

Invariant theory of projective reflection groups, and their Kronecker coefficients

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November 23, 2009



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Theorem (MacMahon)

$$\begin{aligned} W(q) &= \sum_{\sigma \in S_n} q^{\text{maj}(\sigma)} = \sum_{\sigma \in S_n} q^{\text{inv}(\sigma)} \\ &= \prod_{i=1}^n (1 + q + q^2 + \cdots + q^i), \end{aligned}$$

where $\text{inv}(\sigma) = |\{(i, j) : i < j \text{ and } \sigma(i) > \sigma(j)\}|$.

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Theorem (Lusztig, Kraskiewicz-Weyman)

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The polynomial $W(q, t)$ is the bimahonian distribution.

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Theorem (Garsia-Gessel)

$$W(q, t) = \sum_{\sigma \in S_n} q^{\text{maj}(\sigma)} t^{\text{maj}(\sigma^{-1})}$$

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The equality between the first and the last line follows also immediately from the Robinson-Schensted correspondence.

The Robinson-Schensted correspondence

Let $\sigma = 31542$. Then

$$\sigma \xrightarrow{RS} \left[\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 & \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 4 \\ \hline 5 & \\ \hline \end{array} \right]$$

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Use Kronecker coefficients!

$$g_{\lambda, \mu, \nu} := \frac{1}{n!} \sum_{\sigma \in S_n} \chi^\lambda(\sigma) \chi^\mu(\sigma) \chi^\nu(\sigma).$$

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Here X, Y, Z stand for three n -tuples of variables

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Extending this we can also decompose the algebra

$$\mathbb{C}[X, Y, Z]^{\Delta S_n} / \mathbb{C}[X, Y, Z]_+^{S_n \times 3}$$

in homogeneous components whose degrees are triples of partitions with at most n parts.

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So similarly to the case of the total degree we have

$$\begin{aligned}
 W(Q_1, Q_2, Q_3) &= \sum_{T_1, T_2, T_3} g_{\mu(T_1), \mu(T_2), \mu(T_3)} Q_1^{\lambda(T_1)} Q_2^{\lambda(T_2)} Q_3^{\lambda(T_3)} \\
 &= \sum_{\lambda, \mu, \nu} g_{\lambda, \mu, \nu} f^\lambda(Q_1) f^\mu(Q_2) f^\nu(Q_3) \\
 &= \text{Hilb} \left(\frac{\mathbb{C}[X, Y, Z]^{\Delta S_n}}{(\mathbb{C}[X, Y, Z]_+^{S_n^{\times 3}})} \right) (Q_1, Q_2, Q_3) \\
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 \end{aligned}$$

The generalized Robison-Schensted correspondence

Corollary

There is a map RS that associates to every triple of permutations whose product is the identity a triple of standard tableaux of size n such that:

- $|\text{RS}^{-1}(T_1, T_2, T_3)| = g_{\mu(T_1), \mu(T_2), \mu(T_3)}$;
- If $(\sigma_1, \sigma_2, \sigma_3) \mapsto (T_1, T_2, T_3)$ then $\text{Des}(T_i) = \text{Des}(\sigma_i) \forall i$.

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Let $\tilde{g}_{\lambda, \mu, \nu} \in \mathbb{N}$ for all triples of partitions λ, μ, ν of n .

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The Kronecker coefficients are uniquely determined by this!

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From symmetric to complex reflection groups

Vic Reiner observed that one can reobtain (in a non trivial way) all the interpretations of the refined multimahonian distribution using the Stanley-Reisner ring of the barycentric subdivision of an $n - 1$ -dimensional complex instead of the coinvariant algebras, these being isomorphic.

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That's why I was convinced that my approach is better...
...and I was led to introduce projective complex reflection groups.

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Complex reflection groups are subgroups of $GL(n, \mathbb{C})$ generated by reflections, i.e. elements of finite order that fix a hyperplane pointwise.

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Example

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$G(r, p, n)$, the elements in $G(r, n)$ whose permanent is a r/p -th root of unity. The matrix above is an element in $G(4, 2, 4)$.

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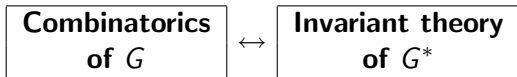
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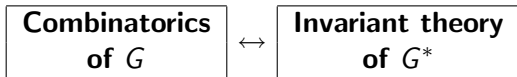
We observe that if G is a complex reflection group then G^* is not in general.

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- If $G = G(r, 1, 1, n)$ then $G^* = G$. This holds in particular for $S_n = G(1, 1, 1, n)$ and $B_n = G(2, 1, 1, n)$.

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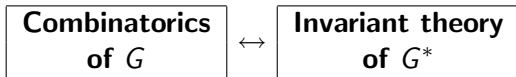


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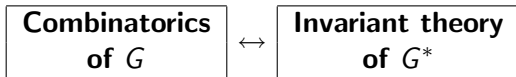


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- If $G = D_n = G(2, 2, 1, n)$, then $G^* = G(2, 1, 2, n) = B_n/\pm I$ and it turns out that the combinatorics of $B_n/\pm I$ describes the invariant theory of D_n , and viceversa.

The combinatorics

$$g = \begin{bmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 1 \\ i & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & \zeta_4^3 & 0 \\ 0 & 0 & 0 & \zeta_4^0 \\ \zeta_4^1 & 0 & 0 & 0 \\ 0 & \zeta_4^2 & 0 & 0 \end{bmatrix},$$

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Let $g \in G(r, p, q, n)$ and $\sigma = |g|$ be its projection in S_n . Let

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$$k_i(g) := \begin{cases} [z_n]_{r/q} & \text{if } i = n \\ k_{i+1} + [z_i - z_{i+1}]_r & \text{if } i \in [n-1]. \end{cases}$$

The projective flag-major index

Letting $\lambda_i(g) := r \cdot h_i(g) + k_i(g)$ then the sequence

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$$\text{fmaj}(g) := |\lambda(g)|$$

for all groups $G(r, p, q, n)$.

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$G = G(r, p, q, n)$ naturally acts on $S_q[X]$, the q -th Veronese subalgebra of $\mathbb{C}[X] := \mathbb{C}[x_1, \dots, x_n]$, i.e. the subalgebra generated in degree q .

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$\text{Fer}(r, p, n) = r$ -tuples of Ferrers diagrams $(\lambda^{(0)}, \dots, \lambda^{(r-1)})$
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Theorem

The irreducible representations of $G(r, p, q, n)$ are naturally parametrized by pairs (μ, ρ) , where $\mu \in \text{Fer}(r, q, p, n)$ and $\rho \in (C_p)_\mu$, the stabilizer of any element in the class μ .

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The descent representations

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Theorem

If $\mu \in \text{Fer}(r, q, p, n)$ the multiplicity of the representation (μ, ρ) in R_λ^G is equal to

$$|\{T \in \text{ST}(r, q, p, n) : \mu(T) = \mu \text{ and } \lambda(T) = \lambda\}|.$$

Tensorial and diagonal action

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This result was known in type A and B only (Garsia-Gessel, F.Bergeron-Lamontagne, F.Bergeron-Biagioli).

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If we consider the Hilbert series with respect to the bipartition degree we have

Corollary

We have

$$\frac{\text{Hilb}(S_q[X, Y]^{\Delta G})}{\text{Hilb}(S_q[X, Y]^{G \times G})}(Q, T) = \sum_{g \in G^*} Q^{\lambda(g)} T^{\lambda(g^{-1})},$$

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and its unrefined version

$$\frac{\text{Hilb}(S_q[X, Y]^{\Delta G})}{\text{Hilb}(S_q[X, Y]^{G \times G})}(q, t) = \sum_{g \in G^*} q^{\text{fmaj}(g)} t^{\text{fmaj}(g^{-1})}.$$

Kronecker coefficients

Let $f^\phi(Q)$ be the polynomial whose coefficient of Q^λ is the multiplicity of the irreducible representation ϕ of G in R_λ^G .

Theorem (C)

$$\frac{\text{Hilb}(S_q[X, Y, Z]^{G \times 3})}{\text{Hilb}(S_q[X, Y, Z]^{\Delta G})} = \sum_{\phi_1, \phi_2, \phi_3} g_{\phi_1, \phi_2, \phi_3} f^{\phi_1}(Q_1) f^{\phi_2}(Q_2) f^{\phi_3}(Q_3)$$

Corollary

$$\sum_{g_1 g_2 g_3 = 1} Q_1^{\lambda(g_1)} Q_2^{\lambda(g_2)} Q_3^{\lambda(g_3)} = \sum_{T_1, T_2, T_3} g_{\mu(T_1), \mu(T_2), \mu(T_3)} Q_1^{\lambda(T_1)} Q_2^{\lambda(T_2)} Q_3^{\lambda(T_3)}$$

where $g_{\mu_1, \mu_2, \mu_3} = \sum_{\rho_1, \rho_2, \rho_3} g_{(\mu_1, \rho_1), (\mu_2, \rho_2), (\mu_3, \rho_3)}$.

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If $\sigma \in \text{Gal}(\mathbb{Q}[\zeta_r], \mathbb{Q})$ then $\sigma \in \text{Aut}(G)$, where $G = G(r, p, q, n)$.

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$$G^\sigma(Q, T) := \text{Hilb} \left(\frac{S_q[X, Y]^{\Delta^\sigma G}}{I_+^{G \times G}} \right) (Q, T) = \sum_{\phi \in \text{Irr}(G)} f^{\sigma\phi}(Q) f^{\bar{\phi}}(T).$$

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The unrefined version of the previous corollary

$$G^\sigma(q, t) = \sum_{g \in G^*} q^{\text{fmaj}(\sigma g)} t^{\text{fmaj}(g^{-1})}$$

is a solution of a problem posed by Barcelo, Reiner and Stanton.