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Designing compression structures by Topological Mapping

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ABSTRACT

The Force Density Method has been traditionally employed to seek the equilibrium shape of tension structures. Recently, the use of a process based on topology has been introduced to provide a first network in which the Force Density Method can be applied. Compression-only structures such as vaults and domes can also be modelled by means of the former approach if this is modified to get a design process similar to the Gaudí's one based on hanging models. The conjunction of Topological Mapping and the Forced Density Method is first explained in the form-finding process of tension structures and, later on, the modifications to that approach are introduced so that an iterative procedure is obtained to get equilibrium shapes of compression-only structures. The versatility of this novel approach is presented by means of the analysis of some representative examples

Keywords: Force Density Method, Topological Mapping, Compression structures, Gaudí, Design techniques.

1. INTRODUCTION

The work of the Spanish architect Antonio Gaudí (1852-1926) is well known by, among other features, the employment of hanging models and graphical analysis in the design of his structures. A clear example: the Colonia Güell church in Barcelona, Fig. 1. In Gaudí's own words, the logic form comes out from the necessities [1] so that design and structural analysis are linked from the very beginning of the process [2].

However, the use of hanging models could be questioned nowadays since architects and structural engineers have powerful and price-affordable computers and software on hand which allow them to create the geometry of their designs and, afterward, to analyze their structural behavior. But, in spite of this, the structural analysis and geometry conception may require large time investments and does not effortlessly allow going back and making changes in the original creation. Precisely this was the problem Gaudí was solving with his hanging models and, in this way, some approaches have been proposed that tackle the design similarly as hanging models do [3]–[5].

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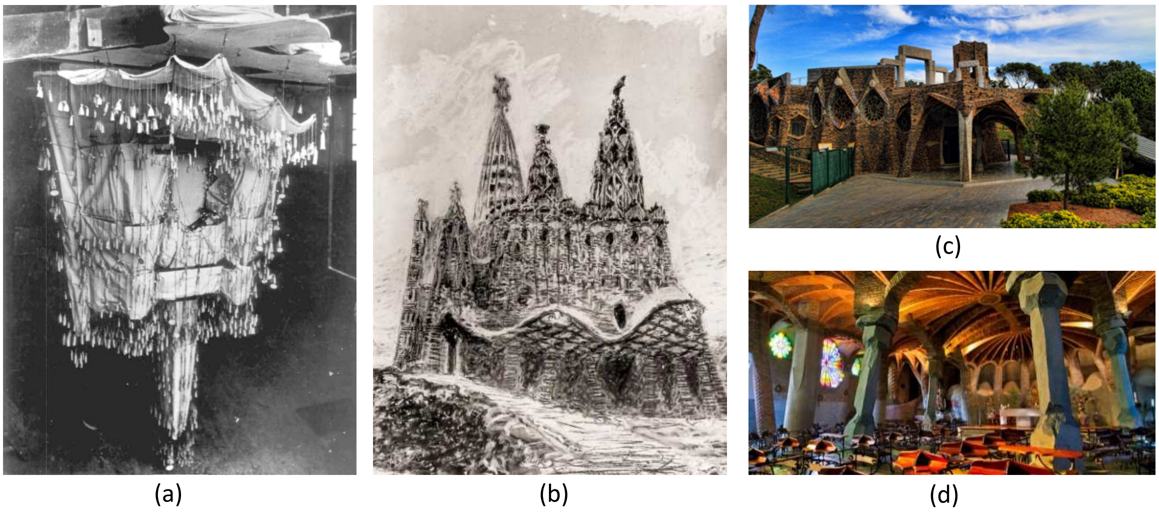


Figure 1. *The church in the Colonia Güell: (a) original hanging model; (b) view of the designed church painted over a 180° turned photo of the hanging model; (c) external view of the built part (crypt) of the church; (d) internal view of the crypt*

The present work proposes a new method in the design of compression structures using the analogy of the Gaudí's hanging models. This approach makes use of the so-called Force Density Method (FDM) [6]–[8], widely employed to seek the equilibrium configuration of tension and tensegrity structures [9], [10]. The proposed method considers the self-weight of the mesh of nodes and branches introduced in FDM as the resulting polygons in the network were built, for instance, in concrete. In this way, a virtual hanging model is obtained and can be turned upside down to get a compression structure. Controlling the force density values in the branches of the mesh, a huge variety of structures can be designed including those with arch ribs.

As it has been above mentioned, FDM needs a mesh of nodes and branches where to be applied. This mesh is provided by the Topological Mapping (TM) technique, as introduced by Hernández-Montes [11]. The main novelty and difference of TM in contrast with other mapping methods is found in the fact that the designer does not need to think about any initial shape and its contour since the mesh is first built in topology and, after this, translated to the space by FDM.

This procedure has been implemented into a computer application that hands over the needed input and provides the resulting form of the compression structure, allowing the user to get many different possible designs modifying just few parameters. The authors of this work have uploaded this computer application for free download [12].

2. THE CONJUNTION BETWEEN FORCE DENSITY METHOD AND TOPOLOGICAL MAPPING

2.1. Force Density Method

The Force Density Method (FDM) was presented by Linkwitz and Schek [6] as a procedure to solve linearly an originally highly non-linear problem [8]: the form-finding of a pin-joint network. This method converts the problem into a set of linear equations by introducing the force-length ratios or force densities for each branch constituting the mesh or network.

Let us consider the following problem (Fig. 2): a load $P = P_3$ has to be held by two bars 1 and 2 which are pinned to points 1 and 2, respectively; the bars are linked one to another at the point where the load is hung.

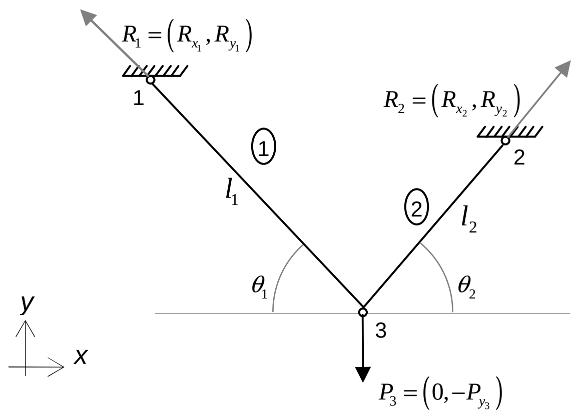


Figure 2. *Simple example to be solved by FDM*

By observation of the geometry of the problem, the equilibrium in FIGURE 2 dictates that:

$$\left.\begin{aligned} S_1\frac{x_1-x_3}{l_1} = R_{x_1} \\ S_1\frac{y_1-y_3}{l_1} = R_{y_1} \end{aligned}\right\} @1$$
$$\left.\begin{aligned} S_2\frac{x_2-x_3}{l_2} = R_{x_2} \\ S_1\frac{y_2-y_3}{l_2} = R_{y_2} \end{aligned}\right\} @2 \tag{1}$$
$$\left.\begin{aligned} S_1\frac{x_1-x_3}{l_1} + S_2\frac{x_2-x_3}{l_2} = 0 \\ S_1\frac{y_1-y_3}{l_1} + S_2\frac{y_2-y_3}{l_2} = P_{y_1} \end{aligned}\right\} @3$$

where, $S_1$and S_2 are the forces in the bars and l_1 and l_2 are their lengths which depend on the position of point 3, (x_3, y_3), so that:

$$\begin{aligned} l_1 &= \sqrt{(x_1 - x_3)^2 + (y_1 - y_3)^2} \\ l_2 &= \sqrt{(x_2 - x_3)^2 + (y_2 - y_3)^2} \end{aligned} \tag{2}$$

A priori, the only known parameters are the position of points 1 and 2, (x_1, y_1) and (x_2, y_2) respectively, and the load $P = P_3$. If the coordinates of point 3 are to be determined, the resulting problem is nonlinear, Eq. (2), and undetermined, 6 equations and 8 unknowns (S_1, S_2, R_{x_1}, R_{x_2}, R_{y_1}, R_{y_2}, x_3, y_3). Therefore, the two parameters need to be introduced in order to determine a particular solution of the problem. The FDM introduces what is called the force densities which in bars 1 and 2 are defined as:

$$\begin{aligned} q_1 &= \frac{S_1}{l_1} \\ q_2 &= \frac{S_2}{l_2} \end{aligned} \tag{3}$$

Introducing the force densities as known parameters in the equilibrium equations at point 3, its position can be solved easily as they constitute a linear and determined system of equations:

$$\left.\begin{aligned}(x_1 - x_3)q_1 + (x_2 - x_3)q_2 = 0 \\ (y_1 - y_3)q_1 + (y_2 - y_3)q_2 = P_{y_1}\end{aligned}\right\} @3 \quad (4)$$

Therefore, in order to solve the problem in this manner, it is essential to provide the q values. The power of the method is found in the fact that for different values of q, the equilibrium will be different so that the position of point 3 will be modified.

The approach can be formulated in a more general way: if $\boldsymbol{u}$ is the difference vector of coordinates x for the extreme nodes in each branch and $\boldsymbol{v}$ the same vector for the y coordinate:

$$\begin{aligned}\boldsymbol{u} &= \begin{pmatrix} x_1 - x_3 \\ x_2 - x_3 \end{pmatrix} \\ \boldsymbol{v} &= \begin{pmatrix} y_1 - y_3 \\ y_2 - y_3 \end{pmatrix}\end{aligned} \quad (5)$$

they can be related to the coordinate vectors of the nodes:

$$\begin{aligned}\boldsymbol{X} &= \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ \boldsymbol{Y} &= \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\end{aligned} \quad (6)$$

through the connectivity matrix $\boldsymbol{C}$, according to:

$$\begin{aligned}\boldsymbol{u} &= \boldsymbol{CX} \\ \boldsymbol{v} &= \boldsymbol{CY}\end{aligned} \quad (7)$$

being $\boldsymbol{C}$

$$\boldsymbol{C} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix} \quad (8)$$

The connectivity matrix $\boldsymbol{C}$ can be generally defined for a mesh with n_n nodes and n_b branches as a $n_b \times n_n$ matrix. If the branch j in the mesh connects the nodes $i(j)$ and $k(j)$, with $i < k$, then $\boldsymbol{C}$ remains:

$$\boldsymbol{C} = C(j, r) = \begin{cases} 1 & if\ i(j) = r \\ -1 & if\ k(j) = r \\ 0 & otherwise \end{cases} \quad (8)$$

On the other hand, the parameters defined as the force densities can be positioned in a n_b-dimensional diagonal matrix $\boldsymbol{Q}$ so that the force density q_j of the branch j is placed at the row and column j. Considering Eqs. (5)-(8) and taking into account that the problem can be set up in three dimensions, if $\boldsymbol{P_x}$, $\boldsymbol{P_y}$ and $\boldsymbol{P_z}$ are the vectors that group the nodal forces in x, y and z directions respectively, Eq. (4) can be rewritten as:

$$\begin{aligned} (\boldsymbol{C}^T\boldsymbol{QC})\boldsymbol{X} + \boldsymbol{P_x} = 0 \\ (\boldsymbol{C}^T\boldsymbol{QC})\boldsymbol{Y} + \boldsymbol{P_y} = 0 \\ (\boldsymbol{C}^T\boldsymbol{QC})\boldsymbol{Z} + \boldsymbol{P_z} = 0 \end{aligned} \quad (9)$$

Therefore, the spatial equilibrium configuration of any mesh subjected to a specific set of external loads (vectors $\boldsymbol{P_x}$, $\boldsymbol{P_y}$ and $\boldsymbol{P_z}$), with a given fixed nodes and a determined connectivity matrix $\boldsymbol{C}$, can be established provided the force densities in the branches composing the network, $\boldsymbol{Q}$, just by means of the resolution of the linear system of equations (9).

2.2. Topological Mapping

The Topological Mapping (TM) was proposed by Hernández-Montes et al. [11] as a novel tool to build and provide the connectivity matrix $\boldsymbol{C}$ to be introduced into the FDM to perform the form-finding of tension structures. The main advantage of the TM is that the network of branches and nodes can be initially constructed without any idea of the final configuration of the mesh, in contrast to other methods that carry out the mapping firstly assuming a determined contour of the tensile structure.

The TM generates two types of triangulated meshes: closed and open. In the first case, the network is built by means of successive concentric rings of branches, Fig. 3 (a); for open networks, successive parallel steps of branches are added, Fig. 3 (b). The way of connecting the nodes of two successive set of branches depends on their basic topological relationship, which can be of type A, B or C. Relationship A corresponds to a pattern in which a node at a determined set of branches is linked to three more nodes at the following set, Fig. 3 (c). For relationships B, the nodes in a set of branches are connected to two nodes in the next set of branches Fig. 3 (d). Finally, in C relationships each node at a given set is linked alternatively to one or three nodes in the following set Fig. 3 (e).

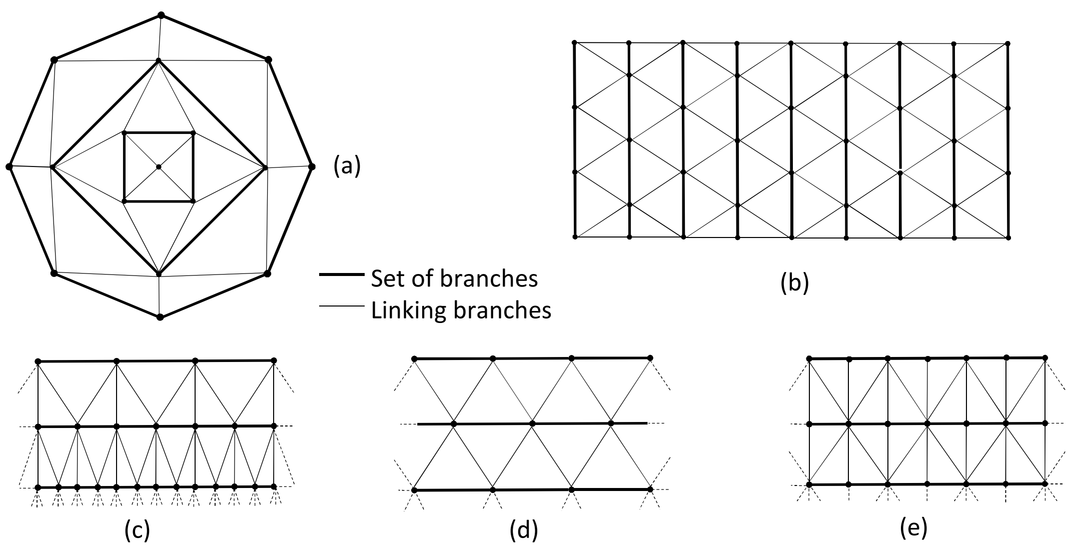


Figure 3. *Types of networks built by the TM: (a) open mesh; (b) open mesh; (c) relationship type A; (d) relationship type B; (e) relationship type C*

Thus, once the type of mesh to be built is chosen (open or closed), the needed inputs to the TM to generate a mesh are:

i. Number of nodes in the first step or ring
ii. Number of total steps or rings
iii. Topological relation between successive steps (can be changed from a ring to another)

Once the mesh is built by the TM, Fig. 4 (a), it is driven from topology to geometry by the FDM, Figs. 4 (b) and (c). Prior to this, it is essential to assign the correspondence between the fixed nodes and the nodes in the last set of branches and this is done distributing those nodes proportionally to the real distance among the fixed points. The resulting tensile structure shown in Fig 4 remarks the ability of the conjunction TM-FDM to perform the form-finding of a tensile structure whose plan view corresponds to a complex polygon, i.e., a polygon which intersects itself. In what follows, the present work only deals with closed networks of type B.

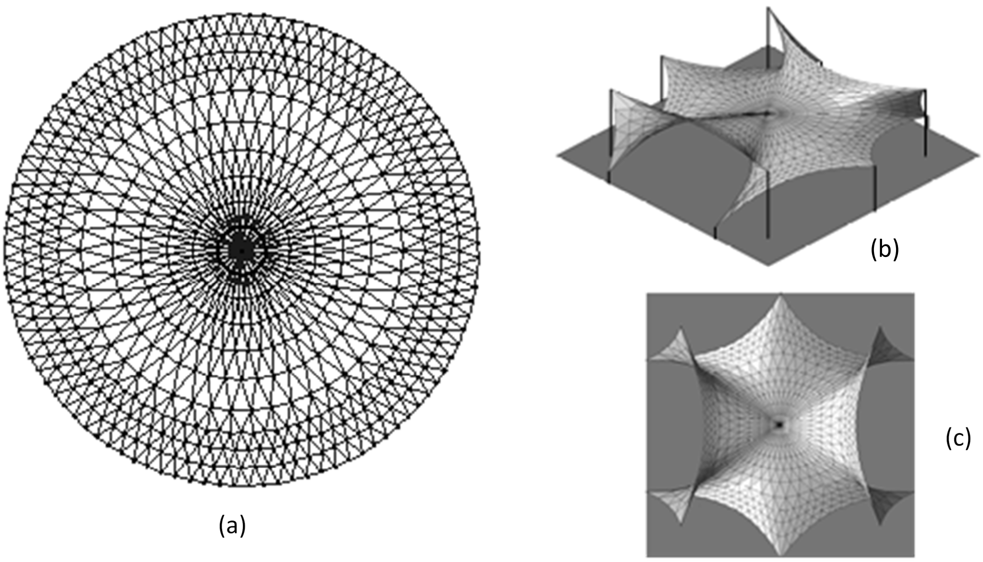


Figure 4. *Results from the FDM-TM: (a) mesh from the TM; (b) general view of the resulting tensile structure; (c) plan view*

3. TM-FDM IN THE DESIGN OF COMPRESSION STRUCTURES

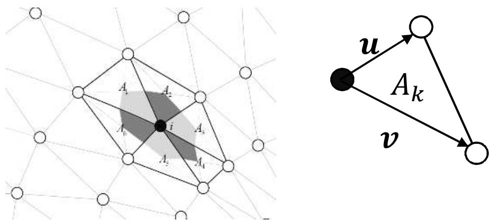
The applicability of the TM together with FDM in the determination of the equilibrium shape of tension structures has been above explained. This approach can be modified in order to perform the design of compression structures based on pseudo hanging models. If the mesh's self-weight is considered as it were built in a specific structural material, for instance concrete, the equilibrium configuration of the mesh as a tension structure can be determined and, afterwards, turned upside down to get a compression only structure.

The consideration of the mesh's self-weight introduces nonlinearities to the original set of equations of the FMD, Eqs. (9), concretely the equilibrium equations in the direction in which gravity acts, z direction, need to be modified to introduce the mesh's weight as a set of vertical nodal loads, $\mathbf{P}_{\mathbf{z,s-w}}$.

Therefore, the new equilibrium equation is nonlinear since the area of the mesh will depend on the position of the nodes in the space:

$$\begin{aligned} &(\boldsymbol{C}^T\boldsymbol{QC})\boldsymbol{X} + \boldsymbol{P_x} = 0 \\ &(\boldsymbol{C}^T\boldsymbol{QC})\boldsymbol{Y} + \boldsymbol{P_y} = 0 \\ &(\mathbf{C}^T\mathbf{QC})\mathbf{z} + \mathbf{P_z} + \mathbf{P_{z,s-w}}(\mathbf{x},\mathbf{y},\mathbf{z}) = 0 \end{aligned} \tag{10}$$

The mesh's weight, which corresponds to the summation of each triangle's self-weight, is applied distributedly to the nodes so that each node is loaded with a third of the weight of every triangle that contains the node as a vertex, Fig. 5. To compute the area of every triangle, the norm of the cross product of two of the vectors composing two connected sides of the triangle is performed.



Node *i*: $P_{z,s-w_i} = \sum_{k=1}^{6} \frac{\gamma}{3} A_k$

where A_k is the area of triangle *k* and γ the density of the shell's material

$$A_k = \frac{1}{2}\|\boldsymbol{u} \times \boldsymbol{v}\|$$

Figure 5. *Weight of triangles affecting a generic node* i

Let us consider the adjacency matrix $\mathbf{A}$ so that the element in that matrix equals to 1 if the node i is linked to the node j and 0 otherwise, Eq. (11). Employing this adjacency matrix $\boldsymbol{A}$, the component i of the vector $\mathbf{P_{z,s-w}}$, Fig. 5 can be computed according to Eq. (12).

$$\boldsymbol{A} = A(i,j) = a_{ij} = \begin{cases} 1 & if\ i\ is\ linked\ to\ j \\ 0 & otherwise, including\ i = j \end{cases} \tag{11}$$

$$\mathrm{P}_{\mathrm{z,s-w}_i} = \sum_{j=1}^{n_n-1} \sum_{k>j}^{n_n} \left[\left(a_{ij} \cdot a_{ik} \cdot a_{jk} \right) \frac{\gamma}{3} \left(\frac{1}{2} \|\boldsymbol{ij} \times \boldsymbol{ik}\| \right) \right] \tag{12}$$

where γ is the specific gravity of the material in which the mesh is built and $\boldsymbol{ij}$ is the vector that starts at node i and ends at j. The cross product in Eq. (12) is the reason why new nonlinearities are introduced in the FDM equations.

Despite the introduced nonlinearities, Eqs. (12) are solved linearly by iterations. Thereby, once the equilibrium configuration of the mesh is found without considering its self-weight, using the size of the triangles in this first iteration, the self-weight is calculated and applied to the nodes, computing again the equilibrium configuration, and so on. The vector $\boldsymbol{z}^{nit}$ contains the z coordinate of the nodes in the mesh for the iteration nit. The process is considered to be finished when

$$\Delta z^{nit} = \left\| \boldsymbol{z}^{nit} - \boldsymbol{z}^{nit-1} \right\| < \Delta \mathrm{z}_{max} \tag{12}$$

where Δz_{max} is the convergence criterion stablished by the designer.

However, the process does not offer convergence in every case but it may not find a possible solution with the given set of the force densities in the branches of the mesh. The reason for the no convergence of the calculations can be found in a very simple example. Let us imagine a spring with a coefficient k determining the relationship between force F and the elongation Δl in the spring, $F = k\Delta l$, and a body whose weight, p, depends linearly on the elongation undergone by the spring according to a specific parameter γ, $p = p_0 + \gamma\Delta l$, Fig. 6. If equilibrium is to be found, $k > \gamma$.

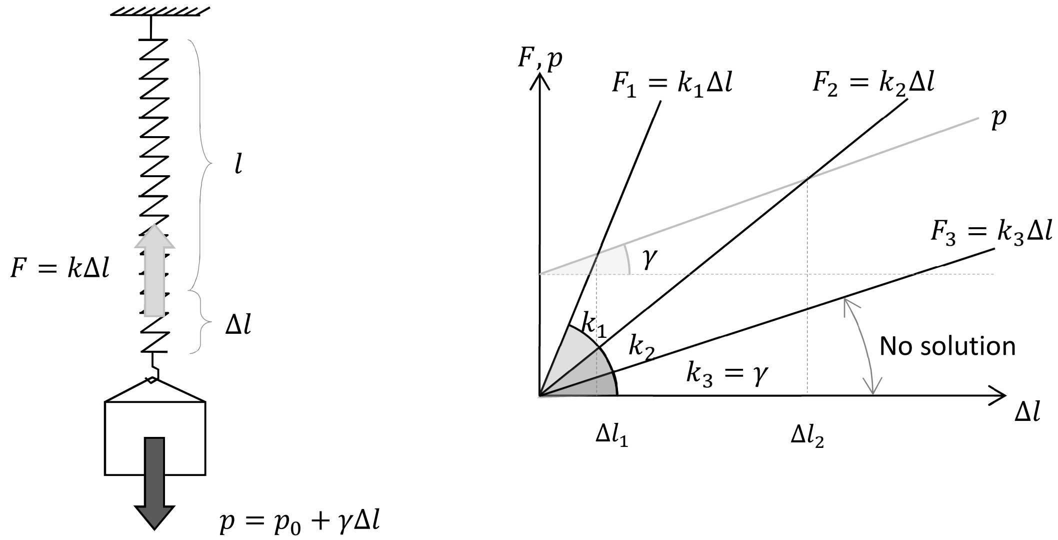


Figure 6. *Analogy of the body hung of the spring to explain the possibility of no convergence in the TM-FDM if an appropriate set of force densities is not assigned to the branches in the mesh*

Similarly to this problem, the set of force densities assigned to each branch in the mesh needs to be such that the resulting forces in the branches increase faster than the weight of the mesh does. If this condition is not met, equilibrium is not ever reached.

4. EXAMPLES PRESENTING THE VERSATILITY OF THE APPROACH

As stated as one of its inventors, the linearization done by the FDM allows making changes back and forth to the design and investigate many different feasible forms with minimal effort and in a playful spirit [8]. Furthermore, with the introduction of the TM as provider of the mesh to be feed to the FDM, the range of possible designs gets much wider. Fig. 7 presents a compression only structure that curls around some of its edges, similarly as done with the tensile structure presented in Fig. 4. This one is a clear example of the capabilities that the conjunction TM-FDM provides.

Another singularity of the method at the authors' knowledge is the capability of designing compression structures with ribs and buttresses. Since the force densities in the branches can be introduced individually for each one, the designer can assign a higher value of force density to a particular set of branches in order to make them to withstand more loads. The following example illustrates the commented capability. Let us consider a pentagonal vault built in concrete – $f_{ck} = 40$ MPa, self-weight $\gamma = 22$kN/m^3 – with a thickness of 0.3 m and with with an irregular basis whose five sides are 11, 10, 12, 12, and 8 m long. The network generated by the TM consists of 10 rings with 43 nodes each. Table 1 shows the coordinates of the fixed nodes.

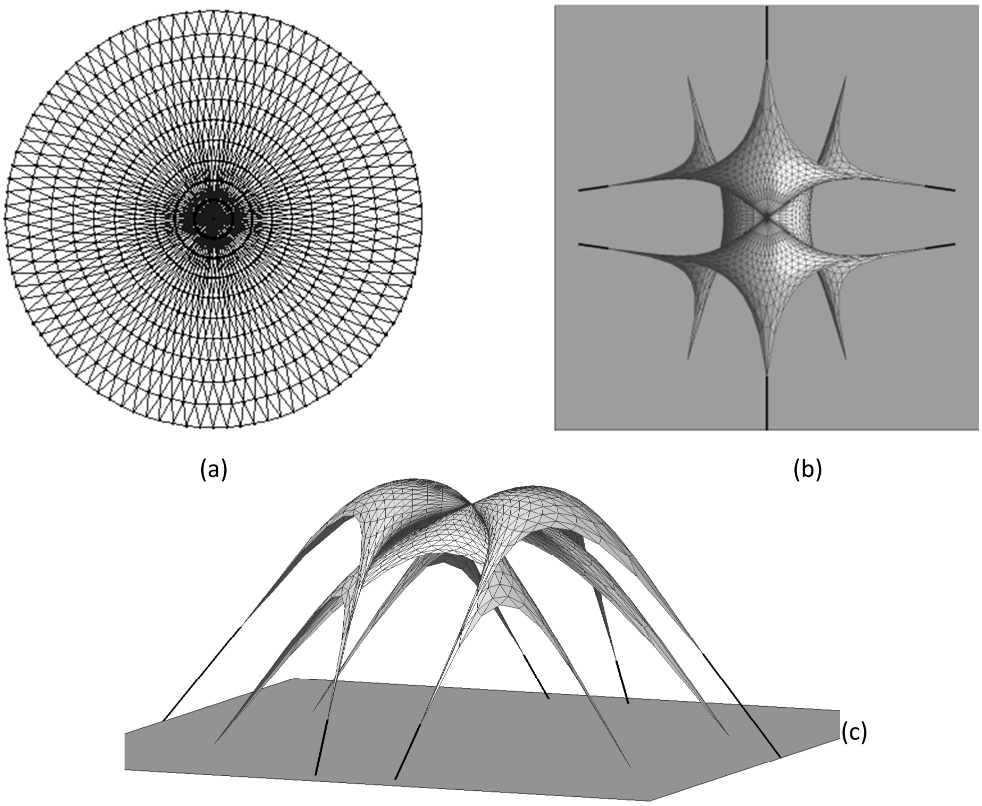


Figure 7. *Curling compression structure: (a) initial mesh from the TM; (b) plan view of the resulting compression structure; (c) perspective*

Table 1. Fixed nodes of the irregular pentagon

Node	X_f (m)	Y_f (m)	Z_f (m)
1	3.4911	0.0000	0.0000
2	14.4911	0.0000	0.0000
3	15.9544	9.8923	0.0000
4	6.5014	17.2843	0.0000
5	0.0000	7.1981	0.0000

To perform an initial analysis of the structure's free form, the interior branches are assigned a force density of 10 kN/m and the branches in the external ring 20 kN/m. Establishing a convergence criterion of $\Delta z_{max} = 10^{-5}$ m, the resulting structure is shown in Fig. 8 (a).

If some ribs are to be created, the force density in a series of branches that go from the fixed points to the central node is increased to 50 kN/m. Since the existence of those ribs provides the structure with a stiffer sustaining system, the structure becomes shorter in height, Fig. 8 (b). In order to avoid that, to get a more voluminous structure, the force density in the interior branches is decreased to 4 kN/m, the force density in three of the ribs is set as 60 kN/m and in the rest of the ribs 40 kN/m. Furthermore, all the branches in the external ring are assigned a force density of 15 kN/m but the branches in the shorter edge that are assigned a q vale of 10 kN/m. With all these modifications from the original attempt, the structure gets a maximum height of 9.64 m, Fig. 8 (c). This new structure is divided in five domes each of the one rests in its corresponding edge and rib arches. The way in which the new structure behaves is completely different from the way in which the original one does, Fig. 8 (a), where the load is driven to the edge arches and from them to the foundation.

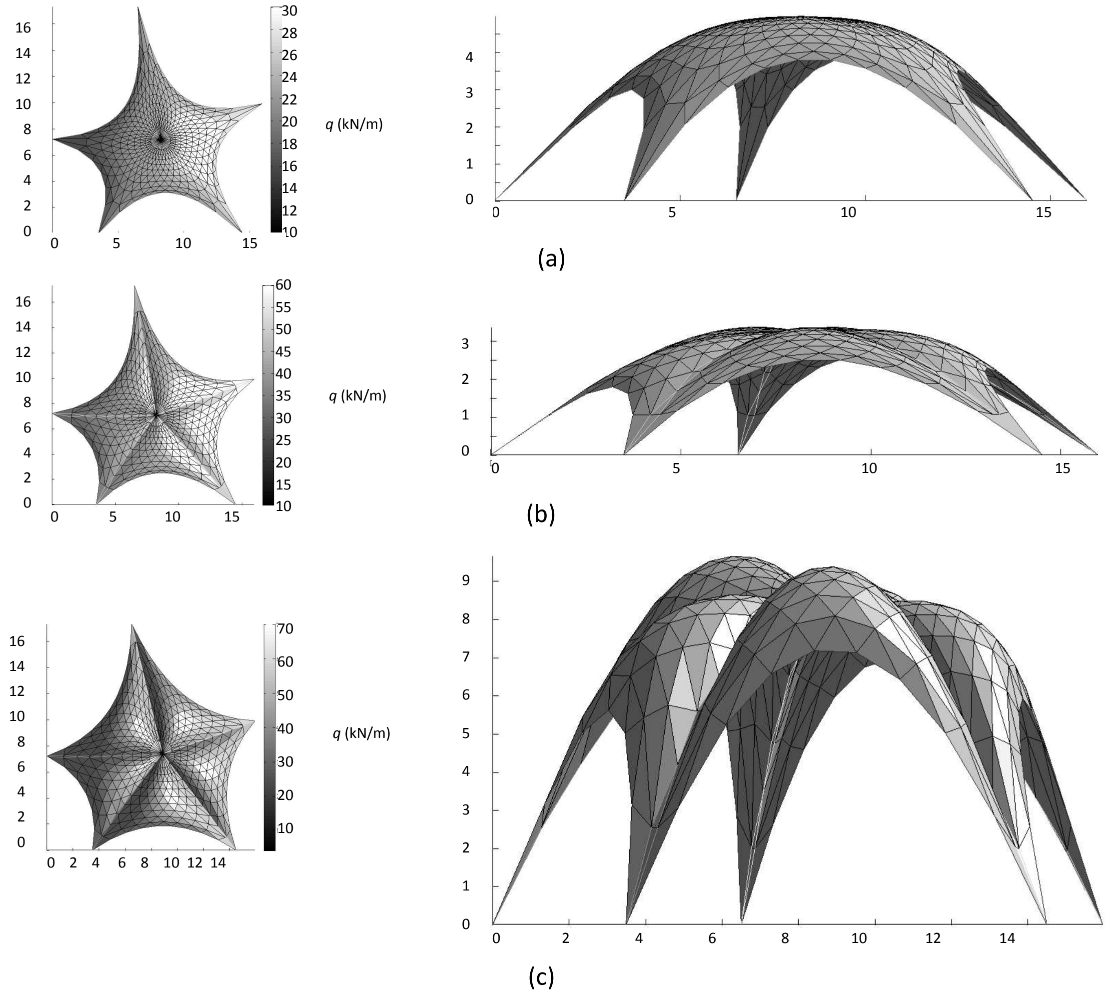


Figure 8. *Developed example: (a) compression structure with the initial input; (b) compression structure obtained increasing the force densities in the branches that compose the diagonal buttresses; (c) obtained structure with the final force density assignment*

5. CONCLUSIONS

The conjunction TM-FDM has been reviewed in the form finding of tension structures. This original method can be modified introducing the mesh's self-weight and computing the equilibrium position. Due to this self-weight consideration, nonlinearities are introduced. However, tackling the problem in an iterative way, it can be again linearly solve as in the original FDM. When the shape of equilibrium is found, the model can be turned upside down just as Gaudì used to do with his hanging chain models.

Thanks to the employment of the TM, it has been shown that a wider range of solutions are provided to the designer. The presented approach can be a very useful and attractive tool for architectural desing: complex vaults, domes and other compression-only structures can be modelled almost effortlessly. After analyzing the obtained structure, its equilibrium configuration can be changed by modifying the force densities in the branches or the mesh's topology. Some examples showing the capabilities of the modified TM-FDM have been presented proving how the structure changes his behavior as the branches' force density assignment is modified.

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