XX Congreso de Ecuaciones Diferenciales y Aplicaciones
X Congreso de Matemática Aplicada
Sevilla, 24-28 septiembre 2007
(pp. 1–8)

Asymptotic behaviour of a singularly perturbed convection-diffusion problem in a rectangle with discontinuous Dirichlet data

J.L. López[1], Ester Pérez Sinusía[1]

[1] *Dpto. de Ingeniería Matemática e Informática, Universidad Pública de Navarra, 31006 Pamplona.*
E-mails: jl.lopez@unavarra.es, ester.perez@unavarra.es.

Palabras clave: Singular perturbation problem, discontinuous boundary data, asymptotic expansions, error function.

Resumen

We consider a singularly perturbed convection-diffusion equation, $-\varepsilon\triangle u+\overrightarrow{v}\cdot\overrightarrow{\nabla}u=0$, defined on a rectangular domain $\Omega\equiv\{(x,y)\,|\,0\le x\le\pi a, 0\le y\le\pi\}$, $a>0$, with Dirichlet-type boundary conditions discontinuous at the points $(0,0)$ and $(\pi a,0)$: $u(x,0)=1$, $u(x,\pi)=u(0,y)=u(\pi a,y)=0$. An asymptotic expansion of the solution is obtained from a a series representation in two limits: a) when the singular parameter $\varepsilon\to 0^+$ (with fixed distance to the points $(0,0)$ and $(\pi a,0)$) and b) when $(x,y)\to(0,0)$ or $(x,y)\to(\pi a,0)$ (with fixed ε). It is shown that the first term of the expansion at $\varepsilon=0$ contains a linear combination of error functions. This term characterizes the effect of the discontinuities on the $\varepsilon-$behaviour of the solution $u(x,y)$ in the boundary or the internal layers. On the other hand, near the points of discontinuity $(0,0)$ and $(\pi a,0)$, the solution $u(x,y)$ is approximated by a linear function of the polar angle.

1. Introduction

The solution of a singularly perturbed convection-diffusion problem usually presents boundary and/or interior layers. The location and shape of these layers depend, among other things, on the discontinuities of the boundary condition. An 'a priori' knowledge of the location of the internal or boundary layers is quite useful to design numerical methods for this kind of problems. This information may be obtained from an asymptotic expansion of the solution [11], [13]. There is an extensive literature devoted to the construction of approximated solutions of singular perturbation problems based on matching of asymptotic expansions (see for example [2], [11], [12] or [13] for a historical survey on the subject). But a perturbative analysis based on an expansion of the solution in powers of the perturbation parameter does not always work for discontinuous Dirichlet boundary conditions [14]. This is so, because the coefficients of the expansion contain derivatives of the boundary condition, whereas the solution of the elliptic problem is smooth inside the domain.

In former works [7, 8, 9], we have studied the asymptotic behaviour of the solution of several singular perturbation convection-diffusion problems with discontinuous data defined on different unbounded domains (quarter plane, an infinite and a semi-infinite strip, a sector). In this work [10], we analyze a problem of the same type but defined on a bounded domain, more interesting for practical purposes. Several authors have obtained some asymptotic information in specific problems defined on bounded domains (see for example [1, 3, 4], [5, p. 537], [6, 12]), but the technique used there (matching of asymptotic expansions) does not proportionate, in general, an expansion uniformly valid in the whole domain and finds some extra difficulties when the boundary conditions are discontinuous.

We consider the problem $-\varepsilon\Delta u + \overrightarrow{v}\cdot\overrightarrow{\nabla}u = 0$ defined in a rectangle with a discontinuous boundary condition at two of the corners of the rectangle. This problem displays boundary and interior layers. We derive the exact solution of the problem by means of the method of separation of variables. The exact representation can be written in terms of a Fourier series. The series is transformed into a series of integrals in the complex plane from which we obtain complete asymptotic expansions. We approximate the solution by deriving asymptotic expansions from this series, not only in the singular limit $\varepsilon \to 0^+$, but also in the limit $r \to 0^+$, where r represents the distance to the points of discontinuity. Then, we approximate the solution on the whole domain, including the neighborhood of the points of discontinuity.

2. The problem and its exact solution

We are interested in approximating the solution of the following singularly perturbed convection-diffusion problem defined in a rectangle $\Omega \equiv (0,\pi a)\times(0,\pi)$ with discontinuous Dirichlet-type boundary conditions:

$$\left\{\begin{array}{ll} -\varepsilon\triangle U + \overrightarrow{v}\cdot\overrightarrow{\nabla}U = 0, & (x,y)\in\Omega, \\ \left| \begin{array}{l} U(x,0) = 1, \\ U(x,\pi) = U(0,y) = U(\pi a,y) = 0 \end{array}\right. & \begin{array}{l} U\in\mathcal{C}(\tilde{\Omega})\cap\mathcal{D}^2(\Omega), \\ \text{U bounded in } \tilde{\Omega}, \end{array} \end{array}\right. \qquad (P)$$

where $\overrightarrow{v} \equiv (\sin\beta,\cos\beta)$ is a constant vector, $0\le\beta<2\pi$, a is a positive constant and $\tilde{\Omega}\equiv\bar{\Omega}\backslash\{(0,0),(\pi a,0)\}$ (observe the discontinuous Dirichlet conditions at the lower corners of the rectangle, see Figure 1(a)).

After the change of the dependent variable $U(x,y) = F(x,y)\exp\left(\overrightarrow{v}\cdot\overrightarrow{r}/(2\varepsilon)\right)$, with $w\equiv 1/(2\varepsilon)$ and $\overrightarrow{r}\equiv(x,y)$, the problem (P) is transformed into the Yukawa equation for $F(x,y)$:

$$\left\{\begin{array}{ll} \triangle F - w^2 F = 0 & (x,y)\in\Omega, \text{ F bounded in } \tilde{\Omega}, \\ \left| \begin{array}{l} F(x,0) = e^{-wx\sin\beta}, \\ F(x,\pi) = F(0,y) = F(\pi a,y) = 0, \end{array}\right. & F\in\mathcal{C}(\tilde{\Omega})\cap\mathcal{D}^2(\Omega). \end{array}\right. \qquad (1)$$

Using a similar technique to the one used in [9], we can show that this problem has a unique solution. In the following proposition we obtain the explicit solution of the problem

(P) by means of a series representation. In what follows, empty sums must be understood as zero.

Proposition 2.1. *For $(x,y) \in \Omega$ and $\beta \in (0, \pi/2]$, the solution $U_\beta(x,y)$ of (P) is:*

$$U_\beta(x,y) = e^{w(x\sin\beta + y\cos\beta)}[G(x,y) + e^{-\pi a w \sin\beta} G(\pi a - x, y)] \tag{2}$$

where

$$G(x,y) \equiv \sum_{n=-\infty}^{\infty} H_\beta(x + 2n\pi a, y) \tag{3}$$

and

$$H_\beta(x,y) \equiv \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{\sinh\left[(\pi - y)\sqrt{w^2+t^2}\right]}{\sinh\left[\pi\sqrt{w^2+t^2}\right]} \frac{t e^{itx}}{w^2 \sin^2\beta + t^2}\, dt. \tag{4}$$

On the other hand, for $\beta = 0$:

$$U_0(x,y) = \sum_{n=-\infty}^{\infty} (-1)^n \tilde{H}_0(x + n\pi a, y), \tag{5}$$

where

$$\tilde{H}_0(x,y) \equiv \frac{e^{wy}}{2\pi i} \int_{-\infty}^{\infty} \frac{\sinh\left[(\pi - y)\sqrt{w^2+t^2}\right]}{\sinh\left[\pi\sqrt{w^2+t^2}\right]} \frac{e^{ixt} - e^{i(x-\pi a)t}}{t}\, dt. \tag{6}$$

Demostración. The exact solution of (1) may be obtained by separation of variables:

$$F(x,y) = \sum_{n=1}^{\infty} \frac{2n}{\pi} \frac{\sinh[(\pi - y)\sqrt{w^2 + n^2/a^2}]}{\sinh\left[\pi\sqrt{w^2+n^2/a^2}\right]} \frac{1 - (-1)^n e^{-w\pi a \sin\beta}}{a^2 w^2 \sin^2\beta + n^2} \sin\left(\frac{nx}{a}\right). \tag{7}$$

Then, the function $U_\beta(x,y) \equiv e^{w(x\sin\beta + y\cos\beta)} F(x,y)$, with $F(x,y)$ defined above, is the solution of (P). It can be rewritten as follows for $0 < \beta \le \pi/2$:

$$U_\beta(x,y) = e^{w(x\sin\beta + y\cos\beta)}[J(x,y) + e^{-\pi a w \sin\beta} J(\pi a - x, y)], \tag{8}$$

with

$$J(x,y) \equiv \frac{1}{i\pi a} \sum_{n=-\infty}^{\infty} \frac{n/a}{w^2\sin^2\beta + n^2/a^2} \frac{\sinh[(\pi-y)\sqrt{w^2+n^2/a^2}]}{\sinh[\pi\sqrt{w^2+n^2/a^2}]} e^{inx/a}. \tag{9}$$

Applying the Poisson summation formula to the series in (9) and inserting the result in (8) we obtain (2)-(4). Formulas (5)-(6) follow after applying the Poisson summation formula directly to the series on the right hand side of (7) with $\beta = 0$. The convergence of the series in (3) and (5) follows from Theorems 3.1 and 3.2 respectively. □

Observation 2.1. The explicit representation given in Proposition 2.1 is only valid when the angle β between the convection vector $\overrightarrow{v}$ and the y-axis is restricted to the interval $[0, \pi/2]$. Nevertheless, an explicit integral representation for the solution $U(x,y)$ of the

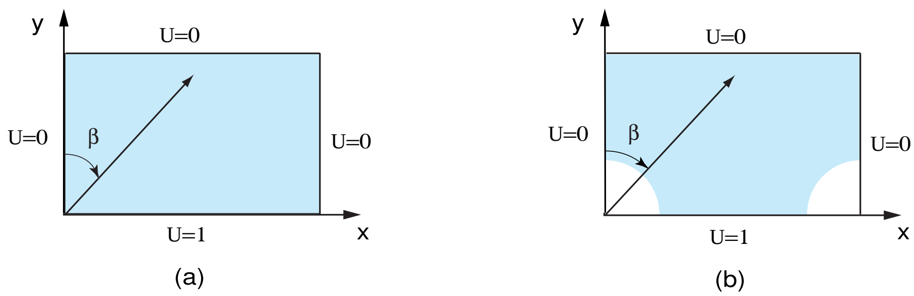


Figura 1: *(a) Domain Ω of problem (P). (b) Indented region Ω^* in Theorems 3.1 and 3.2.*

problem (P) whatever the direction of $\overrightarrow{v}$ is, may be obtained by means of symmetry arguments:

$$U(x,y) = \begin{cases} U_\beta(x,y) & \text{if } 0 \leq \beta \leq \pi/2 \\ e^{2wy\cos\beta}\, U_{\pi-\beta}(x,y) & \text{if } \pi/2 < \beta \leq \pi \\ U_{-\beta}(\pi a - x, y) & \text{if } -\pi/2 \leq \beta < 0 \\ e^{2wy\cos\beta}\, U_{\pi+\beta}(\pi a - x, y) & \text{if } -\pi < \beta < -\pi/2 \end{cases}$$

where $U_\beta(x,y)$ is given in (2) for $0 < \beta \leq \pi/2$ and $U_0(x,y)$ is given in (5). Therefore, in the remainder, we will restrict ourselves to $\beta \in [0, \pi/2]$.

The solution of (P) can not be written in terms of known functions. But, for $\varepsilon \to 0^+$ and (x,y) away from $(0,0)$ and $(\pi a, 0)$, we can approximate $U_\beta(x,y)$ by a combination of error functions plus an asymptotic expansion in powers of ε. For $(x,y) \to (0,0)$ or $(x,y) \to (\pi a, 0)$ (and $\varepsilon \geq \varepsilon_0 > 0$), we can approximate $U_\beta(x,y)$ by an asymptotic expansion in powers of r or of $\sqrt{(x-\pi a)^2 + y^2}$ respectively. This is the subject of the two following sections.

3. Asymptotic expansion of $U(x,y)$ in the singular limit

In this section we denote by Ω^* the rectangular domain indented at the points $(0,0)$ and $(\pi a, 0)$ (see Figure 1b):

$$\Omega^* \equiv \left\{ (x,y) \in \Omega,\ 0 < r_0 < \sqrt{x^2+y^2},\, 0 < r_0 < \sqrt{(x-\pi a)^2 + y^2} \right\}.$$

Theorem 3.1. *For $(x,y) \in \Omega^*$ and $\beta \in (0, \pi/2]$, the solution $U_\beta(x,y)$ of (P) given in Proposition 2.1 is*

$$U_\beta(x,y) = U_\beta^0(x,y) + \frac{1}{\sqrt{w}} U_\beta^1(x,y), \tag{10}$$

where

$$\begin{aligned}
U_\beta^0(x,y) \equiv\ & e^{wy\cos\beta}\frac{\sinh[(\pi-y)w\cos\beta]}{\sinh[\pi w\cos\beta]} \\
& \times\left\{\frac{\left(1+\delta_{\beta,\pi/2}\right)}{2}\left[\operatorname{sign}\left(\beta-\arctan\left(\frac{x}{y}\right)\right)\operatorname{erfc}\sqrt{w\zeta(x,y)}\right.\right. \\
& - e^{2(x-\pi a)w\sin\beta}\operatorname{sign}\left(\beta-\arctan\left(\frac{2\pi a-x}{y}\right)\right)\operatorname{erfc}\sqrt{w\zeta(2\pi a-x,y)} \\
& \left.+e^{2(x-\pi a)w\sin\beta}\operatorname{sign}\left(\beta-\arctan\left(\frac{\pi a-x}{y}\right)\right)\operatorname{erfc}\sqrt{w\zeta(\pi a-x,y)}\right] \\
& \left.+\chi_A(x,y)-e^{2w(x-\pi a)\sin\beta}\chi_B(x,y)\right\}.
\end{aligned} \tag{11}$$

In these formulas, sign(0) *must be understood as zero,*

$$\zeta(x,y)\equiv\sqrt{x^2+y^2}-x\sin\beta-y\cos\beta, \tag{12}$$

the regions A and B are (see Figure 2):

$$\begin{aligned}
A &\equiv\{(x,y)\in\Omega, y<x\cot\beta\}, \\
B &\equiv\{(x,y)\in\Omega, (\pi a-x)\cot\beta<y<(2\pi a-x)\cot\beta\}
\end{aligned} \tag{13}$$

and $\chi_A(x,y)$ and $\chi_B(x,y)$ are the characteristic function of the sets A and B respectively.

For $n=0,1,2,\dots$, $U_\beta^1(x,y)$ has an asymptotic expansion in powers of w^{-1}:

$$U_\beta^1(x,y)=\sum_{k=0}^{n-1}\frac{T_k(x,y)}{w^k}+R_n(x,y), \tag{14}$$

where empty sums must be understood as zero and the coefficients $T_k(x,y)$ are smooth functions of x and y and $\mathcal{O}(1)$ as $w\to\infty$ uniformly for $(x,y)\in\Omega^*$. The remainder $R_n(x,y)$ satisfies the bound

$$|R_n(x,y)|\le M\frac{\Gamma(n+1/2)}{(2wd\widetilde{r})^n}\,e^{-w\widetilde{\zeta}(x,y)}, \tag{15}$$

for some positive constants M and d given below, $\widetilde{\zeta}(x,y)=\text{mín}\{\zeta(x,y),\zeta(\pi a-x,y)\}$ and $\widetilde{r}^2=\text{mín}\{x^2+y^2,(\pi a-x)^2+y^2\}$.

Theorem 3.2. *For $(x,y)\in\Omega^*$ and $\beta=0$, the solution $U_0(x,y)$ of (P) given in Proposition 2.1 is*

$$U_0(x,y)=U_0^0(x,y)+\frac{1}{\sqrt{w}}\,U_0^1(x,y), \tag{16}$$

where

$$U_0^0(x,y)\equiv\left\{1-\operatorname{erfc}\sqrt{w\,\zeta(x,y)}-\operatorname{erfc}\sqrt{w\,\zeta(x-\pi a,y)}\right\}e^{wy}\frac{\sinh[(\pi-y)w]}{\sinh[\pi w]} \tag{17}$$

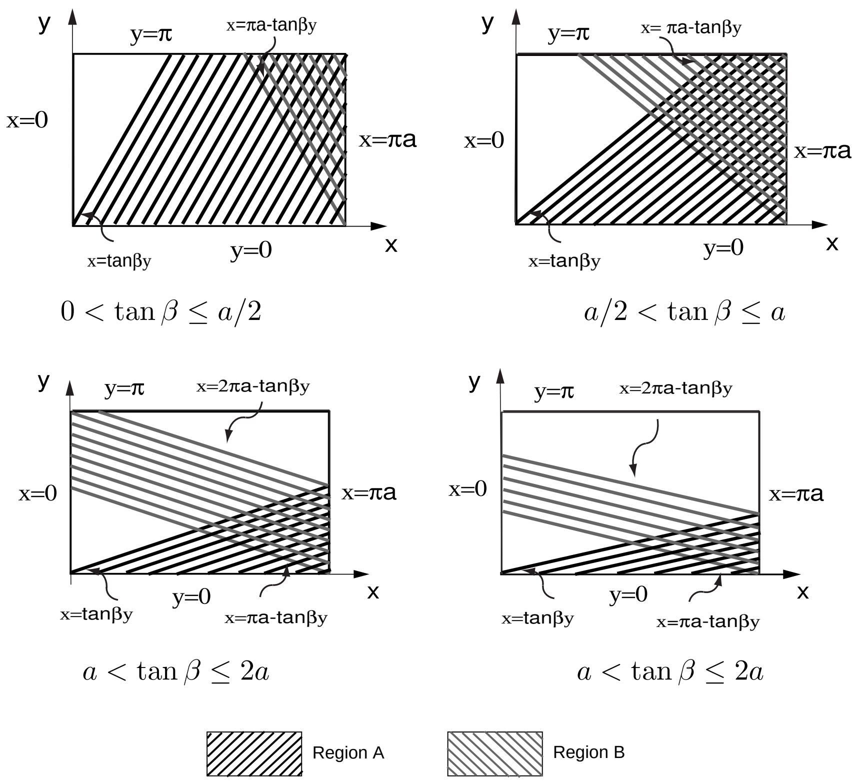


Figura 2: *Different aspects of the regions A and B defined in Theorem 3.1 depending on the relative value between* $\tan\beta$ *and* a.

and $\zeta(x,y) \equiv \sqrt{x^2+y^2}-y$. *The function* $U_0^1(x,y)$ *has an asymptotic expansion in powers of* w^{-1}:

$$U_0^1(x,y) = \sum_{k=0}^{n-1} \frac{\tilde{T}_k(x,y)}{w^k} + R_n(x,y), \tag{18}$$

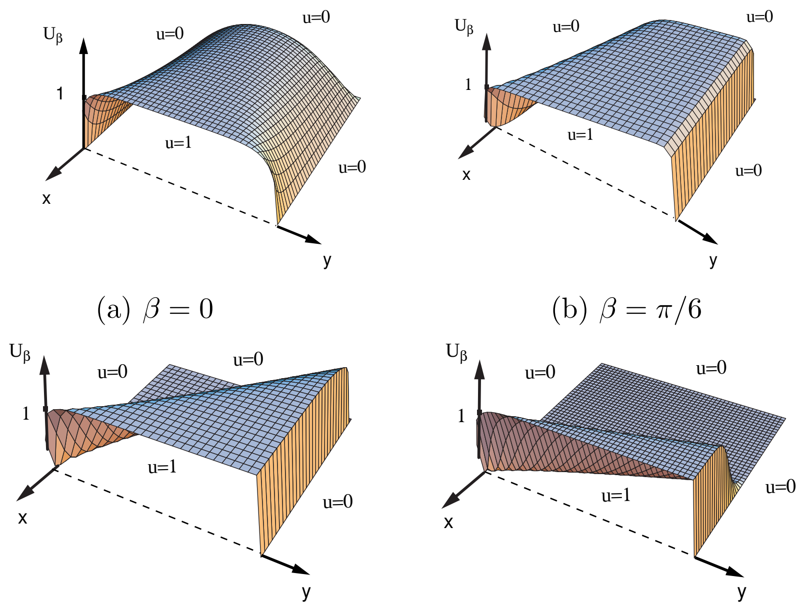
where empty sums are zero. The coefficients $\tilde{T}_k(x,y)$ *are smooth functions of* x *and* y *and* $\mathcal{O}(1)$ *as* $w \to \infty$ *uniformly for* $(x,y) \in \Omega^*$.

The remainder $R_n(x,y)$ *satisfies*

$$|R_n(x,y)| \le M \frac{\Gamma(n+1/2)}{(2wd\tilde{r})^n}, \tag{19}$$

for some positive constants M *and* d.

Remark 3.1. From (10), (14) and (15) we see that $U_\beta(x,y) = U_\beta^0(x,y)[1+\mathcal{O}(\sqrt{\varepsilon})]$ and from (16), (18) and (19) we see that $U_0(x,y) = U_0^0(x,y)[1+\mathcal{O}(\sqrt{\varepsilon})]$ as $\varepsilon \to 0^+$ away from the points $(0,0)$ and $(\pi a, 0)$. Then, the first order approximation to the solution of (P) is a linear combination of error functions and elementary functions. The error functions in (11) and (17) exhibit interior/boundary layers of width $\mathcal{O}(\sqrt{\varepsilon})$. The exponential factors in (11) and (17) exhibit boundary layers of width $\mathcal{O}(\varepsilon)$ (see Figure 3).



(a) $\beta = 0$ (b) $\beta = \pi/6$

(c) $\beta = \pi/3$ (d) $\beta = 4\pi/9$

Figura 3: *Graphs of the first order approximation, $U_\beta^0(x,y)$, to the solution of the problem (P) for different values of β and $\varepsilon = 0{,}1$. The convection vector $\overrightarrow{v}$ "drags" the discontinuity of the boundary condition at $r = 0$ originating a parabolic layer of size $\mathcal{O}(\sqrt{\varepsilon})$ along $\overrightarrow{v}$.*

4. Asymptotic expansion of $U(x, y)$ near the corner singularities

The asymptotic expansions given in Theorem 3.1 and Theorem 3.2 break down when $(x, y) \to (0, 0)$ or $(x, y) \to (\pi a, 0)$ (that is, $\tilde{r} \to 0$ in (15) and (19)). The asymptotic approximation of $U_\beta(x, y)$ near these points requires a different analysis. An asymptotic approximation of $U_\beta(x, y)$ as $(x, y) \to (0, 0)$ or $(x, y) \to (\pi a, 0)$ faster than $\varepsilon \to 0^+$ is given in the following theorem.

Theorem 4.1. *Let $x = r \sin\phi$, $y = r\cos\phi$ and $\pi a - x = \bar{r}\sin\bar{\phi}$, $y = \bar{r}\cos\bar{\phi}$. Then, for $\beta \in [0, \pi/2]$ and $(x, y) \in \Omega$, the solution $U_\beta(x, y)$ of (P) verifies:*

$$U_\beta(x,y) = \frac{2\phi}{\pi} + \mathcal{O}\left(\frac{r}{\varepsilon}\right) + \mathcal{O}(e^{-\alpha/\varepsilon}), \tag{20}$$

as $r, \varepsilon \to 0^+$ with $r/\varepsilon \to 0^+$. And

$$U_\beta(x,y) = \frac{2\bar{\phi}}{\pi} + \mathcal{O}\left(\frac{\bar{r}}{\varepsilon}\right) + \mathcal{O}(e^{-\alpha/\varepsilon}), \tag{21}$$

as $\bar{r}, \varepsilon \to 0^+$ with $\bar{r}/\varepsilon \to 0^+$. The $\mathcal{O}(e^{-\alpha/\varepsilon})$ symbols hold uniformly for $(x, y) \in \Omega$ with $\alpha > 0$.

5. Conclusions

The singularly perturbed convection-diffusion problem (P) has been defined on a rectangle by means of discontinuous Dirichlet boundary conditions with two points of discontinuity located on the two lower corners of the domain. We have obtained a series representation of the solution susceptible of an asymptotic analysis. Then, an asymptotic expansion of the solution have been obtained in the singular limit $\varepsilon \to 0^+$ and away from the points of discontinuity $(0,0)$ and $(\pi a, 0)$ (Theorems 3.1 and 3.2). On the other hand, two asymptotic approximations of the solution near the points of discontinuity $(0,0)$ or $(\pi a, 0)$ (valid for $\varepsilon \geq \varepsilon_0 > 0$) have been derived in Theorem 4.1.

The asymptotic expansion in the singular limit shows that the main contribution from the data's discontinuities to the shape of the solution on the singular layers is contained in a certain combination of error functions, exponential functions and characteristic functions (Equations (11) and (17)). This combination is necessary to approach the behaviour of the solution on the interior/boundary layers of width $\mathcal{O}(\sqrt{\varepsilon})$ or on the boundary layer of width $\mathcal{O}(\varepsilon)$. On the other hand, the asymptotic approximations near the discontinuities (Equations (20) and (21)) show that the points of discontinuity on the boundary is smoothed inside the domain by means of a linear function of the polar angle.

Referencias

[1] L. P. Cook and G.S.S. Ludford, *The behavior as $\varepsilon \to +0$ of solutions to $\varepsilon\nabla^2 w = \partial w/\partial y$ in $|y| \leq 1$ for discontinuous boundary data*, SIAM J. Math. Anal, (1971), 567-594.

[2] W. Eckhaus, *Matched Asymptotic Expansions and Singular Perturbations*, North-Holland, Amsterdam, 1973.

[3] J. Grasman, *On the birth of boundary layers*, MC Tract, *Mathematisch Centrum, Amsterdam*, 36 (1971).

[4] J. Grasman, *An elliptic singular perturbation problem with almost characteristic boundaries*, J. Math. Anal. Appl., 46 (1974), 438-446.

[5] J. Kevorkian, *Partial Differential Equations. Analytical Solution Techniques*, Springer-Verlag, New York, 1990.

[6] J. Kevorkian, J.D. Cole, *Multiple Scale and Singular Perturbation Methods,* Springer-Verlag, New York, 1996.

[7] J.L. López y Ester Pérez Sinusía. *Asymptotic expansions for two singularly perturbed convection-diffusion problems with discontinuous data: the quarter plane and the infinite strip.* Stud. Appl. Math, 113 (2004), 57-89.

[8] J.L. López y Ester Pérez Sinusía. *Analytic Approximations for a singularly perturbed convection-diffusion problem with discontinuous data in a half-infinite strip.* Acta Applicandae Mathimaticae, 82 (2004), 101-117.

[9] J.L. López y Ester Pérez Sinusía. *Asymptotic Approximations for a singularly perturbed convection-diffusion problem with discontinuous data in a sector.* JCAM, 181 (2004), 1-23.

[10] J.L. López y Ester Pérez Sinusía. *The role of the error function in a singularly perturbed convection-diffusion problem in a rectangle with corner singularities.* The Royal Society of Edinburgh Proceedings A (Mathematics), (2006).

[11] R. E. O'Malley, *Introduction to Singular Perturbation*, Academic Press, New York, 1974.

[12] S.-D. Shih and R. B. Kellogg, *Asymptotic analysis of a singular perturbation problem*, SIAM J. Math. Anal., 18, 5 (1987), 1467-1511.

[13] D. R. Smith, *Singular Perturbation Theory*, Cambridge Univ. Press, 1985.

[14] N.M. Temme, *Analytical methods for a selection of elliptic singular perturbation problems*, Recent advances in differential equations, (1997), 131-148.