

Qualitative features of Hamiltonian systems through averaging and reduction

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Abstract

In this work we analyze the existence and stability of periodic solutions to a Hamiltonian vector field which is a small perturbation of a vector field tangent to the fibers of a circle bundle. By averaging the perturbation over the fibers of the circle bundle one obtains a Hamiltonian system on the reduced (orbit) space of the circle bundle. First we state results which have hypotheses on the reduced system and have conclusions about the full system. The second part is devoted to the application of the general results to the spatial lunar problem of celestial mechanics, i.e. the restricted three-body problem where the infinitesimal is close to one of the primaries. After scaling, the lunar problem is a perturbation of the Kepler problem, which after regularization is a circle bundle flow. We prove the existence of four families of periodic solutions for any small regular perturbation of the spatial Kepler problem: we find the classical near circular periodic solutions and the near rectilinear periodic solutions for all values of the small parameter. Finally we compute their approximate multipliers.

1 Introduction

In this work we present some of the results obtained in [12]. In the early 1950s two ground breaking papers by Reeb [9] and Seifert [10] investigated the existence of periodic solutions to a vector field which is a small perturbation of a vector field tangent to the fibers of a circle bundle. Seifert considers perturbations of the Hopf foliation of the three sphere and uses an index theory argument. Reeb considers the general perturbation of a circle bundle field and pays particular attention to Hamiltonian flows. By averaging the perturbation over the fibers of the circle bundle Reeb obtains a vector field on the base (reduced) space.

If the original system is Hamiltonian then so is the system on the base space. He is able to give necessary conditions for the existence of periodic solutions by looking at the system on the base alone.

Since then there has been a multitude of papers which analyze a system by looking at the reduced system only, see [12] for some references. One starts with a small parameter which is a measure of the perturbation of an integrable system where all the solutions are periodic. Then one averages the perturbation term by term in the small parameter. After a finite number of terms have been averaged the higher order perturbations are truncated thus obtaining an approximation of the full system. This approximation is well defined on the lower dimensional reduced space. Being lower dimensional the system on the reduced space is easier to understand. But, not all the features of the reduced system accurately portray the original full system. It typically does not see the breakdown of invariant tori, ergodic regions or solenoids.

2 Averaging Theorems

Let (M, Ω) be a symplectic manifold of dimension $2n$, $\mathcal{H}_0 : M \rightarrow \mathbb{R}$ a smooth Hamiltonian, which defines a Hamiltonian vector field $Y_0 = (d\mathcal{H}_0)^\#$ with symplectic flow ϕ_0^t . Let $\mathbb{I} \subset \mathbb{R}$ be an interval such that each $h \in \mathbb{I}$ is a regular value of \mathcal{H}_0 and $\mathcal{N}_0(h) = \mathcal{H}_0^{-1}(h)$ is a compact connected circle bundle over a base space $B(h)$ with projection $\pi : \mathcal{N}_0(h) \rightarrow B(h)$. Assume the vector field Y_0 is everywhere tangent to the fibers of $\mathcal{N}_0(h)$, i.e. assume that all the solutions of Y_0 in $\mathcal{N}_0(h)$ are periodic. There is no loss of generality by assuming that all these periodic solutions have periods smoothly depending only on the value of the Hamiltonian, i.e. the period is a smooth function $T(h)$.

2.1 Reeb's Theorems

Here we state two of Reeb's theorems in more modern terminology.

Theorem 2.1 *The base space B inherits a symplectic structure ω from (M, Ω) , i.e. (B, ω) is a symplectic manifold.*

This is the original reduction theorem. Now let us look at a perturbation of this situation. Let ε be a small parameter, $\mathcal{H}_1 : M \rightarrow \mathbb{R}$ smooth, $\mathcal{H}_\varepsilon = \mathcal{H}_0 + \varepsilon\mathcal{H}_1$, $Y_\varepsilon = Y_0 + \varepsilon Y_1 = d\mathcal{H}_\varepsilon^\#$, $\mathcal{N}_\varepsilon(h) = \mathcal{H}_\varepsilon^{-1}(h)$, and ϕ_ε^t the flow defined by Y_ε .

Let the average of \mathcal{H}_1 be

$$\bar{\mathcal{H}} = \frac{1}{T} \int_0^T \mathcal{H}_1(\phi_0^t) dt,$$

which is a smooth function on $B(h)$, and let $\bar{\phi}^t$ be the flow on $B(h)$ defined by $\bar{Y} = d\bar{\mathcal{H}}^\#$.

A critical point of $\bar{\mathcal{H}}$ is *nondegenerate* if the Hessian at the critical point is nonsingular and the function $\bar{\mathcal{H}}$ is a *Morse function* if all its critical points are nondegenerate. The *index* of a nondegenerate critical point p of $\bar{\mathcal{H}}$ is the dimension of the maximal linear subspace where the Hessian of $\bar{\mathcal{H}}$ at p is negative definite.

Theorem 2.2 *If $\bar{\mathcal{H}}$ has a nondegenerate critical point at $\pi(p) = \bar{p} \in B$ with $p \in \mathcal{N}_0$, then there are smooth functions $p(\varepsilon)$ and $T(\varepsilon)$ for ε small with $p(0) = p$, $T(0) = T$, $p(\varepsilon) \in \mathcal{N}_\varepsilon$ and the solution of Y_ε through $p(\varepsilon)$ is $T(\varepsilon)$ -periodic.*

If $\bar{\mathcal{H}}$ is a Morse function then Y_ε has at least $\chi(B)$ periodic solutions, where $\chi(B)$ is the Euler-Poincaré characteristic of B .

Lemma 2.1 is the key for an original direct proof of Reeb's Theorems using symplectic geometry arguments [12]. These alternative proofs lead to further applications. The essence of the proof of the local part of Theorem 2.2 is the existence of symplectic coordinates for a tubular neighborhood of the orbit through p , which is the result provided by Lemma 2.1.

Lemma 2.1 *Let $p \in \mathcal{N}_0(h)$, with $h \in \mathbb{I}$ fixed. Then there are symplectic coordinates (I, θ, y) valid in a tubular neighborhood of the periodic solution $\phi_0^t(p)$ of $Y_0(h)$ where (I, θ) are action-angle coordinates and $y \in \mathbb{N}$ where \mathbb{N} is an open neighborhood of the origin in \mathbb{R}^{2n-2} . The point p corresponds to $(I, \theta, y) = (0, 0, 0)$.*

In these coordinates $\mathcal{H}_0 = \mathcal{H}_0(I)$. A local cross section is $\theta = \alpha$ and a local cross section in an energy level is $\theta = \alpha, I = \beta$, where α, β are constants. In addition to that, $y \in \mathbb{N}$ are coordinates in the cross section in the energy level.

The Hamiltonian is

$$\mathcal{H}_\varepsilon(I, \theta, y) = \mathcal{H}_0(I) + \varepsilon \mathcal{H}_1(I, \theta, y) = \mathcal{H}_0(I) + \varepsilon \bar{\mathcal{H}}(I, y) + O(\varepsilon^2). \quad (1)$$

For the proof of this lemma and the proofs of Theorems 2.1 and 2.2, which follow from the lemma, see [12].

2.2 Corollaries

Only the last sentence in Theorem 2.2 gives a truly global result. Those conversant with Morse theory will see there is a sharper global result.

Corollary 2.1 *Let $\bar{\mathcal{H}}$ be a Morse function, let β_j be the j^{th} Betti number of B and let C_j be the number of critical points of index j . Then $C_j \geq \beta_j$ or better yet*

$$\begin{aligned} C_0 &\geq \beta_0 \\ C_1 - C_0 &\geq \beta_1 - \beta_0 \\ C_2 - C_1 + C_0 &\geq \beta_2 - \beta_1 + \beta_0 \\ &\dots \\ C_k - C_{k-1} + C_{k+2} - \dots \pm C_0 &\geq \beta_k - \beta_{k-1} + \beta_{k+2} - \dots \pm \beta_0 \quad (k < 2n - 2) \\ C_0 - C_1 + C_2 - \dots - C_{2n-3} + C_{2n-2} &= \beta_0 - \beta_1 + \beta_2 - \dots - \beta_{2n-3} + \beta_{2n-2} = \chi(B). \end{aligned} \quad (2)$$

For these better inequalities on a Morse function see [6]. The lower estimate on the number of periodic solutions in Theorem 2.2 is $\chi(B)$ the alternating sum of the Betti numbers which could be 0 or negative, whereas, the Morse inequalities give a lower estimate which is the sum of the Betti numbers. Moreover, the estimates give some information on the number of critical points of various indices.

The nontrivial characteristic multipliers of the periodic solution given in Theorem 2.2 are the eigenvalues of

$$\mathcal{P} = \frac{\partial P}{\partial y}(\bar{y}(\varepsilon)) = E + \varepsilon T J \frac{\partial^2 \bar{\mathcal{H}}}{\partial y^2}(0, 0) + O(\varepsilon^2),$$

where E is the identity matrix. The eigenvalues of Hamiltonian matrix

$$A = J \frac{\partial^2 \bar{\mathcal{H}}}{\partial y^2}(0, 0) \quad (3)$$

are the characteristic exponents of the critical point of \bar{Y} at \bar{p} on B . Thus, the lemma also yields:

Corollary 2.2 *Let p be as in Theorem 2.2 and let the characteristic exponents of $\bar{Y}(\bar{p})$ be $\lambda_1, \lambda_2, \dots, \lambda_{2n-2}$, then the characteristic multipliers of the periodic solution through $p(\varepsilon)$ are*

$$1, 1, 1 + \varepsilon \lambda_1 T + O(\varepsilon^2), 1 + \varepsilon \lambda_2 T + O(\varepsilon^2), \dots, 1 + \varepsilon \lambda_{2n-2} T + O(\varepsilon^2).$$

This result was used in [5]. We shall say that a periodic solution is *elliptic* or *linearly stable* if the monodromy matrix is diagonalizable and all the eigenvalues have absolute value 1.

The solution of this problem lies on the Krein-Gel'fand concept of parametric stability [11] which we will briefly summarize.

Consider the linear constant coefficient Hamiltonian system

$$\dot{y} = Cy = J\nabla H(y), \quad H = \frac{1}{2}y^T S y, \quad (4)$$

where S is a symmetric matrix and $C = JS$ is a Hamiltonian matrix. System (4) (or the Hamiltonian matrix C) is *stable* if all its solutions are bounded for all t and it is *parametrically stable* or *strongly stable* if it and all sufficiently small linear constant coefficient Hamiltonian perturbations of it are stable. System (4) is parametrically stable implies it is stable and (4) is stable if and only if C is diagonalizable and has only purely imaginary eigenvalues.

Let $\pm\alpha_1 i, \pm\alpha_2 i, \dots, \pm\alpha_s i$ be the eigenvalues of the stable matrix C and V_j , $j = 1, \dots, s$ be the maximal real linear subspace where C has eigenvalues $\pm\alpha_j i$. So V_j is a C invariant symplectic subspace, C restricted to V_j has eigenvalues $\pm\alpha_j i$, and $\mathbb{R}^{2n} = V_1 \oplus V_2 \oplus \dots \oplus V_s$. Let H_j be the restriction of H to V_j .

Theorem 2.3 [11] *System (4) is parametrically stable if and only if: (i) all the eigenvalues of C are purely imaginary; (ii) C is nonsingular; (iii) C is diagonalizable over the complex numbers; (iv) the Hamiltonian H_j is positive or negative definite for each j .*

Thus, $2H = (u_1^2 + v_1^2) + (u_2^2 + v_2^2)$ and $2H = (u_1^2 + v_1^2) - 4(u_2^2 + v_2^2)$ are parametrically stable, but $2H = (u_1^2 + v_1^2) - (u_2^2 + v_2^2)$ is not parametrically stable.

Consider now the linear T -periodic Hamiltonian system

$$\dot{y} = D(t)y = J\nabla H(y), \quad H = \frac{1}{2}y^T R(t)y, \quad (5)$$

where $R(t) = R(t + T)$ is symmetric and $D(t) = JR(t)$ is Hamiltonian. The periodic system (5) is *stable* if all its solutions are bounded for all t and it is *parametrically stable* or *strongly stable* if it and all sufficiently small linear T -periodic Hamiltonian perturbations of it are stable. The monodromy matrix is $M = \mathbf{Z}(T)$ where $\mathbf{Z}(t)$ is a fundamental matrix solution of (5). If the system is parametrically stable then it is stable and (5) is stable if

and only if its monodromy matrix is diagonalizable and has only eigenvalues (multipliers) of unit modulus.

Let $\beta_1^{\pm 1}, \beta_2^{\pm 1}, \dots, \beta_s^{\pm 1}$ be the eigenvalues of M and $V_j, j = 1, \dots, s$ be the maximal real linear subspace where M has eigenvalues $\beta_j^{\pm 1}$. So V_j is an M invariant symplectic subspace, M restricted to V_j (denoted by M_j) is symplectic and has eigenvalues $\beta_j^{\pm 1}$, and $\mathbb{R}^{2n} = V_1 \oplus V_2 \oplus \dots \oplus V_s$.

For periodic systems we need to define the analog of the quadratic form H_j . We use Cayley transformation to do this. The particular Möbius transformation

$$\Psi : z \rightarrow w = (z - 1)(z + 1)^{-1}, \quad \Psi^{-1} : w \rightarrow z = (1 + w)(1 - w)^{-1}$$

is known as the *Cayley transformation*. One checks that $\Psi(1) = 0, \Psi(i) = i, \Psi(-1) = \infty$ and so Ψ takes the unit circle in the z -plane to the imaginary axis in the w -plane, the interior of the unit circle in the z -plane to the left half w -plane, etc. Transformation Ψ can be applied to any matrix B which does not have -1 as an eigenvalue and λ is an eigenvalue of B if and only if $\Psi(\lambda)$ is an eigenvalue of $\Psi(B)$.

Lemma 2.2 *Let M be a symplectic matrix which does not have the eigenvalue -1 then $C = \Psi(M)$ is a Hamiltonian matrix. Moreover, if M has only eigenvalues of unit modulus and is diagonalizable, then $C = \Psi(M)$ has only purely imaginary eigenvalues and is diagonalizable.*

Matrix M_j is the restriction of M to V_j and is symplectic, so $C_j = \Psi(M_j)$ is a Hamiltonian matrix and $S_j = JC_j$ is a symmetric matrix.

Theorem 2.4 [11] *System (5) is parametrically stable if and only if: (i) all the eigenvalues of M have unit modulus; (ii) M does not have the eigenvalue $+1$ nor -1 ; (iii) M is diagonalizable over the complex numbers; (iv) the symmetric matrix S_j is positive or negative definite for each j .*

Corollary 2.3 *If one or more of the λ_j of Corollary 2.2 is real or has a real part then the periodic solution through $p(\varepsilon)$ is unstable. If the matrix A in (3) is the coefficient matrix of a parametrically stable system then the periodic solution through $p(\varepsilon)$ is elliptic. In particular, if \bar{p} is a nondegenerate maximum or minimum of $\bar{\mathcal{H}}$, then the periodic solution through $p(\varepsilon)$ is elliptic. If $\bar{\mathcal{H}}$ is a Morse function then there are at least $2 = \beta_0 + \beta_{2n-2}$ elliptic periodic solutions.*

We believe this application of Krein-Gel'fand theory to be new. For the proof, see [12].

3 The Spatial Lunar Problem

3.1 The Hamiltonians

The Hamiltonian of the spatial problem is given in the rotating frame by:

$$\mathcal{H} = \frac{1}{2}(y_1^2 + y_2^2 + y_3^2) - (x_1y_2 - x_2y_1) - \frac{\mu}{\sqrt{(x_1 - 1 + \mu)^2 + x_2^2 + x_3^2}} - \frac{1 - \mu}{\sqrt{(x_1 + \mu)^2 + x_2^2 + x_3^2}}.$$

We change variables, scale time and scale the Hamiltonian in order to arrive at the lunar case of the spatial restricted circular three body problem, see [4]. After expansion in powers of the small parameter we end up with the system

$$\mathcal{H}_\varepsilon = \frac{1}{2}(y_1^2 + y_2^2 + y_3^2) - \frac{1}{\sqrt{x_1^2 + x_2^2 + x_3^2}} - \varepsilon^3(x_1y_2 - x_2y_1) + \frac{1}{2}\varepsilon^6\mu(-2x_1^2 + x_2^2 + x_3^2) + \dots$$

Now we have a perturbation of the spatial Kepler problem. Moser has shown that the three-dimensional Kepler problem can be regularized and the regularized flow is equivalent to the geodesic flow on S^3 [7, 12].

The following step consists in expressing \mathcal{H}_ε in such a way that we can perform Lie transformations conveniently, see [3]. We use polar-nodal $(r, \vartheta, \nu, R, G, N)$ and Delaunay coordinates (ℓ, g, ν, L, G, N) . The angle ℓ corresponds to the mean anomaly, g to the argument of the pericenter, ν is the argument of the node, L the square of the semimajor axis, G is the third component of angular momentum vector \mathbf{G} and N is the third component of the angular momentum, so $0 \leq |N| \leq G \leq L$. The coordinate R is the momentum conjugate to the radial variable r and the angle ϑ is the argument of the latitude. Expressing \mathcal{H}_ε in these variables we get

$$\begin{aligned} \mathcal{H}_\varepsilon = & -\frac{1}{2L^2} - \varepsilon^3 N + \frac{1}{8}\varepsilon^6\mu r^2 \left(1 - 3c^2 - 3(1 - c^2)\cos(2\vartheta)\right. \\ & \left. - 3(1 - c^2 + (1 + c^2)\cos(2\vartheta))\cos(2\nu) + 6c\sin(2\nu)\sin(2\vartheta)\right) + \dots, \end{aligned}$$

where $c = N/G$. After performing the normalization of Delaunay to a fixed finite order we arrive at the Hamiltonian

$$\begin{aligned} \mathcal{H}_\varepsilon = & -\frac{1}{2L^2} - \varepsilon^3 N + \frac{1}{16}\varepsilon^6\mu L^4 \left((2 + 3e^2)(1 - 3c^2 - 3(1 - c^2)\cos(2\nu))\right. \\ & \left. - 15e^2\cos(2g)(1 - c^2 + (1 + c^2)\cos(2\nu)) + 30ce^2\sin(2g)\sin(2\nu)\right) + \dots \end{aligned} \quad (6)$$

This normal form Hamiltonian has been calculated previously in [8]. The transformed Hamiltonian, after truncating higher order terms, depends on the two angles g and ν plus their associated momenta G and N respectively, whereas L is an integral of motion. Applying reduction theory, once higher order terms have been dropped, \mathcal{H}_ε is defined on the orbit space, or base space, which is the four dimensional space $S^2 \times S^2$ [7].

We can use the set of variables given by $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ with the constraints $a_1^2 + a_2^2 + a_3^2 = L^2$ and $b_1^2 + b_2^2 + b_3^2 = L^2$ to parameterize $S^2 \times S^2$, where $\mathbf{a} = \mathbf{G} + LA$ and $\mathbf{b} = \mathbf{G} - LA$. We recall that \mathbf{G} corresponds to the angular momentum vector and A is the Laplace-Runge-Lenz vector; moreover, $|\mathbf{a}| = |\mathbf{b}| = L$. Notice that the a_i and b_i belong to the interval $[-L, L]$. The explicit expressions for \mathbf{a} and \mathbf{b} in terms of Delaunay variables are found in Cushman [2]. We remark that the introduction of the invariants extends the use of the Delaunay variables as we can include equatorial, circular and rectilinear orbits.

After several simplifications and manipulations over \mathcal{H}_ε including the dropping of the constant term $-1/(2L^2)$ and the division by ε^3 , we arrive at

$$\bar{\mathcal{H}} = -\frac{1}{2}(a_3 + b_3) - \frac{1}{8}\varepsilon^3\mu L^2(3a_1^2 - 3a_2^2 - 3a_3^2 - 12a_1b_1 + 3b_1^2 + 6a_2b_2 - 3b_2^2 + 6a_3b_3 - 3b_3^2) + \dots \quad (7)$$

The corresponding equations of motion are

$$\begin{aligned}
 \dot{a}_1 &= a_2 - \frac{3}{2}\varepsilon^3 \mu L^2 (a_3 b_2 - a_2 b_3) + \dots, \\
 \dot{a}_2 &= -a_1 + \frac{3}{2}\varepsilon^3 \mu L^2 (2a_1 a_3 - 2a_3 b_1 - a_1 b_3) + \dots, \\
 \dot{a}_3 &= -\frac{3}{2}\varepsilon^3 \mu L^2 (2a_1 a_2 - 2a_2 b_1 - a_1 b_2) + \dots, \\
 \dot{b}_1 &= b_2 + \frac{3}{2}\varepsilon^3 \mu L^2 (a_3 b_2 - a_2 b_3) + \dots, \\
 \dot{b}_2 &= -b_1 - \frac{3}{2}\varepsilon^3 \mu L^2 (a_3 b_1 + 2a_1 b_3 - 2b_1 b_3) + \dots, \\
 \dot{b}_3 &= \frac{3}{2}\varepsilon^3 \mu L^2 (a_2 b_1 + 2a_1 b_2 - 2b_1 b_2) + \dots.
 \end{aligned} \tag{8}$$

We stress that the equations of motion are global in the whole base space B . Including terms of order ε^3 is enough to determine the relative equilibria of $\bar{\mathcal{H}}$.

3.2 Analysis of Equilibria

The Hamiltonian (7) starts as $\bar{\mathcal{H}} = -\frac{1}{2}(a_3 + b_3) + \dots$ so it has a nondegenerate maximum at $(\mathbf{a}, \mathbf{b}) = (0, 0, -L, 0, 0, -L)$ and a nondegenerate minimum at $(\mathbf{a}, \mathbf{b}) = (0, 0, L, 0, 0, L)$. The index of $(0, 0, L, 0, 0, L)$ is 0 whereas $(0, 0, -L, 0, 0, -L)$ has index 4. Now by Reeb's Theorem 2.2 and Corollary 2.2, the points $(0, 0, \pm L, 0, 0, \pm L)$ correspond to elliptic periodic solutions of the spatial restricted three-body problem of period $T(\varepsilon) = T + O(\varepsilon^3)$. These are the circular equatorial motions already present in the planar case [12].

Hamiltonian $\bar{\mathcal{H}}$ also has two other nondegenerate critical points of index 2 at $(\mathbf{a}, \mathbf{b}) = (0, 0, \pm L, 0, 0, \mp L)$ which correspond to rectilinear motions whose projection in the coordinate space leads to periodic solutions in the vertical axis x_3 . They generalize the rectilinear trajectories found by Belbruno [1] for small μ . However the minimax critical points at $(0, 0, \pm L, 0, 0, \mp L)$ are not parametrically stable as their corresponding linearization is of the type $\mp \frac{1}{2}(u_1^2 + v_1^2) \pm \frac{1}{2}(u_2^2 + v_2^2)$ and so small linear perturbations can lead to unstable periodic solutions. Thus we cannot conclude at this point that these equilibria give rise to elliptic periodic solutions. A deeper analysis is needed to decide about the stability of those periodic solutions arising from $(0, 0, \pm L, 0, 0, \mp L)$.

The Betti numbers of $S^2 \times S^2$ are $\beta_0 = \beta_4 = 1$, $\beta_2 = 2$ and all the others are zero. Besides, $C_0 = C_4 = 1$, $C_2 = 2$ and $C_j = 0$ for $j \notin \{0, 2, 4\}$, hence $C_j = \beta_j$ for all j . As we have seen $\bar{\mathcal{H}}$ is a Morse function, and has the minimum number of critical points consistent with the Morse inequalities found in Corollary 2.1.

Near the critical points we can use (a_1, a_2, b_1, b_2) as coordinates on $B = S^2 \times S^2$. From the equations (8) one sees that the characteristic exponents of all the four critical points of Y_ε at the four equilibria are $\pm i$ (double). Thus, by Corollary 2.2, the characteristic multipliers of the corresponding periodic solutions are: $1, 1, 1 + \varepsilon^3 T i, 1 + \varepsilon^3 T i, 1 - \varepsilon^3 T i, 1 - \varepsilon^3 T i$ plus terms of order ε^6 . As we have said the maximum and minimum at $(0, 0, \pm L, 0, 0, \pm L)$ give rise to elliptic periodic solutions, following Corollary 2.3.

A final remark is that we have proven that any small perturbation of the spatial Kepler problem has at least four periodic solutions, two of them of circular type in the equatorial plane and these solutions are elliptic. The other two solutions are of rectilinear type but their stability analysis is out of the scope of the present paper. We need to undo the linear changes of coordinates to go back to Hamiltonian \mathcal{H} . Thus, the four periodic solutions of

\mathcal{H} have approximate period $T(\varepsilon) \approx 2\pi\varepsilon^3L^3$. Moreover, the near circular solutions have radii $|x| \approx \varepsilon^2L^2$ whereas the near rectilinear solutions have their radii bounded between $\varepsilon^2(1 - e)L^2$ and $\varepsilon^2(1 + e)L^2$ where e is a parameter (the eccentricity) close to 1.

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References

- [1] E. A. Belbruno, *A new family of periodic orbits for the restricted problem*, Celestial Mech. 25, 1981, 195–217.
- [2] R. Cushman, *Reduction, Brouwer's Hamiltonian, and the critical inclination*, Celestial Mech. 31, 1983, 401–429.
- [3] A. Deprit, *Canonical transformations depending on a small parameter*, Celestial Mech. 1, 1969, 12–30.
- [4] R. C. Howison and K. R. Meyer, *Doubly-symmetric periodic solutions of the spatial restricted three-body problem*, J. Differential Equations 163, 2000, 174–197.
- [5] K. R. Meyer, *Symmetries and integrals in mechanics*, in Dynamical Systems (Ed. M. M. Peixoto) Acad. Press, New York, 1973, 259–272.
- [6] J. Milnor, *Morse Theory*, Annals of Math. Studies 51, Princeton Univ. Press, Princeton, NJ, 1963.
- [7] J. Moser, *Regularization of Kepler's problem and the averaging method on a manifold*, Comm. Pure Appl. Math. 23, 1970, 609–636.
- [8] J. F. Palacián and P. Yanguas, *Invariant manifolds of spatial restricted three-body problems: the lunar case*, in New Advances in Celestial Mechanics and Hamiltonian Systems (Eds: J. Delgado, E. A. Lacomba, J. Llibre, E. Pérez-Chavela) Kluwer Academic/Plenum Publishers, New York, 2004.
- [9] G. Reeb, *Sur certaines propriétés topologiques des trajectoires des systèmes dynamiques*, Acad. Roy. Belgique. Cl. Sci. Mém. Coll. in 8°, Ser. 2, 27(9), 1952.
- [10] H. Seifert, *Closed integral curves in 3-space and isotopic two-dimensional deformations*, Proc. Amer. Math. Soc. 1, 1950, 287–302.
- [11] V. A. Yakubovich and V. M. Starzhinskii, *Linear Differential Equations with Periodic Coefficients 1 and 2*, John Wiley & Sons, New York, 1975.
- [12] P. Yanguas, J. Palacián, K. Meyer and H. S. Dumas, *Averaging, regular reduction and reconstruction in Hamiltonian systems*, submitted, 2007.