

Optimal control problem for the generalized bioconvective flow

M.A. ROJAS-MEDAR¹, R. DE AGUIAR², J. ORTEGA¹, M.D.
ROJAS-MEDAR³

¹ *Universidad del Bío-Bío Facultad de Ciencias, Departamento de Ciencias Básicas, Campus Fernando May, Casilla 447, Chillán, Chile. E-mails: marko@ueubiobio.cl, jortega@dim.uchile.cl.*

² *DMAT-UDESC-JOINVILLE, Campus Avelino Marcante S/N, CEP 89223-100, Joinville-SC, Brasil. E-mail: dma2ra@joinville.udesc.br.*

³ *Departamento de Matemáticas, Facultad de Ciencias Básicas, Universidad de Antofagasta, Casilla 170, Antofagasta, Chile. E-mail: mrojas@uantof.cl*

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Abstract

In this work, we consider an optimal control problem for the generalized bioconvective flow, which is a well known model to describe the convection caused by the concentration of upward swimming microorganisms in a fluid. Firstly, we study the existence and uniqueness of weak solutions for this model, moreover we prove the existence of the optimal control and we establish the minimum principle by using Dubovitskii-Milyutin's formalism.

1 Introduction

The DM formalism (see [3]) turns to be an operational writing separation of the convex sets in Banach spaces, hence for instance an abstract formulation of the Lagrange multiplier theorem can be obtained. As far as we know, this approach to optimal control in PDE has been very little used. The first work, to our knowledge, in which the DM formalism was applied in the context of PDE, corresponds to the work of A. Papageorgiou and N. S. Papageorgiou [12]. Some extensions of this formalism can be found in the works of U. Ledzewicz [8, 9], Y. Censor [2], W. Kotarski [5] and S. Walczak [13]. An interesting extension of the DM formalism was presented by I. Lasiecka (see [6, 7]), where the author introduce a weaker type of conical approximation called the external cone. By applying

this approximation to one set and the internal approximation of Neustadt to the other, an appropriate separation theorem is proved, and then from this theorem a generalization of the DM theorem is proved. On the other hand, the *bioconvective flow* is a well known model to describe the convection caused by the concentration of upward swimming microorganisms in a fluid. This model was derived by M. Levandodovsky, W. S. Hunter and E. A. Spiegel [10] and independently by Y. Moribe [11].

The first work, from the mathematical point of view, is due to Y. Kan-On, K. Narukawa and Y. Teramoto [4]. In this work, the authors proved the existence of solutions and the positivity of the concentration for the stationary problem. They also studied the nonstationary case when the viscosity of the fluid is constant. We note that A. Later, A. Căpătina and R. Strave [1] studied a control problem for the stationary bioconvective fluid, also considering a constant viscosity.

In our case, we consider a generalized bioconvective fluid, in which the viscosity is non-homogeneous and it is a function of the concentration of the microorganisms, which is a more realistic model.

The corresponding stationary model which describes this phenomenon is the following: Let $\Omega \subset \mathbb{R}^3$ be a bounded domain, with regular boundary $\partial\Omega$, where the fluid is contained, $\mathbf{u}(x) = (u_1(x), u_2(x), u_3(x)) \in \mathbb{R}^3$ denotes the velocity of the fluid, $p(x)$ denotes the hydrostatic pressure of the fluid and $c(x)$ denotes the concentration of the microorganism at the point $x \in \Omega$, then we have

$$\left. \begin{aligned} -2 \operatorname{div}(\nu(c)D(\mathbf{u})) + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p &= -g(1 + \rho c)\mathbf{i}_3 + \mathbf{f}, & \text{in } \Omega \\ \operatorname{div} \mathbf{u} &= 0, & \text{in } \Omega \\ -\theta \Delta c + \mathbf{u} \cdot \nabla c + U \frac{\partial c}{\partial x_3} &= 0, & \text{in } \Omega, \end{aligned} \right\} \quad (1)$$

where $\nu(\cdot) > 0$ is the kinematic viscosity, which is a function of the concentration, θ is the constant that give the diffusion of the microorganisms, g is the gravitational coefficient (we assume constant), \mathbf{f} is the density by unit of mass of the external force acting on the fluid, $\mathbf{i}_3 = (0, 0, 1)$ is the unitary vector in the vertical direction, U denotes the mean velocity of natation of the microorganism, in the vertical direction, γ is a positive constant, given by $\gamma = \rho_0 \backslash \rho_m - 1$, where ρ_0 and ρ_m are the density of one microorganism and the density of the cultivate in the fluid, respectively.

The expressions grad , Δ and div denote the gradient, Laplacian and divergence operators, respectively (we also denotes the gradient operator by ∇); $\mathbf{u} \cdot \nabla \mathbf{u}$ denotes the convective operator, and the expression $D(\mathbf{u}) = \frac{1}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})^t)$ denotes the stress tensor.

It is important to note that if \mathbf{u}_0 is a solution of the Navier-Stokes equation, we have that the pair $(\mathbf{u}_0, 0)$ is a solution of our stationary problem which does not describe in a right form our phenomenon, for this reason we include a new condition on the concentration of the microorganism which is

$$\int_{\Omega} c(x) dx = \alpha. \quad (2)$$

In order to complete our model, we consider the following boundary conditions

$$\left. \begin{aligned} \mathbf{u}(x) &= 0, & \text{on } S \\ \mathbf{u}(x) \cdot \mathbf{n} &= 0, & \text{on } \Gamma \\ \nu(c)[D(\mathbf{u})\mathbf{n} - (\mathbf{n} \cdot D(\mathbf{u})\mathbf{n})\mathbf{n}] &= b_1, & \text{on } \Gamma \\ \theta \frac{\partial c}{\partial \mathbf{n}} - Ucn_3 &= 0, & \text{on } \partial\Omega \end{aligned} \right\} \quad (3)$$

where the boundary $\partial\Omega = S \cup \Gamma$ is composed for a rigid boundary S and Γ is a free boundary; $\mathbf{n}(x) = (n_1(x), n_2(x), n_3(x))$ is the exterior unitary normal vector at the point $x \in \partial\Omega$ and $\partial/\partial\mathbf{n}$ is the normal derivative on $\partial\Omega$.

Thus, we are interested in the existence and uniqueness of weak solutions of (1)-(3). On the other hand, we are interested in the study of an optimal control problem for the above system. Let us assume that $\nu(\cdot)$ is a continuous function which satisfies some suitable bounds and the velocity of natation of the microorganism U is small enough. We consider also that the viscosity $\nu(c(x))$ verifies that $\nu(c) \in H^1(\Omega)$.

Let us define the functional $J : K \times H^1(\Omega) \rightarrow \mathbb{R}$, where

$$J(\alpha, c) = \frac{1}{2} \int_{\Omega} (c - c_d)^2 dx + \frac{N}{2} \alpha^2, \quad (4)$$

$K \subset [0, +\infty)$ being a closed interval nonempty non degenerated, N is a non negative constant and $c_d \in L^2(\Omega)$ is a given function. Therefore, our optimal control problem is the following:

$$\min \{ J(\alpha, c) \mid (\alpha, c) \in T \}, \quad (5)$$

where $T = \{(\alpha, c) \in K \times H^1(\Omega) : (u, c) \in J_0 \times H^1(\Omega) \text{ is a weak solution of (1)-(3)}\}$.

Thus, we are interested in to obtain the mean concentration α of microorganisms which lead us to a suitable concentration c_d . In this work we prove the existence of optimal control for (4)-(5), moreover we find necessary conditions of the optimality (minimum principle) by using the Dubovitskii-Milyutin's formalism.

2 Main Results

In this section we will present our main results. Firstly we will present an existence and uniqueness result of weak solutions for our stationary model. In what follows we will consider the space of functions

$$J_0 = \{ \mathbf{z} \in H^1(\Omega)^3 : \operatorname{div} \mathbf{z} = 0 \text{ in } \Omega, \mathbf{z} = 0 \text{ on } S \text{ and } \mathbf{z} \cdot \mathbf{n} = 0 \text{ on } \Gamma \}.$$

Let us consider the operators:

$$B_0 : J_0 \times J_0 \times J_0 \longrightarrow \mathbb{R}, \quad B_0(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \int_{\Omega} ((\mathbf{u} \cdot \nabla) \cdot \mathbf{v}) \cdot \mathbf{w}.$$

$$B : J_0 \times H^1(\Omega) \times H^1(\Omega) \longrightarrow \mathbb{R}, \quad B(\mathbf{u}, c, r) = \int_{\Omega} ((\mathbf{u} \cdot \nabla) c) r$$

$$a : J_0 \times J_0 \longrightarrow \mathbb{R}, \quad a(\mathbf{u}, \mathbf{v}) = \int_{\Omega} \nabla \mathbf{u} \cdot \nabla \mathbf{v}.$$

Definition Given $\mathbf{f} \in L^2(\Omega)^3$, the pair of functions $(\mathbf{u}, c) \in J_0 \times H^1(\Omega)$ is called a **weak solution** of the system (3) if the following identities hold $\forall \mathbf{z} \in J_0, \forall r \in H^1(\Omega)$:

$$\begin{aligned} (2\nu(c)D(\mathbf{u}), D(\mathbf{z})) + B_0(\mathbf{u}, \mathbf{u}, \mathbf{z}) &= 2(b_1 \cdot \mathbf{z})_{\Gamma} - k(\mathbf{c}\mathbf{i}_3, \mathbf{z}) + (\mathbf{f}, \mathbf{z}), \\ \theta a(c, r) + B(\mathbf{u}, c, r) - U \int_{\Omega} c \frac{\partial r}{\partial x_3} dx &= 0. \end{aligned}$$

Thus, our uniqueness and existence result reads as follow.

Theorem Let ν be a Lipschitz continuous function and we assume that there exists positive numbers ν_0 and ν_1 , such that

$$\begin{aligned} \nu_0 &= \inf \{ \nu(s) \mid s \in \mathbb{R} \} > 0 \\ \nu_1 &= \sup \{ \nu(s) \mid s \in \mathbb{R} \} < +\infty \end{aligned} \tag{6}$$

Let $\mathbf{f} \in L^2(\Omega)$ and U be small enough. Then there exists a unique weak solution of the problem (1)-(3).

On the other hand, we are interested in the study of an optimal control result for the generalized bioconvective fluid and to obtain some necessary conditions of optimality by using the Dubovyt'ski-Miluytin's formalism. In what follows we assume that $\nu(\cdot)$ is a continuous function satisfying the condition (6) and the following inequality

$$U < \frac{\theta}{C}$$

holds, for a suitable positive constant C . We consider also that the cinematic viscosity depends on the concentration of the microorganism, that is, $\nu(c)$ and moreover we have that $\nu(c) \in H^1(\Omega)$ if $\nu'_1 = \sup\{\nu'(t) \mid t \in \mathbb{R}\} < \infty$.

We will consider the functional $J : K \times H^1(\Omega) \rightarrow \mathbb{R}$, defined by

$$J(\alpha, c) = \frac{1}{2} \int_{\Omega} (c - c_d)^2 dx + \frac{N}{2} \alpha^2$$

where $K \subset [0, +\infty)$ is a closed interval nonempty, non degenerated, N is a non negative constant and $c_d \in L^2(\Omega)$ is a given function.

We consider the following optimal control problem:

$$\min \{ J(\alpha, c) \mid (\alpha, c) \in T \}, \tag{7}$$

where

$$T = \{ (\alpha, c) \in K \times H^1(\Omega) \mid (\mathbf{u}, c) \text{ is a weak solution of (1) - (3)} \}$$

Therefore we have the following result.

The notations will be established in the next sections.

Theorem *The problem (7) has an optimal solution (α^*, c^*) and there exist elements $\mathbf{u}^* \in J_0$, $(\mathbf{p}^*, q^*) \in J_0 \times \widetilde{H}^1(\Omega)$ and $\lambda \in \{0, 1\}$ such that*

$$\begin{cases} (2v(c^*)D(\mathbf{u}^*), D(\mathbf{z})) + B_0(\mathbf{u}^*, \mathbf{u}^*, \mathbf{z}) = 2(b_1, \mathbf{z})_\Gamma - k(\tilde{c}^* \mathbf{i}_3, \mathbf{z}) + (\mathbf{f}, \mathbf{z}) \\ \theta a(\tilde{c}^*, r) + B(\mathbf{w}_\epsilon, \tilde{c}^*, r) - U(\tilde{c}^*, \frac{\partial r}{\partial x_3}) = \frac{U\alpha^*}{|\Omega|}(\frac{\partial r}{\partial x_3}, 1) \\ \forall \mathbf{z} \in J_0, \quad \forall r \in \widetilde{H}^1(\Omega) \end{cases} \quad (8)$$

$$\begin{cases} (2v(c^*)D(\mathbf{p}^*), D(\mathbf{z})) + B_0(\mathbf{u}^*, \mathbf{p}^*, \mathbf{z}) = B_0(\mathbf{z}, \mathbf{p}^*, \mathbf{u}^*) + B(\mathbf{z}, q^*, \tilde{c}^*) \\ \theta a(r, q^*) - B(\mathbf{u}^*, q^*, r) = k(r \mathbf{i}_3, \mathbf{p}^*) + U(r, \frac{\partial q^*}{\partial x_3}) \\ \quad - (2v'(c^*)rD(\mathbf{u}^*), D(\mathbf{p}^*)) + \lambda(c_\epsilon - c_d, r) \\ \forall \mathbf{z} \in J_0, \quad \forall r \in \widetilde{H}^1(\Omega) \end{cases} \quad (9)$$

$$\left[U(\frac{\partial q^*}{\partial x_3}, 1) - (2v'(c^*)D(\mathbf{u}^*), D(\mathbf{p}^*)) + \lambda[(c_\epsilon - c_d, 1) + N\alpha_\epsilon |\Omega|] \right] (\alpha - \alpha^*) \geq 0 \quad (10)$$

$$\lambda + \|q^*\|_{\widetilde{H}^1} > 0. \quad (11)$$

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