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Reduction methods for quasilinear differential-algebraic equations

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Resumen

Geometric reduction methods for differential-algebraic equations (DAEs) aim at an iterative reduction of the problem to an explicit ODE on a lower-dimensional submanifold of the so-called semistate space. This approach usually relies on certain algebraic (typically constant-rank) conditions holding at every reduction step. When these conditions are met the DAE is called *regular*. We discuss in this contribution several recent results concerning the use of reduction techniques in the analysis of quasilinear DAEs, not only for regular systems but also for *singular* ones, in which the above-mentioned conditions fail.

1. Outline

Quasilinear autonomous differential-algebraic equations (DAEs) are implicit ODEs of the form

$$A(x)x' = f(x), \tag{1}$$

where $A \in C^\infty(W_0, \mathbb{R}^{n\times n})$ is a rank-deficient matrix-valued mapping, $f \in C^\infty(W_0, \mathbb{R}^n)$, and W_0 is an open set in $\mathbb{R}^n$. We summarize in Section 2 below the geometric reduction approach of Rabier and Rheinboldt for the analysis of quasilinear systems of the form (1). In Section 3 we recast this framework in a local manner, aimed at the analysis of singular problems carried out in Section 4. In order to provide the reader with a self-contained discussion, we address here the main ideas and refer him/her to [4] and to the forthcoming title [8] for details, specially concerning several results which are stated without proof in this communication.

2. The reduction framework of Rabier and Rheinboldt

Stemming from the seminal paper [7] by Rheinboldt, reduction methods are essentially based on the work of Rabier and Rheinboldt [1, 4], and Reich [5, 6]. We summarize below the approach of Rabier and Rheinboldt for the quasilinear DAE (1) following [4].

Any C^1 solution to (1) must obviously lie on the set

$$W_1 = \{x \in W_0 \ / \ f(x) \in \operatorname{im} A(x)\}. \tag{2}$$

Define $F : TW_0 \simeq W_0 \times \mathbb{R}^n \to \mathbb{R}^n$ as $F(x,p) = A(x)p - f(x)$, and consider the set $M_0 = F^{-1}(0) = \{(x,p) \in TW_0 \ / \ A(x)p - f(x) = 0\}$. Denoting by $\pi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ the projection onto the first factor we have $W_1 = \pi(M_0)$. If $x(t)$ solves (1), it follows that the pair $(x(t), x'(t))$ must belong to M_0.

Via the subimmersion theorem, the following global conditions make W_1 an r-dimensional submanifold of W_0 (cf. [4]):

(G1) $A(x)$ has constant rank $r_1 \leq n$ for all $x \in W_1$.

(G2) $F(x,p)$ is a submersion on its zero set M_0.

Now, provided that $x(t)$ is a solution to (1), thereby lying entirely on W_1, the pair $(x(t), x'(t))$ must also belong to the tangent bundle TW_1. This means that $x'(t)$ must be tangent not only to W_0 but also to W_1 itself. Hence, the pair $(x(t), x'(t))$ needs to be in the intersection $M_1 = TW_1 \cap M_0$ and, in particular, $x(t)$ must lie on $W_2 = \pi(M_1)$. Letting $F_1 = F|_{TW_1}$, we can describe $M_1 = TW_1 \cap M_0$ as $F_1^{-1}(0)$, whereas the set W_2 reads $W_2 = \{x \in W_1 \ / \ f(x) \in \operatorname{im} A(x)|_{T_xW_1}\}$. If the analogs of assumptions G1 and G2 hold when applied to F_1 and $A(x)|_{T_xW_1}$, W_2 will be an r_2-dimensional manifold with $r_2 = \operatorname{rk} A(x)|_{T_xW_1}$, and the same reasoning can be performed one step further.

This way, if the the above-mentioned working assumptions hold at every step, the procedure yields two sequences of smooth manifolds which will eventually stabilize, namely, $M_0 \supset M_1 \supset \ldots \supset M_\nu = M_{\nu+1}$ and

$$W_0 \supset W_1 \supset \ldots \supset \ldots \supset W_\nu \supseteq W_{\nu+1} = W_{\nu+2}. \tag{3}$$

The dimensions of these manifolds are given by the ranks $n > r_1 > \ldots > r_\nu = r_{\nu+1} = r_{\nu+2}$. The smallest integer ν such that either $M_\nu = \emptyset$ or $M_\nu \neq \emptyset$ and $r_\nu = r_{\nu+1}$ is the *geometric index* of (1). In index-ν problems with $M_\nu \neq \emptyset$, the manifold $W_{\nu+1}$ turns out to be open in W_ν. Always under the above-stated working assumptions, this manifold comprises all the smooth solutions of the DAE, which can be described in terms of a vector field uniquely defined on $W_{\nu+1}$. Details can be found in Theorems 23.2 and 24.1 of [4]. Via local parametrizations, solutions of index-ν DAEs can be also locally described in terms of reduced equations, in a way similar to the one detailed in Section 3 below.

3. A local approach

The above-summarized framework provides a nice approach for the analysis of quasilinear DAEs when the global assumptions G1 and G2 above hold at every reduction step. But its obvious drawback is the exclusion of DAEs for which these assumptions are not met. In order to accommodate these *singular* problems, let us recast the construction above in a local manner.

A point $x^* \in W_0$ is called *regular with geometric index zero* for the DAE (1) if $A(x^*)$ is a non-singular matrix. The behavior on the (open) set of index zero points trivially amounts to that of the explicit ODE $x' = A^{-1}(x)f(x)$.

Definition 1 *A point $x^* \in W_0$ is said to be* 0-regular *for the DAE (1) if $x^* \in W_1$ and the following two conditions hold:*

(R1) *$A(x)$ has constant rank $r_1 \leq n$ in some neighborhood of x^*.*

(R2) *F is a submersion at (x^*, p^*), for some p^* satisfying $A(x^*)p^* = f(x^*)$.*

The set of 0-regular points will be denoted by W_1^{reg}.

By construction W_1^{reg} is open in W_1 and therefore the sets W_1^{reg} and W_1 coincide locally around any 0-regular point. More precisely, for all $x^* \in W_1^{\mathrm{reg}}$ there exist a neighborhood U of x^* such that $W_1^{\mathrm{reg}} \cap U = W_1 \cap U$. Henceforth we will abbreviate this kind of relation as $W_1^{\mathrm{reg}} \overset{\mathrm{loc}}{=} W_1$.

The constant rank condition in R1 above is a local version of G1 in page 2, with the slightly stronger requirement that the rank is constant within a whole neighborhood (say $\tilde{U}_0$) of x^* in W_0. If this locally constant rank r_1 verifies $r_1 < n$, then there will exist another open neighborhood $\hat{U}_0 \subseteq \tilde{U}_0 \cap U$ of x^* and a smooth matrix-valued map $H \in C^\infty(\hat{U}_0, \mathbb{R}^{(n-r_1)\times n})$ such that $\mathrm{ke}\, H(x) = \mathrm{im}\, A(x) \;\; \forall x \in \hat{U}_0$, see e.g. Lemma 22.1 in [4]. Note that $H(x)A(x) = 0$ and $\mathrm{rk}\, H(x) = n - r_1$ on $\hat{U}_0$. Now $v \in \mathrm{im}\, A(x) \Leftrightarrow H(x)v = 0$ for $x \in \hat{U}_0$, allowing for the local implicit description of $W_1^{\mathrm{reg}} \overset{\mathrm{loc}}{=} W_1$ as $W_1^{\mathrm{reg}} \cap \hat{U}_0 = W_1 \cap \hat{U}_0 = \{x \in \hat{U}_0 \;/\; H(x)f(x) = 0\}$.

The submersion condition R2 requires $\mathrm{rk}\, F'(x^*, p^*) = n$. This is a key hypothesis because it characterizes the situations in which the product $H(x)f(x)$ is a submersion, as stated in Lemma 1 below.

Lemma 1 *Let $x^* \in W_1$. Assume that $A(x)$ has constant rank r_1, with $0 < r_1 < n$, in some open neighborhood $\tilde{U}_0$ of x^*, and let the matrix-valued map $H \in C^\infty(\hat{U}_0, \mathbb{R}^{(n-r_1)\times n})$ verify $\mathrm{ke}\, H(x) = \mathrm{im}\, A(x)$ $\forall x \in \hat{U}_0 \subseteq \tilde{U}_0$. Then $H(x)f(x)$ is a submersion at x^* if and only if $F(x,p)$ is a submersion at (x^*, p^*) for some (hence any) p^* satisfying $A(x^*)p^* = f(x^*)$.*

In this setting, the local description of $W_1^{\mathrm{reg}} \overset{\mathrm{loc}}{=} W_1$ as the zero set of $H(x)f(x)$ locally yields a smooth structure on this set and paves the way for the following local reduction.

Theorem 1 *Let $x^* \in W_0$ be a 0-regular point for (1), and denote by r_1 the locally constant rank of $A(x)$ around x^*. If $r_1 > 0$, then there exists an open neighborhood $U_0 \subseteq W_0 \subseteq \mathbb{R}^n$ of x^* such that*

(i) *$W_1^{\mathrm{reg}} \cap U_0 = W_1 \cap U_0$ admits a smooth r_1-dimensional parametrization $x = \varphi_1(\xi)$ with surjective $\varphi_1 : \Omega_1 \to W_1^{\mathrm{reg}} \cap U_0$;*

(ii) *there exists a C^∞ matrix-valued mapping $P_1 : U_0 \to \mathbb{R}^{r_1 \times n}$ verifying that $P_1(x)\big|_{\mathrm{im}\, A(x)}$ is an isomorphism $\mathrm{im}\, A(x) \leftrightarrow \mathbb{R}^{r_1}$ for all $x \in U_0$.*

For any such φ_1, P_1, $x(t)$ is a solution of (1) within U_0 if and only if $x(t) \in W_1^{\text{reg}} \overset{\text{loc}}{=} W_1$ for all t and $\xi(t) = \varphi_1^{-1}(x(t))$ is a solution of

$$A_1(\xi)\xi' = f_1(\xi), \ \ \xi \in \Omega_1 \subseteq \mathbb{R}^{r_1} \tag{4}$$

with $A_1(\xi) = P_1(\varphi_1(\xi))A(\varphi_1(\xi))\varphi_1'(\xi)$, $f_1(\xi) = P_1(\varphi_1(\xi))f(\varphi_1(\xi))$.

In the index one setting described by item (a) of Definition 2 below, the reduction (4) can be rewritten as a explicit ODE in some neighborhood of ξ^*, possibly smaller than Ω_1.

Definition 2 *A point $x^* \in W_0$ is called* regular with geometric index one *for (1)*

(a) either if it is 0-regular with $n > r_1 > 0$ and $A_1(\xi^*)$ is non-singular for some (hence any) reduction pair (P_1, φ_1) satisfying $x^* = \varphi_1(\xi^*)$;

(b) or if it is 0-regular with $r_1 = 0$.

The set of index one points will be denoted by $W^{\text{ind}1}$.

For points which are not index one, the same procedure can be applied to $(A_1(\xi), f_1(\xi))$ in (4). Introduce $r_2 = \text{rk}\, A_1$, $W_2 = \{x \in W_1^{\text{reg}} \ / \ f(x) \in \text{im}\, A(x)|_{T_x W_1^{\text{reg}}}\} \subseteq W_1^{\text{reg}} \subseteq W_1$ or, in local coordinates, $V_2 = \{\xi \in \Omega_1 \ / \ f_1(\xi) \in \text{im}\, A_1(\xi)\} \subseteq \Omega_1 \subseteq \mathbb{R}^{r_1}$ which yields a local description of W_2 as $\varphi_1(V_2)$. A 0-regular point ξ^* of (A_1, f_1) will define $x^* = \varphi_1(\xi^*)$ as a *1-regular point* of (A, f). Recursively, we are naturally led to the following notion.

Definition 3 *A point $x^* \in W_0$ is called* regular with geometric index ν, *$\nu \geq 1$, for (1)*

(a) either if it is $(\nu-1)$-regular with $n > r_1 > r_2 > \ldots > r_\nu > 0$, and the matrix $A_\nu(u^*)$ is non-singular, for some (hence any) reduction sequence $(P_1, \varphi_1), \ldots, (P_\nu, \varphi_\nu)$ satisfying $x^* = \varphi_1 \circ \cdots \circ \varphi_\nu(u^*)$;

(b) or if it is $(\nu-1)$-regular with $n > r_1 > \ldots > r_\nu = 0$.

The set of index-ν points will be denoted by $W^{\text{ind}\nu}$.

Solutions of the original DAE (1) near a given $(\nu-1)$-regular point with $r_\nu > 0$ are mapped bijectively into those of the reduced equation $A_\nu(u)u' = f_\nu(u)$, as stated in Theorem 2 below which naturally extends Theorem 1. We denote as U the neighborhood of x^* given by $\varphi_1 \circ \ldots \circ \varphi_{\nu-1}(U_{\nu-1})$, where $U_{\nu-1}$ is such that the last-step parametrization φ_ν is onto $V_\nu \cap U_{\nu-1}$. In particular, under an index-ν assumption an explicit ODE reduction is possible and thereby local unique solvability properties follow from the corresponding theory for explicit ODEs.

Theorem 2 *Assume that $x^* \in W_0$ is a $(\nu-1)$-regular point for (1) with $r_\nu > 0$, $\nu \geq 1$, and let*

$$A_\nu(u)u' = f_\nu(u), \ \ u \in \Omega_\nu \subseteq \mathbb{R}^{r_\nu} \tag{5}$$

be a ν-th step reduction of (1), given by a sequence of reduction pairs $(P_1, \varphi_1), \ldots, (P_\nu, \varphi_\nu)$, on a neighborhood Ω_ν of $u^* = (\varphi_1 \circ \cdots \circ \varphi_\nu)^{-1}(x^*)$. Then $x(t)$ is a solution of (1) within U if and only if $x(t) \in W_\nu^{\text{reg}} \overset{\text{loc}}{=} W_\nu$ for all t and $u(t) = (\varphi_1 \circ \ldots \circ \varphi_\nu)^{-1}(x(t))$ solves (5).

Moreover, if x^* is index ν, then $A_\nu(u)$ is non-singular on some neighborhood of u^* within Ω_ν, and in that neighborhood the reduction (5) can be rewritten in the explicit form

$$u' = A_\nu^{-1}(u)f_\nu(u). \tag{6}$$

Equation (6) is a local state space description of the DAE behavior. Of course, different reduction pairs will yield different state-space descriptions, although all of them can be proved to be C^∞-conjugate.

4. Singularities

In the light of the result in Section 3, the manifold sequence (3) can be replaced by

$$W_0 \supseteq W_1^{\mathrm{reg}} \supseteq W_2^{\mathrm{reg}} \supseteq \ldots \supseteq W_n^{\mathrm{reg}}, \tag{7}$$

which, around an index-ν point, will stabilize as $W_0 \supset W_1^{\mathrm{reg}} \supset \ldots \supset W_\nu^{\mathrm{reg}} \overset{\mathrm{loc}}{=} W_{\nu+1}^{\mathrm{reg}} \overset{\mathrm{loc}}{=} W^{\mathrm{ind}\nu}$. This point of view allows us to accommodate *singular points* within this framework, under the working assumptions S1 and S2 below. Points in $W_{k+1} - W_{k+1}^{\mathrm{reg}}$ are called *inner k-singular points*, whereas those in $\overline{W_{k+1}} - W_{k+1}$ are called *boundary k-singular points.*

Assumption S1 below is aimed to cover cases in which the constant rank assumption R1 in Definition 1 fails after the k-th reduction step, that is, on $A_k(\zeta)$, $k = 0$ standing for rank deficiencies in $A(x)$. This can be the case for both inner and boundary k-singularities. Assumption S1 describes situation very often found in practice in which, despite the rank deficiency, $\mathrm{im}\, A_k(\zeta)$ admits a smooth extension or continuation $L_k(\zeta)$ on a neighborhood of the singularity, with $L_k(\zeta) = \mathrm{im}\, A_k(\zeta)$ on some dense subset of that neighborhood. By an r-dimensional C^∞-space $L(x)$ on U we mean an x-dependent linear space which is spanned by r basis mappings depending smoothly on $x \in U$ or, equivalently, such that $\bigcup_{x\in U}\{x\} \times L(x)$ has an r-dimensional vector bundle structure.

Assumption S1. *Let x^* be a k-singularity for (1), with $k \geq 0$, and consider the k-th local reduction $A_k(\zeta)\zeta' = f_k(\zeta)$. Write $x^* = \varphi_1 \circ \ldots \circ \varphi_k(\zeta^*)$. There exists an open neighborhood $\tilde{U}_k \subseteq \Omega_k \subseteq \mathbb{R}^{r_k}$ of ζ^* and, for some $\tilde{r}_{k+1} \leq r_k$, an $\tilde{r}_{k+1}$-dimensional C^∞-space $L_k(\zeta)$ defined on $\tilde{U}_k$ such that* $\mathrm{im}\, A_k(\zeta) = L_k(\zeta)$ *on some dense subset of $\tilde{U}_k$.*

It may happen in particular that $\tilde{r}_{k+1} = r_k$: in this case Assumption S1 expresses that A_k is non-singular on a dense subset of $\tilde{U}_k$, since $L_k(\zeta) = \mathbb{R}^{r_k}$ meets the requirements. We may speak in this situation of a "last-step" singularity. This is essentially the context considered by Rabier and Rheinboldt in [2, 3, 4]. There is no need for further reduction of the DAE, and Theorem 4 will apply. This will be a particular instance of a singular index k problem.

If $\tilde{r}_{k+1} < r_k$, from the structure of $L_k(\zeta)$ there must exist an open neighborhood $\hat{U}_k \subseteq \tilde{U}_k$ of $\zeta^* \in \Omega_k \subseteq \mathbb{R}^{r_k}$ and a smooth, maximal rank matrix-valued map $H_k(\zeta) \in \mathbb{R}^{(r_k-\tilde{r}_{k+1})\times r_k}$ with $\mathrm{ke}\, H_k(\zeta) = L_k(\zeta)$ on $\hat{U}_k$, so that $v \in L_k(\zeta)$ if and only if $H_k(\zeta)v = 0$ for $\zeta \in \hat{U}_k$. Note that, if x^* is an inner k-singular point, the set $V_{k+1} = \{\zeta \in \Omega_k \ / \ f_k(\zeta) \in \mathrm{im}\, A_k(\zeta)\}$ cannot be guaranteed to admit a local parametrization near ζ^*, in the terms holding at k-regular points. Near (inner or boundary) k-singularities we will work instead with the set

$$\tilde{V}_{k+1} = \{\zeta \in \hat{U}_k \ / \ f_k(\zeta) \in L_k(\zeta)\} = \{\zeta \in \hat{U}_k \ / \ H_k(\zeta) f_k(\zeta) = 0\} \subseteq \Omega_k \tag{8}$$

which not even locally can be identified with V_{k+1}. But in the setting defined by Assumption S1, we have

$$V_{k+1} \cap \hat{U}_k \subseteq \overline{V_{k+1}} \cap \hat{U}_k \subseteq \tilde{V}_{k+1}. \tag{9}$$

Indeed, by the density hypothesis in Assumption S1 we have $\operatorname{im} A_k(\zeta) \subseteq L_k(\zeta) = \operatorname{ke} H_k(\zeta)$ for all $\zeta \in \hat{U}_k$, showing that $V_{k+1} \cap \hat{U}_k \subseteq \tilde{V}_{k+1}$. The relations (9) then follow from the fact that $\tilde{V}_{k+1}$ is closed in $\hat{U}_k$. Set $\tilde{W}_{k+1} = \varphi_1 \circ \ldots \circ \varphi_k(\tilde{V}_{k+1})$, the obvious analog of (9) holding for W_{k+1}, $\overline{W_{k+1}}$ and $\tilde{W}_{k+1}$.

The relation depicted in (9) suggests that $\tilde{V}_{k+1}$ may also accommodate a reduction around boundary k-singularities. Indeed, under Assumption S2 below, $\tilde{V}_{k+1}$ will admit a local $\tilde{r}_{k+1}$-dimensional parametrization.

Assumption S2. *Let $x^* = \varphi_1 \circ \ldots \circ \varphi_k(\zeta^*)$ be a k-singularity for (1), with $k \geq 0$. If Assumption S1 holds with $\tilde{r}_{k+1} < r_k$, let $\hat{U}_k \subseteq \tilde{U}_k \subseteq \Omega_k$ be an open neighborhood of ζ^* such that $H_k \in C^\infty(\hat{U}_k, \mathbb{R}^{(r_k - \tilde{r}_{k+1}) \times r_k})$ verifies $\operatorname{ke} H_k(\zeta) = L_k(\zeta)$ $\forall \zeta \in \hat{U}_k$. Then $H_k(\zeta) f_k(\zeta)$ is a submersion at ζ^*.*

As stated above, Assumption S2 applies to both inner and boundary k-singular points. Inner ones verify $\zeta^* \in V_{k+1}$, and hence they admit solutions p^* to $A_k(\zeta^*)p^* - f_k(\zeta^*) = 0$: Assumption S2 then holds if the submersion condition in item R2 of Definition 1 is met in the current context. More precisely, if x^* is an inner k-singularity for (1), Assumption S1 is met with $\tilde{r}_{k+1} < r_k$, and $F_k(\zeta, p) = A_k(\zeta)p - f_k(\zeta)$ is a submersion at (ζ^*, p^*) for some p^* satisfying $A_k(\zeta^*)p^* = f_k(\zeta^*)$, then Assumption S2 can be checked to hold.

Assumptions S1 and S2 make it possible to perform a reduction of singular quasilinear DAEs, as detailed below.

Theorem 3 *Let $x^* = \varphi_1 \circ \ldots \circ \varphi_k(\zeta^*)$ be a k-singularity for (1) satisfying Assumptions S1 and S2 with $0 < \tilde{r}_{k+1} < r_k$. Then there exists an open neighborhood $U_k \subseteq \hat{U}_k \subseteq \Omega_k \subseteq \mathbb{R}^{r_k}$ of ζ^* such that*

- (i) *$\tilde{V}_{k+1} \cap U_k$ admits an $\tilde{r}_{k+1}$-dimensional parametrization $\zeta = \tilde{\varphi}_{k+1}(\eta)$ with surjective $\tilde{\varphi}_{k+1} : \Omega_{k+1} \to \tilde{V}_{k+1} \cap U_k$;*
- (ii) *there exists a C^∞ matrix-valued map $\tilde{P}_{k+1} : U_k \to \mathbb{R}^{\tilde{r}_{k+1} \times r_k}$ verifying that $\tilde{P}_{k+1}(\zeta)\big|_{L_k(\zeta)}$ yields an isomorphism $L_k(\zeta) \to \mathbb{R}^{\tilde{r}_{k+1}}$ for all $\zeta \in U_k$.*

For any such $\tilde{\varphi}_{k+1}$, $\tilde{P}_{k+1}$, $\zeta(t)$ is a solution of the k-th reduction

$$A_k(\zeta)\zeta' = f_k(\zeta), \; \zeta \in \Omega_k \subseteq \mathbb{R}^{r_k} \tag{10}$$

within U_k if and only if $\zeta(t) \in \tilde{V}_{k+1}$ for all t and $\eta(t) = \tilde{\varphi}_{k+1}^{-1}(\zeta(t))$ is a solution of

$$\tilde{A}_{k+1}(\eta)\eta' = \tilde{f}_{k+1}(\eta), \; \eta \in \Omega_{k+1} \subseteq \mathbb{R}^{\tilde{r}_{k+1}} \tag{11}$$

with $\tilde{A}_{k+1}(\eta) = \tilde{P}_{k+1}(\tilde{\varphi}_{k+1}(\eta))A_k(\tilde{\varphi}_{k+1}(\eta))\tilde{\varphi}'_{k+1}(\eta)$, $\tilde{f}_{k+1}(\eta) = \tilde{P}_{k+1}(\tilde{\varphi}_{k+1}(\eta))f_k(\tilde{\varphi}_{k+1}(\eta))$.

Proof: The existence of the smooth parametrization $\tilde{\varphi}_{k+1}$ follows from (8) together with Assumption S2, whereas that of $\tilde{P}_{k+1}$ is due to the smooth structure of $L_k(\zeta)$ in Assumption S1.

Assume that $\zeta(t)$ solves (10). Then $f_k(\zeta(t)) \in \operatorname{im} A_k(\zeta(t))$, that is, $\zeta(t) \in V_{k+1}$ and thus $\zeta(t) \in \tilde{V}_{k+1}$ for all t by (9). This means that $\eta(t)$ is well-defined by $\zeta(t) = \tilde{\varphi}_{k+1}(\eta(t))$: premultiplying (10) by $\tilde{P}_{k+1}(\tilde{\varphi}_{k+1}(\eta(t)))$ and inserting $\zeta(t) = \tilde{\varphi}_{k+1}(\eta(t))$, $\zeta'(t) = \tilde{\varphi}'_{k+1}(\eta(t))\eta'(t)$ in the resulting equation, we obtain (11).

Conversely, the assumption that (11) holds can be written as

$$\tilde{P}_{k+1}(\tilde{\varphi}_{k+1}(\eta))A_k(\tilde{\varphi}_{k+1}(\eta))\tilde{\varphi}'_{k+1}(\eta)\eta' = \tilde{P}_{k+1}(\tilde{\varphi}_{k+1}(\eta))f_k(\tilde{\varphi}_{k+1}(\eta))$$

or, in terms of $\zeta = \tilde{\varphi}_{k+1}(\eta)$,

$$\tilde{P}_{k+1}(\zeta)A_k(\zeta)\zeta' = \tilde{P}_{k+1}(\zeta)f_k(\zeta). \tag{12}$$

If we show that $A_k(\zeta)\zeta' \in L_k(\zeta)$, $f_k(\zeta) \in L_k(\zeta)$, the identity (12) would yield (10) due to the isomorphism $\tilde{P}_{k+1}(\zeta)\big|_{L_k(\zeta)} : L_k(\zeta) \to \mathbb{R}^{\tilde{r}_{k+1}}$. Indeed, the relation $A_k(\zeta)\zeta' \in L_k(\zeta)$ holds trivially due to $\operatorname{im} A_k(\zeta) \subseteq L_k(\zeta)$, whereas $\zeta = \tilde{\varphi}_{k+1}(\eta) \in \tilde{V}_{k+1}$ means $f_k(\zeta) \in L_k(\zeta)$ by (8). □

This result generalizes the one-step local reduction of Theorem 1 to singular points as long as they meet Assumptions S1 and S2. In the setting defined by Theorem 3, a one-step singular reduction is again suitable for assessment for the reduction (11). Defining $V^s_{k+2} = \{\eta \in \Omega_{k+1} \ / \ \tilde{f}_{k+1}(\eta) \in \operatorname{im} \tilde{A}_{k+1}(\eta)\}$, $W^s_{k+2} = \varphi_1 \circ \ldots \circ \varphi_k \circ \tilde{\varphi}_{k+1}(V^s_{k+2}) \subseteq \tilde{W}_{k+1}$, we may naturally extend the singular reduction process beyond the $(k+1)$-th step.

This way, instead of the sequence of manifolds $W_1 \supset W_2 \supset W_3 \ldots$ constructed in the regular setting, we build up a sequence of the form

$$W_0 \supset W_1^{\mathrm{reg}} \supset \ldots \supset W_k^{\mathrm{reg}} \supset \tilde{W}_{k+1} \supset \ldots \supset \tilde{W}_\nu \stackrel{\mathrm{loc}}{=} \tilde{W}_{\nu+1}, \tag{13}$$

the local stabilization after the ν-th step holding in the setting of Theorem 4 below. The importance of this construction stems from the fact W_{k+1} and later on W^s_{k+2} and subsequent sets may fail to have a C^∞ structure near an inner k-singularity, whereas the extensions $\tilde{W}_{k+1}$, $\tilde{W}_{k+2}$, etc., display a local C^∞ structure, allowing for a local reduction of the DAE. These manifolds comprise in addition the closures $\overline{W_{k+1}}$, $\overline{W^s_{k+2}}$, etc., and therefore may also accommodate boundary singularities.

The repeated application of the one-step singular reduction in Theorem 3 yields the following analog of Theorem 2; the meaning of U parallelizes exactly the one explained there. In the particular case $k = \nu$, the symbols $\tilde{r}_\nu$, $\tilde{A}_\nu$, $\tilde{f}_\nu$ and $\tilde{W}_\nu$ below must be replaced by r_ν, A_ν, f_ν and W_ν. Since no singular reduction is required for these last-step singular points, in this situation Theorem 4 virtually amounts to Theorem 2, consistently with the fact that the setting of Rabier and Rheinboldt discussed in [2, 3, 4] accommodates last-step singularities.

Theorem 4 *Let $x^* \in W_0$ be a k-singularity for (1), $k \geq 0$. Suppose that Assumptions S1 and S2 hold in steps $k+1$, $k+2, \ldots, \nu$ of the singular reduction process described above with*

$$n = r_0 > r_1 > \ldots > r_k > \tilde{r}_{k+1} > \tilde{r}_{k+2} > \ldots > \tilde{r}_\nu > 0, \tag{14}$$

and that Assumption S1 is met in step $\nu + 1$ with $\tilde{r}_\nu = \tilde{r}_{\nu+1}$. Let

$$\tilde{A}_\nu(u)u' = \tilde{f}_\nu(u), \ \ u \in \Omega_\nu \subseteq \mathbb{R}^{\tilde{r}_\nu} \tag{15}$$

be a ν-th step reduction of (1) given by a sequence of reduction pairs (P_1, φ_1), $\ldots$, (P_k, φ_k), $(\tilde{P}_{k+1}, \tilde{\varphi}_{k+1})$, $\ldots$, $(\tilde{P}_\nu, \tilde{\varphi}_\nu)$ on a neighborhood Ω_ν of $u^* = (\varphi_1 \circ \cdots \circ \varphi_k \circ \tilde{\varphi}_{k+1} \circ \ldots \circ \tilde{\varphi}_\nu)^{-1}(x^*)$.

Then $x(t)$ is a solution of (1) within U if and only if $x(t) \in \tilde{W}_\nu$ for all t and $u(t) = (\varphi_1 \circ \ldots \circ \varphi_k \circ \tilde{\varphi}_{k+1} \circ \ldots \circ \tilde{\varphi}_\nu)^{-1}(x(t))$ solves (15).

The requirement that Assumption S1 holds in the last step with $\tilde{r}_\nu = \tilde{r}_{\nu+1} > 0$ amounts to saying that $\tilde{A}_\nu$ (or A_k if $\nu = k$) is non-singular on some dense subset of $\tilde{U}_\nu \subseteq \Omega_\nu$. This means that points in this dense subset are regular with index ν. We speak of a k-singularity x^* as a *singular index* ν point when the hypotheses of this Theorem hold. In these situations the DAE (1) can be locally thought of as a singular index ν problem.

The difference between Theorem 4 and the regular index ν statement within Theorem 2 is that now $\tilde{A}_\nu(u^*)$ will typically be a singular matrix. This may be due to a rank-deficiency arising at any reduction step, not necessarily at the last one. Theorem 4 hence drives the local analysis of a broad family of singular quasilinear DAEs not to the context of explicit ODEs but to the quasilinear ODE setting. This way, not only impasse points but the whole analysis of singular phenomena in [9] can by systematically tackled in the DAE context.

Finally, it is worth emphasizing that all the notions introduced above are invariant with respect to *local equivalence*. Two quasilinear DAEs $A(x)x' = f(x)$, $B(y)y' = g(y)$ defined on W_0^a, W_0^b open in $\mathbb{R}^n$, are said to be C^∞-equivalent locally around x^*, y^* if there exist open neighborhoods $U_b \subseteq W_0^b$ of y^*, $U_a \subseteq W_0^a$ of x^*, a C^∞-diffeomorphism $\phi : U_b \to U_a$ with $\phi(y^*) = x^*$, and a C^∞ non-singular matrix-valued mapping $E : U_b \to \mathbb{R}^{n\times n}$, such that $B(y) = E(y)A(\phi(y))\phi'(y)$, $g(y) = E(y)f(\phi(y))$ for all $y \in U_b$. The relation $g(y) = E(y)f(\phi(y))$ is a *contact equivalence* between f and g. Note that, for the equivalence of the quasilinear systems, the pair (E, ϕ) is required to link additionally the matrix mappings A and B. This equivalence relation amounts to a C^∞-conjugacy for index-0 cases, that is, for explicit ODEs, thereby explaining the fact that any two local reductions of a quasilinear DAE are C^∞-conjugate.

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