Computational Aspects of Invariants of Multipartite Quantum Systems

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Main Problem

Characterize the non-local properties of quantum states.

Various approaches

- entanglement measures:
  (real) functions on the state space, *e. g.* distance to product/separable states

- local equivalence:
  Given two quantum states

$$|\psi\rangle \text{ and } |\phi\rangle \qquad (\rho \text{ and } \rho')$$

on n particles (qudits), is there a local *unitary*[a] transformation $U = U_1 \otimes U_2 \otimes \ldots \otimes U_n$ with

$$U|\psi\rangle = |\phi\rangle \qquad (U\rho U^{-1} = \rho')?$$

[a] We do not consider SLOCC here.

Our Approach

Use the polynomial invariants of the groups

- $SU(d_1) \otimes \ldots \otimes SU(d_n)$
- $U(d_1) \otimes \ldots \otimes U(d_n)$

operating on

- pure states $|\psi\rangle$
- mixed states ρ

to describe multi-particle entanglement.

This gives a *complete* description:

Theorem:
The orbits of a compact linear group acting in a *real* vector space are separated by the (polynomial) invariants.

(A. L. Onishchik, *Lie groups and algebraic groups*, Springer, 1990, Ch. 3, §4)

Operation of $GL(d, \mathbb{K})$

pure quantum states:

linear operation on polynomials $f \in \mathbb{K}[x_1, \ldots, x_d] =: \mathbb{K}[\boldsymbol{x}]$

$$f(\boldsymbol{x})^g := f(\boldsymbol{x}^g) \qquad \text{where } \boldsymbol{x}^g = (x_1, \ldots, x_d) \cdot g \text{ and } g \in GL(d, \mathbb{K})$$

mixed quantum states:

operation on polynomials $f \in \mathbb{K}[x_{11}, \ldots, x_{dd}] =: \mathbb{K}[X]$ via conjugation

$$f(X)^g := f(X^g) \quad \text{where}$$

$$X^g = g^{-1} \cdot \begin{pmatrix} x_{11} & \cdots & x_{1d} \\ \vdots & \ddots & \vdots \\ x_{d1} & \cdots & x_{dd} \end{pmatrix} \cdot g$$

Polynomial Invariants

Properties of $\mathbb{K}[\boldsymbol{x}]^G := \{f(\boldsymbol{x}) \in \mathbb{K}[\boldsymbol{x}] | \forall g \in G\colon f(\boldsymbol{x})^g = f(\boldsymbol{x})\}$

- Homogeneous polynomials remain homogeneous
  $\Longrightarrow$ homogeneous generators.
- Any linear combination of invariants is an invariant.
- The product of invariants is an invariant.
- For reductive groups $\mathbb{K}[\boldsymbol{x}]^G$ is finitely generated.
- Some invariants are algebraically independent (primary invariants).
- The other invariants obey some polynomial relations.
- In special cases: the invariant ring can be decomposed as a free module (generated by the secondary invariants) over the primary invariants.

Reynolds Operator

finite groups

$$\begin{array}{rccc} R_G\colon & \mathbb{K}[\boldsymbol{x}] & \to & \mathbb{K}[\boldsymbol{x}]^G \\ & f(\boldsymbol{x}) & \mapsto & \displaystyle\frac{1}{|G|}\sum_{g\in G} f(\boldsymbol{x})^g \end{array}$$

R_G is a linear projection operator

$\Rightarrow$ compute $R_G(\boldsymbol{m})$ for all monomials $\boldsymbol{m} \in \mathbb{K}[\boldsymbol{x}]$ of degree $k = 1, 2, \ldots$

compact groups

$$\begin{array}{rccc} R_G\colon & \mathbb{K}[\boldsymbol{x}] & \to & \mathbb{K}[\boldsymbol{x}]^G \\ & f(\boldsymbol{x}) & \mapsto & \displaystyle\int_{g\in G} f(\boldsymbol{x})^g d\mu_G(g) \end{array}$$

where $\mu_G(g)$ is the normalized Haar measure of G

Problem: computing the integral is very difficult

Invariant Polynomials and Commuting Matrices

Every homogeneous polynomial $f(X) \in \mathbb{K}[x_{11}, \ldots, x_{dd}]$ of degree k can be expressed as

$$f_F(X) := \operatorname{tr}(F \cdot X^{\otimes k}) \quad \text{where } F \in \mathbb{K}^{kd \times kd}$$

(since $X^{\otimes k}$ contains all monomials of degree k).

Example ($n = 2$, $k = 2$):

$$X = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$$

$$X^{\otimes 2} = \begin{pmatrix} x_{11}^2 & x_{11}x_{12} & x_{12}x_{11} & x_{12}^2 \\ x_{11}x_{21} & x_{11}x_{22} & x_{12}x_{21} & x_{12}x_{22} \\ x_{21}x_{11} & x_{21}x_{12} & x_{22}x_{11} & x_{22}x_{12} \\ x_{21}^2 & x_{21}x_{22} & x_{22}x_{21} & x_{22}^2 \end{pmatrix}$$

Invariant Polynomials and Commuting Matrices

$$\begin{aligned} f_F(X)^g &= \mathrm{tr}(F \cdot (g^{-1} \cdot X \cdot g)^{\otimes k}) \\ &= \mathrm{tr}(F \cdot (g^{-1})^{\otimes k} \cdot X^{\otimes k} \cdot g^{\otimes k}) \\ &= \mathrm{tr}(g^{\otimes k} \cdot F \cdot (g^{-1})^{\otimes k} \cdot X^{\otimes k}) \\ &= \mathrm{tr}(F^{(g^{-1})^{\otimes k}} \cdot X^{\otimes k}) \end{aligned}$$

$$f_F(X)^g = f_F(X) \Longleftrightarrow f_F(X) = f_{F'}(X) \quad \text{and} \quad F' \cdot g^{\otimes k} = g^{\otimes k} \cdot F'$$

transformed question

Which matrices commute with each $g^{\otimes k}$ for $g \in G$?

R. Brauer (1937):

The algebra $\mathcal{A}_{d,k}$ of matrices that commute with each $U^{\otimes k}$ for $U \in U(d)$ is generated by a certain representation of S_k.

One Particle

- Hilbert space $\mathcal{H}$ of dimension d
- $G = U(d)$
- representation of S_k:
  S_k operates on a tensor product of k Hilbert spaces $\mathcal{H}_i$ of dimension d by permuting the spaces:

$$T_{d,k}(\pi) \cdot (\mathcal{H}_1 \otimes \ldots \otimes \mathcal{H}_k) = \mathcal{H}_{\pi(1)} \otimes \ldots \otimes \mathcal{H}_{\pi(k)}$$

- "permuting k copies of $\mathcal{H}$"

N Particles

- Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \ldots \otimes \mathcal{H}_N$
- $G = U(d)^{\otimes N}$, $g = U_1 \otimes \ldots \otimes U_N$,
  $g^{\otimes k} = (U_1 \otimes \ldots \otimes U_N) \otimes \ldots \otimes (U_1 \otimes \ldots \otimes U_N)$
- N permutations $\pi_\nu \in S_k$
- representation of $(S_k)^N$:
  $\pi = (\pi_1, \ldots, \pi_N)$, π_ν permutes the copies of the ν^{th} particle:

$$T_{d,k}^{(N)}(\pi) \cdot \big((\mathcal{H}_{1,1} \otimes \ldots \otimes \mathcal{H}_{N,1}) \otimes \ldots \otimes (\mathcal{H}_{1,k} \otimes \ldots \otimes \mathcal{H}_{N,k})\big) =$$

$$(\mathcal{H}_{1,\pi_1(1)} \otimes \ldots \otimes \mathcal{H}_{N,\pi_N(1)}) \otimes \ldots \otimes (\mathcal{H}_{1,\pi_1(k)} \otimes \ldots \otimes \mathcal{H}_{N,\pi_N(k)})\big)$$

Computing Invariants

(see E. Rains, quant-ph/9704042[a]; Grassl et al., quant-ph/9712040[b])

Computing the homogeneous polynomial invariants of degree k for an N particle system with density operator ρ:

for each N tuple $\pi = (\pi_1, \ldots, \pi_N)$ of permutations $\pi_\nu \in S_k$ compute

$$f_{\pi_1,\ldots,\pi_N}(\rho_{ij}) := \operatorname{tr}\left(T_{d,k}^{(N)}(\pi) \cdot \rho^{\otimes k}\right)$$

- all homogeneous polynomial invariant of degree k
- in general, $(k!)^N$ invariants to compute
- not necessarily linearly independent, not even distinct
- it is sufficient to consider certain tuples of permutations

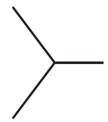
[a] IEEE Transactions on Information Theory, vol. 46, no. 1, pp. 54–59 (2000)
[b] Physical Review A 58, 1833–1839 (1998)

Invariant Tensors

- use local basis for the density matrix:

$$\rho = \frac{1}{4} I + \sum_{i=x,y,z} s_i \sigma_i \otimes I + \sum_{j=x,y,z} p_j I \otimes \sigma_j + \sum_{i,j=x,y,z} \beta_{ij} \sigma_i \otimes \sigma_j$$

- $SU(2) \otimes SU(2)$ acts as $SO(3) \times SO(3)$ on the coefficient vectors s, p and the coefficient matrix β
- contract copies of the coefficient tensors with tensors that are invariant under $SO(3)$ resp. $SO(3) \times SO(3)$

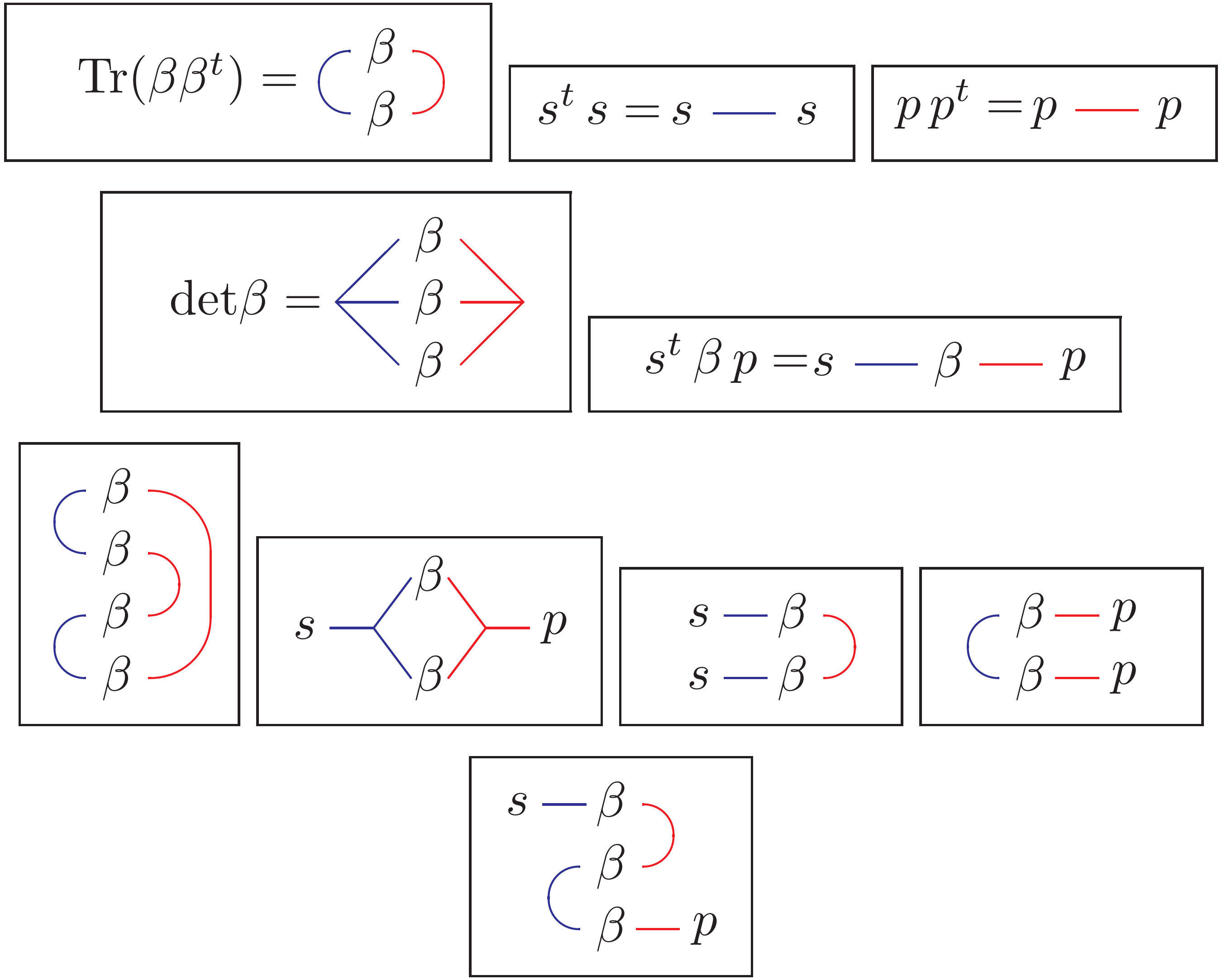
δ_{ij}	inner product	
ϵ_{ijk}	determinant	

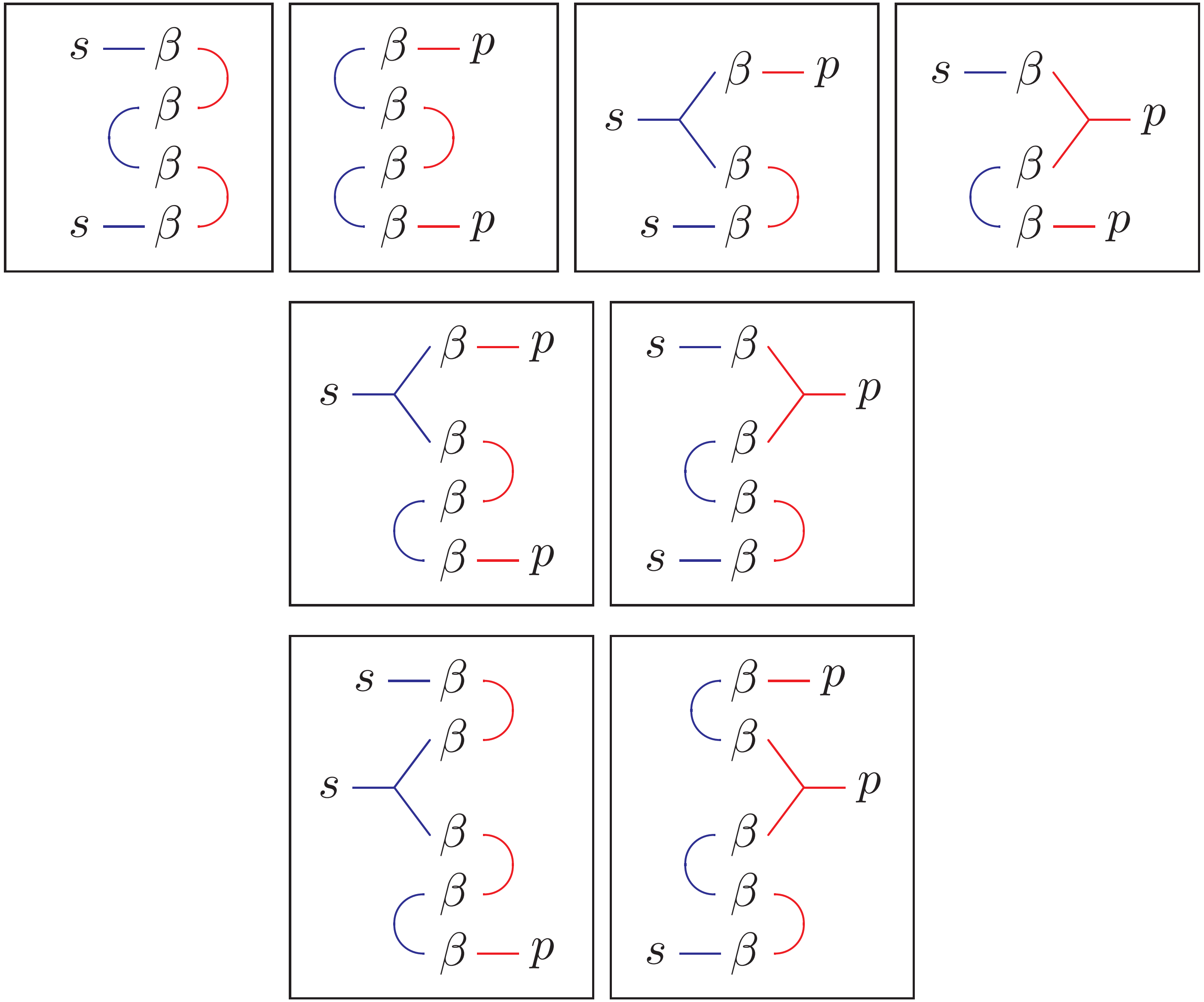
- create all possible contractions modulo the relations of the tensors

for two qubits, there is only a finite number of such contractions

$\Longrightarrow$ complete set of invariants, resp. a set of generators for all invariants

Fundamental Invariants (I)

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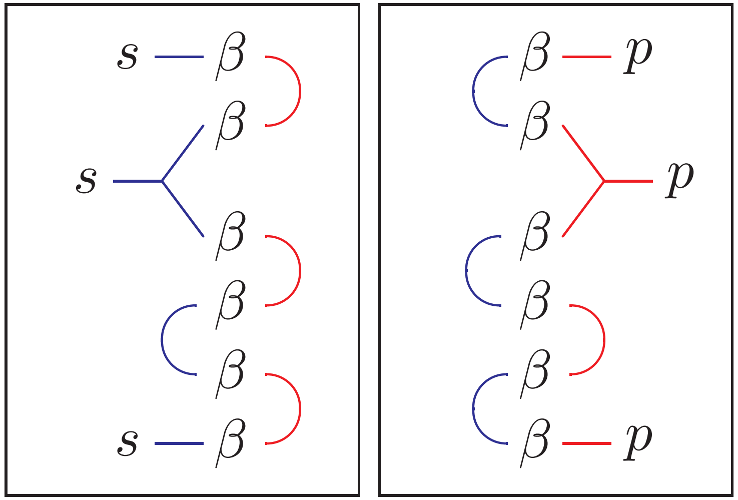


References

Makhlin, Nonlocal properties of two-qubit gates and mixed states and optimization of quantum computations, Quantum Info. Proc. 1, 243–252, (2000).

Grassl, Entanglement and Invariant Theory, Quantum Computation and Information Seminar, UC Berkeley, 19.11.2002.

King, Welsh & Jarvis, The mixed two-qubit system and the structure of its ring of local invariants, J. Phys. A. 40, 10083–10108 (2007).

Hilbert Series

- encodes the vector space dimension d_k of the homogeneous invariants of degree k as a formal power series with non–negative integer coefficients:

$$M(z) := \sum_{k\geq 0} d_k z^k \in \mathbb{Z}[\![z]\!]$$

- a rational function (for finitely generated algebras)
- general formula (for linear operation)

$$M(z) = \int_{g\in G} \mathrm{d}\mu_G(g) \frac{1}{\det(id - z\cdot g)}$$

1. applies only to the case of linear operation
   $\Longrightarrow$ "linearize" the operation by conjugation via the adjoint representation
2. integral is very difficult to compute

Hilbert Series via Kronecker Coefficients

(see King *et al.*)

- the number of invariants d_m of degree m corresponds to the multiplicity of the trivial representation in the m-th symmetric power
- via branching rules for the restricted representation one obtains

$$d_m = \sum_{\lambda \vdash m; \ell(\lambda) \leq 4} \left(\sum_{\mu \vdash m; \ell(\mu) \leq 2} k_{\mu\mu}^{\lambda} \right)^2$$

$$= \sum_{\lambda \vdash m; \ell(\lambda) \leq 4} \left(\sum_{\mu,\nu \vdash m; \ell(\mu),\ell(\nu) \leq 2} (k_{\mu\nu}^{\lambda})^2 \right)$$

Example: Two Qubits

pure state

$$|\psi\rangle = x_{00}|00\rangle + x_{01}|01\rangle + x_{10}|10\rangle + x_{11}|11\rangle$$

Invariants

$$\operatorname{tr}(|\psi\rangle\langle\psi|) = x_{00}\overline{x}_{00} + x_{01}\overline{x}_{01} + x_{10}\overline{x}_{10} + x_{11}\overline{x}_{11}$$

$$\begin{aligned}\operatorname{tr}\left((\operatorname{tr}_i|\psi\rangle\langle\psi|)^2\right) &= x_{00}^2\overline{x}_{00}^2 + x_{01}^2\overline{x}_{01}^2 + x_{10}^2\overline{x}_{10}^2 + x_{11}^2\overline{x}_{11}^2 \\ &\quad +2x_{00}x_{01}\overline{x}_{00}\overline{x}_{01} + 2x_{00}x_{10}\overline{x}_{00}\overline{x}_{10} + 2x_{00}x_{11}\overline{x}_{01}\overline{x}_{10} \\ &\quad +2x_{01}x_{10}\overline{x}_{00}\overline{x}_{11} + 2x_{01}x_{11}\overline{x}_{01}\overline{x}_{11} + 2x_{10}x_{11}\overline{x}_{10}\overline{x}_{11}\end{aligned}$$

Problem

We have to introduce new variables which are the “complex conjugated variables”.

Multivariate Hilbert Series

- operation on polynomials $f(\boldsymbol{x}, \overline{\boldsymbol{x}})$ in variables x_i and $\overline{x}_i$ with the representation $g \oplus \overline{g}$
- bi-degree

$$(\deg_{x_1,\ldots,x_d} f,\ \deg_{\overline{x}_1,\ldots,\overline{x}_d} f)$$

- invariant ring admits bi-graduation with Hilbert series

$$M(z, \overline{z}) := \sum_{k,\ell \geq 0} d_{k,\ell}\, z^k\, \overline{z}^\ell \in \mathbb{Z}[\![z, \overline{z}]\!]$$

- general formula (for linear operation)

$$M(z, \overline{z}) = \int_G \mathrm{d}\mu_G(g) \frac{1}{\det(id - z \cdot g)} \frac{1}{\det(id - \overline{z} \cdot \overline{g})}$$

Three Qubits: Ansatz for Series of $SU(2)^{\otimes 3}$

$$H_{SU}(\overline{z}, z) = \int_{U\in G} \mathrm{d}\mu_G(U) \frac{1}{\det(id - z\cdot U)} \frac{1}{\det(id - \overline{z}\cdot U^t)}$$

$$= \frac{1}{(2\pi i)^3} \oint_{\Gamma_v} \oint_{\Gamma_w} \oint_{\Gamma_x} \frac{(1-v^2)(1-w^2)(1-x^2)}{\prod_{a,b,c\in\{1,-1\}} (1 - z\cdot v^a w^b x^c)\,(1-\overline{z}\cdot v^a w^b x^c)} \frac{\mathrm{d}v}{v}\frac{\mathrm{d}w}{w}\frac{\mathrm{d}x}{x}$$

($G = SU(2)^{\otimes 3}$, $U = U_1 \otimes U_2 \otimes U_3$, $\Gamma =$complex unit circle)

Computation of the integral using the theorem of residues

- symbolic computation of singularities and residues
- data type: factored rational functions implemented in MAGMA
  (back in 1997, Maple fails: "object too large")

Three Qubits: Series for $SU(2)^{\otimes 3}$ and $U(2)^{\otimes 3}$

$$H_{SU}(z,\overline{z}) = \frac{z^5\overline{z}^5 + z^3\overline{z}^3 + z^2\overline{z}^2 + 1}{(1-z\overline{z})(1-z^4)(1-\overline{z}^4)(1-z^2\overline{z}^2)^2(1-z\overline{z}^3)(1-z^3\overline{z})}$$

$$= 1 + z\overline{z} + z^4 + z^3\overline{z} + 4z^2\overline{z}^2 + z\overline{z}^3 + \overline{z}^4 + z^5\overline{z} + z^4\overline{z}^2 + 5z^3\overline{z}^3 + z^2\overline{z}^4 + z\overline{z}^5$$
$$+ z^8 + z^7\overline{z} + 5z^6\overline{z}^2 + 5z^5\overline{z}^3 + 12z^4\overline{z}^4 + 5z^3\overline{z}^5 + 5z^2\overline{z}^6 + z\overline{z}^7 + \overline{z}^8$$
$$+ z^9\overline{z} + z^8\overline{z}^2 + 6z^7\overline{z}^3 + 6z^6\overline{z}^4 + 15z^5\overline{z}^5 + z\overline{z}^9 + z^2\overline{z}^8 + 6z^3\overline{z}^7 + 6z^4\overline{z}^6$$
$$+ z^{12} + z^{11}\overline{z} + 5z^{10}\overline{z}^2 + 6z^9\overline{z}^3 + 16z^8\overline{z}^4 + 16z^7\overline{z}^5 + 30z^6\overline{z}^6$$
$$+ \overline{z}^{12} + z\overline{z}^{11} + 5z^2\overline{z}^{10} + 6z^3\overline{z}^9 + 16z^4\overline{z}^8 + 16z^5\overline{z}^7$$
$$+ \ldots$$

$$H_U(z) = \frac{z^{12}+1}{(1-z^2)(1-z^4)^3(1-z^6)(1-z^8)}$$
$$= 1 + z^2 + 4z^4 + 5z^6 + 12z^8 + 15z^{10} + 30z^{12} + 37z^{14} + 65z^{16} + 80z^{18}$$
$$+ 128z^{20} + 156z^{22} + 234z^{24} + 282z^{26} + 402z^{28} + 480z^{30} + \ldots$$

Three Qubits: Invariant Ring of $SU(2)^{\otimes 3}$

Coefficient vector:

$$\boldsymbol{x} = \Big(\underbrace{x_{000}, x_{001}}_{00}, \underbrace{x_{010}, x_{011}}_{01}, \underbrace{x_{100}, x_{101}}_{10}, \underbrace{x_{110}, x_{111}}_{11}\Big)$$

Invariants of $I_4 \otimes SU(2)$:

brackets	$[i,j]$	$:=$	$x_{i0}x_{j1} - x_{i1}x_{j0}$	invariant of $SL(2) \supset SU(2)$
inner products	$\langle i,j\rangle$	$:=$	$x_{i0}\overline{x}_{j0} + x_{i1}\overline{x}_{j1}$	

Invariants of $U(1) \otimes SU(2) \otimes SU(2) \otimes SU(2)$:

correspond to permutations (π_1, π_2, π_3):

$$f_{\pi_1,\pi_2,\pi_3} = \sum_{i,j,\ldots} x_{i_1,i_2,i_3}\overline{x}_{\pi_1(i_1),\pi_2(i_2),\pi_3(i_3)} \cdot x_{j_1,j_2,j_3}\overline{x}_{\pi_1(j_1),\pi_2(j_2),\pi_3(j_3)} \cdot \ldots$$

Three Qubits: Invariant Ring of $SU(2)^{\otimes 3}$

Generators:

	bi-degree	permutations (π_1, π_2, π_3), brackets, inner products	#terms
f_1	$(1,1)$	(id, id, id)	8
f_2	$(2,2)$	$((1,2),(1,2),id)$	36
f_3	$(2,2)$	$((1,2),id,(1,2))$	36
s_1	$(4,0)$	$[1,2]^2 - 2[0,1][2,3] - 2[0,2][1,3] + [0,3]^2$	12
$\overline{s_1}$	$(0,4)$	$\overline{[1,2]}^2 - 2\overline{[0,1]}\ \overline{[2,3]} - 2\overline{[0,2]}\ \overline{[1,3]} + \overline{[0,3]}^2$	12
s_2	$(3,1)$	$[3,0]\langle 0,0\rangle - [3,0]\langle 3,3\rangle + [3,1]\langle 0,1\rangle + [3,2]\langle 0,2\rangle$ $+2[3,2]\langle 1,3\rangle - 2[1,0]\langle 2,0\rangle - [1,0]\langle 3,1\rangle - [2,0]\langle 3,2\rangle$ $-[2,1]\langle 0,0\rangle - [2,1]\langle 1,1\rangle + [2,1]\langle 2,2\rangle + [2,1]\langle 3,3\rangle$	40
$\overline{s_2}$	$(1,3)$		40
f_4	$(2,2)$	$(id,(1,2),(1,2))$	36
f_5	$(3,3)$	$((1,2),(2,3),(1,3))$	176
$f_4 f_5$	$(5,5)$		3760

Three Qubits: Invariant Ring of $U(2)^{\otimes 3}$

Generators of the invariant ring:

	degree	permutations (π_1, π_2, π_3)	#terms
f_1	2	(id, id, id)	8
f_2	4	$((1,2),(1,2),id)$	36
f_3	4	$((1,2),id,(1,2))$	36
f_4	4	$(id,(1,2),(1,2))$	36
f_5	6	$((1,2),(2,3),(1,3))$	176
f_6	8	$s_1\overline{s}_1$	144
f_7	12	$\overline{s}_1 s_2^2$	5988

f_1,...,f_6 are algebraic independent; relation for f_7:

$$f_7^2 + c_1(f_1,\ldots,f_6)f_7 + c_0(f_1,\ldots,f_6) \qquad \text{where } c_0, c_1 \in \mathbb{Q}[f_1,\ldots,f_6]$$

completeness can be shown using the fact that there is only one algebraic relation

Four Qubits: Ansatz for Series of $SU(2)^{\otimes 4}$

$$H_{SU}(\overline{z}, z) = \int_{U\in G} \mathrm{d}\mu_G(U) \frac{1}{\det(id - z\cdot U)} \frac{1}{\det(id - \overline{z}\cdot U^t)}$$

$$= \alpha \oint_{\Gamma_u} \oint_{\Gamma_v} \oint_{\Gamma_w} \oint_{\Gamma_x} \frac{(1-u^2)(1-v^2)(1-w^2)(1-x^2)}{\prod\limits_{a,b,c,d\in\{1,-1\}} (1 - z\cdot u^a v^b w^c x^d)\,(1-\overline{z}\cdot u^a v^b w^c x^d)} \frac{\mathrm{d}u}{u}\frac{\mathrm{d}v}{v}\frac{\mathrm{d}w}{w}\frac{\mathrm{d}x}{x}$$

Four Qubits: Hilbert Series of $SU(2)^{\otimes 4}$

$$
\begin{aligned}
H_{SU}(z,\overline{z}) = {} & \big(z^{36}\overline{z}^{36} - z^{35}\overline{z}^{33} + 2z^{34}\overline{z}^{34} + 6z^{34}\overline{z}^{32} + 9z^{34}\overline{z}^{30} + 4z^{34}\overline{z}^{28} + \\
& 3z^{34}\overline{z}^{26} - z^{33}\overline{z}^{35} + 7z^{33}\overline{z}^{33} + 12z^{33}\overline{z}^{31} + \ldots + 12z^{3}\overline{z}^{5} + 7z^{3}\overline{z}^{3} - \\
& z^{3}\overline{z} + 3z^{2}\overline{z}^{10} + 4z^{2}\overline{z}^{8} + 9z^{2}\overline{z}^{6} + 6z^{2}\overline{z}^{4} + 2z^{2}\overline{z}^{2} - z\overline{z}^{3} + 1\big)\Big/ \\
& \big((1-\overline{z}^{6})(1-\overline{z}^{4})(1-\overline{z}^{4})(1-\overline{z}^{2})(1-z^{6})(1-z^{4})(1-z^{4})(1-z^{2}) \\
& (1-z^{3}\overline{z}^{3})(1-z^{2}\overline{z}^{2})^{4}(1-z\overline{z})(1-z^{5}\overline{z})(1-z^{3}\overline{z})^{3}(1-z^{4}\overline{z}^{2}) \\
& (1-\overline{z}^{5}z)(1-\overline{z}^{3}z)^{3}(1-\overline{z}^{4}z^{2})\big) \\
= {} & 1+z^{2}+z\overline{z}+\overline{z}^{2}+3z^{4}+3z^{3}\overline{z}+8z^{2}\overline{z}^{2}+3z\overline{z}^{3}+3\overline{z}^{4}+4z^{6}+6z^{5}\overline{z}+19z^{4}\overline{z}^{2} \\
& +20z^{3}\overline{z}^{3}+19z^{2}\overline{z}^{4}+6z\overline{z}^{5}+4\overline{z}^{6}+7z^{8}+11z^{7}\overline{z}+47z^{6}\overline{z}^{2}+62z^{5}\overline{z}^{3}+98z^{4}\overline{z}^{4} \\
& +62z^{3}\overline{z}^{5}+47z^{2}\overline{z}^{6}+11z\overline{z}^{7}+7\overline{z}^{8}+9z^{10}+18z^{9}\overline{z}+81z^{8}\overline{z}^{2}+150z^{7}\overline{z}^{3} \\
& +278z^{6}\overline{z}^{4}+293z^{5}\overline{z}^{5}+278z^{4}\overline{z}^{6}+150z^{3}\overline{z}^{7}+81z^{2}\overline{z}^{8}+18z\overline{z}^{9}+9\overline{z}^{10} \\
& +14z^{12}+27z^{11}\overline{z}+143z^{10}\overline{z}^{2}+299z^{9}\overline{z}^{3}+669z^{8}\overline{z}^{4}+900z^{7}\overline{z}^{5}+1128z^{6}\overline{z}^{6} \\
& +900z^{5}\overline{z}^{7}+669z^{4}\overline{z}^{8}+299z^{3}\overline{z}^{9}+143z^{2}\overline{z}^{10}+27z\overline{z}^{11}+14\overline{z}^{12} + \ldots
\end{aligned}
$$

Four Qubits: Hilbert Series of $U(2)^{\otimes 4}$

$$
\begin{aligned}
H_U(z) = \Big(&z^{76}+6z^{70}+46z^{68}+110z^{66}+344z^{64}+844z^{62}+2154z^{60}+4606z^{58}+9397z^{56}\\
&+16848z^{54}+28747z^{52}+44580z^{50}+65366z^{48}+88036z^{46}+111909z^{44}\\
&+131368z^{42}+145676z^{40}+149860z^{38}+145676z^{36}+131368z^{34}\\
&+111909z^{32}+88036z^{30}+65366z^{28}+44580z^{26}+28747z^{24}+16848z^{221}\\
&+9397z^{20}+4606z^{18}+2154z^{16}+844z^{14}+344z^{12}+110z^{10}+46z^{8}+6z^{6}\\
&+1\Big) \Big/ \Big(\left(1-z^{10}\right)\left(1-z^{8}\right)^{4}\left(1-z^{6}\right)^{6}\left(1-z^{4}\right)^{7}\left(1-z^{2}\right)\Big)\\
= \ & 1 + z^2 + 8z^4 + 20z^6 + 98z^8 + 293z^{10} + 1\,128z^{12} + 3\,409z^{14}\\
&+ 10\,846z^{16} + 30\,480z^{18} + 84\,652z^{20} + 217\,677z^{22} + 544\,312z^{24}\\
&+ 1\,289\,225z^{26} + 2\,961\,626z^{28} + 6\,528\,284z^{30} + 13\,980\,717z^{32}\\
&+ 28\,963\,980z^{34} + 58\,464\,510z^{36} + 114\,806\,429z^{38} + \ldots
\end{aligned}
$$

Four Qubits: Invariants of $U(2)^{\otimes 4}$

$$
\begin{aligned}
H_U(z) = \; & 1 + z^2 + 8z^4 + 20z^6 + 98z^8 + 293z^{10} + 1\,128z^{12} + 3\,409z^{14} \\
& + 10\,846z^{16} + 30\,480z^{18} + 84\,652z^{20} + 217\,677z^{22} + 544\,312z^{24} \\
& + 1\,289\,225z^{26} + 2\,961\,626z^{28} + 6\,528\,284z^{30} + 13\,980\,717z^{32} \\
& + 28\,963\,980z^{34} + 58\,464\,510z^{36} + 114\,806\,429z^{38} + \ldots
\end{aligned}
$$

intermediate results:

1 invariant of degree 2	
7 invariants of degree 4	these 109 invariants generate a (sub)ring
12 invariants of degree 6	with series
50 invariants of degree 8	$1 + z^2 + 8z^4 + 20z^6 + 98z^8 + 221z^{10} + \ldots$
39 invariants of degree 10	

$\Longrightarrow$ even more invariants are required to generate the whole invariant ring

Relation Ideal

Problem:

Given some invariants $f_1, \ldots, f_m$, do they generate the full invariant ring?

evaluation homorphism: $\mathbb{K}[y_1, \ldots, y_m] \to \mathbb{K}[x_1, \ldots, x_d]$

$$g(y_1, \ldots, y_m) \mapsto g(f_1, \ldots, f_m)$$

relation ideal:

$$\operatorname{Rel}(f_1, \ldots, f_m) = \{g(y_1, \ldots, y_m) \colon g(f_1, \ldots, f_m) = 0\} \trianglelefteq \mathbb{K}[y_1, \ldots, y_m]$$

$$\mathcal{A} = \langle f_1, \ldots, f_m \rangle \cong \mathbb{K}[y_1, \ldots, y_m] / \operatorname{Rel}(f_1, \ldots, f_m)$$

Hilbert series: $\operatorname{Hilb}(\mathcal{A}) = \operatorname{Hilb}(\operatorname{Rel})$

computed (in principle) as

$$\operatorname{Rel}(f_1, \ldots, f_m) = \langle f_1 - y_1, \ldots, f_m - y_m \rangle \cap \mathbb{K}[y_1, \ldots, y_m]$$

SAGBI Bases

Subalgebra **A**nalogue to **G**röbner **B**asis for **I**deals[a]

- basis $B = \{g_1, \ldots, g_\ell\}$ of a subalgebra $\mathcal{A} = \langle f_1, \ldots, f_m \rangle \subset \mathbb{K}[x_1, \ldots, x_n]$
- depends on a term ordering $>$ for polynomials, *e. g.*, lexicographic ordering $x_1 > x_2 > \cdots > x_n$
- the semigroup $\mathrm{LM}(\mathcal{A})$ of leading monomials of $\mathcal{A}$ is generated by $\mathrm{LM}(B)$, *i. e.* $\mathrm{LM}(\mathcal{A}) = \langle \mathrm{LM}(g_1), \ldots, \mathrm{LM}(g_\ell) \rangle$
- allows membership test for $\mathcal{A}$ via top reduction:

$$h \xrightarrow{B} h - c g_{i_1}^{e_1} \cdots g_{i_k}^{e_k} \quad \text{if } \mathrm{LT}(h) = c\,\mathrm{LT}(g_{i_1})^{e_1} \cdots \mathrm{LT}(g_{i_k})^{e_k}$$

- need not be finite, even if $\mathcal{A}$ is finitely generated

[a] Kapur & Madlener 1989, Robbiano & Sweedler, 1990

Computing SAGBI Bases

semi-algorithm to compute a SAGBI basis

0. set $B \leftarrow \{f_1, \ldots, f_m\}$
1. compute the relation ideal $\mathrm{Rel}(\mathrm{LM}(B))$ of the leading monomials of B
2. for all generators $r(y_1, \ldots, y_m)$ of $\mathrm{Rel}(\mathrm{LM}(B))$, compute $r(f_1, \ldots, f_m) \xrightarrow{B} h$
3. if $h \neq 0$, add h to B
4. repeat from Step 1. until no new element has been added to B

Computing SAGBI Bases

semi-algorithm to compute a SAGBI basis

0. set $B \leftarrow \{f_1, \ldots, f_m\}$
1. compute the relation ideal $\mathrm{Rel}(\mathrm{LM}(B))$ of the leading monomials of B *up to degree* d
2. for all generators $r(y_1, \ldots, y_m)$ of $\mathrm{Rel}(\mathrm{LM}(B))$, compute $r(f_1, \ldots, f_m) \xrightarrow{B} h$
3. if $h \neq 0$, add h to B
4. repeat from Step 1. *with increased bound* d until no new element has been added to B

Using SAGBI Bases

assume $B = \{g_1, \ldots, g_\ell\}$ is a SAGBI basis of the polynomial algebra $\mathcal{A}$

all relevant information is given by the leading monomials

- $\mathrm{Hilb}(\mathcal{A}) = \mathrm{Hilb}(\langle \mathrm{LM}(g_1), \ldots, \mathrm{LM}(g_\ell)\rangle)$
- the Hilbert series can be computed from the ideal

$$\mathrm{Rel}(\mathrm{LM}(B)) = \langle \mathrm{LM}(g_1) - t_1, \ldots, \mathrm{LM}(g_\ell) - t_\ell \rangle \cap \mathbb{K}[t_1, \ldots, t_\ell]$$

- if B has been been computed only up to degree d, we can still compare the Hilbert series

$\Longrightarrow$ direct proof of completeness for two-qubits mixed state < 1 min

$\Longrightarrow$ proof of completeness for $SU(2)^{\otimes 3}$
("private communication" in Luque, Thibon & Toumazet (2007))

Three Qubits

(joint work with Robert Zeier, work in progress)

- action of $U(2)^{\otimes 3}$ on density matrices ρ (or Hamiltonians) via conjugation
- adjoint representation of $SU(2)$ decomposes as $1 \oplus 3$
  $\Longrightarrow (1 \oplus 3)^3 = 1 \oplus 3 \times 3 \oplus 3 \times 3^2 \oplus 3^3$
- corresponds to the action on

$$
\begin{gathered}
I_2 \otimes I_2 \otimes I_2 \\
\oplus\, (\mathfrak{su}(2) \otimes I_2 \otimes I_2) \oplus (I_2 \otimes \mathfrak{su}(2) \otimes I_2) \oplus (I_2 \otimes I_2 \otimes \mathfrak{su}(2)) \\
\oplus\, (\mathfrak{su}(2) \otimes \mathfrak{su}(2) \otimes I_2) \oplus (\mathfrak{su}(2) \otimes I_2 \otimes \mathfrak{su}(2)) \oplus (I_2 \otimes \mathfrak{su}(2) \otimes \mathfrak{su}(2)) \\
\oplus\, (\mathfrak{su}(2) \otimes \mathfrak{su}(2) \otimes \mathfrak{su}(2))
\end{gathered}
$$

- invariant ring (excluding the trivial rep.) admits 7-fold grading
- Hilbert series $H(z_1, z_2, z_3,\ z_{12}, z_{13}, z_{23},\ z_{123})$
- consider only some of the irreducible components

Three Qubits: Partial Results

univariate Hilbert series

$$H(z) = (z^{206} + \ldots + 1)/(1 - \ldots - z^{270})$$
$$= 1 + z + 8\,z^2 + 24\,z^3 + 148\,z^4 + 649\,z^5 + 3.576\,z^6 + 17.206\,z^7$$
$$+ 84.320\,z^8 + 386.599\,z^9 + 1.720.880\,z^{10} + 7.302.550\,z^{11} + 29.864.124\,z^{12}$$
$$+ 117.329.840\,z^{13} + 444.769.448\,z^{14} + 1.627.560.935\,z^{15} + \ldots$$

computed up to degree 8000 in about 4.5 days via two-fold integration and (Laurent) series expansion using `LazySeries` in MAGMA

Three Qubits: Partial Results

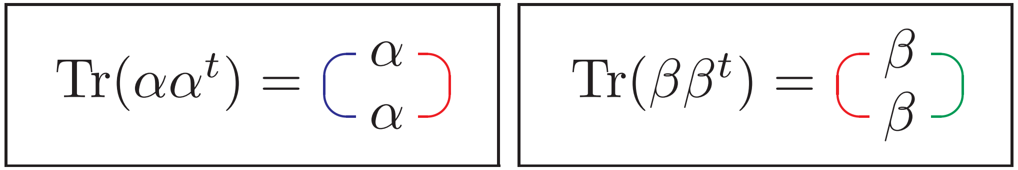
action on two components of dimension 9

- Hilbert series

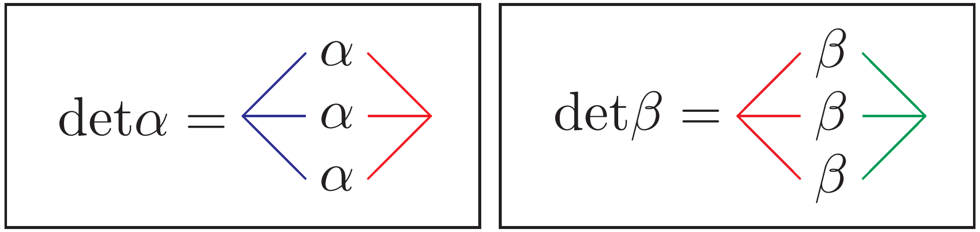
$$H_{9\oplus 9}(z) = \frac{1+z^8+z^{16}}{(1-z^2)^2(1-z^3)^2(1-z^4)^3(1-z^6)^2}$$
$$= 1 + 2\,z^2 + 2\,z^3 + 6\,z^4 + 4\,z^5 + 15\,z^6 + 12\,z^7 + 31\,z^8 + 28\,z^9$$
$$+ 62\,z^{10} + 58\,z^{11} + 120\,z^{12} + 112\,z^{13} + 213\,z^{14} + 212\,z^{15}$$
$$+ 370\,z^{16} + 368\,z^{17} + 622\,z^{18} + 628\,z^{19} + 1006\,z^{20} + \ldots$$

- generated by 9 primary invariants and 1 additional invariant
- completeness follows from the fact that there is only one additional invariant

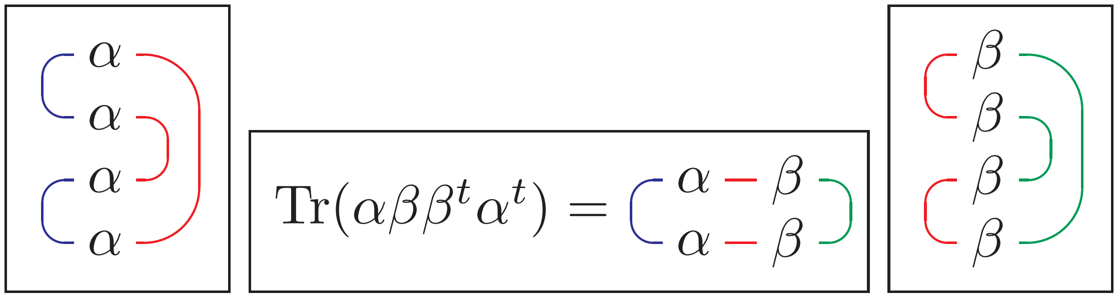
degree 2:



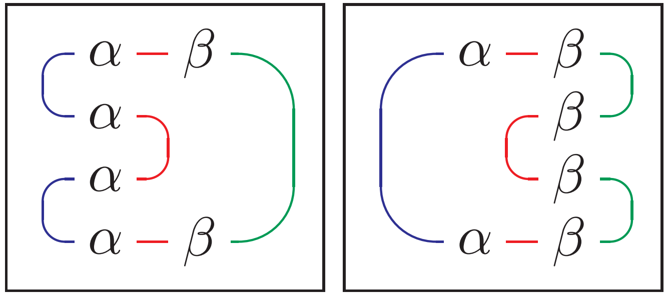
degree 3:



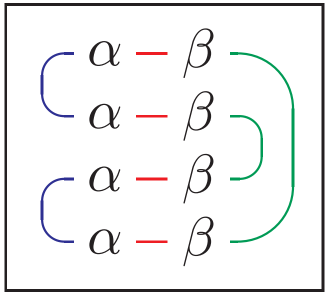
degree 4:



degree 6:



degree 8:



Are there any relations for these tensors like Cayley-Hamilton?

Three Qubits: Partial Results

action on three components of dimension 9

$$H_{3\times 9}(z) = \big(z^{36} - z^{35} - z^{34} + z^{33} + 4\,z^{32} + 6\,z^{30} - 2\,z^{29} + 12\,z^{28} + 12\,z^{27} + 33\,z^{26}$$
$$+ 28\,z^{25} + 69\,z^{24} + 45\,z^{23} + 82\,z^{22} + 73\,z^{21} + 116\,z^{20} + 86\,z^{19} + 134\,z^{18}$$
$$+ 86\,z^{17} + 116\,z^{16} + 73\,z^{15} + 82\,z^{14} + 45\,z^{13} + 69\,z^{12} + 28\,z^{11} + 33\,z^{10}$$
$$+ 12\,z^{9} + 12\,z^{8} - 2\,z^{7} + 6\,z^{6} + 4\,z^{4} + z^{3} - z^{2} - z + 1\big)/$$
$$\big((z-1)^{18}(z+1)^{11}(z^2-z+1)^2(z^2+1)^5(z^2+z+1)^6(z^4+z^3+z^2+z+1)^2\big)$$
$$= 1 + 3\,z^2 + 4\,z^3 + 15\,z^4 + 18\,z^5 + 63\,z^6 + 90\,z^7 + 240\,z^8 + 386\,z^9 + 882\,z^{10}$$
$$+ 1.479\,z^{11} + 3.093\,z^{12} + 5.247\,z^{13} + 10.179\,z^{14} + 17.299\,z^{15} + 31.695\,z^{16}$$
$$+ 53.133\,z^{17} + 93.143\,z^{18} + 153.354\,z^{19} + 258.852\,z^{20} + \ldots$$

- computed 178 invariants with max. degree 12
- verified up to degree 20 using triple-grading, max. dimension 6.281