Mathematical Model for Soil-Structure-Interaction in inverted pendulum type structures

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ABSTRACT

Considering that mass rotational inertia in a structure with an inverted pendulum shape increases period of vibration of structure. Such increase depends on location of mass center, on relation of considered mass inertia, on relation of rigidity between structure and foundation, and on relation of system stiffness. In this work expressions that allow to identify importance of such parameters are proposed, for cases in which the oscillator is supported on hard soil or soft soil. It is defined the magnitude error that occurs when not considering mass rotational inertia in calculation for period is defined.

keywords: Soil-structure interaction, inverted pendulum, bridge.

1. INTRODUCTION

Inverted pendulum type structures have three common characteristics: (1) the upper section mainly concentrates its mass, (2) the development of one single plastic articulation in a column is enough to produce its collapse, and (3) the influence of gravitational forces decreases its capacity to withstand lateral stress.

Flexural momentums at superior ends of columns in this type of structures have special concern. To a great extent its design is ruled by such moments induced by seismic loads, as well as in column as in pavement or cover.

In some of these type of structures, besides horizontal flexural moments also torsion arise in relation to an horizontal axe in roof system, both due to horizontal inertial strengths. By these reasons, specially as stability depends on resistance of one single plastic articulation, inverted pendulums had been very vulnerable to earthquakes [1].

If torsion effect is restricted, as occurs in viaduct type structures, the effect of rotational inertia around bridge longitudinal axe is more important than the one produced around vertical axe. If superstructure width is equal or greater than the height of column, mass center displacements of a column stack as depicted on Fig. 1, can have vertical axe displacements of superstructure which are

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compared in magnitude with lateral displacements. This system cannot be properly represented with one degree of freedom system because the rotational inertia of superstructure could modify the fundamental period, and induce important magnitude moments in mass center, such moments can be positive, or negative, in relation to the base moment at any time, as consequence of response of two degrees of freedom system.

An incorrect modeling of these type of systems can cause to underestimate shear strengths in the column during design process [2].

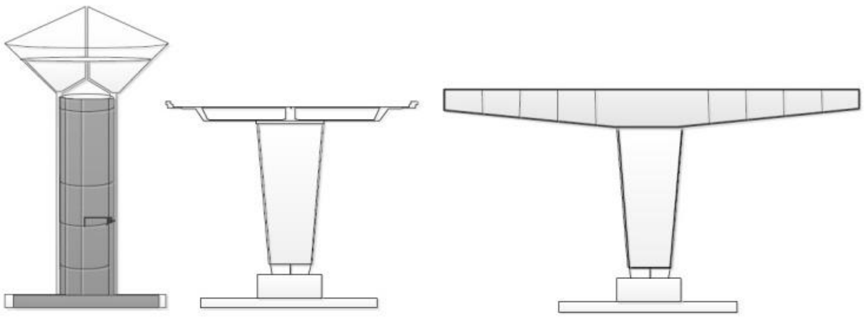


Figure 1. Inverted pendulum type structural systems.

2. DYNAMIC EQUATIONS

Considering an oscillator with mass m supported on an element that provides lateral rigidity with height, L, deep foundation, and mass center position, yc, measured from joint between elements that provides lateral rigidity to system (Figure 2), the restitution strength referred to an inertial system, is

$$F_R = m\ddot{U}_g + m\ddot{z} + m y_c \ddot{\theta} \tag{1}$$

where $\ddot{U}_g$ is ground acceleration and θ is rotation angle of mass center.

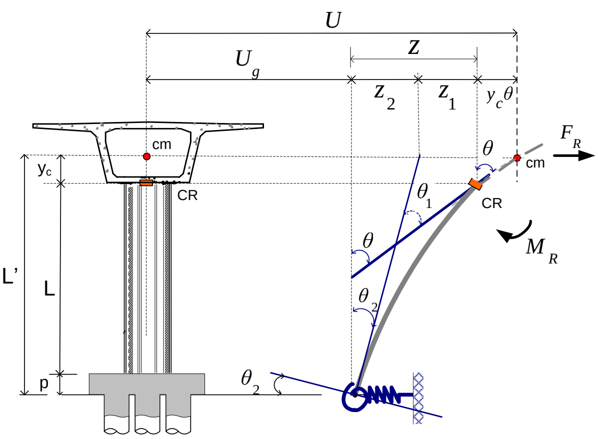


Figure 2. Two DOF structure on flexible base.

Restitution momentum due to mass rotation is:

$$M_R = I_{cm}\ddot{\theta} + F_R\, y_c \quad (2)$$

where I_{cm} is mass inertial momentum in relation to the mass center. Replacing equation (1) in equation (2), is obtained,

$$M_R = \left[I_{cm} + m\, y_c^2 \right] \ddot{\theta} + m\, \ddot{U}_g\, y_c + m\, \ddot{z}\, y_c \quad (3)$$

And as inertia mass moment, in relation to mass base, is,

$$I_b = I_{cm} + m\, y_c^2 \quad (4)$$

Equation (3) can be expressed,

$$M_R = I_b\, \ddot{\theta} + m\, \ddot{U}_g\, y_c + m\, \ddot{z}\, y_c \quad (5)$$

Without considering foundation mass, and considering ground and foundation with properties of translation rigidity, k_c, and with rotational rigidity, k_φ, such that,

$$z = z_1 + z_2 \quad (6)$$

where z_1 is mass lateral displacement and z_2 is mass lateral displacement due to base rotation, and

$$\theta = \theta_1 + \theta_2 \quad (7)$$

Where θ is total angular displacement of mass center and θ_2 is base rotation. Replacing equations (6) and (7) in (1) and (5), there is an equations system that represents oscillator dynamic balance, for the case that it is founded in flexible soil,

$$F_R = m\ddot{U}_g + m\, \ddot{z}_1 + m\, \ddot{z}_2 + m\, y_c \ddot{\theta}_1 + m\, y_c \ddot{\theta}_2 \quad (8)$$

$$M_R = I_b\, \ddot{\theta}_1 + I_b\, \ddot{\theta}_2 + \ddot{U}_g m\, y_c + \ddot{z}_1\, m\, y_c + \ddot{z}_2\, m\, y_c \quad (9)$$

Considering a harmonic excitation,

$$\begin{aligned} \ddot{z}_1 &= \omega^2 z_1 \\ \ddot{z}_2 &= \omega^2 z_2 \\ \ddot{\theta}_1 &= \omega^2 \theta_1 \\ \ddot{\theta}_2 &= \omega^2 \theta_2 \\ \ddot{U}_g &= \omega^2 U_g \end{aligned} \quad (10)$$

Equations (8) and (9) can be expressed like

$$F_R = m\omega^2 \left\lfloor U_g + z_2 + z_1 \right\rfloor + y_c m\omega^2 \left[\theta_1 + \theta_2 \right] \quad (11)$$

$$M_R = I_{cm}\,\omega^2\left[\theta_1+\theta_2\right] + y_c m\,\omega^2\left\lfloor U_g + z_2 + z_1 + y_c\theta\right\rfloor \quad (12)$$

Considering that,

$$\alpha = \frac{F_R}{k} \quad (13)$$

$$\beta = \frac{M_R}{k_r} \quad (14)$$

Equations (6) and (7), are

$$z_1 = \frac{F_R}{k} + \frac{M_R}{k_r}\delta \quad (15)$$

$$\theta_1 = \frac{F_R}{k}\varepsilon + \frac{M_R}{k_r} \quad (16)$$

Due to foundation spin,

$$\theta_2 = \frac{M_0}{k_\varphi} \quad (17)$$

Where k_φ represents foundation angular rigidity and M_0 is foundation momentum. If small displacements are considered, mass center displacement due to foundation spin is,

$$z_2 = \theta_2 L^{'} \quad (18)$$

where,

$$L^{'} = p + L + y_c \quad (19)$$

The foundation horizontal strength is,

$$F_0 = F_R \quad (20)$$

And foundation momentum,

$$M_0 = M_R + F_R L^{'} \quad (21)$$

equations (17) and (18) lead to:

$$\theta_2 = \frac{M_R}{k_\varphi} + F_R\frac{L^{'}}{k_\varphi} \quad (22)$$

and

$$z_2 = M_R \frac{L'}{k_\varphi} + F_R \frac{(L')^2}{k_\varphi} \tag{23}$$

The ground horizontal displacement is,

$$U_g = \frac{F_0}{k_c} = \frac{F_R}{k_c} \tag{24}$$

where k_c represents the translation rigidity of the foundation. The equation (11) can be written,

$$\frac{1}{m\omega^2} = \left[\frac{1}{k} + \frac{1}{k_c} + \frac{L'^2}{k_\varphi}\right] + y_c\left[\frac{\varepsilon}{k} + \frac{L'}{k_\varphi}\right] + \\ +\left[\frac{M_R}{F_R}\right]\left[\left[\frac{\delta}{k_r} + \frac{L'}{k_\varphi}\right] + y_c\left[\frac{1}{k_r} + \frac{1}{k_\varphi}\right]\right] \tag{25}$$

From equation (12),

$$\left[\frac{M_R}{F_R}\right] = \frac{I_b\omega^2\left[\left[\frac{\varepsilon}{k} + \frac{L'}{k_\varphi}\right] + \frac{m^*}{I_b}\left[\frac{1}{k} + \frac{1}{k_c} + \frac{L'^2}{k_\varphi}\right]\right]}{1 - I_b\omega^2\left[\left[\frac{1}{k_r} + \frac{1}{k_\varphi}\right] + \frac{m^*}{I_b}\left[\frac{\delta}{k_r} + \frac{L'}{k_\varphi}\right]\right]} \tag{26}$$

where $m^* = m\,y_c$. Replacing (26) in (25) and sorting terms is obtained the equation that allows to calculate oscillator circular frequencies considering soil-structure interaction (SSI),

$$\omega_{1,2}^2 = \frac{1}{2}\left[E \pm \sqrt{E^2 - \frac{4}{D}}\right] \tag{27}$$

Where,

$$D = I_b m\left[AB - C\right] \tag{28}$$

and

$$E = \frac{Am + I_b B}{D} \tag{29}$$

$$A = \left[\frac{1}{k} + \frac{1}{k_c} + \frac{L'^2}{k_\varphi}\right] + y_c\left[\frac{\varepsilon}{k} + \frac{L'}{k_\varphi}\right] \tag{30}$$

$$B = \left[\frac{1}{k_r} + \frac{1}{k_\varphi} \right] + y_c \frac{m}{I_b} \left[\frac{\delta}{k_r} + \frac{L'}{k_\varphi} \right] \quad (31)$$

$$C = \left\{ \left[\frac{\delta}{k_r} + \frac{L'}{k_\varphi} \right] + y_c \left[\frac{1}{k_r} + \frac{1}{k_\varphi} \right] \right\} \left\{ \left[\frac{\varepsilon}{k} + \frac{L'}{k_\varphi} \right] + y_c \frac{m}{I_b} \left[\frac{1}{k} + \frac{1}{k_c} + \frac{L'^2}{k_\varphi} \right] \right\} \quad (32)$$

Considering that the oscillator is supported on hard soil, then:

$$A = \left[\frac{1}{k} \right] + y_c \left[\frac{\varepsilon}{k} \right] \quad (33a)$$

$$B = \left[\frac{1}{k_r} \right] + y_c \frac{m}{I_b} \left[\frac{\delta}{k_r} \right] \quad (33b)$$

$$C = \left\{ \frac{\delta}{k_r} + y_c \frac{1}{k_r} \right\} \left\{ \frac{\varepsilon}{k} + y_c \frac{m}{I_b} \frac{1}{k} \right\} \quad (33c)$$

It can be verified that results that are obtained with equations (33) are equal to the ones obtained when oscillator is founded in hard soil. Alternatively, the equation (27) can be written,

$$\Omega_{1,2}^2 = \frac{1}{2} \left[F_1 \pm \sqrt{F_1^2 - F_2} \right] \quad (34)$$

Where,

$$F_1 = 1 + 3r_1 + 3r_1^2 + 3r_2 + \alpha + \alpha s \left[1 + 2\left(\frac{r_1}{\tau} \right) + \left(\frac{r_1}{\tau} \right)^2 + \frac{r_2}{\tau^2} \right] \quad (35)$$

$$F_2 = 3r_2 \left[1 + 4\alpha + 4\alpha s \left[1 - \frac{1}{2\tau} \right] \right] + r_2 \left[\frac{\alpha s}{\tau^2} \right] \left[4 + 4\alpha - 6\tau \right] \quad (36)$$

Consider the following relationships,

$$r_1 = \frac{y_c}{L} \quad (37)$$

$$r_2 = \frac{r^2}{L^2} \quad (38)$$

where r is mass spin radius measured in relation to mass center, and the stiffness relationship is represented by

$$s = L'^2 \frac{k_c}{k_\varphi} \quad (39)$$

And relationship of translation rigidity by,

$$\alpha = \frac{k}{k_c} \quad (40)$$

It can be considered that deep of foundation is:

$$p = 0.10L \quad (41)$$

so that,

$$\tau = \frac{L'}{L} = r_1 + 1.10 \quad (42)$$

Periods of vibration relationship can be written as,

$$\frac{T_{1e}}{T} = \Omega_1 \quad (43)$$

$$\frac{T_{2e}}{T} = \Omega_2 \quad (44)$$

Where T represents oscillator period of vibration considering it like SDOF in translation supported on hard soil, T_{1e} is oscillator period of vibration coupled in translation and T_{2e} is oscillator period of vibration coupled in rotation; both considering effects of rotational inertia and with SSI.

As can be seen on equation (37) oscillator period of vibration where effects of rotational mass besides effects of soil-structure interaction are considered, depends on position of mass center, y_c, on the relationship between mass spin radius and column height; on the relationships between oscillator and foundation rigidities, as well as on the deep of foundation.

It will be analyzed the case in which mass is considered as concentrated in the upper section of column, $r = 0$ ($r_2 = 0$) and mass center coincides with the junction point between column and mass base, $y_c = 0$ ($r_1 = 0$), then oscillator periods of vibration considering SSI are,

$$T_{1e} = \sqrt{1 + \alpha(1+s)} = T_{0e} \quad (45)$$

$$T_{2e} = 0 \quad (46)$$

Where T_{0e} is oscillator fundamental period considering mass concentrated and with SSI. In Figure 3 are shown the values of equation (45) by changing $s = 0,1,2,3$; and α from 0 to 2.

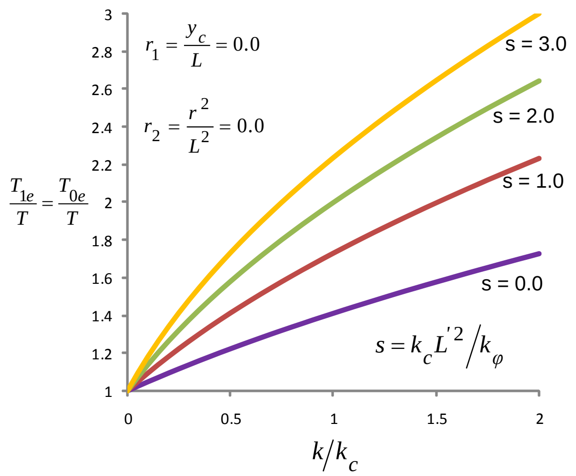
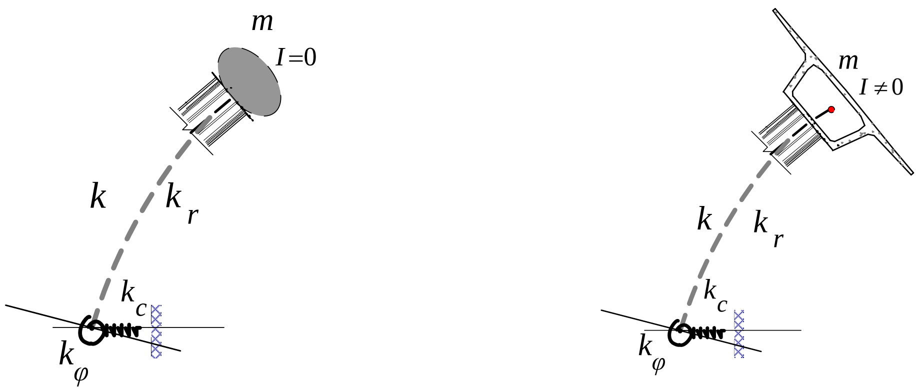


Figure 3. Effect of SSI on oscillator fundamental period without considering rotational inertia.

Results shown on Figure 3 correspond to the ones reported by Mylonakis [3] for the case of an oscillator with concentrated mass where the effect of Soil-Structure Interaction is considered, in other words, without considering the effects of mass rotational inertia (Figure 4a).



(a) Traditional model. (b) Model considering $I \neq 0$

Figure 4. Models that consider SSI.

Considering values suggested by Mylonakis [3] of $s = L'^2 k_c / k_\varphi$ of 0, 1 and 3; and of $\alpha = k/k_c$ varying from 0.0 to 2.0; for different values of relationship mass spin radius-resistant element height, $r_2 = r^2/L^2$ from 0.0 to 0.50.

Figure 5 shows the relationship between oscillator period considered like 2 DOF system with SSI, T_{1e}, and oscillator period of vibration considered like SDOF system, T founded on hard soil. Such relationship increases as the value of $s = L'^2 \frac{k_c}{k_\varphi}$ is increased. Being greater for case of $s = 3$. Such relationship also increases as rigidity relationship increases $\alpha = \frac{k}{k_c}$.

Being greater for $k/k_c = 2$. Dotted line, $r = r_2 = 0.0$ represents the case of oscillator with SSI, with concentrated mass, that is to say, like a SDOF oscillator (Figure 3a), whose period corresponds to T_{0e}. On the right side, the relationship between oscillator lateral period with SSI and considering mass inertia, T_{1e}, and period with SSI, but considering concentrated mass, T_{0e} is shown. The relationship T_{1e}/T_{0e} is greater for small values of $\alpha = \frac{k}{k_c}$, and decreases as this value increases; being smaller than or equal to 1.10 for: $s = 0$ and $\alpha = 2$.

When $s = 3$ and $\alpha = 2$ such relationship is smaller than or equal to 1.18, for values of r_2 analyzed. The smallest value of relationship T_{1e}/T_{0e} that can be seen is 1.06 and the greatest is 1.52.

In Figure 6 the relationship between oscillator period of vibration in rotation considering SSI and rotational mass, T_{e2} is shown, (Figure 4b) regarding that oscillator period is considered like SDOF founded in hard soil. The increase of period, T_{e2}, regarding T is barely sensitive to variation of $s = L'^2 k_c/k_\varphi$, then the plots of each one of the analyzed values are almost the same.

It is observed that oscillator period of rotation considering effect of soil-structure interaction, mainly depends from relationship k/k_c increasing as such relation increases. Periods relationship, T_{e2}/T, is from 0.18 to 0.40 for small values of k/k_c and for values of $k/k_c = 2$, the increase of such relationship varies from 0.33 for $r_2 = 0.05$, up to 0.97 for values of $r_2 = 0.50$.

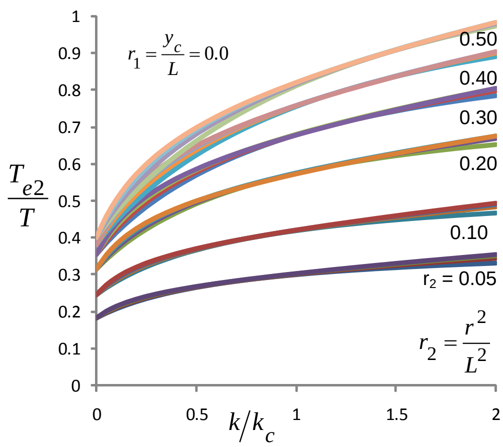


Figure 6. Relationship of period of rotation with SSI and oscillator transversal period in hard soil.

In Figure 7 on left side the relationship between periods T_{1e}/T for the case that portion of mass center is at a distance of $y_c = 0.20L$ measured from the joint of oscillator resistant lateral element is shown. The relation of periods increases as value of s increases, being greater for case of $s = 3$. Such increase is also greater as value of $\alpha = k/k_c$ being greater for $k/k_c = 2$. In the analyzed cases there is always an increase in lateral period if the influence of inertial rotation in movement equation is considered.

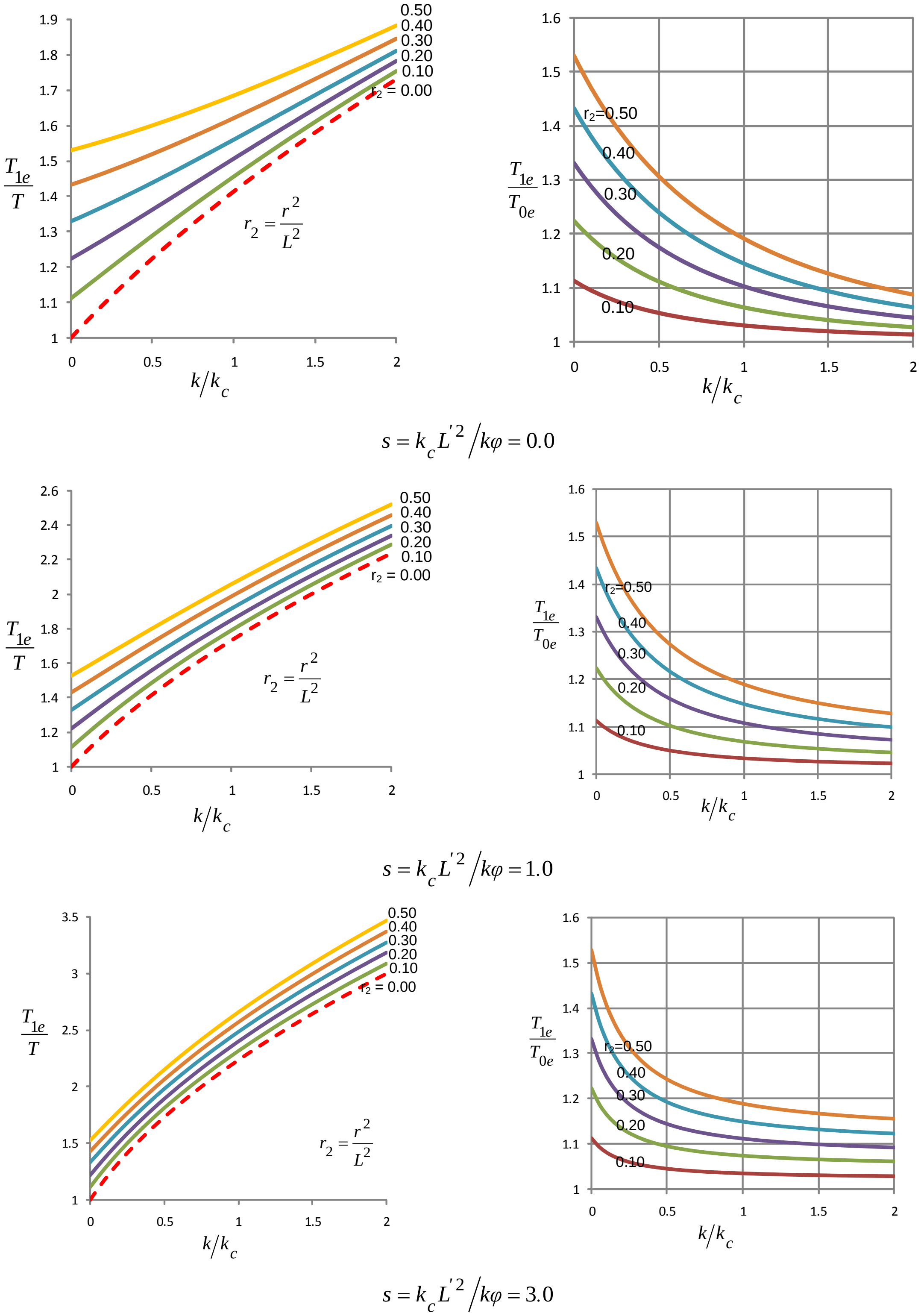
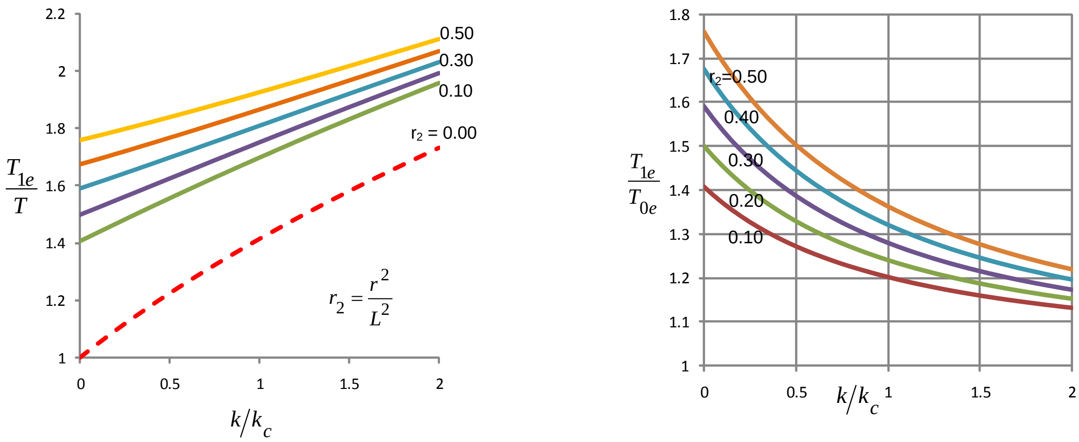
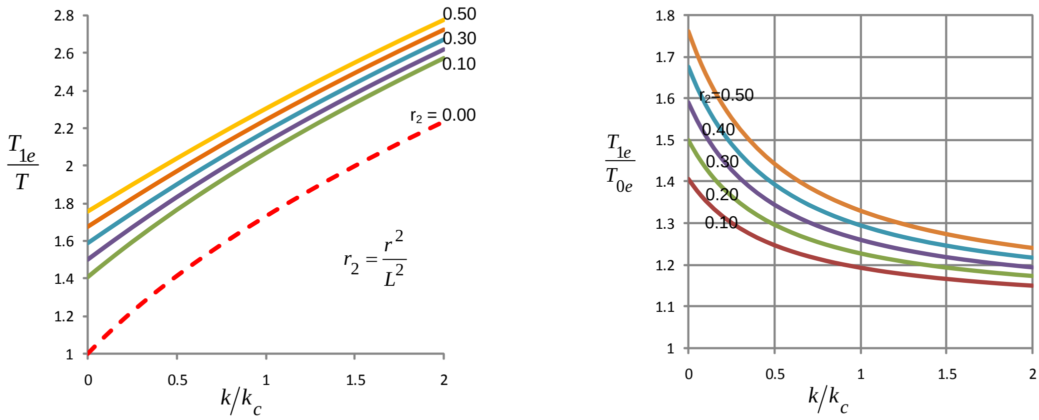


Figure 5. Relationship of periods T_{1e}/T and T_{1e}/T_{0e} considering $r_1 = y_c/L = 0.0$.

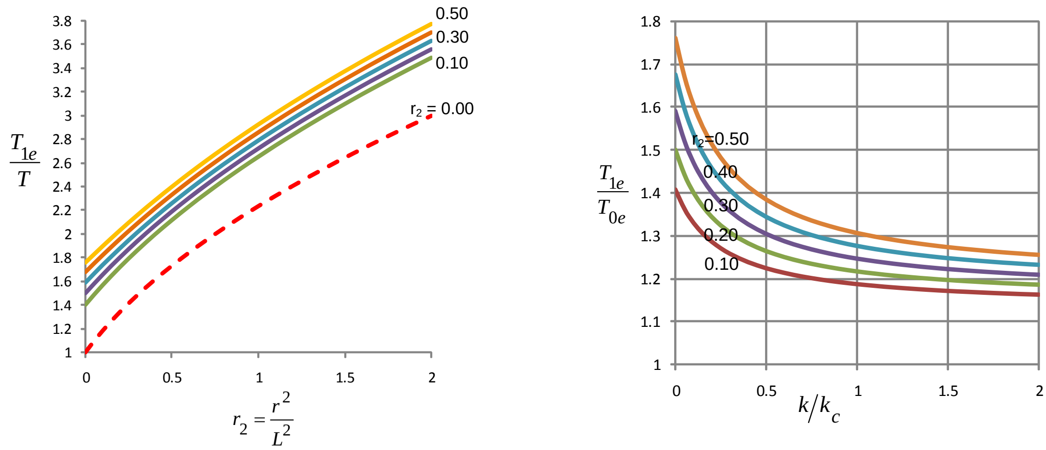
Comparing values of relationship T_{1e}/T shown in Figure 5, $y_c = 0.0$ and values shown on Figure 7, $y_c = 0.20L$, it can be observed that these last ones are greater, in other words, the relation between transversal periods, T_{1e} and T is greater if mass center is further from upper node of oscillator resistant lateral element.



$s = k_c L'^2 / k\varphi = 0.0$



$s = k_c L'^2 / k\varphi = 1.0$



$s = k_c L'^2 / k\varphi = 3.0$

Figure 7. Relationship of periods T_{1e}/T and T_{1e}/T_{0e} considering $y_c = 0.20L$.

In Figure 7, on right side, the relation between periods T_{1e} and T_{0e} for the case in which $y_c = 0.20L$ is shown. Such relation is greater for small values of $\alpha = k/k_c$ and decreases as this value increases; being smaller than or equal to 1.22 for $s = 0$ and $\alpha = 2$. When $s = 3$ and $\alpha = 2$ such relation is smaller than or equal to 1.26. The minimum value of this relationship of periods is 1.13 and the maximum is 1.76.

Comparing the values on the right side of Figures 5 and 7, it is observed that not considering the effects of inertial mass, in calculation of lateral period is more critical if oscillator mass center, y_c, is located higher than the upper point of oscillator resistant lateral element.

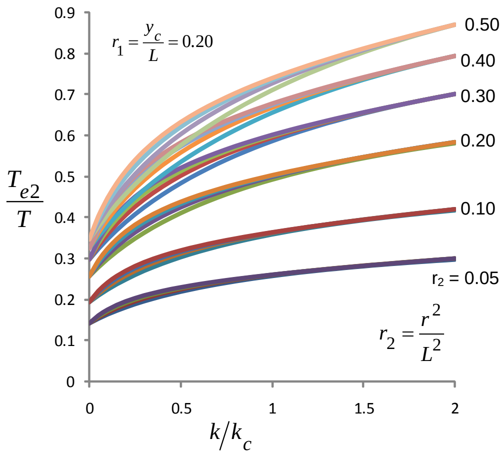


Figure 8. Relationship of periods of rotation with SSI and transversal without SSI; $r_1 = 0.20$.

In Figure 8 the relationship of oscillator period of vibration in rotation considering SSI, T_{e2} and oscillator period considered as SDOF founded in hard soil is shown. The increase in period, T_{e2} regarding, T, is barely sensitive to variation of $s = L'^2 k_c/k_\varphi$; therefore, the plots of each one of the analyzed values is almost the same. It is observed that relationship T_{2e}/T depends directly on the relation k/k_c, increasing as this relation increases. The relation of periods, for small values of k/k_c is from 0.14 to 0.34, and for values of $k/k_c = 2$ the increase of such relation varies from 0.30 for $r_2 = 0.05$, up to 0.87 for values of $r_2 = 0.50$.

3. CONCLUSIONS

In this work expressions that allow to estimate importance of rotational mass in calculation of periods of vibration in translation and rotation, in an inverted pendulum type oscillator supported on hard or soft soil were developed. For cases in which oscillator is supported on hard soil, the translation and rotation periods are coupled and depend on the position of mass center and on the relation rotational inertia-height of resistant element.

It is considered in all cases that mass rotational inertia increases the period of vibration for oscillator. When inverted pendulum type structures rest on soft soil, the phenomenon of ground-structure dynamic interaction plays a predominant role. In this case, translation and rotation periods are also coupled, and also depend on parameters that define an unstable soil, on relation of stiffness between oscillator and soil, and on relation of oscillator stiffness. In all analyzed cases there is an increase on lateral and rotation periods, if mass rotational inertia is considered in movement equation. Similar results are obtained for case of viaduct type structures.

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