Invariant theory of projective reflection groups, and their Kronecker coefficients

FABRIZIO CASELLI

November 23, 2009

ALMA MATER STUDIORUM
UNIVERSITÀ DI BOLOGNA

Preliminaries

Given $\sigma \in S_n$ let

$$\text{Descents of } \sigma = \mathrm{Des}(\sigma) = \{i | \sigma(i) > \sigma(i+1)\}$$

Preliminaries

Given $\sigma \in S_n$ let

$$\begin{aligned} \text{Descents of } \sigma &= \mathrm{Des}(\sigma) = \{i | \sigma(i) > \sigma(i+1)\} \\ \text{Major index of } \sigma &= \mathrm{maj}(\sigma) = \sum_{i \in \mathrm{Des}(\sigma)} i \end{aligned}$$

Preliminaries

Given $\sigma \in S_n$ let

$$\begin{aligned} \text{Descents of } \sigma &= \mathrm{Des}(\sigma) = \{i | \sigma(i) > \sigma(i+1)\} \\ \text{Major index of } \sigma &= \mathrm{maj}(\sigma) = \sum_{i \in \mathrm{Des}(\sigma)} i \end{aligned}$$

Example

If $\sigma = 35241 \in S_5$ we have $\mathrm{Des}(\sigma) = \{2, 4\}$ and $\mathrm{maj}(\sigma) = 6$.

Preliminaries

Given $\sigma \in S_n$ let

$$\begin{aligned} \text{Descents of } \sigma &= \mathrm{Des}(\sigma) = \{i | \sigma(i) > \sigma(i+1)\} \\ \text{Major index of } \sigma &= \mathrm{maj}(\sigma) = \sum_{i \in \mathrm{Des}(\sigma)} i \end{aligned}$$

Example

If $\sigma = 35241 \in S_5$ we have $\mathrm{Des}(\sigma) = \{2, 4\}$ and $\mathrm{maj}(\sigma) = 6$.

Theorem (MacMahon)

$$\begin{aligned} W(q) &= \sum_{\sigma \in S_n} q^{\mathrm{maj}(\sigma)} = \sum_{\sigma \in S_n} q^{\mathrm{inv}(\sigma)} \\ &= \prod_{i=1}^{n} (1 + q + q^2 + \cdots + q^i), \end{aligned}$$

where $\mathrm{inv}(\sigma) = |\{(i,j) : i < j \text{ and } \sigma(i) > \sigma(j)\}|$.

The coinvariant algebra

The coinvariant algebra of S_n is

$$R^{S_n} := \mathbb{C}[X]/\mathbb{C}[X]_+^{S_n},$$

The coinvariant algebra

The coinvariant algebra of S_n is

$$R^{S_n} := \mathbb{C}[X]/\mathbb{C}[X]_+^{S_n},$$

where

- $\mathbb{C}[X] = \mathbb{C}[x_1, \ldots, x_n]$;

The coinvariant algebra

The coinvariant algebra of S_n is

$$R^{S_n} := \mathbb{C}[X]/\mathbb{C}[X]_+^{S_n},$$

where

- $\mathbb{C}[X] = \mathbb{C}[x_1, \ldots, x_n]$;
- $\mathbb{C}[X]_+^{S_n}$ is the ideal generated by the symmetric polynomials with zero constant term;

The coinvariant algebra

The coinvariant algebra of S_n is

$$R^{S_n} := \mathbb{C}[X]/\mathbb{C}[X]^{S_n}_+,$$

where

- $\mathbb{C}[X] = \mathbb{C}[x_1, \ldots, x_n]$;
- $\mathbb{C}[X]^{S_n}_+$ is the ideal generated by the symmetric polynomials with zero constant term;

The algebra R^{S_n} is graded in $\mathbb{N}$ by total degree and

The coinvariant algebra

The coinvariant algebra of S_n is

$$R^{S_n} := \mathbb{C}[X]/\mathbb{C}[X]_+^{S_n},$$

where

- $\mathbb{C}[X] = \mathbb{C}[x_1, \ldots, x_n]$;
- $\mathbb{C}[X]_+^{S_n}$ is the ideal generated by the symmetric polynomials with zero constant term;

The algebra R^{S_n} is graded in $\mathbb{N}$ by total degree and

Theorem

We have

$$\mathrm{Hilb}(R^{S_n}) = W(q).$$

Fake-degree polynomials

Let μ be an irreducible representation of S_n and

Fake-degree polynomials

Let μ be an irreducible representation of S_n and $f^{\mu}(q)$ be the polynomial whose coefficient of q^k is the multiplicity of μ in $R_k^{S_n}$, i.e.

$$f^{\mu}(q) = \sum \langle \chi^{\mu}, \chi(R_k^{S_n}) \rangle q^k.$$

Fake-degree polynomials

Let μ be an irreducible representation of S_n and $f^\mu(q)$ be the polynomial whose coefficient of q^k is the multiplicity of μ in $R_k^{S_n}$, i.e.

$$f^\mu(q) = \sum \langle \chi^\mu, \chi(R_k^{S_n}) \rangle q^k.$$

$$T = \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 6 & \\ \cline{1-2} 4 & & \\ \cline{1-1} 7 & & \\ \cline{1-1} \end{array}$$

- $\mathrm{Des}(T) = \{1, 3, 5, 6\}$

Fake-degree polynomials

Let μ be an irreducible representation of S_n and $f^\mu(q)$ be the polynomial whose coefficient of q^k is the multiplicity of μ in $R_k^{S_n}$, i.e.

$$f^\mu(q) = \sum \langle \chi^\mu, \chi(R_k^{S_n}) \rangle q^k.$$

$$T = \begin{array}{|c|c|c|} \hline 1 & 3 & 5 \\ \hline 2 & 6 \\ \cline{1-2} 4 \\ \cline{1-1} 7 \\ \cline{1-1} \end{array}$$

- $\mathrm{Des}(T) = \{1, 3, 5, 6\}$
- $\mathrm{maj}(T) = 15$

Fake-degree polynomials

Let μ be an irreducible representation of S_n and
$f^\mu(q)$ be the polynomial whose coefficient of q^k is the multiplicity of μ in $R_k^{S_n}$, i.e.

$$f^\mu(q) = \sum \langle \chi^\mu, \chi(R_k^{S_n}) \rangle q^k.$$

$T =$

1	3	5
2	6	
4		
7		

- $\mathrm{Des}(T) = \{1, 3, 5, 6\}$
- $\mathrm{maj}(T) = 15$
- $\mu(T) = (3, 2, 1, 1)$

Fake-degree polynomials

Let μ be an irreducible representation of S_n and $f^\mu(q)$ be the polynomial whose coefficient of q^k is the multiplicity of μ in $R_k^{S_n}$, i.e.

$$f^\mu(q) = \sum \langle \chi^\mu, \chi(R_k^{S_n}) \rangle q^k.$$

$$T = \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 6 & \multicolumn{1}{c}{} \\ \cline{1-2} 4 & \multicolumn{2}{c}{} \\ \cline{1-1} 7 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array}$$

- $\mathrm{Des}(T) = \{1, 3, 5, 6\}$
- $\mathrm{maj}(T) = 15$
- $\mu(T) = (3, 2, 1, 1)$

Theorem (Lusztig, Kraskiewicz-Weyman)

We have

$$f^\mu(q) = \sum_{\{T : \mu(T) = \mu\}} q^{\mathrm{maj}(T)}$$

The 2-dimensional case

$R^{S_n} \cong \mathbb{C}S_n$ as S_n-modules. So

The 2-dimensional case

$R^{S_n} \cong \mathbb{C}S_n$ as S_n-modules. So

$$W(q) = \sum_{\mu \vdash n} f^{\mu}(1) f^{\mu}(q).$$

The 2-dimensional case

$R^{S_n} \cong \mathbb{C}S_n$ as S_n-modules. So

$$W(q) = \sum_{\mu \vdash n} f^{\mu}(1) f^{\mu}(q).$$

The generalization

$$W(q,t) := \sum_{\mu \vdash n} f^{\mu}(q) f^{\mu}(t)$$

is again a Hilbert polynomial

The 2-dimensional case

$R^{S_n} \cong \mathbb{C}S_n$ as S_n-modules. So

$$W(q) = \sum_{\mu \vdash n} f^{\mu}(1) f^{\mu}(q).$$

The generalization

$$W(q,t) := \sum_{\mu \vdash n} f^{\mu}(q) f^{\mu}(t)$$

is again a Hilbert polynomial

Theorem (Solomon)

$W(q,t)$ is the Hilbert polynomial of

$$\mathbb{C}[X,Y]^{\Delta S_n} / \mathbb{C}[X,Y]_+^{S_n \times S_n}.$$

The 2-dimensional case

$R^{S_n} \cong \mathbb{C}S_n$ as S_n-modules. So

$$W(q) = \sum_{\mu \vdash n} f^{\mu}(1) f^{\mu}(q).$$

The generalization

$$W(q,t) := \sum_{\mu \vdash n} f^{\mu}(q) f^{\mu}(t)$$

is again a Hilbert polynomial

Theorem (Solomon)

$W(q,t)$ is the Hilbert polynomial of

$$\mathbb{C}[X,Y]^{\Delta S_n} / \mathbb{C}[X,Y]_+^{S_n \times S_n}.$$

The polynomial $W(q,t)$ is the bimahonian distribution.

Other interpretations

The algebra $\mathbb{C}[X, Y]^{\Delta S_n}$ is a Cohen-Macauley algebra

Other interpretations

The algebra $\mathbb{C}[X,Y]^{\Delta S_n}$ is a Cohen-Macauley algebra so it is a free module on the subalgebra $\mathbb{C}[X,Y]^{S_n \times S_n}$.

Other interpretations

The algebra $\mathbb{C}[X,Y]^{\Delta S_n}$ is a Cohen-Macauley algebra so it is a free module on the subalgebra $\mathbb{C}[X,Y]^{S_n \times S_n}$. So

$$W(q,t) = \frac{\mathrm{Hilb}(\mathbb{C}[X,Y]^{\Delta S_n})}{\mathrm{Hilb}(\mathbb{C}[X,Y]^{S_n \times S_n})}.$$

Other interpretations

The algebra $\mathbb{C}[X, Y]^{\Delta S_n}$ is a Cohen-Macauley algebra so it is a free module on the subalgebra $\mathbb{C}[X, Y]^{S_n \times S_n}$. So

$$W(q, t) = \frac{\mathrm{Hilb}(\mathbb{C}[X, Y]^{\Delta S_n})}{\mathrm{Hilb}(\mathbb{C}[X, Y]^{S_n \times S_n})}.$$

These Hilbert series can be studied using the theory of bipartite partitions.

Other interpretations

The algebra $\mathbb{C}[X,Y]^{\Delta S_n}$ is a Cohen-Macauley algebra so it is a free module on the subalgebra $\mathbb{C}[X,Y]^{S_n \times S_n}$. So

$$W(q,t) = \frac{\mathrm{Hilb}(\mathbb{C}[X,Y]^{\Delta S_n})}{\mathrm{Hilb}(\mathbb{C}[X,Y]^{S_n \times S_n})}.$$

These Hilbert series can be studied using the theory of bipartite partitions.

Theorem (Garsia-Gessel)

$$W(q,t) = \sum_{\sigma \in S_n} q^{\mathrm{maj}(\sigma)} t^{\mathrm{maj}(\sigma^{-1})}$$

We can summarize these facts in the following sequence of identities:

$$W(q,t) = \sum_{\{S,T:\mu(S)=\mu(T)\}} q^{\mathrm{maj}(S)} t^{\mathrm{maj}(T)}$$

The many faces of the bimahonian distribution

We can summarize these facts in the following sequence of identities:

$$\begin{aligned} W(q,t) &= \sum_{\{S,T:\mu(S)=\mu(T)\}} q^{\mathrm{maj}(S)} t^{\mathrm{maj}(T)} \\ \text{(Lusztig, Kraskiewicz-Weyman)} &= \sum_{\mu \vdash n} f^{\mu}(q) f^{\mu}(t) \end{aligned}$$

We can summarize these facts in the following sequence of identities:

$$\begin{aligned} W(q,t) &= \sum_{\{S,T:\mu(S)=\mu(T)\}} q^{\mathrm{maj}(S)} t^{\mathrm{maj}(T)} \\ \text{(Lusztig, Kraskiewicz-Weyman)} &= \sum_{\mu\vdash n} f^{\mu}(q) f^{\mu}(t) \\ \text{(Solomon)} &= \mathrm{Hilb}\left(\frac{\mathbb{C}[X,Y]^{\Delta S_n}}{\mathbb{C}[X,Y]_+^{S_n\times S_n}}\right) \end{aligned}$$

The many faces of the bimahonian distribution

We can summarize these facts in the following sequence of identities:

$$\begin{aligned} W(q,t) &= \sum_{\{S,T:\mu(S)=\mu(T)\}} q^{\mathrm{maj}(S)} t^{\mathrm{maj}(T)} \\ \text{(Lusztig, Kraskiewicz-Weyman)} &= \sum_{\mu \vdash n} f^{\mu}(q) f^{\mu}(t) \\ \text{(Solomon)} &= \mathrm{Hilb}\left(\frac{\mathbb{C}[X,Y]^{\Delta S_n}}{\mathbb{C}[X,Y]_+^{S_n \times S_n}} \right) \\ \text{(Cohen-Macauley)} &= \frac{\mathrm{Hilb}(\mathbb{C}[X,Y]^{\Delta S_n})}{\mathrm{Hilb}(\mathbb{C}[X,Y]^{S_n \times S_n})} \end{aligned}$$

We can summarize these facts in the following sequence of identities:

$$\begin{aligned}
W(q,t) &= \sum_{\{S,T:\mu(S)=\mu(T)\}} q^{\mathrm{maj}(S)} t^{\mathrm{maj}(T)} \\
\text{(Lusztig, Kraskiewicz-Weyman)} &= \sum_{\mu \vdash n} f^{\mu}(q) f^{\mu}(t) \\
\text{(Solomon)} &= \mathrm{Hilb}\left(\frac{\mathbb{C}[X,Y]^{\Delta S_n}}{\mathbb{C}[X,Y]_+^{S_n \times S_n}}\right) \\
\text{(Cohen-Macauley)} &= \frac{\mathrm{Hilb}(\mathbb{C}[X,Y]^{\Delta S_n})}{\mathrm{Hilb}(\mathbb{C}[X,Y]^{S_n \times S_n})} \\
\text{(Garsia-Gessel)} &= \sum_{\sigma \in S_n} q^{\mathrm{maj}(\sigma)} t^{\mathrm{maj}(\sigma^{-1})}
\end{aligned}$$

The many faces of the bimahonian distribution

We can summarize these facts in the following sequence of identities:

$$\begin{aligned}
W(q,t) &= \sum_{\{S,T:\mu(S)=\mu(T)\}} q^{\mathrm{maj}(S)} t^{\mathrm{maj}(T)} \\
\text{(Lusztig, Kraskiewicz-Weyman)} &= \sum_{\mu \vdash n} f^{\mu}(q) f^{\mu}(t) \\
\text{(Solomon)} &= \mathrm{Hilb}\left(\frac{\mathbb{C}[X,Y]^{\Delta S_n}}{\mathbb{C}[X,Y]_+^{S_n \times S_n}}\right) \\
\text{(Cohen-Macauley)} &= \frac{\mathrm{Hilb}(\mathbb{C}[X,Y]^{\Delta S_n})}{\mathrm{Hilb}(\mathbb{C}[X,Y]^{S_n \times S_n})} \\
\text{(Garsia-Gessel)} &= \sum_{\sigma \in S_n} q^{\mathrm{maj}(\sigma)} t^{\mathrm{maj}(\sigma^{-1})}
\end{aligned}$$

The equality between the first and the last line follows also immediately from the Robinson-Schensted correspondence.

The Robinson-Schensted correspondence

Let $\sigma = 31542$. Then

$$\sigma \overset{RS}{\mapsto} \left[\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 \\ \cline{1-1} \end{array} , \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 4 \\ \hline 5 \\ \cline{1-1} \end{array} \right]$$

Let $\sigma = 31542$. Then

$$\sigma \overset{RS}{\mapsto} \left[\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 \\ \cline{1-1} \end{array}, \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline 5 \\ \cline{1-1} \end{array} \right]$$

Theorem (Robinson-Schensted correspondence)

The correspondence $\sigma \overset{RS}{\mapsto} [P(\sigma), Q(\sigma)]$ *is a bijection between* S_n *and pairs of tableaux having the same shape*

The Robinson-Schensted correspondence

Let $\sigma = 31542$. Then

$$\sigma \overset{RS}{\mapsto} \left[\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 \\ \cline{1-1} \end{array} , \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 4 \\ \hline 5 \\ \cline{1-1} \end{array} \right]$$

Theorem (Robinson-Schensted correspondence)

The correspondence $\sigma \overset{RS}{\mapsto} [P(\sigma), Q(\sigma)]$ is a bijection between S_n and pairs of tableaux having the same shape and

- $\mathrm{Des}(\sigma) = \mathrm{Des}(Q(\sigma))$;

Let $\sigma = 31542$. Then

$$\sigma \overset{RS}{\mapsto} \left[\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 & \\ \cline{1-1} \end{array} , \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 4 \\ \hline 5 & \\ \cline{1-1} \end{array} \right]$$

Theorem (Robinson-Schensted correspondence)

The correspondence $\sigma \overset{RS}{\mapsto} [P(\sigma), Q(\sigma)]$ is a bijection between S_n and pairs of tableaux having the same shape and

- $\mathrm{Des}(\sigma) = \mathrm{Des}(Q(\sigma))$;
- $\mathrm{Des}(\sigma^{-1}) = \mathrm{Des}(P(\sigma))$.

The many faces of the bimahonian distribution

$$
\begin{aligned}
W(q,t) &= \sum_{\{S,T:\mu(S)=\mu(T)\}} q^{\mathrm{maj}(S)} t^{\mathrm{maj}(T)} \\
\text{(Lusztig, Kraskiewicz-Weyman)} &= \sum_{\mu} f^{\mu}(q) f^{\mu}(t) \\
\text{(Solomon)} &= \mathrm{Hilb}\left(\frac{\mathbb{C}[X,Y]^{\Delta S_n}}{\mathbb{C}[X,Y]_{+}^{S_n \times S_n}}\right) \\
\text{(Cohen-Macauley)} &= \frac{\mathrm{Hilb}(\mathbb{C}[X,Y]^{\Delta S_n})}{\mathrm{Hilb}(\mathbb{C}[X,Y]^{S_n \times S_n})} \\
\text{(Garsia-Gessel)} &= \sum_{\sigma \in S_n} q^{\mathrm{maj}(\sigma)} t^{\mathrm{maj}(\sigma^{-1})}
\end{aligned}
$$

How to generalize to the 3-dimensional case?

Multimahonian distributions

How to generalize to the 3-dimensional case?
Use Kronecker coefficients!

$$g_{\lambda,\mu,\nu} := \frac{1}{n!} \sum_{\sigma \in S_n} \chi^{\lambda}(\sigma)\chi^{\mu}(\sigma)\chi^{\nu}(\sigma).$$

Multimahonian distributions

How to generalize to the 3-dimensional case?
Use Kronecker coefficients!

$$g_{\lambda,\mu,\nu} := \frac{1}{n!} \sum_{\sigma \in S_n} \chi^\lambda(\sigma)\chi^\mu(\sigma)\chi^\nu(\sigma).$$

Theorem (Solomon)

We have

$$\mathrm{Hilb}\left(\frac{\mathbb{C}[X,Y,Z]^{\Delta S_n}}{\mathbb{C}[X,Y,Z]_+^{S_n^{\times 3}}}\right) = \sum_{\lambda,\mu,\nu} g_{\lambda,\mu,\nu} f^\lambda(q_1) f^\mu(q_2) f^\nu(q_3).$$

Here X, Y, Z stand for three n-tuples of variables

Descents representations

How to refine to multivariate degrees?

Descents representations

How to refine to multivariate degrees?

Use descent representations!

Descents representations

How to refine to multivariate degrees?
Use descent representations!

Theorem (Solomon, Adin-Brenti-Roichman)

We have

$$R_k^{S_n} \cong \bigoplus_{\lambda \vdash k} R_\lambda,$$

as S_n-modules, by means of a canonical isomorphism.

Descents representations

How to refine to multivariate degrees?
Use descent representations!

Theorem (Solomon, Adin-Brenti-Roichman)

We have

$$R_k^{S_n} \cong \bigoplus_{\lambda \vdash k} R_\lambda,$$

as S_n-modules, by means of a canonical isomorphism.

We say that R_λ is the component of R^{S_n} of multidegree λ

Descents representations

How to refine to multivariate degrees?
Use descent representations!

Theorem (Solomon, Adin-Brenti-Roichman)

We have

$$R_k^{S_n} \cong \bigoplus_{\lambda \vdash k} R_\lambda,$$

as S_n-modules, by means of a canonical isomorphism.

We say that R_λ is the component of R^{S_n} of multidegree λ

$$\mathrm{Hilb}(R^{S_n})(q_1, \ldots, q_n) = \sum_\lambda (\dim R_\lambda) q_1^{\lambda_1} \cdots q_n^{\lambda_n}.$$

Descents representations

How to refine to multivariate degrees?
Use descent representations!

Theorem (Solomon, Adin-Brenti-Roichman)

We have

$$R_k^{S_n} \cong \bigoplus_{\lambda \vdash k} R_\lambda,$$

as S_n-modules, by means of a canonical isomorphism.

We say that R_λ is the component of R^{S_n} of multidegree λ

$$\mathrm{Hilb}(R^{S_n})(q_1, \ldots, q_n) = \sum_{\lambda} (\dim R_\lambda) q_1^{\lambda_1} \cdots q_n^{\lambda_n}.$$

Extending this we can also decompose the algebra

$$\mathbb{C}[X, Y, Z]^{\Delta S_n} / \mathbb{C}[X, Y, Z]_+^{S_n^{\times 3}}$$

in homogeneous components whose degrees are triples of partitions with at most n parts.

Therefore the Hilbert series will depend on 3 n-tuples of variables Q_1, Q_2, Q_3, where $Q_i = (q_{i,1}, \ldots, q_{i,n})$.

Refined fake-degree polynomials

Therefore the Hilbert series will depend on 3 n-tuples of variables Q_1, Q_2, Q_3, where $Q_i = (q_{i,1}, \dots, q_{i,n})$.
Let $f^\mu(q_1, \dots, q_n)$ be the polynomial whose coefficient of $q_1^{\lambda_1} \cdots q_n^{\lambda_n}$ is the multiplicity of the representation μ in R_λ.

Refined fake-degree polynomials

Therefore the Hilbert series will depend on 3 n-tuples of variables Q_1, Q_2, Q_3, where $Q_i = (q_{i,1}, \dots, q_{i,n})$.
Let $f^\mu(q_1, \dots, q_n)$ be the polynomial whose coefficient of $q_1^{\lambda_1} \cdots q_n^{\lambda_n}$ is the multiplicity of the representation μ in R_λ.

Theorem

We have

$$\mathrm{Hilb}\Big(\frac{\mathbb{C}[X,Y,Z]^{\Delta S_n}}{\mathbb{C}[X,Y,Z]_+^{S_n^{\times 3}}}\Big)(Q_1,Q_2,Q_3) = \sum_{\lambda,\mu,\nu} g_{\lambda,\mu,\nu} f^\lambda(Q_1) f^\mu(Q_2) f^\nu(Q_3).$$

Refined fake-degree polynomials

For X a permutation or a tableau we let

$$(\lambda(X))_i = |\mathrm{Des}(X) \cap \{i, \ldots, n\}|$$

Refined fake-degree polynomials

For X a permutation or a tableau we let

$$(\lambda(X))_i = |\mathrm{Des}(X) \cap \{i, \ldots, n\}| \quad \text{so } \lambda(X) \vdash \mathrm{maj}(X)$$

Refined fake-degree polynomials

For X a permutation or a tableau we let

$$(\lambda(X))_i = |\mathrm{Des}(X) \cap \{i, \ldots, n\}| \ \text{ so } \lambda(X) \vdash \mathrm{maj}(X)$$

Theorem (Adin-Brenti-Roichman)

We have

$$f^{\mu}(q_1, \ldots, q_n) = \sum_{\{T : \mu(T) = \mu\}} Q^{\lambda(T)}.$$

Partition-degree on polynomials

And from the point of view of permutations?

Partition-degree on polynomials

And from the point of view of permutations?
$\mathbb{C}[X]$ is multigraded by exponent partition

Partition-degree on polynomials

And from the point of view of permutations?
$\mathbb{C}[X]$ is multigraded by exponent partition

$$\deg(x_1^3 x_2^5 x_3^4) = (5, 4, 3).$$

Partition-degree on polynomials

And from the point of view of permutations?
$\mathbb{C}[X]$ is multigraded by exponent partition

$$\deg(x_1^3 x_2^5 x_3^4) = (5, 4, 3).$$

The algebra of polynomials in $3n$ variables $\mathbb{C}[X, Y, Z]$ is multigraded by triples of partitions.

And from the point of view of permutations?
$\mathbb{C}[X]$ is multigraded by exponent partition

$$\deg(x_1^3 x_2^5 x_3^4) = (5,4,3).$$

The algebra of polynomials in $3n$ variables $\mathbb{C}[X,Y,Z]$ is multigraded by triples of partitions.
The action of $S_n^{\times 3}$ respects this grading and

Partition-degree on polynomials

And from the point of view of permutations?
$\mathbb{C}[X]$ is multigraded by exponent partition

$$\deg(x_1^3 x_2^5 x_3^4) = (5, 4, 3).$$

The algebra of polynomials in $3n$ variables $\mathbb{C}[X, Y, Z]$ is multigraded by triples of partitions.
The action of $S_n^{\times 3}$ respects this grading and

Theorem

We have

$$\frac{\mathrm{Hilb}(\mathbb{C}[X, Y, Z]^{\Delta S_n})(Q_1, Q_2, Q_3)}{\mathrm{Hilb}(\mathbb{C}[X, Y, Z]^{S_n^{\times 3}})(Q_1, Q_2, Q_3)} = \sum_{\sigma_1 \sigma_2 \sigma_3 = 1} Q_1^{\lambda(\sigma_1)} Q_2^{\lambda(\sigma_2)} Q_3^{\lambda(\sigma_3)}$$

Refined multimahonian distribution

So similarly to the case of the total degree we have

$$\begin{aligned}
W(Q_1, Q_2, Q_3) &= \sum_{T_1, T_2, T_3} g_{\mu(T_1),\mu(T_2),\mu(T_3)} Q_1^{\lambda(T_1)} Q_2^{\lambda(T_2)} Q_3^{\lambda(T_3)} \\
&= \sum_{\lambda,\mu,\nu} g_{\lambda,\mu,\nu} f^{\lambda}(Q_1) f^{\mu}(Q_2) f^{\nu}(Q_3) \\
&= \mathrm{Hilb}\left(\frac{\mathbb{C}[X, Y, Z]^{\Delta S_n}}{(\mathbb{C}[X, Y, Z]_+^{S_n^{\times 3}})} \right)(Q_1, Q_2, Q_3) \\
&= \frac{\mathrm{Hilb}(\mathbb{C}[X, Y, Z]^{\Delta S_n})(Q_1, Q_2, Q_3)}{\mathrm{Hilb}(\mathbb{C}[X, Y, Z]^{S_n^{\times 3}})(Q_1, Q_2, Q_3)} \\
&= \sum_{\sigma_1\sigma_2\sigma_3=1} Q_1^{\lambda(\sigma_1)} Q_2^{\lambda(\sigma_2)} Q_3^{\lambda(\sigma_3)}
\end{aligned}$$

The generalized Robison-Schensted correspondence

Corollary

There is a map RS *that associates to every triple of permutations whose product is the identity a triple of standard tableaux of size* n *such that:*

- $|\mathrm{RS}^{-1}(T_1, T_2, T_3)| = g_{\mu(T_1),\mu(T_2),\mu(T_3)}$;
- *If* $(\sigma_1, \sigma_2, \sigma_3) \mapsto (T_1, T_2, T_3)$ *then* $\mathrm{Des}(T_i) = \mathrm{Des}(\sigma_i)\ \forall i$.

The generalized Robison-Schensted correspondence

Corollary

There is a map RS *that associates to every triple of permutations whose product is the identity a triple of standard tableaux of size* n *such that:*

- $|\mathrm{RS}^{-1}(T_1, T_2, T_3)| = g_{\mu(T_1),\mu(T_2),\mu(T_3)}$;
- *If* $(\sigma_1, \sigma_2, \sigma_3) \mapsto (T_1, T_2, T_3)$ *then* $\mathrm{Des}(T_i) = \mathrm{Des}(\sigma_i)\ \forall i$.

The Kronecker coefficients are uniquely determined by this!

The generalized Robison-Schensted correspondence

Corollary

There is a map RS *that associates to every triple of permutations whose product is the identity a triple of standard tableaux of size* n *such that:*

- $|\mathrm{RS}^{-1}(T_1, T_2, T_3)| = g_{\mu(T_1),\mu(T_2),\mu(T_3)}$;
- *If* $(\sigma_1, \sigma_2, \sigma_3) \mapsto (T_1, T_2, T_3)$ *then* $\mathrm{Des}(T_i) = \mathrm{Des}(\sigma_i)\ \forall i$.

The Kronecker coefficients are uniquely determined by this!

Corollary

Let $\tilde{g}_{\lambda,\mu,\nu} \in \mathbb{N}$ *for all triples of partitions* λ, μ, ν *of* n.

The generalized Robison-Schensted correspondence

Corollary

There is a map RS *that associates to every triple of permutations whose product is the identity a triple of standard tableaux of size* n *such that:*

- $|\mathrm{RS}^{-1}(T_1, T_2, T_3)| = g_{\mu(T_1),\mu(T_2),\mu(T_3)}$;
- *If* $(\sigma_1, \sigma_2, \sigma_3) \mapsto (T_1, T_2, T_3)$ *then* $\mathrm{Des}(T_i) = \mathrm{Des}(\sigma_i)\ \forall i$.

The Kronecker coefficients are uniquely determined by this!

Corollary

Let $\tilde{g}_{\lambda,\mu,\nu} \in \mathbb{N}$ *for all triples of partitions* λ, μ, ν *of* n*. If* ϕ *is a map that associate to every triple of permutations whose product is the identity a triple of standard tableaux such that*

The generalized Robison-Schensted correspondence

Corollary

There is a map RS *that associates to every triple of permutations whose product is the identity a triple of standard tableaux of size* n *such that:*

- $|\mathrm{RS}^{-1}(T_1, T_2, T_3)| = g_{\mu(T_1),\mu(T_2),\mu(T_3)}$;
- *If* $(\sigma_1, \sigma_2, \sigma_3) \mapsto (T_1, T_2, T_3)$ *then* $\mathrm{Des}(T_i) = \mathrm{Des}(\sigma_i)\ \forall i$.

The Kronecker coefficients are uniquely determined by this!

Corollary

Let $\tilde{g}_{\lambda,\mu,\nu} \in \mathbb{N}$ *for all triples of partitions* λ, μ, ν *of* n. *If* ϕ *is a map that associate to every triple of permutations whose product is the identity a triple of standard tableaux such that*

- $|\phi^{-1}(T_1, T_2, T_3)| = \tilde{g}_{\mu(T_1),\mu(T_2),\mu(T_3)}$;

The generalized Robison-Schensted correspondence

Corollary

There is a map RS *that associates to every triple of permutations whose product is the identity a triple of standard tableaux of size* n *such that:*

- $|\mathrm{RS}^{-1}(T_1, T_2, T_3)| = g_{\mu(T_1),\mu(T_2),\mu(T_3)}$;
- *If* $(\sigma_1, \sigma_2, \sigma_3) \mapsto (T_1, T_2, T_3)$ *then* $\mathrm{Des}(T_i) = \mathrm{Des}(\sigma_i)\ \forall i$.

The Kronecker coefficients are uniquely determined by this!

Corollary

Let $\tilde{g}_{\lambda,\mu,\nu} \in \mathbb{N}$ *for all triples of partitions* λ, μ, ν *of* n. *If* ϕ *is a map that associate to every triple of permutations whose product is the identity a triple of standard tableaux such that*

- $|\phi^{-1}(T_1, T_2, T_3)| = \tilde{g}_{\mu(T_1),\mu(T_2),\mu(T_3)}$;
- *If* $(\sigma_1, \sigma_2, \sigma_3) \mapsto (T_1, T_2, T_3)$ *then* $\mathrm{Des}(T_i) = \mathrm{Des}(\sigma_i)\ \forall i$.

The generalized Robison-Schensted correspondence

Corollary

There is a map RS *that associates to every triple of permutations whose product is the identity a triple of standard tableaux of size n such that:*

- $|\mathrm{RS}^{-1}(T_1, T_2, T_3)| = g_{\mu(T_1),\mu(T_2),\mu(T_3)}$;
- *If* $(\sigma_1, \sigma_2, \sigma_3) \mapsto (T_1, T_2, T_3)$ *then* $\mathrm{Des}(T_i) = \mathrm{Des}(\sigma_i)\ \forall i$.

The Kronecker coefficients are uniquely determined by this!

Corollary

Let $\tilde{g}_{\lambda,\mu,\nu} \in \mathbb{N}$ *for all triples of partitions* λ, μ, ν *of n. If* ϕ *is a map that associate to every triple of permutations whose product is the identity a triple of standard tableaux such that*

- $|\phi^{-1}(T_1, T_2, T_3)| = \tilde{g}_{\mu(T_1),\mu(T_2),\mu(T_3)}$;
- *If* $(\sigma_1, \sigma_2, \sigma_3) \mapsto (T_1, T_2, T_3)$ *then* $\mathrm{Des}(T_i) = \mathrm{Des}(\sigma_i)\ \forall i$.

Then $\tilde{g}_{\lambda,\mu,\nu} = g_{\lambda,\mu,\nu}$.

A combinatorial algorithm

Using the previous corollary one can easily define a combinatorial algorithm for the computation of the Kronecker coefficients.

A combinatorial algorithm

Using the previous corollary one can easily define a combinatorial algorithm for the computation of the Kronecker coefficients.

The recursion step expresses a generic $g_{\lambda,\mu,\nu}$ in terms of

A combinatorial algorithm

Using the previous corollary one can easily define a combinatorial algorithm for the computation of the Kronecker coefficients.

The recursion step expresses a generic $g_{\lambda,\mu,\nu}$ in terms of

- the number of triples of permutations $(\sigma_1, \sigma_2, \sigma_3)$ such that $\sigma_1\sigma_2\sigma_3 = 1$ and $\mathrm{Des}(\sigma_1) = \{\lambda_1, \lambda_1 + \lambda_2, \ldots\}$ and similarly for the descent sets of σ_2 and σ_3;

A combinatorial algorithm

Using the previous corollary one can easily define a combinatorial algorithm for the computation of the Kronecker coefficients.

The recursion step expresses a generic $g_{\lambda,\mu,\nu}$ in terms of

- the number of triples of permutations $(\sigma_1, \sigma_2, \sigma_3)$ such that $\sigma_1\sigma_2\sigma_3 = 1$ and $\mathrm{Des}(\sigma_1) = \{\lambda_1, \lambda_1 + \lambda_2, \ldots\}$ and similarly for the descent sets of σ_2 and σ_3;
- Some Kronecker coefficients $g_{\lambda',\mu',\nu'}$ where $\lambda' \triangleright \lambda$, $\mu' \triangleright \mu$ and $\nu' \triangleright \nu$, where $\triangleright$ denotes dominance order on partitions.

A combinatorial algorithm

Using the previous corollary one can easily define a combinatorial algorithm for the computation of the Kronecker coefficients.

The recursion step expresses a generic $g_{\lambda,\mu,\nu}$ in terms of

- the number of triples of permutations $(\sigma_1, \sigma_2, \sigma_3)$ such that $\sigma_1\sigma_2\sigma_3 = 1$ and $\mathrm{Des}(\sigma_1) = \{\lambda_1, \lambda_1 + \lambda_2, \ldots\}$ and similarly for the descent sets of σ_2 and σ_3;
- Some Kronecker coefficients $g_{\lambda',\mu',\nu'}$ where $\lambda' \rhd \lambda$, $\mu' \rhd \mu$ and $\nu' \rhd \nu$, where $\rhd$ denotes dominance order on partitions.

This algorithm is certainly less efficient than the one shown by Derksen based on the Murnagham-Nakajama rule...

A combinatorial algorithm

Using the previous corollary one can easily define a combinatorial algorithm for the computation of the Kronecker coefficients.

The recursion step expresses a generic $g_{\lambda,\mu,\nu}$ in terms of

- the number of triples of permutations $(\sigma_1, \sigma_2, \sigma_3)$ such that $\sigma_1\sigma_2\sigma_3 = 1$ and $\mathrm{Des}(\sigma_1) = \{\lambda_1, \lambda_1 + \lambda_2, \dots\}$ and similarly for the descent sets of σ_2 and σ_3;
- Some Kronecker coefficients $g_{\lambda',\mu',\nu'}$ where $\lambda' \rhd \lambda$, $\mu' \rhd \mu$ and $\nu' \rhd \nu$, where $\rhd$ denotes dominance order on partitions.

This algorithm is certainly less efficient than the one shown by Derksen based on the Murnagham-Nakajama rule... but maybe one can improve it.

Vic Reiner observed that one can reobtain (in a non trivial way) all the interpretations of the refined multimahonian distribution using the Stanley-Reisner ring of the baricentric subdivision of an $n-1$-dimensional complex instead of the coinvariant algebras, these being isomorphic.

From symmetric to complex reflection groups

Vic Reiner observed that one can reobtain (in a non trivial way) all the interpretations of the refined multimahonian distribution using the Stanley-Reisner ring of the baricentric subdivision of an $n-1$-dimensional complex instead of the coinvariant algebras, these being isomorphic.

These are no longer isomorphic for other Weyl groups

From symmetric to complex reflection groups

Vic Reiner observed that one can reobtain (in a non trivial way) all the interpretations of the refined multimahonian distribution using the Stanley-Reisner ring of the baricentric subdivision of an $n-1$-dimensional complex instead of the coinvariant algebras, these being isomorphic.

These are no longer isomorphic for other Weyl groups and the Stanley-Reisner ring is not defined at all for complex reflection groups.

From symmetric to complex reflection groups

Vic Reiner observed that one can reobtain (in a non trivial way) all the interpretations of the refined multimahonian distribution using the Stanley-Reisner ring of the baricentric subdivision of an $n-1$-dimensional complex instead of the coinvariant algebras, these being isomorphic.

These are no longer isomorphic for other Weyl groups and the Stanley-Reisner ring is not defined at all for complex reflection groups.

That's why I was convinced that my approach is better...

From symmetric to complex reflection groups

Vic Reiner observed that one can reobtain (in a non trivial way) all the interpretations of the refined multimahonian distribution using the Stanley-Reisner ring of the baricentric subdivision of an $n-1$-dimensional complex instead of the coinvariant algebras, these being isomorphic.

These are no longer isomorphic for other Weyl groups and the Stanley-Reisner ring is not defined at all for complex reflection groups.

That's why I was convinced that my approach is better...
...and I was led to introduce projective complex reflection groups.

Complex reflection groups

Complex reflection groups are subgroups of $GL(n, \mathbb{C})$ generated by reflections, i.e. elements of finite order that fix a hyperplane pointwise.

Complex reflection groups

Complex reflection groups are subgroups of $GL(n, \mathbb{C})$ generated by reflections, i.e. elements of finite order that fix a hyperplane pointwise.

Example

$G(r, n)$, the group of $n \times n$ monomial matrices whose non-zero entries are r-th roots of 1.

$$\begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & i \\ -i & 0 & 0 & 0 \end{bmatrix} \in G(4,4)$$

Complex reflection groups

Complex reflection groups are subgroups of $GL(n, \mathbb{C})$ generated by reflections, i.e. elements of finite order that fix a hyperplane pointwise.

Example

$G(r, n)$, the group of $n \times n$ monomial matrices whose non-zero entries are r-th roots of 1.

$$\begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & i \\ -i & 0 & 0 & 0 \end{bmatrix} \in G(4,4)$$

Example

$G(r, p, n)$, the elements in $G(r, n)$ whose permanent is a r/p-th root of unity. The matrix above is an element in $G(4, 2, 4)$.

Let C_q be the cyclic group of scalar matrices generated by $e^{\frac{2\pi i}{q}} I$.

Projective reflection groups

Let C_q be the cyclic group of scalar matrices generated by $e^{\frac{2\pi i}{q}} I$.

Definition

If $C_q \subset G(r,p,n)$ we define the projective reflection group $G(r,p,q,n) = G(r,p,n)/C_q$.

Projective reflection groups

Let C_q be the cyclic group of scalar matrices generated by $e^{\frac{2\pi i}{q}} I$.

Definition

If $C_q \subset G(r, p, n)$ we define the projective reflection group $G(r, p, q, n) = G(r, p, n)/C_q$.

Definition

If $G = G(r, p, q, n)$ we say that the group $G^* = G(r, q, p, n)$ is the dual of G.

Projective reflection groups

Let C_q be the cyclic group of scalar matrices generated by $e^{\frac{2\pi i}{q}} I$.

Definition

If $C_q \subset G(r,p,n)$ we define the projective reflection group $G(r,p,q,n) = G(r,p,n)/C_q$.

Definition

If $G = G(r,p,q,n)$ we say that the group $G^* = G(r,q,p,n)$ is the dual of G.

We observe that if G is a complex reflection group then G^* is not in general.

The duality

We will see many occurrences of the following relationship

Combinatorics of G	$\longleftrightarrow$	**Invariant theory of** G^*

The duality

We will see many occurrences of the following relationship

Combinatorics of G	$\longleftrightarrow$	**Invariant theory of G^***

Example

- If $G = G(r, 1, 1, n)$ then $G^* = G$. This holds in particular for $S_n = G(1, 1, 1, n)$ and $B_n = G(2, 1, 1, n)$.

The duality

We will see many occurrences of the following relationship

Combinatorics of G	$\leftrightarrow$	**Invariant theory of G^***

Example

- If $G = G(r,1,1,n)$ then $G^* = G$. This holds in particular for $S_n = G(1,1,1,n)$ and $B_n = G(2,1,1,n)$.
- If $G = D_n = G(2,2,1,n)$,

We will see many occurrences of the following relationship

Combinatorics of G	$\leftrightarrow$	**Invariant theory of G^***

Example

- If $G = G(r,1,1,n)$ then $G^* = G$. This holds in particular for $S_n = G(1,1,1,n)$ and $B_n = G(2,1,1,n)$.
- If $G = D_n = G(2,2,1,n)$, then $G^* = G(2,1,2,n) = B_n/\pm I$

The duality

We will see many occurrences of the following relationship

Combinatorics of G	$\leftrightarrow$	**Invariant theory of G^***

Example

- If $G = G(r, 1, 1, n)$ then $G^* = G$. This holds in particular for $S_n = G(1, 1, 1, n)$ and $B_n = G(2, 1, 1, n)$.
- If $G = D_n = G(2, 2, 1, n)$, then $G^* = G(2, 1, 2, n) = B_n/\pm I$ and it turns out that the combinatorics of $B_n/\pm I$ describes the invariant theory of D_n, and viceversa.

The combinatorics

$$g = \begin{bmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 1 \\ i & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & \zeta_4^{3} & 0 \\ 0 & 0 & 0 & \zeta_4^{0} \\ \zeta_4^{1} & 0 & 0 & 0 \\ 0 & \zeta_4^{2} & 0 & 0 \end{bmatrix},$$

where $\zeta_r = e^{\frac{2\pi i}{r}} \in G(r, p, q, n)$.

The combinatorics

$$g = \begin{bmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 1 \\ i & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & \zeta_4^{3} & 0 \\ 0 & 0 & 0 & \zeta_4^{0} \\ \zeta_4^{1} & 0 & 0 & 0 \\ 0 & \zeta_4^{2} & 0 & 0 \end{bmatrix},$$

where $\zeta_r = e^{\frac{2\pi i}{r}} \in G(r,p,q,n)$.
We let $z(g) = (3, 0, 1, 2)$, the color vector of g.
Let $g \in G(r,p,q,n)$ and $\sigma = |g|$ be its projection in S_n. Let

$$\mathrm{HDes}(g) \quad := \quad \{i \in [n-1] : z_i(g) = z_{i+1}(g) \text{ and } \sigma_i > \sigma_{i+1}\}$$

The combinatorics

$$g = \begin{bmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 1 \\ i & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & \zeta_4^{3} & 0 \\ 0 & 0 & 0 & \zeta_4^{0} \\ \zeta_4^{1} & 0 & 0 & 0 \\ 0 & \zeta_4^{2} & 0 & 0 \end{bmatrix},$$

where $\zeta_r = e^{\frac{2\pi i}{r}} \in G(r,p,q,n)$.
We let $z(g) = (3,0,1,2)$, the color vector of g.
Let $g \in G(r,p,q,n)$ and $\sigma = |g|$ be its projection in S_n. Let

$$\begin{aligned} \mathrm{HDes}(g) &:= \{i \in [n-1] : z_i(g) = z_{i+1}(g) \text{ and } \sigma_i > \sigma_{i+1}\} \\ h_i(g) &:= \#\{j \geq i : j \in \mathrm{HDes}(g)\} \end{aligned}$$

The combinatorics

$$g = \begin{bmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 1 \\ i & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & \zeta_4^3 & 0 \\ 0 & 0 & 0 & \zeta_4^0 \\ \zeta_4^1 & 0 & 0 & 0 \\ 0 & \zeta_4^2 & 0 & 0 \end{bmatrix},$$

where $\zeta_r = e^{\frac{2\pi i}{r}} \in G(r,p,q,n)$.
We let $z(g) = (3,0,1,2)$, the color vector of g.
Let $g \in G(r,p,q,n)$ and $\sigma = |g|$ be its projection in S_n. Let

$$\begin{aligned} \mathrm{HDes}(g) &:= \{i \in [n-1] : z_i(g) = z_{i+1}(g) \text{ and } \sigma_i > \sigma_{i+1}\} \\ h_i(g) &:= \#\{j \geq i : j \in \mathrm{HDes}(g)\} \\ k_i(g) &:= \begin{cases} [z_n]_{r/q} & \text{if } i = n \\ k_{i+1} + [z_i - z_{i+1}]_r & \text{if } i \in [n-1]. \end{cases} \end{aligned}$$

The projective flag-major index

Letting $\lambda_i(g) := r \cdot h_i(g) + k_i(g)$ then the sequence

$$\lambda(g) := (\lambda_1(g), \ldots, \lambda_n(g))$$

is a partition.

The projective flag-major index

Letting $\lambda_i(g) := r \cdot h_i(g) + k_i(g)$ then the sequence

$$\lambda(g) := (\lambda_1(g), \ldots, \lambda_n(g))$$

is a partition.
If $p = q = 1$ we have $|\lambda(g)| = \mathrm{fmaj}(g)$, the flag-major index defined by Adin and Roichman

The projective flag-major index

Letting $\lambda_i(g) := r \cdot h_i(g) + k_i(g)$ then the sequence

$$\lambda(g) := (\lambda_1(g), \ldots, \lambda_n(g))$$

is a partition.
If $p = q = 1$ we have $|\lambda(g)| = \mathrm{fmaj}(g)$, the flag-major index defined by Adin and Roichman so we define

$$\mathrm{fmaj}(g) := |\lambda(g)|$$

for all groups $G(r, p, q, n)$.

The coinvariant algebra

$G = G(r,p,q,n)$ naturally acts on $S_q[X]$, the q-th Veronese subalgebra of $\mathbb{C}[X] := \mathbb{C}[x_1, \ldots, x_n]$, i.e. the subalgebra generated in degree q.

The coinvariant algebra

$G = G(r, p, q, n)$ naturally acts on $S_q[X]$, the q-th Veronese subalgebra of $\mathbb{C}[X] := \mathbb{C}[x_1, \ldots, x_n]$, i.e. the subalgebra generated in degree q.

The coinvariant algebra is

$$R^G := S_q[X]/I_+^G.$$

The coinvariant algebra

$G = G(r,p,q,n)$ naturally acts on $S_q[X]$, the q-th Veronese subalgebra of $\mathbb{C}[X] := \mathbb{C}[x_1,\ldots,x_n]$, i.e. the subalgebra generated in degree q.

The coinvariant algebra is

$$R^G := S_q[X]/I_+^G.$$

Theorem (C)

R^G affords the regular representation of G.

The coinvariant algebra

$G = G(r,p,q,n)$ naturally acts on $S_q[X]$, the q-th Veronese subalgebra of $\mathbb{C}[X] := \mathbb{C}[x_1, \ldots, x_n]$, i.e. the subalgebra generated in degree q.

The coinvariant algebra is

$$R^G := S_q[X]/I_+^G.$$

Theorem (C)

R^G affords the regular representation of G.

If $g \in G$ we consider the monomial

$$a_g := x_{\sigma(1)}^{\lambda_1(g)} \cdots x_{\sigma(n)}^{\lambda_n(g)}, \text{ (where } \sigma = |g|)$$

The coinvariant algebra

$G = G(r, p, q, n)$ naturally acts on $S_q[X]$, the q-th Veronese subalgebra of $\mathbb{C}[X] := \mathbb{C}[x_1, \ldots, x_n]$, i.e. the subalgebra generated in degree q.

The coinvariant algebra is

$$R^G := S_q[X]/I_+^G.$$

Theorem (C)

R^G affords the regular representation of G.

If $g \in G$ we consider the monomial

$$a_g := x_{\sigma(1)}^{\lambda_1(g)} \cdots x_{\sigma(n)}^{\lambda_n(g)}, \text{ (where } \sigma = |g|)$$

Generalizing Garsia-Stanton, Adin-Brenti-Roichman, Bagno-Biagioli

The coinvariant algebra

$G = G(r,p,q,n)$ naturally acts on $S_q[X]$, the q-th Veronese subalgebra of $\mathbb{C}[X] := \mathbb{C}[x_1,\ldots,x_n]$, i.e. the subalgebra generated in degree q.
The coinvariant algebra is

$$R^G := S_q[X]/I_+^G.$$

Theorem (C)

R^G affords the regular representation of G.

If $g \in G$ we consider the monomial

$$a_g := x_{\sigma(1)}^{\lambda_1(g)} \cdots x_{\sigma(n)}^{\lambda_n(g)}, \text{ (where } \sigma = |g|)$$

Generalizing Garsia-Stanton, Adin-Brenti-Roichman, Bagno-Biagioli

Theorem (C)

The set of monomial $\{a_g : g \in G^*\}$ is a linear basis for R^G.

The irreducible representations

$\mathrm{Fer}(r, p, n) =$ r-tuples of Ferrers diagrams $(\lambda^{(0)}, \ldots, \lambda^{(r-1)})$ having a total of n cells and $\sum_i i|\lambda^{(i)}| \equiv 0 \mod p$.

The irreducible representations

$\mathrm{Fer}(r, p, n) =$ r-tuples of Ferrers diagrams $(\lambda^{(0)}, \ldots, \lambda^{(r-1)})$ having a total of n cells and $\sum_i i|\lambda^{(i)}| \equiv 0 \mod p$.
$\mathrm{ST}(r, p, n) =$ standard tableaux with shape in $\mathrm{Fer}(r, p, n)$.

Example

$$\left[\begin{array}{|c|c|}\hline 1 & 4 \\ \hline 5 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 2 & 8 \\ \hline 3 & 9 \\ \hline\end{array}, \begin{array}{|c|c|}\hline 6 & 7 \\ \hline\end{array}\right] \in \mathrm{ST}(3, 1, 9)$$

The irreducible representations

$\mathrm{Fer}(r,p,n)$ = r-tuples of Ferrers diagrams $(\lambda^{(0)},\ldots,\lambda^{(r-1)})$ having a total of n cells and $\sum_i i|\lambda^{(i)}| \equiv 0 \mod p$.
$\mathrm{ST}(r,p,n)$ = standard tableaux with shape in $\mathrm{Fer}(r,p,n)$.

Example

$$\left[\begin{array}{|c|c|}\hline 1 & 4 \\ \hline 5 \\ \cline{1-1}\end{array}\;,\; \begin{array}{|c|c|}\hline 2 & 8 \\ \hline 3 & 9 \\ \hline\end{array}\;,\; \begin{array}{|c|c|}\hline 6 & 7 \\ \hline\end{array}\right] \in \mathrm{ST}(3,1,9)$$

C_q acts on both $\mathrm{Fer}(r,p,n)$ and $\mathrm{ST}(r,p,n)$.

The irreducible representations

$\mathrm{Fer}(r,p,n) =$ r-tuples of Ferrers diagrams $(\lambda^{(0)},\ldots,\lambda^{(r-1)})$ having a total of n cells and $\sum_i i|\lambda^{(i)}| \equiv 0 \mod p$.
$\mathrm{ST}(r,p,n) =$ standard tableaux with shape in $\mathrm{Fer}(r,p,n)$.

Example

$$\left[\ \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 5 \\ \cline{1-1}\end{array},\ \begin{array}{|c|c|}\hline 2 & 8 \\ \hline 3 & 9 \\ \hline\end{array},\ \begin{array}{|c|c|}\hline 6 & 7 \\ \hline\end{array}\ \right] \in \mathrm{ST}(3,1,9)$$

C_q acts on both $\mathrm{Fer}(r,p,n)$ and $\mathrm{ST}(r,p,n)$.
Denote by $\mathrm{Fer}(r,p,q,n)$ and $\mathrm{ST}(r,p,q,n)$ the quotient sets.

The irreducible representations

$\mathrm{Fer}(r, p, n)$ = r-tuples of Ferrers diagrams $(\lambda^{(0)}, \ldots, \lambda^{(r-1)})$ having a total of n cells and $\sum_i i|\lambda^{(i)}| \equiv 0 \mod p$.
$\mathrm{ST}(r, p, n)$ = standard tableaux with shape in $\mathrm{Fer}(r, p, n)$.

Example

$$\left[\begin{array}{|c|c|}\hline 1 & 4 \\ \hline 5 & \\ \cline{1-1} \end{array}, \begin{array}{|c|c|}\hline 2 & 8 \\ \hline 3 & 9 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 6 & 7 \\ \hline \end{array} \right] \in \mathrm{ST}(3, 1, 9)$$

C_q acts on both $\mathrm{Fer}(r, p, n)$ and $\mathrm{ST}(r, p, n)$.
Denote by $\mathrm{Fer}(r, p, q, n)$ and $\mathrm{ST}(r, p, q, n)$ the quotient sets.

Theorem

The irreducible representations of $G(r, p, q, n)$ are naturally parametrized by pairs (μ, ρ), where $\mu \in \mathrm{Fer}(r, q, p, n)$ and $\rho \in (C_p)_\mu$, the stabilizer of any element in the class μ.

The descent representations

Let $G = G(r, p, q, n)$.

The descent representations

Let $G = G(r, p, q, n)$.

Theorem (C)

$$R_k^G \cong \bigoplus_{\lambda \vdash k} R_\lambda^G$$

as G-modules and $\{a_g : g \in G^* \text{ and } \lambda(g) = \lambda\}$ is a basis of R_λ^G.

The descent representations

Let $G = G(r, p, q, n)$.

Theorem (C)

$$R_k^G \cong \bigoplus_{\lambda \vdash k} R_\lambda^G$$

as G-modules and $\{a_g : g \in G^* \text{ and } \lambda(g) = \lambda\}$ is a basis of R_λ^G.

If $T \in \mathrm{ST}(r, p, q, n)$

The descent representations

Let $G = G(r,p,q,n)$.

Theorem (C)

$$R_k^G \cong \bigoplus_{\lambda \vdash k} R_\lambda^G$$

as G-modules and $\{a_g : g \in G^* \text{ and } \lambda(g) = \lambda\}$ is a basis of R_λ^G.

If $T \in \mathrm{ST}(r,p,q,n)$

- $\mathrm{HDes}(T) = \{i : i \text{ is above and in the same tableau of } i+1\}$;

The descent representations

Let $G = G(r, p, q, n)$.

Theorem (C)

$$R_k^G \cong \bigoplus_{\lambda \vdash k} R_\lambda^G$$

as G-modules and $\{a_g : g \in G^* \text{ and } \lambda(g) = \lambda\}$ is a basis of R_λ^G.

If $T \in \mathrm{ST}(r, p, q, n)$

- $\mathrm{HDes}(T) = \{i : i \text{ is above and in the same tableau of } i+1\}$;
- $z_i(T) = j$ if i appears in the $(j+1)$-th tableau of T.

The descent representations

Let $G = G(r, p, q, n)$.

Theorem (C)

$$R_k^G \cong \bigoplus_{\lambda \vdash k} R_\lambda^G$$

as G-modules and $\{a_g : g \in G^* \text{ and } \lambda(g) = \lambda\}$ *is a basis of* R_λ^G.

If $T \in \mathrm{ST}(r, p, q, n)$

- $\mathrm{HDes}(T) = \{i : i \text{ is above and in the same tableau of } i+1\}$;
- $z_i(T) = j$ if i appears in the $(j+1)$-th tableau of T.
- $h_i(T), k_i(T), \lambda(T)$ are defined as for elements in $G(r, p, q, n)$.

Let $G = G(r,p,q,n)$.

Theorem (C)

$$R_k^G \cong \bigoplus_{\lambda \vdash k} R_\lambda^G$$

as G-modules and $\{a_g : g \in G^* \text{ and } \lambda(g) = \lambda\}$ *is a basis of* R_λ^G.

If $T \in \mathrm{ST}(r,p,q,n)$

- $\mathrm{HDes}(T) = \{i : i \text{ is above and in the same tableau of } i+1\}$;
- $z_i(T) = j$ if i appears in the $(j+1)$-th tableau of T.
- $h_i(T), k_i(T), \lambda(T)$ are defined as for elements in $G(r,p,q,n)$.

Theorem

If $\mu \in \mathrm{Fer}(r,q,p,n)$ *the multiplicity of the representation* (μ,ρ) *in* R_λ^G *is equal to*

$$|\{T \in \mathrm{ST}(r,q,p,n) : \mu(T) = \mu \text{ and } \lambda(T) = \lambda\}|.$$

Tensorial and diagonal action

Consider the algebra $S_q[X, Y] := S_q[X] \otimes S_q[Y]$ in $2n$ variables. We consider the natural action of $G \times G$ and of its diagonal subgroup ΔG on $S_q[X, Y]$.

Tensorial and diagonal action

Consider the algebra $S_q[X,Y] := S_q[X] \otimes S_q[Y]$ in $2n$ variables.
We consider the natural action of $G \times G$ and of its diagonal subgroup ΔG on $S_q[X,Y]$.
We let

$$a_g(X,Y) = \frac{1}{|G|} \sum_{h \in \Delta G} h(x_1^{\lambda_1(g)} \cdots x_n^{\lambda_n(g)} y_{\sigma(1)}^{\lambda_1(g^{-1})} \cdots y_{\sigma(n)}^{\lambda_n(g^{-1})})$$

Tensorial and diagonal action

Consider the algebra $S_q[X, Y] := S_q[X] \otimes S_q[Y]$ in $2n$ variables.
We consider the natural action of $G \times G$ and of its diagonal subgroup ΔG on $S_q[X, Y]$.
We let

$$a_g(X, Y) = \frac{1}{|G|} \sum_{h \in \Delta G} h(x_1^{\lambda_1(g)} \cdots x_n^{\lambda_n(g)} y_{\sigma(1)}^{\lambda_1(g^{-1})} \cdots y_{\sigma(n)}^{\lambda_n(g^{-1})})$$

Main theorem

Let $G = G(r, p, q, n)$. The set $\{a_g(X, Y) : g \in G^*\}$ is a basis for $S_q[X, Y]^{\Delta G}$ as $S_q[X, Y]^{G \times G}$-module.

Tensorial and diagonal action

Consider the algebra $S_q[X,Y] := S_q[X] \otimes S_q[Y]$ in $2n$ variables.
We consider the natural action of $G \times G$ and of its diagonal subgroup ΔG on $S_q[X,Y]$.
We let

$$a_g(X,Y) = \frac{1}{|G|} \sum_{h \in \Delta G} h(x_1^{\lambda_1(g)} \cdots x_n^{\lambda_n(g)} y_{\sigma(1)}^{\lambda_1(g^{-1})} \cdots y_{\sigma(n)}^{\lambda_n(g^{-1})})$$

Main theorem

Let $G = G(r,p,q,n)$. The set $\{a_g(X,Y) : g \in G^*\}$ is a basis for $S_q[X,Y]^{\Delta G}$ as $S_q[X,Y]^{G \times G}$-module.

This result was known in type A and B only (Garsia-Gessel, F.Bergeron-Lamontagne, F.Bergeron-Biagioli).

Mahonian distributions

If we consider the Hilbert series with respect to the bipartition degree we have

Corollary

We have

$$\frac{\mathrm{Hilb}(S_q[X,Y]^{\Delta G})}{\mathrm{Hilb}(S_q[X,Y]^{G\times G})}(Q,T)=\sum_{g\in G^*} Q^{\lambda(g)}T^{\lambda(g^{-1})},$$

where $Q^\lambda := q_1^{\lambda_1}\cdots q_n^{\lambda_n}$ *and similarly for* T*.*

If we consider the Hilbert series with respect to the bipartition degree we have

Corollary

We have

$$\frac{\mathrm{Hilb}(S_q[X,Y]^{\Delta G})}{\mathrm{Hilb}(S_q[X,Y]^{G\times G})}(Q,T)=\sum_{g\in G^*} Q^{\lambda(g)}T^{\lambda(g^{-1})},$$

where $Q^\lambda := q_1^{\lambda_1}\cdots q_n^{\lambda_n}$ *and similarly for* T.

and its unrefined version

$$\frac{\mathrm{Hilb}(S_q[X,Y]^{\Delta G})}{\mathrm{Hilb}(S_q[X,Y]^{G\times G})}(q,t)=\sum_{g\in G^*} q^{\mathrm{fmaj}(g)}t^{\mathrm{fmaj}(g^{-1})}.$$

Kronecker coefficients

Let $f^{\phi}(Q)$ be the polynomial whose coefficient of Q^{λ} is the multiplicity of the irreducible representation ϕ of G in R^G_{λ}.

Theorem (C)

$$\frac{\mathrm{Hilb}(S_q[X,Y,Z]^{G^{\times 3}})}{\mathrm{Hilb}(S_q[X,Y,Z]^{\Delta G})} = \sum_{\phi_1,\phi_2,\phi_3} g_{\phi_1,\phi_2,\phi_3} f^{\phi_1}(Q_1) f^{\phi_2}(Q_2) f^{\phi_3}(Q_3)$$

Corollary

$$\sum_{g_1 g_2 g_3 = 1} Q_1^{\lambda(g_1)} Q_2^{\lambda(g_2)} Q_3^{\lambda(g_3)} = \sum_{T_1,T_2,T_3} g_{\mu(T_1),\mu(T_2),\mu(T_3)} Q_1^{\lambda(T_1)} Q_2^{\lambda(T_2)} Q_3^{\lambda(T_3)}$$

where $g_{\mu_1,\mu_2,\mu_3} = \sum_{\rho_1,\rho_2,\rho_3} g_{(\mu_1,\rho_1),(\mu_2,\rho_2),(\mu_3,\rho_3)}$.

A problem of Barcelo-Reiner-Stanton

If $\sigma \in \mathrm{Gal}(\mathbb{Q}[\zeta_r], \mathbb{Q})$ then $\sigma \in \mathrm{Aut}(G)$, where $G = G(r, p, q, n)$.

A problem of Barcelo-Reiner-Stanton

If $\sigma \in \mathrm{Gal}(\mathbb{Q}[\zeta_r], \mathbb{Q})$ then $\sigma \in \mathrm{Aut}(G)$, where $G = G(r, p, q, n)$.
$\Delta^\sigma G = \{(g, \sigma g) : g \in G\} \subseteq G \times G$.

Theorem

$$G^\sigma(Q, T) := \mathrm{Hilb}\left(\frac{S_q[X, Y]^{\Delta^\sigma G}}{I_+^{G \times G}}\right)(Q, T) = \sum_{\phi \in \mathrm{Irr}(G)} f^{\sigma\phi}(Q) f^{\bar{\phi}}(T).$$

A problem of Barcelo-Reiner-Stanton

If $\sigma \in \mathrm{Gal}(\mathbb{Q}[\zeta_r], \mathbb{Q})$ then $\sigma \in \mathrm{Aut}(G)$, where $G = G(r, p, q, n)$.
$\Delta^\sigma G = \{(g, \sigma g) : g \in G\} \subseteq G \times G$.

Theorem

$$G^\sigma(Q, T) := \mathrm{Hilb}\left(\frac{S_q[X, Y]^{\Delta^\sigma G}}{I_+^{G \times G}}\right)(Q, T) = \sum_{\phi \in \mathrm{Irr}(G)} f^{\sigma\phi}(Q) f^{\bar{\phi}}(T).$$

Corollary

$$G^\sigma(Q, T) = \sum_{g \in G^*} Q^{\lambda(\sigma g)} T^{\lambda(g^{-1})}.$$

A problem of Barcelo-Reiner-Stanton

If $\sigma \in \mathrm{Gal}(\mathbb{Q}[\zeta_r], \mathbb{Q})$ then $\sigma \in \mathrm{Aut}(G)$, where $G = G(r, p, q, n)$.
$\Delta^\sigma G = \{(g, \sigma g) : g \in G\} \subseteq G \times G$.

Theorem

$$G^\sigma(Q, T) := \mathrm{Hilb}\left(\frac{S_q[X, Y]^{\Delta^\sigma G}}{I_+^{G\times G}}\right)(Q, T) = \sum_{\phi \in \mathrm{Irr}(G)} f^{\sigma\phi}(Q) f^{\bar{\phi}}(T).$$

Corollary

$$G^\sigma(Q, T) = \sum_{g \in G^*} Q^{\lambda(\sigma g)} T^{\lambda(g^{-1})}.$$

The unrefined version of the previous corollary

$$G^\sigma(q, t) = \sum_{g \in G^*} q^{\mathrm{fmaj}(\sigma g)} t^{\mathrm{fmaj}(g^{-1})}$$

is a solution of a problem posed by Barcelo, Reiner and Stanton.