Some qualitative results on magnetic vector fields

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Resumen

In this work I will study magnetic vector fields generated by thin current lines. The existence of first integrals in some symmetric situations as well as their effect on particles motion will be examined. Some examples of current distributions giving rise to quasi-periodic and chaotic magnetic lines will be shown. Finally some results on the existence of periodic orbits and polynomial first integrals of magnetic fields will be provided.

1. Introduction

I will focus on a class of vector fields which is particularly important in Physics and multidisciplinary applications, i.e. magnetic fields created by DC (direct current) flows. The mathematical description of these fields involve a smooth curve L in $\mathbb{R}^3$, parametrized by the embedding map $\tau : \mathbb{R} \to \mathbb{R}^3$, which represents the electric wire, and a constant J which stands for the current intensity. The magnetic field B evaluated at the point $r \in \mathbb{R}^3$ is obtained from Biot-Savart law, as was discovered in the XVIII century:

$$B(r) = \frac{\mu_0 J}{4\pi} \int_I \frac{\dot{\tau}(t) \wedge (r - \tau(t))}{|r - \tau(t)|^3} \, , \tag{1}$$

where μ_0 denotes the magnetic permeability constant, the dot over τ stands for the derivative with respect to t and $\wedge$ and $|\cdot|$ represent the standard vector product and Euclidean norm in $\mathbb{R}^3$. $I \subset \mathbb{R}$ is the (finite or infinite) interval of parametrization of the curve L.

Eq. (1) is the main formula of magnetostatics and defines the magnetic field created by the current distribution (L, J). According to the superposition principle the magnetic field created by n wires $(L_1, J_1), \ldots, (L_n, J_n)$ is given by the sum $B = \sum_{i=1}^n B_i$ of the individual magnetic fields B_i obtained from Biot-Savart law.

In the following sections I will summarize the results on this topic that I have obtained in collaboration with other authors [1, 2, 3, 4].

2. Symmetries and first integrals [4]

When the magnetic field has a Euclidean symmetry then there exist two independent first integrals (possibly multi-valued in $\mathbb{R}^3$). We apply this result to magnetic fields created by several current distributions:

1. N straight line wires parallel to the z-axis.

2. N circular wires on the planes $\{z = z_i\}$ all of whose centers are lined up along the z-axis.

3. A wire with the shape of a standard helix.

As a spin-off we get a description of the phase portraits of all these magnetic vector fields. In particular we get that magnetic lines are closed near the currents (except for the helix) but critical points and homoclinic orbits arise in the global picture.

We also study a configuration exhibiting radial symmetry: the magnetic field created by N straight line wires concurrent at the origin. Again two first integrals are computed and the phase structure is obtained, concluding that magnetic lines are closed near the wires (excluding the origin point).

3. The non-swallowing property [3, 4]

The magnetic fields with Euclidean symmetry studied in the preceding section give rise to an interesting phenomenon when studying the motion of charged particles. The Newton-Lorenz equation of the motion of a unit-mass, unit-charge test particle subjected to the magnetic field $B(r)$ is:

$$\ddot{r} = \dot{r} \wedge B(r). \tag{2}$$

We prove that for any initial condition $(r_0, \dot{r}_0)$ the trajectory of the particle cannot reach the wires and in fact there are some confinement spatial regions.

The key of the proof is that the first integrals of B transmit a first integral to the equations of motion, and this new first integral prevents from collision with wires and escape to any region. The Euclidean symmetry is essential and in fact the new conservation law is a generalized momentum.

This property is in strong contrast with the motion of particles subjected to electric or gravitational fields created by charges or masses, where collisions usually appear.

4. Chaotic magnetic fields [2, 1]

It is widely believed that magnetic vector fields created by current lines are "simple", specifically that they cannot be chaotic and that magnetic lines near wires are closed.

We show that this belief is wrong studying several configurations of wires which give rise to the main features of Hamiltonian chaos: quasi-periodic orbits, KAM islands and homoclinic tangles. A rigorous proof of chaos (using KAM and Poincaré-Birkhoff theorems) is provided in the case of a configuration formed by a straight line wire and small

perturbations of a circular wire. Other configurations, formed by piecewise rectilinear currents, are studied using numerical simulations, and chaos, as well as open "complicated" orbits are found.

We also analyze a transition chaos-order for a configuration formed by a straight line finite wire of length L and a square wire. When $L \to \infty$ chaotic regions, as well as the highest Lyapunov exponent, grow, while in the limit $L \to 0$ we get a non-chaotic, completely integrable phase portrait.

5.   Periodic orbits and Stefanescu's conjecture [1]

In this section we restrict ourselves to configurations of wires which lie on a plane. In this setting we prove that magnetic lines are closed near current distributions (except at possibly non-smooth points) and we compute two (local) first integrals in a small enough neighborhood of each wire. To prove this result we use that the magnetic field at a point P close enough to the wire goes as ρ^{-1} and is orthogonal to the current, ρ standing for the distance between P and the wire. Although this is common knowledge we have not found its proof in the literature so we provide a demonstration, which turns out not to be trivial and some hypotheses are required.

The most remarkable result that we get is two counterexamples to Stefanescu's conjecture: any magnetic field created by a planar rectilinear distribution possesses a polynomial first integral. Indeed, using onfigurations formed by three straight line wires and four straight line wires we produce magnetic fields which are shown to lack of algebraic first integrals, thus contradicting the conjecture.

Agradecimientos

Referencias

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