of S_n.
- Product: $\chi_\rho^\lambda \, \chi_\rho^\mu = \sum_\nu g_{\lambda\mu}^\nu \, \chi_\rho^\nu$ with $\lambda, \mu, \nu, \rho \vdash n$

  Kronecker coefficients $g_{\lambda\mu}^\nu \in \mathbb{Z}_{\geq 0}$
- Note $g_{\lambda\mu}^\nu = \sum_\rho \frac{1}{z_\rho} \chi_\rho^\lambda \, \chi_\rho^\nu \, \chi_\rho^\mu$ is symmetric in λ, μ, ν

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- Inner product: $s_\lambda * s_\mu = \sum_\nu g^\nu_{\lambda\mu} \, s_\nu$
- Inner coproduct: $\delta : s_\nu = \sum_{\lambda,\mu} g^\nu_{\lambda\mu} \, s_\lambda \otimes s_\mu$
- $s_n * s_\mu = s_\mu$ and $\delta : s_n = \sum_{\mu \vdash n} s_\mu \otimes s_\mu$
- $s_{(n-1,1)} * s_\mu = s_{\mu/1} \, s_1 - s_\mu = \cdots + (\mathrm{dp}(\mu) - 1) \, s_\mu + \cdots$ where $\mathrm{dp}(\mu)$ is the number of non-zero distinct parts of μ

Evaluation of Kronecker coefficients

Theorem see for example [Robinson, 1961]

$$g^{\nu}_{\lambda\mu} =$$
$$\sum_{\kappa} B_{\lambda,\kappa} \sum_{\rho\vdash\kappa_1} \sum_{\sigma\vdash\kappa_2} \cdots \sum_{\tau\vdash\kappa_n} \left(\sum_{\mu} c^{\mu}_{\rho\sigma\cdots\tau}\right) \left(\sum_{\nu} c^{\nu}_{\rho\sigma\cdots\tau}\right)$$

Proof:

- $$s_\lambda = |\, h_{\lambda_i - i + j} \,| = \sum_{\kappa} B_{\lambda,\kappa}\, h_\kappa = \sum_{r \geq s \geq \cdots \geq t} B_{\lambda,\kappa}\, h_r\, h_s \,\cdots\, h_t$$
- $\delta : h_r = \sum_{\rho\vdash r} s_\rho \otimes s_\rho$ and $s_\rho\, s_\sigma \,\cdots\, s_\tau = \sum_{\mu} c^{\mu}_{\rho\sigma\cdots\tau}\, s_\mu$

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Corollary For all $p \geq 0$ and $\mu \vdash n$

$$s_{(n-p,p)} * s_\mu = \sum_{\sigma\vdash p} s_{\mu/\sigma}\, s_\sigma - \sum_{\tau\vdash p-1} s_{\mu/\tau}\, s_\tau$$

Special case $\lambda = (n-p, p)$

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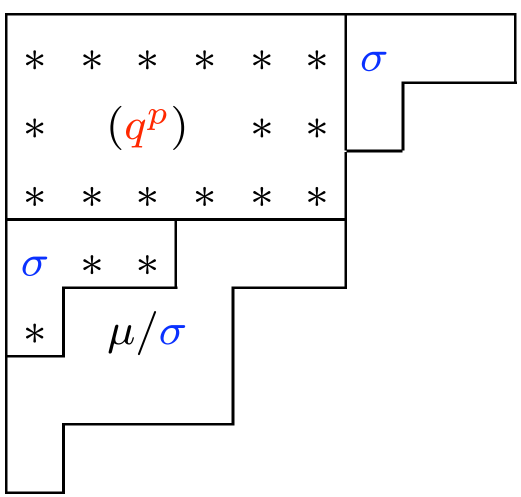
Note: for any $q \geq \mu_1$

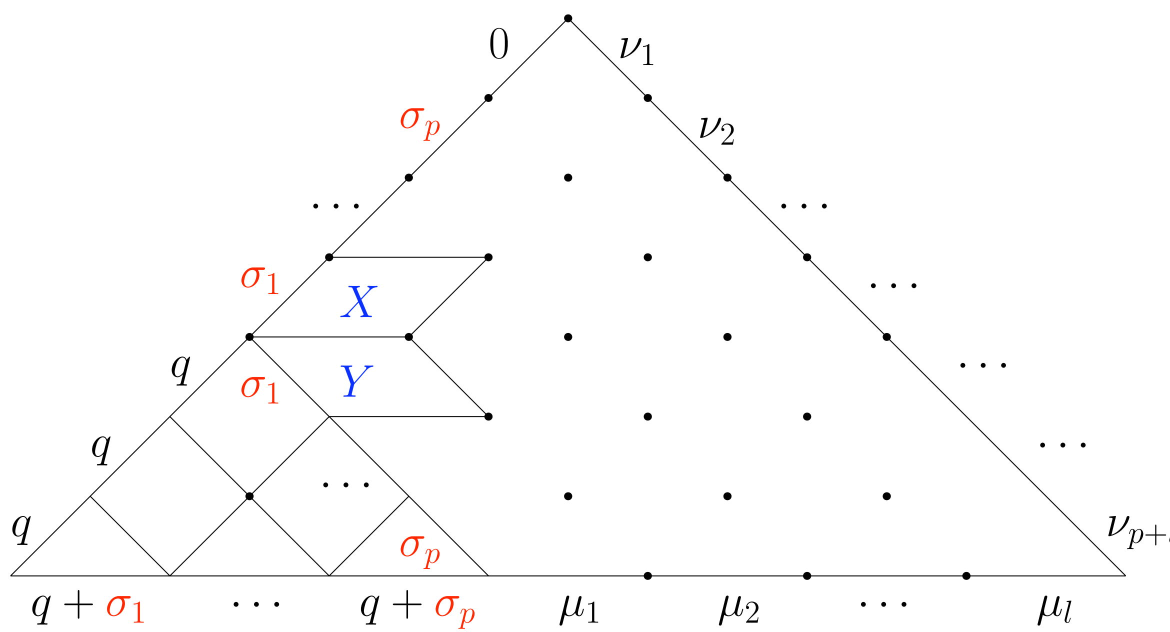


$$\Rightarrow \qquad s_{\mu/\sigma}\, s_\sigma = s_{(q^p+\sigma,\mu)/(q^p,\sigma)}$$

Combinatorial model in case $\lambda = (n-p, p)$

Theorem [Ballantine and Orellana, 2006] expressed here in terms of hives

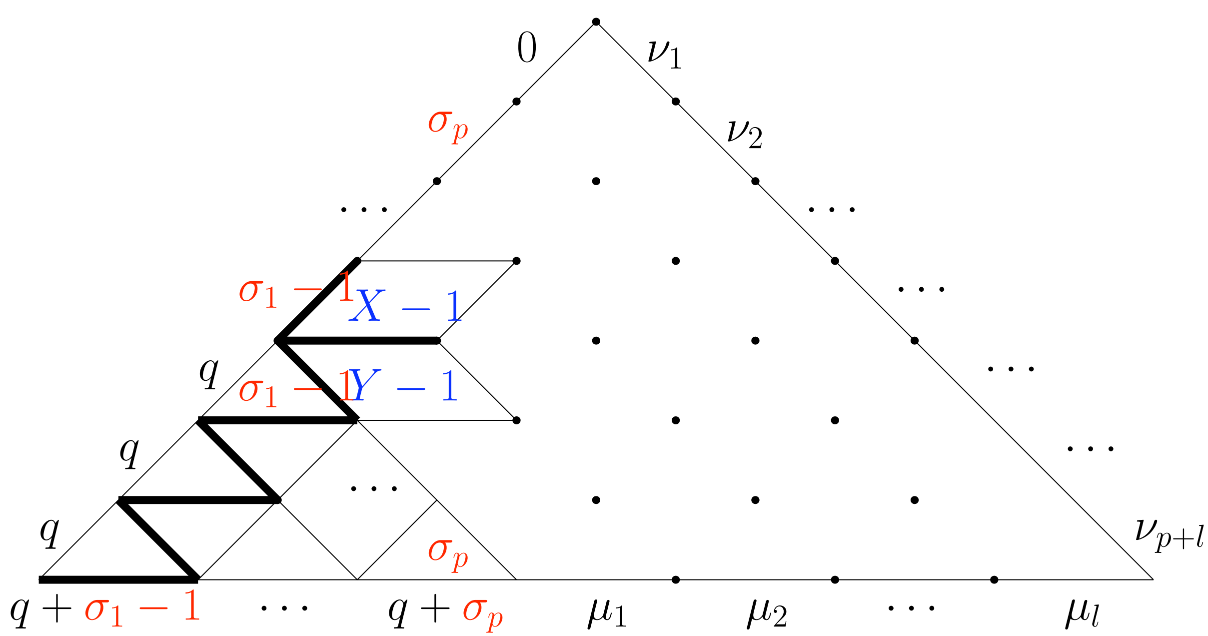
If $\mu_1 \geq 2p-1$ then $g_{(n-p,p),\mu,\nu}$ is the number of hives with boundary as shown for all $\sigma \vdash p$ with $q = \mu_1$ and $XY = 0$



This is a subset of the hives contributing to $\left(s_{(q^p+\sigma,\mu)/(q^p,\sigma)} \,\middle|\, s_\nu\right)$

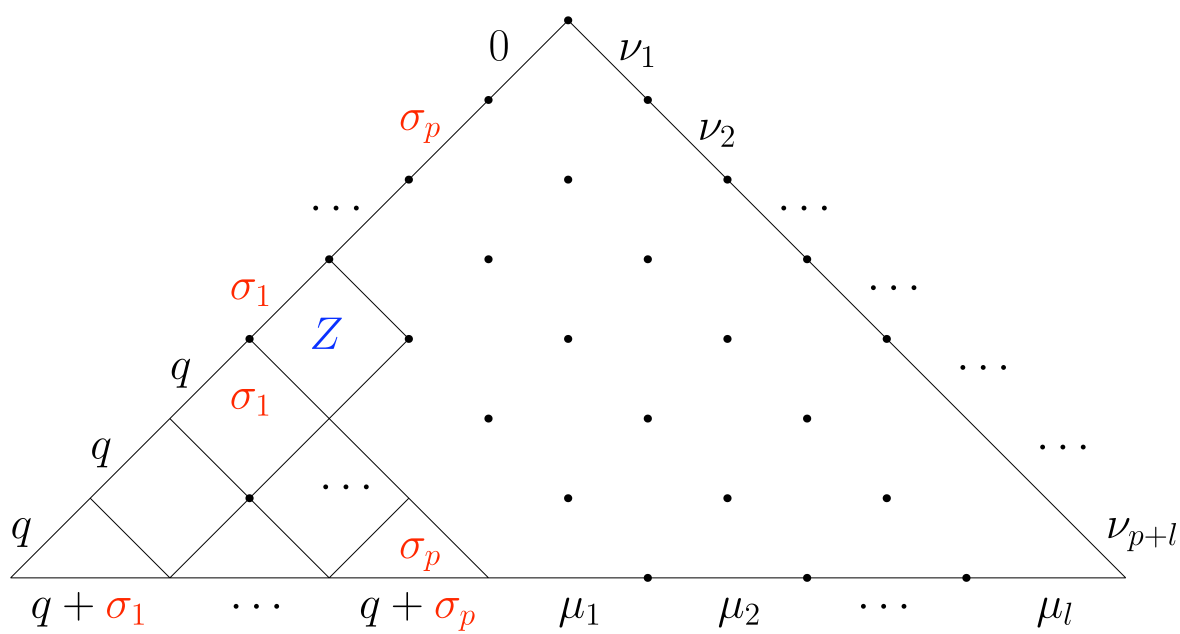
Proof

- For given σ let $\tau = (\sigma_1 - 1, \sigma_2, \ldots, \sigma_p)$
- Let ψ be a map from a σ-hive to a potential τ-hive obtained by deleting 1 from all thick edge labels
- Under this map $\psi : X \mapsto X - 1$ and $\psi : Y \mapsto Y - 1$
- The result is a τ-hive if and only if $X > 0$ and $Y > 0$



Proof contd.

- Under this map $\psi : Z \mapsto Z+1$
- All τ-hives are obtained other than those with $Z+1=0$
- There exists no such τ-hive unless $\mu_1 \leq 2\tau_1 = 2\sigma_1 - 2$
- All τ-hives are obtained if $\mu_1 \geq 2p-1 \geq 2\sigma_1 - 1$



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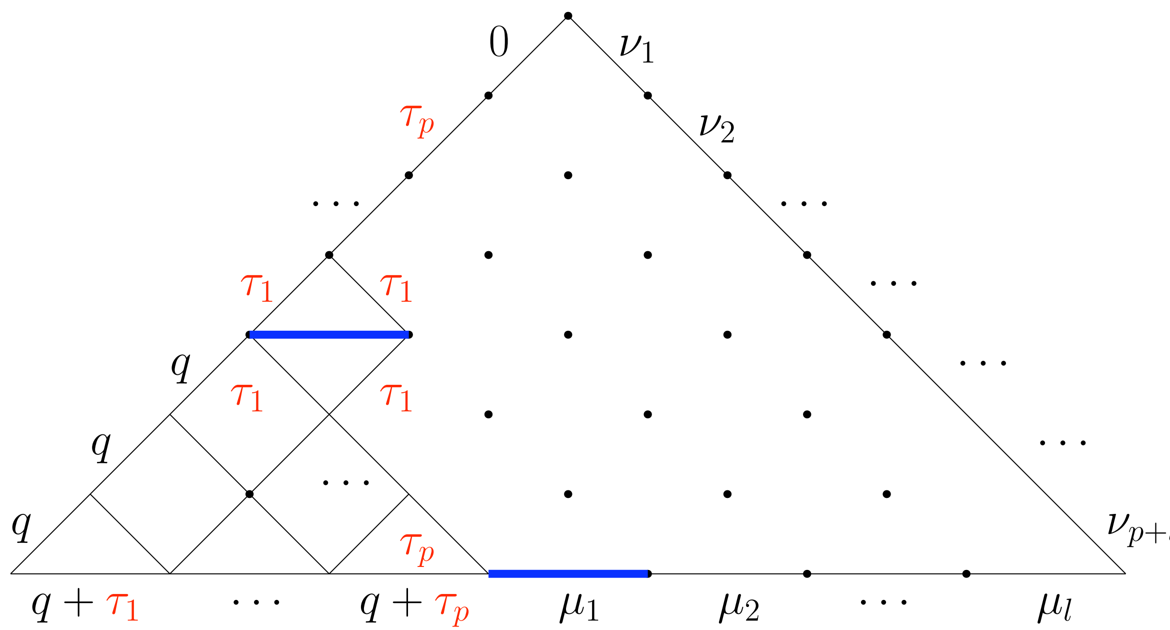
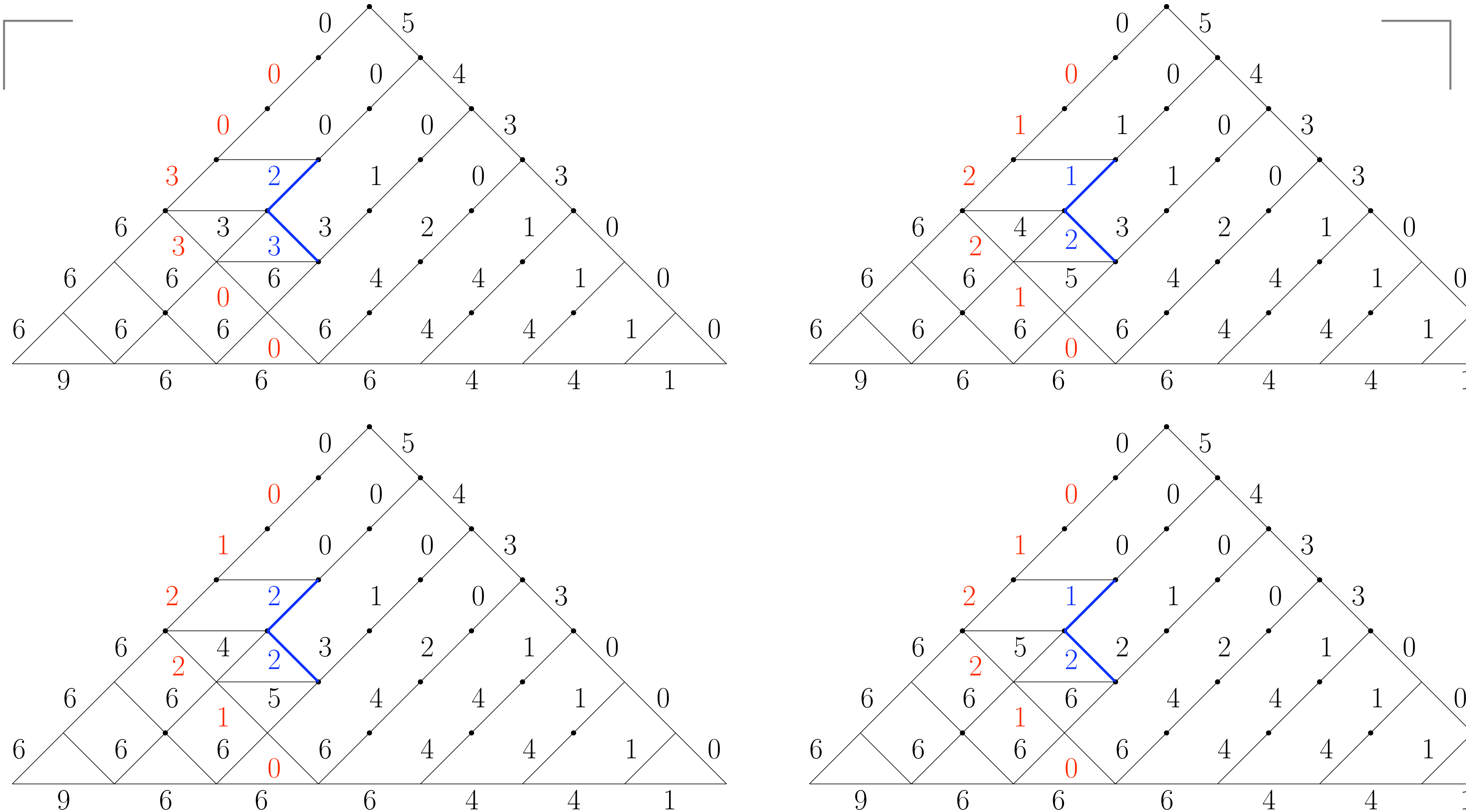
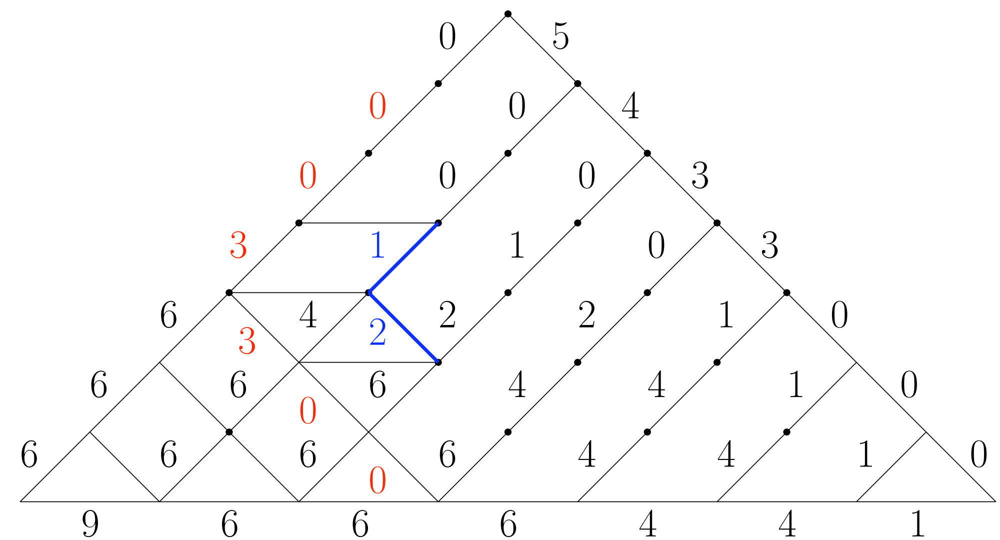


Ex: $p = 3, q = 6, \lambda = (12, 3), \mu = (5433), \nu = (6441), \quad g^{\nu}_{\lambda\mu} = 4$

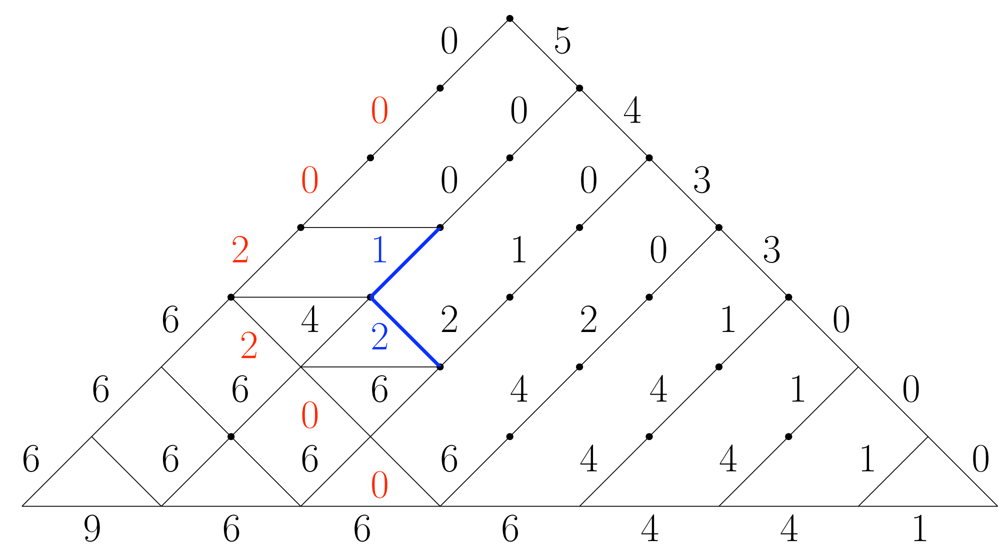


Ex: $(3,0,0)$-hive mapped to a $(2,0,0)$-hive

ψ :



$\mapsto$



Reduced Kronecker coefficients

Theorem: [Murnaghan, 1938]

Let $\lambda = (n-r,\rho)$, $\mu = (n-s,\sigma)$, $\nu = (n-t,\tau)$

with $n \in \mathbb{Z}_{\geq 0}$ and $\rho \vdash r$, $\sigma \vdash s$, $\tau \vdash t$. Then

$$s_\lambda * s_\mu = \sum_\nu \, \overline{g}^{\,\tau}_{\rho\sigma} \, s_\nu$$

reduced Kronecker coefficients $\overline{g}^{\,\tau}_{\rho\sigma} \in \mathbb{Z}_{\geq 0}$

with $\overline{g}^{\,\tau}_{\rho\sigma}$ independent of n

Note For n sufficiently large we have $g^{\nu}_{\lambda\mu} = \overline{g}^{\,\tau}_{\rho\sigma}$

but for $n-t < \tau_1$ modification rules are required

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Theorem: [Thibon, 1991] For all partitions ρ, σ, τ

$$\langle\rho\rangle * \langle\sigma\rangle = \sum_\tau \overline{g}^{\,\tau}_{\rho\sigma}\, \langle\tau\rangle \qquad \text{reduced inner product}$$

Relation between non-reduced and reduced cases

Theorem [Briand, Orellana and Rosas, 2008]

Let λ, μ, ν be partitions of n with

$\lambda = (n-r, \rho)$, $\mu = (n-s, \sigma)$, $\nu = (n-t, \tau)$. Then

$$g_{\lambda\mu}^{\nu} = \sum_{i \geq 1} (-1)^{i-1} \, g_{\rho\sigma}^{\tau(i)}$$

where $\tau(1) = \tau$

and $\tau(i) = (n-t+1, \tau_1 + 1, \ldots, \tau_{i-2} + 1, \tau_i, \ldots, \tau_n)$ for $i \geq 2$

Proof Use reduced inner product and select all terms of weight n together with the modifications following from

$$s_{(n-t,\tau)} = \left| \begin{array}{c} s_{n-t-1+j} \\ \\ s_{\tau_{i-1}-i+j} \end{array} \right|$$

and moving ith row to top

Littlewood's formula

Littlewood [1958] (modern derivation by Thibon [1991])

$$\begin{aligned}\langle\rho\rangle * \langle\sigma\rangle &= \sum_{\xi,\eta,\zeta} \left\langle \left(s_{\rho/(\xi\zeta)}\right) \left(s_{\sigma/(\eta\zeta)}\right) \left(s_\xi * s_\eta\right) \right\rangle \\ &= \sum_{\xi,\eta,m} \left\langle \left(s_{\rho/\xi}\right) \left(s_{\sigma/\eta}\right) D_{s_m}\left(s_\xi * s_\eta\right) \right\rangle\end{aligned}$$

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$$= \sum_{\xi,\eta,m} \langle \left(s_{\rho/\xi}\right) \left(s_{\sigma/\eta}\right) D_{s_m}\left(s_\xi * s_\eta\right) \rangle$$

Special cases

- $\langle 1\rangle * \langle\sigma\rangle = \langle s_1\, s_\sigma\rangle + \langle s_{\sigma/1}\, s_1\rangle + \langle s_{\sigma/1}\rangle$
- $\langle 2\rangle * \langle\sigma\rangle = \langle s_2\, s_\sigma\rangle + \langle s_1\, s_{\sigma/1}\left(s_1 + s_0\right)\rangle$
  $+ \langle s_{\sigma/2}\left(s_2 + s_1 + s_0\right)\rangle + \langle s_{\sigma/1^2}\left(s_{1^2} + s_1\right)\rangle$
- $\langle r\rangle * \langle\sigma\rangle = \sum_{x\geq 0} \sum_{\xi\vdash x} \langle\, s_{r-x}\, s_{\sigma/\xi} \left(\sum_{y\geq 0}\, s_{\xi/y}\right) \rangle$

Special case $\langle r\rangle * \langle s\rangle = \sum_\tau g_{r,s}^\tau \langle \tau\rangle$

$$\begin{aligned}
\langle r\rangle * \langle s\rangle &= \sum_{x\geq y\geq 0} \langle\, s_{r-x}\, s_{s-x}\, s_{x-y}\rangle \\
&= \sum_{x\geq y\geq 0} \langle\, h_{r-x}\, h_{s-x}\, h_{x-y}\,\rangle \\
&= \sum_{x\geq y\geq 0} \sum_{\tau} K_{\tau,(r-x,s-x,x-y)}\,\langle\tau\rangle
\end{aligned}$$

where the Kostka coefficient $K_{\tau,(a,b,c)} =$

$$1+\min_{\geq -1}\{\tau_1-\tau_2, \tau_2-\tau_3, \tau_1-a, \tau_1-b, \tau_1-c, a-\tau_3, b-\tau_3, c-\tau_3\}$$

with

- $\min_{\geq -1}\{\cdots\} = \min\{\cdots\}$ if all arguments are ≥ 0
- and $\min_{\geq -1}\{\cdots\} = -1$ if any argument is < 0

Stretched Kronecker coefficients

- Kronecker coefficients $g_{\lambda\mu}^{\nu}$
- Stretched Kronecker coefficients $Q_{\lambda\mu}^{\nu}(t) = g_{t\lambda,t\mu}^{t\nu}$

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Ex 1: For $n = 2$, $\lambda = (11)$, $\mu = (11)$, $\nu = (11)$

$$g^{\nu}_{\lambda\mu} = 0, \qquad Q^{\nu}_{\lambda\mu}(t) = \begin{cases} 1 & t \equiv 0(mod2) \\ 0 & t \equiv 1(mod2) \end{cases}$$

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Ex 2: For $n = 3$, $\lambda = (21)$, $\mu = (21)$, $\nu = (21)$

$$g^{\nu}_{\lambda\mu} = 2, \qquad Q^{\nu}_{\lambda\mu}(t) = \begin{cases} \frac{1}{2}(t+2) & t \equiv 0(mod2) \\ \frac{1}{2}(t+1) & t \equiv 1(mod2) \end{cases}$$

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Ex 3: Briand, Orellana and Rosas [2008]

For $n = 12$, $\lambda = (66)$, $\mu = (75)$, $\nu = (642)$

$$g_{\lambda\mu}^{\nu} = 0, \qquad Q_{\lambda\mu}^{\nu}(t) = \begin{cases} \frac{1}{2}(t+2) & t \equiv 0(mod 2) \\ \frac{1}{2}(t-1) & t \equiv 1(mod 2) \end{cases}$$

Further examples of $Q^{\nu}_{\lambda\mu}(t) = g^{t\nu}_{t\lambda,t\mu}$

Ex 4: For $n = 6$, $\lambda = (21)$, $\mu = (111)$, $\nu = (111)$, $g^{\nu}_{\lambda\mu} = 0$

$$Q^{\nu}_{\lambda\mu}(t) = \begin{cases} = \frac{1}{6}(t+6) & t \equiv 0(mod6) \\ = \frac{1}{6}(t-1) & t \equiv 1(mod6) \\ = \frac{1}{6}(t+4) & t \equiv 2(mod6) \\ = \frac{1}{6}(t+3) & t \equiv 3(mod6) \\ = \frac{1}{6}(t+2) & t \equiv 4(mod6) \\ = \frac{1}{6}(t+1) & t \equiv 5(mod6) \end{cases}$$

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Ex 5: Briand, Orellana and Rosas [2008]

For $n = 18$, $\lambda = (10, 8)$, $\mu = (11, 7)$, $\nu = (10, 6, 2)$, $g^{\nu}_{\lambda\mu} = 3$

$$Q^{\nu}_{\lambda\mu}(t) = \begin{cases} = \frac{1}{4}(7t^2 + 6t + 4) & t \equiv 0(mod2) \\ = \frac{1}{4}(7t^2 + 6t - 1) & t \equiv 1(mod2) \end{cases}$$

Remarks on $Q^{\nu}_{\lambda\mu}(t)$

- By definition $Q^{\nu}_{\lambda\mu}(t) = g^{t\nu}_{t\lambda,t\mu} \in \mathbb{Z}_{\geq 0}$
- Theorem [Mulmuley 2007] $Q^{\nu}_{\lambda\mu}(t)$ is a quasi-polynomial in t of some quasi-period k
  ie. there exist polynomials $P_l(t)$ for $l = 0, 1, \ldots, k-1$
  such that $Q^{\nu}_{\lambda\mu}(t) = P_l(t)$ for all $t \equiv l (\bmod k)$
- Observation [Briand, Orellana,Rosas, 2008]
  $Q^{\nu}_{\lambda\mu}(t)$ does not satisfy the Saturation Hypothesis
  ie. there exists λ, μ, ν such that
  $$Q^{\nu}_{\lambda\mu}(1) > 0 \not\Leftarrow Q^{\nu}_{\lambda\mu}(t) > 0 \text{ for all } t \in \mathbb{N}$$
  - Ex 1: $Q^{\nu}_{\lambda\mu}(1) = 0 \not\Rightarrow Q^{\nu}_{\lambda\mu}(t) = 0$ for all $t > 0$
  - Ex 3: $Q^{\nu}_{\lambda\mu}(1) = 0 \not\Rightarrow Q^{\nu}_{\lambda\mu}(t) = 0$ for all $t \equiv 1 (\bmod k)$
  - Ex 3: $P_1(1) = 0 \not\Rightarrow P_1(t) = 0$ for all t

Remarks on $Q^{\nu}_{\lambda\mu}(t)$

- Observation [Briand, Orellana,Rosas, 2008]
  $Q^{\nu}_{\lambda\mu}(t)$ does not satisfy the Positivity Hypothesis
  ie. there exists λ, μ, ν and l such that at least one coefficient of $P_l(t)$ is negative. cf. Ex 3,4,5

- Remark $Q^{\nu}_{\lambda\mu}(t)$ is not necessarily an Ehrhart quasi-polynomial since there exists λ, μ, ν such that
  - $Q^{\nu}_{\lambda\mu}(1) = 0 \not\Rightarrow Q^{\nu}_{\lambda\mu}(-1) = 0$ or more generally
  - $Q^{\nu}_{\lambda\mu}(m) \not\geq (-1)^d Q^{\nu}_{\lambda\mu}(-m) = 0$ for any $m \geq 1$
    ie. a violation of Ehrhart reciprocity. cf. Ex 3

  In such cases $Q^{\nu}_{\lambda\mu}(t)$ cannot count the number of integral points of any expanded rational convex polytope

$Q^{\nu}_{\lambda\mu}(t)$ with $\lambda = (n-1,1)$ and $\mu = \nu$

Cases with $a > 0$, $a > b > 0$ and $a > b > c > 0$ as appropriate for $t = 0, 1, \ldots, 12$

$\mu = \nu$	0	1	2	3	4	5	6	7	8	9	10	11	12
a	1	0	0	0	0	0	0	0	0	0	0	0	0
aa	1	0	1	0	1	0	1	0	1	0	1	0	1
ab	1	1	2	2	3	3	4	4	5	5	6	6	7
aaa	1	0	1	1	1	1	2	1	2	2	2	2	3
aab, abb	1	1	3	4	7	9	14	17	24	29	38	45	57
abc	1	2	6	12	24	42	72	114	177	262	380	534	738

Conjecture The results are independent of a, b, c

$Q^{\nu}_{\lambda\mu}(t)$ with $\lambda = (n-1, 1)$ and $\mu = \nu$

Cases with $a > b > c > d > 0$ for $t = 0, 1, \ldots, 10$

$\mu = \nu$	0	1	2	3	4	5	6	7	8	9	10
$aaaa$	1	0	1	1	2	1	3	2	4	3	5
$abbb$	1	1	3	5	9	13	22	30	45	61	85
$aabb$	1	1	4	6	14	20	40	56	98	136	218
$abcc$	1	2	7	16	38	77	157	291	533	922	1566
$abcd$	1	3	12	36	102	258	616	1368	2892	5812	11220

Conjecture The results are independent of a, b, c, d

The cases $abbb$ and $aaab$ are identical

The cases $abcc$, $abbc$ and $aabc$ are identical

$Q^{\nu}_{\lambda\mu}(t)$ with $\lambda = (n-1,1)$ and $\mu = \nu$

Cases with $a > b > c > d > e > 0$ for $t = 0, 1, \ldots, 10$

$\mu = \nu$	0	1	2	3	4	5	6	7	8	9	10
$aaaaa$	1	0	1	1	2	2	3	3	5	5	7
$abbbb$	1	1	3	5	10	15	26	38	60	85	125
$aabbb$	1	1	4	7	16	27	54	88	158	253	421
$abccc$	1	2	7	17	42	91	196				
$abbcc$	1	2	8	20	55	128	304				
$abcdd$	1	3	13	42	133	378	1029				
$abcde$	1	4	20	80	300	1020					

Conjecture The results are independent of a, b, c, d, e

The cases $abbbb$ and $aaaab$ are identical, etc.

Nature of $Q^{\nu}_{\lambda\mu}(t)$

- A quasi-polynomial of degree d and quasi-period k
  - $Q^{\nu}_{\lambda\mu}(t) = a_d(t)\, t^d + \cdots + a_1(t)\, t + a_0(t)$ with $a_d(t) \neq 0$
  - $a_i(t)$ is either constant or has some integer period $m|k$
  - lowest common period k
  - $a_i(t) = [c_0, c_1, \ldots, c_{m-1}]_m$ signifying that
    $a_i(t) = c_r$ for all $t \equiv r \pmod m$ for $r = 0, 1, \ldots, m-1$

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- Ex $Q^{1^3}_{21,1^3}(t) = \dfrac{1}{6}\left(t + [6, -1, 4, 3, 2, 1]_6\right)$
- Ex $Q^{1^4}_{31,1^4}(t) = \dfrac{1}{48}\left(t^2 + [12, 6]_2\, t + [48, -7, 20, 21, 32, -7, 36, 5, 32, 9, 20, 5]_{12}\right)$

Observations contd.

Generating function:

- $G^{\nu}_{\lambda\mu}(z) \;=\; \sum_{t=0}^{\infty} Q^{\nu}_{\lambda\mu}(t)\, z^t \;=\; N(z)/D(z)$
  - $D(z) = (1-z^{p_1})^{n_1}\,(1-z^{p_2})^{n_2}\cdots(1-z^{p_m})^{n_m}$
    with all $n_j > 0$ and $\sum_{j=1}^{m} n_j = d+1$
  - $k = \mathrm{lcm}\{p_1, p_2, \ldots, p_m\}$,
  - $N(z) = \sum_{i=0}^{n} b_i\, z^i$ is a polynomial of degree $n \leq d$
  - $b_0 = 1$ so that $G^{\nu}_{\lambda\mu}(0) = Q^{\nu}_{\lambda\mu}(0) = 1$
  - Conjecture b_i is an integer for all i

Observations contd.

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  - Conjecture b_i is an integer for all i

Ex: For $n = 6$, $\lambda = (51)$, $\mu = (321)$, $\nu = (321)$, $g^{\nu}_{\lambda\mu} = 2$,
degree $d = 5$, quasi-period $k = 6$.

$$G^{\nu}_{\lambda\mu}(z) = \frac{1 - z + z^2}{(1-z)^3(1-z^2)^2(1-z^3)}$$

Stretched reduced Kronecker coefficients

- reduced coeffs $\overline{g}_{\rho\sigma}^{\tau}$, stretched reduced coeffs $\overline{g}_{t\rho,t\sigma}^{t\tau}$.
- Theorem $\overline{Q}_{\rho\sigma}^{\tau}(t) := \overline{g}_{t\rho,t\sigma}^{,t\tau}$ is quasi-polynomial in t.

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- Theorem $\overline{Q}_{\rho\sigma}^{\tau}(t) := \overline{g}_{t\rho,t\sigma}^{,t\tau}$ is quasi-polynomial in t.

Ex: $\rho = (1)$, $\sigma = (21)$, $\nu = (21)$, $\overline{g}_{1,21}^{21} = 2$, $\overline{G}_{1,21}^{21}(t)$

$$= \frac{1}{4320}(t+6)(3t^4 + 42t^3 + 238t^2 + 612t + 720) \quad t \equiv 0(mod6)$$

$$= \frac{1}{4320}(t+5)(3t^4 + 45t^3 + 265t^2 + 715t + 412) \quad t \equiv 1(mod6)$$

$$= \frac{1}{4320}(t+4)(3t^4 + 48t^3 + 298t^2 + 848t + 1000) \quad t \equiv 2(mod6)$$

$$= \frac{1}{4320}(t+3)(3t^4 + 51t^3 + 337t^2 + 1029t + 900) \quad t \equiv 3(mod6)$$

$$= \frac{1}{4320}(t+2)(3t^4 + 54t^3 + 382t^2 + 1276t + 1840) \quad t \equiv 4(mod6)$$

$$= \frac{1}{4320}(t+1)(t+4)(t+7)(3t^2 + 24t + 85) \quad t \equiv 5(mod6)$$

Example continued

Alternatively,

$$\overline{Q}_{1,21,21}(t) \;=\; \frac{1}{4320}\Big(3\,t^5+60\,t^4+490\,t^3+2043\,t^2 \\ +\;[4392,3987]_2\,t \\ +\;[4320,2060,4000,2700,3680,2380]_6\Big)$$

where each $[c_0,c_1,\ldots,c_{m-1}]_m$ signifies that the coefficients take the values c_r for $t\equiv r(\mathrm{mod}\,m)$ and $r=0,1,\ldots,m$

Example continued

Alternatively,

$$\overline{Q}_{1,21,21}(t) \;=\; \frac{1}{4320}\Big(3\,t^5 + 60\,t^4 + 490\,t^3 + 2043\,t^2 + [4392, 3987]_2\,t + [4320, 2060, 4000, 2700, 3680, 2380]_6\Big)$$

where each $[c_0, c_1, \ldots, c_{m-1}]_m$ signifies that the coefficients take the values c_r for $t \equiv r (\operatorname{mod} m)$ and $r = 0, 1, \ldots, m$

Note: Degree $d = 5$, and quasi-period $k = 6$.

Observations

Stretched reduced Kronecker coefficients:

- $\overline{Q}^{\tau}_{\rho\sigma}(t) = \sum_{i=0}^{d} a_i(t)\, t^i$ is a quasi-polynomial of degree d
  - $a_0(0) = 1$ so that $\overline{Q}^{\tau}_{\rho\sigma}(0) = 1$
  - each $a_i(t)$ has integer period
  - lowest common period k
  - for all $t \equiv m (\operatorname{mod} k)$ all $a_i(t)$ are constant and $\overline{Q}^{\tau}_{\rho\sigma}(t)$ is a polynomial
  - Conjecture all $a_i(t) \geq 0$
  - for some m and $t \equiv -m (\operatorname{mod} k)$ $\overline{Q}^{\tau}_{\rho\sigma}(t)$ contains factor $(t+m)$ Problem - predict which m
- Ex. $\overline{Q}^{21}_{1,21}(t) = 0$ for $t = -1, -2, -3, -4, -5, -6, -7$.

Data on $\overline{Q}^{\tau}_{1,\sigma}(t)$ with $a > b > 0$

$\sigma = \tau$	N	$N * \overline{Q}^{\sigma}_{1,\sigma}(t)$
a	$1/2$	t $+[2, 1]_2$
aa	$1/72$	$t^3 + 12t^2$ $+t\,[48, 39]_2$ $+[72, 20, 64, 36, 56, 28]_6$
ab	$1/4320$	$3t^5 + 60t^4 + 490t^3 + 2043t^2$ $+t\,[4392, 3987]_2$ $+[4320, 2060, 4000, 2700, 3680, 2380]_6$

Data on $\overline{Q}^{\tau}_{1,\sigma}(t)$ with $a > b > c > 0$

$\sigma = \tau$	N	$N * \overline{Q}^{\sigma}_{1,\sigma}(t)$
aaa	$1/103680$	$6t^5 + 225t^4 + 3160t^3 + t^2\,[*,*]_2$ $+t\,[*,*,*,*,*,*]_6 + [*,*,\cdots,*]_{12}$
aab	$1/209018880$	$2t^9 + 135t^8 + 4092t^7 + 72900t^6 + 845712t^5$ $+t^4\,[*,*]_2 + t^3\,[*,*]_2 + t^2\,[*,*]_2$ $+t\,[*,*,*,*,*,*]_6 + [*,*,\cdots,*]_{12}$
abc	$1/45984153600$	$12t^{11} + 990t^{10} + 36740t^9 + 809325t^8$ $+11770176t^7 + 11897424t^6 + 85655328t^5$ $+t^4\,[*,*]_2 + t^3\,[*,*]_2 + t^2\,[*,*]_2$ $+t\,[*,*,*,*,*,*]_6 + [*,*,\cdots,*]_{12}$

Data on $\overline{Q}^{\tau}_{1,\sigma}(t)$ with $a > b > c > d > 0$

$\sigma = \tau$	N	$N * \overline{Q}^{\sigma}_{1,\sigma}(t)$
$aaaa$	$1/1451520$	$t^7 + 84t^6 + 2877t^5 + 51660t^4$ $+t^3\,[*,*]_2 + t^2\,[*,*]_2$ $+t\,[*,*,*,*,*,*]_{12} + [*,*,\cdots,*]_{60}$

Observations contd.

Generating function:

- $\overline{G}^{\,\tau}_{\rho\sigma}(z) \; = \; \sum_{t=0}^{\infty} \overline{Q}^{\,\tau}_{\rho\sigma}(t)\, z^t \; = \; N(z)/D(z)$
  - $D(z) = (1 - z^{p_1})^{n_1} (1 - z^{p_2})^{n_2} \cdots (1 - z^{p_m})^{n_m}$
    with all $n_j > 0$ and $\sum_{j=1}^{m} n_j = d + 1$
  - $k = \mathrm{lcm}\{p_1, p_2, \ldots, p_m\}$,
  - $N(z) = \sum_{i=0}^{n} b_i\, z^i$ is a polynomial of degree $n \leq d$
  - $b_0 = 1$ so that $\overline{G}^{\,\tau}_{\rho\sigma}(0) = \overline{Q}^{\,\tau}_{\rho\sigma}(0) = 1$
  - Conjecture b_i is an integer for all i

Observations contd.

Generating function:

- $\overline{G}^{\,\tau}_{\rho\sigma}(z) \;=\; \sum_{t=0}^{\infty} \overline{Q}^{\,\tau}_{\rho\sigma}(t)\, z^t \;=\; N(z)/D(z)$
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  - Conjecture b_i is an integer for all i

Ex: $\rho = (1)$, $\sigma = (21)$, $\tau = (21)$, $\overline{g}^{21}_{1,21} = 2$,

degree $d = 5$, period $k = 6$.

$$\overline{G}^{21}_{1,21}(z) = \frac{1 - z + z^2}{(1-z)^3(1-z^2)^2(1-z^3)}$$

Further data on $\overline{G}^{\tau}_{1,\sigma}(z)$ with $a > b > c > 0$

ρ	$\sigma = \tau$	$\overline{G}^{\sigma}_{1,\sigma}(z)$	d	k
1	a	$1/(1-z)(1-z^2)$	1	2
1	aa	$1/(1-z)(1-z^2)^2(1-z^3)$	3	6
1	ab	$(1-z+z^2)$		
		$/(1-z)^3(1-z^2)^2(1-z^3)$	5	6
1	aaa	$1/(1-z)(1-z^2)^2(1-z^3)^2(1-z^4)$	5	12
1	abb	$(1-z+2z^3-z^5+z^6)$		
		$/(1-z)^3(1-z^2)^4(1-z^3)^2(1-z^4)$	9	12
1	abc	$(1-z+z^2)(1-z+z^2+4z^3+z^4-z^5+z^6)$		
		$/(1-z)^5(1-z^2)^4(1-z^3)^2(1-z^4)$	11	12
1	$aaaa$	$1/(1-z)(1-z^2)^2(1-z^3)^2(1-z^4)^2(1-z^5)$	7	60

Data on $\overline{G}^{\tau}_{6,5}(z)$ and $\overline{Q}^{\tau}_{6,5}(t)$

τ	$\overline{G}^{\tau}_{6,5}(z)$	$\overline{Q}^{\tau}_{6,5}(t)$
11	$1/(1-z)$	1
10	$1/(1-z)(1-z^2)$	$(t+[2,1]_2)/2$
9	$1/(1-z)^2$	$t+1$
8	$(1+z+z^2)/(1-z)(1-z^2)$	$(3t+[2,1]_2)/2$
7	$(1+z)/(1-z)^2$	$2t+1$
6	$(1+2z+2z^2)/(1-z)(1-z^2)$	$(5t+[2,1]_2)/2$
5	$(1+z)/(1-z)^2$	$2t+1$
4	$1/(1-z)$	1
3	$1/(1-z^2)$	$t+1$
2	$1/(1-z)(1-z^2)$	$(t+[2,1]_2)$
1	$1/(1-z)$	1

Data on $\overline{G}^{\tau}_{6,5}(z)$ and $\overline{Q}^{\tau}_{6,5}(t)$

τ	$\overline{G}^{\tau}_{6,5}(z)$	$\overline{Q}^{\tau}_{6,5}(t)$
92	$1/(1-z)$	1
82	$1/(1-z)^2(1-z^2)$	$(t^2+4t+[4,3]_2)4$
72	$(1+z)/(1-z)^3)$	t^2+2t+1
62	$(1+3z+4z^2)/((1-z)^2(1-z^2))$	$(4t^2+5t+[2,1]_2)/2$
52	$(1+3z+3z^2)/(1-z)^3)$	$(5t^2+5t+2)/2$
42	$(1+2z+4z^2)/((1-z)^2(1-z^2))$	$(7t^2+8t+[4,1]_2)/4$
32	$(1+z+z^2)/((1-z)^2(1-z^2))$	$(3t^2+6t+[4,1]_2)/4$
22	$1/(1-z)$	1

Data on $\overline{G}^{\tau}_{6,5}(z)$ and $\overline{Q}^{\tau}_{6,5}(t)$

τ	$\overline{G}^{\tau}_{6,5}(z)$	$\overline{Q}^{\tau}_{6,5}(t)$
621	$1/((1-z)^2(1-z^2))$	$(t^2+4t+[4,1]_2)/4$
531	$1/(1-z)^3$	$(t^2+3t+[2,1]_2)/2$
521	$(1+z+z^2)/((1-z)^2(1-z^2))$	$(3t^2+6t+[4,1]_2)/4$
432	$1/(1-z)$	1

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531	$1/(1-z)^3$	$(t^2+3t+[2,1]_2)/2$
521	$(1+z+z^2)/((1-z)^2(1-z^2))$	$(3t^2+6t+[4,1]_2)/4$
432	$1/(1-z)$	1

Summary

- $\overline{Q}^{\tau}_{r,s}(t)$ all of degree ≤ 2 and quasi-period ≤ 2
- $\overline{Q}^{\tau}_{r,s}(t) = (At^2 + Bt + [4, C]_2)/4$ Conjecture $A, B, C \in \mathbb{Z}_{\geq 0}$
- $\overline{G}^{\tau}_{r,s}(z) = (1 + pz + qz^2)/((1-z)^d(1-z^2)^{deg-d+1})$

Stability of stretched Kronecker coeficients

λ	μ	ν	n	0	1	2	3	4	5	6	7	8	9
66	75	642	$n = 12$	1	0	2	1	3	2	4	3	5	4
76	85	742	$n = 13$	1	2	6	10	17	24	34	44		
86	95	842	$n = 14$	1	3	10	18	31	45	64			
$\langle 6 \rangle$	$\langle 5 \rangle$	$\langle 42 \rangle$	$n \geq 15$	1	4	12	22	37	54	76	100	129	

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$\langle 6\rangle$	$\langle 5\rangle$	$\langle 42\rangle$	$n \geq 15$	1	4	12	22	37	54	76	100	129	

λ	μ	ν	$G(z)$
66	75	642	$(1-z+z^2)/(1-z)(1-z^2)$
76	85	742	$(1+2z^2)/(1-z)^2(1-z^2)$
86	95	842	$(1+z+4z^2)/(1-z)^2(1-z^2)$
$\langle 6\rangle$	$\langle 5\rangle$	$\langle 42\rangle$	$(1+2z+4z^2)/(1-z)^2(1-z^2)$

Stability of stretched Kronecker coeficients

λ	μ	ν	$Q(t)$
66	75	642	$(t + [2, -1]_2)/2$
76	85	742	$(3t^2 + 4t + [4, 1]_2)/4$
86	95	842	$(3t^2 + 3t + [2, 0]_2)/2$
$\langle 6 \rangle$	$\langle 5 \rangle$	$\langle 42 \rangle$	$(7t^2 + 8t + [4, 1]_2)/4$

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λ	μ	ν	$Q(t)$
66	75	642	$(0t^2 + 1t + [4, -2]_2)/4$
76	85	742	$(3t^2 + 4t + [4, 1]_2)/4$
86	95	842	$(6t^2 + 6t + [4, 0]_2)/4$
$\langle 6 \rangle$	$\langle 5 \rangle$	$\langle 42 \rangle$	$(7t^2 + 8t + [4, 1]_2)/4$

Stability of Kronecker coeficients

Stability

- Let $\lambda = (n-r, \rho)$, $\mu = (n-s, \sigma)$, $\nu = (n-t, \tau)$ with $\rho \vdash r$, $\sigma \vdash s$, $\tau \vdash t$.
- There exists $b \in \mathbb{N}$ such that $g_{\lambda\mu}^{\nu} = \overline{g}_{\rho\sigma}^{\tau}$ for all $n \geq b$.
- Problem determine b

Stability of Kronecker coeficients

Stability

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- There exists $b \in \mathbb{N}$ such that $g_{\lambda\mu}^{\nu} = \overline{g}_{\rho\sigma}^{\tau}$ for all $n \geq b$.
- Problem determine b

Example

- $\langle 6 \rangle * \langle 5 \rangle = \cdots + 4\langle 42 \rangle + 6\langle 52 \rangle + 5\langle 62 \rangle + 4\langle 72 \rangle + 2\langle 82 \rangle + \langle 92 \rangle + \cdots$
- $\langle 6 \rangle * \langle 42 \rangle = \cdots + 4\langle 5 \rangle + 6\langle 6 \rangle + \cdots + 2\langle 9 \rangle + \langle 10 \rangle + \cdots$
- $\langle 5 \rangle * \langle 42 \rangle = \cdots + 4\langle 6 \rangle + 4\langle 7 \rangle + \cdots + 2\langle 8 \rangle + \langle 9 \rangle + \cdots$
- $\langle 6 \rangle * \langle 5 \rangle * \langle 42 \rangle = 4\langle 0 \rangle + 39\langle 1 \rangle + \cdots + 3\langle 15 \rangle + \cdots$
- $b = 6 + 9 = 5 + 10 = 6 + 9 = 15$