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Connecting steady states of a discrete diffusive energy balance climate model

J.I.Díaz[1], V.García[2]

[1] *Dpto. Matemática Aplicada, Universidad Complutense de Madrid, C.P. 28040, Madrid. E-mail:* `diaz.racefyn@insde.es`.
[2] *Dpto. Matemática Aplicada, Universidad Complutense de Madrid, C.P. 28040, Madrid. E-mail:* `victor.garcia.garrido@estumail.ucm.es`.

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Resumen

In this communication we consider a discrete version of some simple Budyko-Sellers climate model. Our main goal is to consider the problem of transferring the system (through some sufficiently large time T) from a stationary state to another one. Our study is divided into two parts: firstly we obtain a result on a connected branch of stationary solutions (for instance, as a function of the parameter Q and in the absence of any control; secondly we will use some controllability techniques of nonlinear systems of ODEs to analyze the transferring question by means of suitable controls.

1. Introduction

In this communication we consider some simple Budyko-Sellers climate model of the type

$$(P)\begin{cases} y_t-(k(1-x^2)y_x)_x=R_a(x,y,v)-R_e(y,x,u) & x\in(-1,1),\quad t>0 \\ y(x,0)=y_0(x) & x\in(-1,1)\end{cases} \tag{1}$$

where $k>0$, $R_a(x,y,v)$ is a bounded increasing function on y (the absorbed energy due to the co-albedo) and $R_e(y,x,u)$ is a strictly increasing function on y of the type $R_e(y,x,u)=u\,|y|^3\,y+f(x)$ related to the Stefan-Boltzman radiation law with an emissivity u. Here u and v are taken as control variables that take into account the effects of the anthropogenerated actions on the rate of emissions of the greenhouse gases. This kind of methods were introduced, independently, in 1969 by M.I. Budyko and W.D. Sellers.

These models have a diagnostic character and intended to understand the evolution of global climate on long time scales. Their main characteristic is the high sensitivity to the variation of solar and terrestrial parameters. This kind of models have been used in the study of the Milankovitch theory of the ice-ages.

For some purposes it is useful to assume the presence of possible localized controls of the form $u(t)\chi_{(l_1,l_2)}$ and $v(t)\chi_{(l_1,l_2)}$ for some given latitude control interval $(l_1,l_2) \subset (-1,1)$. We shall assume here that $R_a(x,y,v)$ is closer to the model proposed by Sellers and so $R_a = v(t)\chi_{(l_1,l_2)}QS(x)\beta(y)$ with β a Lipschitz continuous, as for instance,

$$\beta(y) = \begin{cases} m & y < y_i, \\ m + (\frac{y-y_i}{y_w-y_i})(M-m) & y_i \leq y \leq y_w, \\ M & y > y_w, \end{cases} \tag{2}$$

where y_i and y_w are fixed temperatures closed to -10^0C and $m = \beta_i$ and $M = \beta_w$ represent the coalbedo in the ice-covered zone and the free-ice zone respectively and $0 < \beta_i < \beta_w < 1$. Moreover, $S(x)$ is the *insolation function* and Q is the so-called *solar constant*. We assume $S : [-1,1] \to \mathbb{R}$, $S \in C^0([-1,1])$, $S_1 \geq S(x) \geq S_0 > 0$ for any $x \in [-1,1]$ and that $R_e = u(t)\chi_{(l_1,l_2)}\mathcal{G}(y) - f(x)$ with $\mathcal{G} : \mathbb{R} \to \mathbb{R}$ a continuous strictly increasing function such that $\mathcal{G}(0) = 0$, $\text{lím}_{|s|\to\infty} |\mathcal{G}(s)| = +\infty$ and $f \in C^0([-1,1])$.

Our main goal is to consider the problem of transferring the system (through some sufficiently large time T) from a stationary state to another one. This type of problem was raised by J. von Neumann in another general context ([14]: see also [13] and [10]). Our study is divided into two parts: firstly we obtain a result on a connected branch of stationary solutions (for instance, as a function of the parameter Q and in the absence of any control $((l_1,l_2) = (-1,1)$ and $u(t) = v(t) \equiv 1)$; secondly we will use some controllability techniques of nonlinear systems of ODEs to analyze thetransferring question by means of suitable controls.

As a matter of fact, we shall consider here only a simplified version of problem (P). We will concentrate our attention in the discrete version of (P) arising by a spatial difference scheme discretization (for a discretization by finite elements see [3]). There are several possible discrete simplified problems. For instance, to avoid technicalities concerning the degenerate diffusion, as in other precedent papers ([6]), we can replace the degenerate linear diffusion operator by the usual uniform diffusion expression and then add Neumann boundary conditions

$$(P_L) \begin{cases} y_t - ky_{xx} = R_a(x,y,v) - R_e(y,x,u) & x \in (-1,1), t > 0 \\ y_x(1,t) = y_x(-1,t) = 0 & t > 0 \\ y(x,0) = y_0(x) & x \in (-1,1) \end{cases} \tag{3}$$

Then, a spatial difference scheme discretization of problem (P_L) can be generated in the usual way: given $N \in \mathbb{N}$, $N > 1$, we define $h = 2/(N-1)$ and we denote by $y_i(t)$ the approximation of $y(-1+ih,t)$. Then, we consider the discrete algorithm

$$(\mathbf{P}_h) \begin{cases} \dot{\mathbf{y}}(t) + \mathbf{A}\mathbf{y}(t) + \mathbf{R}_e(\mathbf{y}(t), u(t)) - \mathbf{R}_a(\mathbf{y}(t), v(t)) = \mathbf{0}, \\ \mathbf{y}(0) = \mathbf{y}^0, \end{cases} \tag{4}$$

where $\mathbf{y}(t) := (y_1(t), y_2(t), ..., y_N(t))^T$, $u(t) \in \mathbb{R}$, with $u(t)$ and $v(t)$ appearing only in some coordinates associated to some $m \in \mathbb{N}$, $1 < m \leq N$ (the discretized control interval (l_1, l_2) is here represented by an interval of length $(m-1)h$). In $(\mathbf{P}_h)$, $\mathbf{A}$ is the symmetric positive definite matrix of $\mathbb{R}^{N\times N}$ given by

$$\mathbf{A}_N = \frac{k}{h^2}\begin{pmatrix} 1 & -1 & 0 & ... & 0 \\ -1 & 2 & -1 & 0 & ... \\ 0 & -1 & 2 & -1 & 0 \\ ... & 0 & -1 & 2 & -1 \\ 0 & ... & 0 & -1 & 1 \end{pmatrix}, \tag{5}$$

$\mathbf{R}_a$:$\{-1, -1+h, ..., +1\} \times \mathbb{R}^N \times \mathbb{R}^m \to \mathbb{R}^N$ is given by

$$\mathbf{R}_a(x_1, \dots, x_N, y_1, \dots, y_N, v_1, \dots, v_N) = (R_a(x_1, y_1, v_1(t)), \dots, R_a(x_N, y_N, v_N(t)))^T \tag{6}$$

and $\mathbf{R}_e$:$\{-1, -1+h, ..., +1\} \times \mathbb{R}^N \times \mathbb{R}^m \to \mathbb{R}^N$ is given by

$$\mathbf{R}_e(x_1, \dots, x_N, y_1, \dots, y_N, u_1, \dots, u_N) = (R_e(x_1, y_1, u_1(t)), \dots, R_e(x_N, y_N, u_N(t)))^T \tag{7}$$

where we used the following notation: $u_j(t) \equiv 1$ if j is not one of the m coordinates where the control is located and $u_j(t) \equiv u(t)$ otherwise (and analogously for $v_j(t)$) and $x_i = -1 + (j-1)h$.

A different discrete approximation of problem (P), which maintains the peculiar degeneracy of the diffusion leads also to the formulation $(\mathbf{P}_h)$ but with a different matrix of $\mathbb{R}^{N\times N}$ given by

$$\mathbf{A}_D = \frac{k}{h^2}\begin{pmatrix} 0 & 0 & 0 & ... & 0 \\ -(1-x_2^2) & 2(1-x_2^2) & -(1-x_2^2) & 0 & ... \\ 0 & ... & ... & ... & 0 \\ ... & 0 & -(1-x_{N-1}^2) & 2(1-x_{N-1}^2) & -(1-x_{N-1}^2) \\ 0 & ... & 0 & 0 & 0 \end{pmatrix}, \tag{8}$$

which results from the identity $(k(1-x^2)y_x)_x = k(1-x^2)y_{xx} - 2kxy_x$ when we neglect the transport term $2kxy_x$. Note that in that case the first and the last equations of $(\mathbf{P}_h)$ are uncoupled.

Although our results are true for a general value of $N \in \mathbb{N}$, for the sake of this exposition, we shall consider the following illustrative case: $N = 3$ and $m = 1$. This leads to the vectorial formulation

$$(\mathbf{P}_Q)\begin{cases} \dot{\mathbf{y}}(t) = \mathbf{f}(\mathbf{y}(t), u(t), v(t), Q), \\ \mathbf{y}(0) = \mathbf{y}^0 \end{cases} \tag{9}$$

with $\mathbf{f}$:$\mathbb{R}^3 \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^3$ given by (when $\mathbf{A} = \mathbf{A}_N$)

$$\mathbf{f}(\mathbf{y}, u, v, Q) = \begin{pmatrix} \frac{k}{h^2}(y_2 - y_1) + QS(-1)\beta(y_1) - \mathcal{G}(y_1) + f(-1) \\ \frac{k}{h^2}(y_3 - 2y_2 + y_1) + vQS(0)\beta(y_2) - u\mathcal{G}(y_2) + f(0) \\ \frac{k}{h^2}(-y_3 + y_2) + QS(1)\beta(y_3) - \mathcal{G}(y_3) + f(1) \end{pmatrix} \tag{10}$$

and when $\mathbf{A} = \mathbf{A}_D$

$$\mathbf{f}(\mathbf{y},u,v,Q)=\left(\begin{array}{c} QS(-1)\beta(y_1)-\mathcal{G}(y_1)+f(-1) \\ \frac{k}{h^2}(y_3-2y_2+y_1)+vQS(0)\beta(y_2)-u\mathcal{G}(y_2)+f(0) \\ QS(1)\beta(y_3)-\mathcal{G}(y_3)+f(1) \end{array}\right). \tag{11}$$

2. A connected set of stationary solutions depending on Q

In this Section we will assume the absence of any control: $(l_1,l_2)=(-1,1)$ and $u(t)=v(t)\equiv 1$. Our main goal is to adapt the results of [7] and [2] to show that the set of stationary solutions $(\mathbf{y}^\infty,Q)\in\mathbb{R}^3\times\mathbb{R}$ satisfying

$$(\mathbf{P}_Q^\infty) \qquad \mathbf{f}(\mathbf{y}^\infty,1,1,Q)=\mathbf{0}, \tag{12}$$

is very large (depending on the parameter Q) and, for instance, it leads to a bifurcation diagram with a principal branch which is S-shaped containing at least one turning point to the right and another one to the left.

We make the additional assumptions

- (H_{f_∞}): there exists $C_f>0$ such that $f(x_i)\le -C_f$.
- (H_β) β is Lipschitz increasing function and there exists $0<m<M$ and $\epsilon>0$ such that $\beta(r)=\{m\}$ for any $r\in(-\infty,-10-\epsilon)$ and $\beta(r)=\{M\}$ for any $r\in(-10+\epsilon,+\infty)$.

We note that since the matriz $\mathbf{A}$ is symmetric (and, at least, semdefinite positve) the strict monotonicity and the coercivedness assumed on $\mathcal{G}$ implies the existence of a unique $\mathbf{y}_m$ (respect. $\mathbf{y}_M$) solution of the problem $(\mathbf{P}_Q^\infty)_m$ (respect. $(\mathbf{P}_Q^\infty)_M$) given by $(\mathbf{P}_Q^\infty)$ but replacing $\beta(y_i)$ by m (respect. by M). In the rest of the Section we shall use several comparison arguments on $\mathbb{R}^3$. Here we shall use the following notation

$$\mathbf{y}=\left(\begin{array}{c} y_1\\ y_2\\ y_3\end{array}\right)\le\overline{\mathbf{y}}=\left(\begin{array}{c} \overline{y}_1\\ \overline{y}_2\\ \overline{y}_3\end{array}\right) \text{ if and only if } y_1\le\overline{y}_1, y_2\le\overline{y}_2 \text{ and } y_3\le\overline{y}_3. \tag{13}$$

Analogously, the use of the strict inequality $<$ among vectors means that the strict inequality holds among all the components of the vectors. Finally, if $\alpha\in\mathbb{R}$ the notation $\alpha\le\mathbf{y}$ means that $\alpha\le(\mathbf{y})_i$ for $i=1,2,3$.

We start by proving the existence of at least three solutions for suitable Q.

Theorem 1. *Let $\mathbf{y}_m$ (respect.$\mathbf{y}_M$) be the (unique) solutions of the problem $(\mathbf{P}_Q^\infty)_m$ (respect. $(\mathbf{P}_Q^\infty)_M$). Then:*
i) for any $Q>0$ there is a minimal solution $\underline{\mathbf{y}}$ (resp. a maximal solution $\overline{\mathbf{y}}$) of $(\mathbf{P}_Q^\infty)$. Moreover, any other solution $\mathbf{y}$ must satisfy

$$\mathbf{y}_m\ \le\ \underline{\mathbf{y}}\ \le\ \mathbf{y}\ \le\ \overline{\mathbf{y}}\ \le\ \mathbf{y}_M \tag{14}$$

$$\mathcal{G}^{-1}(QS_0m+\min f)\le(\mathbf{y}_m)_i\le\mathcal{G}^{-1}(QS_1m-C_f), \tag{15}$$

$$\mathcal{G}^{-1}(QS_0M + \text{mín} f) \ \leq (\mathbf{y}_M)_i \leq \mathcal{G}^{-1}(QS_1M - C_f) \text{ for } i = 1, 2, 3. \tag{16}$$

If we assume, in addition,

$$(H_{C_f}) \ \ \mathcal{G}(-10-\epsilon) + C_f > 0 \ \textit{and} \ \frac{\mathcal{G}(-10+\epsilon) - \text{mín} f}{\mathcal{G}(-10-\epsilon) + C_f} \leq \frac{S_0M}{S_1m} \tag{17}$$

and define

$$Q_1 = \frac{\mathcal{G}(-10-\epsilon) + C_f}{S_1M} Q_2 = \frac{\mathcal{G}(-10+\epsilon) - \text{mín} f}{S_0M} \tag{18}$$

$$Q_3 = \frac{\mathcal{G}(-10-\epsilon) + C_f}{S_1m} Q_4 = \frac{\mathcal{G}(-10+\epsilon) - \text{mín} f}{S_0m}. \tag{19}$$

then:

ii) if $0 < Q < Q_1($ *repect.* $Q > Q_4)$ *then* $(\mathbf{P}_Q^\infty)$ *has a unique solution* $\mathbf{y} = \mathbf{y}_m$*,* $(\mathbf{y}_m)_i < -10($*respect.* $\mathbf{y} = \mathbf{y}_M$*,* $(\mathbf{y}_M)_i > -10)$ *and*

$$\mathcal{G}^{-1}(\text{mín} f) \leq \lim_{Q\searrow 0} \inf \|y\|_\infty \leq \lim_{Q\searrow 0} \sup \|y\|_\infty \leq \mathcal{G}^{-1}(-C_f), \tag{20}$$

iii) if $Q_2 < Q < Q_3$*, then* $(\mathbf{P}_Q^\infty)$ *has at least three solutions,* $\mathbf{y}_i$*,* $i = 1, 2, 3$ *with* $\mathbf{y}_1 = \mathbf{y}_M$*,* $\mathbf{y}_2 = \mathbf{y}_m$*, and* $\mathbf{y}_1 \geq \mathbf{y}_3 \geq \mathbf{y}_2$*.*

Idea of the Proof. i) and ii) are consequence of the fact that the comparison principle holds for problems $(\mathbf{P}_Q^\infty)_m$, $(\mathbf{P}_Q^\infty)_M$ (since the systems are of cooperative type) and then the method of sub and supersolutions can be applied (see e.g. Pao [15]). The proof of iii) is divided into several steps. Firstly, we construct two constant subsolutions $\mathbf{V}_i$ and two constant supersolutions $\mathbf{U}_i$ for $i = 1, 2$ such that $\mathbf{V}_2 < \mathbf{U}_2 < -10-\epsilon < -10+\epsilon < \mathbf{V}_1 < \mathbf{U}_1$, proving the existence of, at least, two solutions of $(\mathbf{P}_Q^\infty)$. The existence of a third solution of $(\mathbf{P}_Q^\infty)$ is obtained by a topological fixed point argument. Let us show the convergence of the mentioned solution of $(\mathbf{P}_Q^\infty)$ to a third solution of $(P_{Q,f})$. For $\lambda < \lambda_0$ (a certain positive parameter) $\mathbf{U}_1$, $\mathbf{U}_2$ are supersolutions of $(\mathbf{P}_Q^\infty)$ and $\mathbf{V}_1$, $\mathbf{V}_2$ are subsolutions of $(\mathbf{P}_Q^\infty)$. So, arguing as in i) we obtain two solutions $\mathbf{y}_1$ and $\mathbf{y}_2$ of $(\mathbf{P}_Q^\infty)$ such that $-10+\epsilon+\lambda_0 M < \mathbf{V}_1 \leq \mathbf{y}_1 \leq \mathbf{U}_1$, $V_2 \leq \mathbf{y}_2 \leq \mathbf{U}_2 < -10-\epsilon$.
In order to prove that $(\mathbf{P}_Q^\infty)$ has a third solution u_3 different to u_1^λ and u_2^λ we apply a result due to Amann [1] which is justified since the operator $\mathbf{F}(\mathbf{z}) := (\mathbf{A}+\mathbf{v}\mathcal{G})^{-1}(\mathbf{u}Q\mathbf{S}(\cdot)\beta(z)+\mathbf{f})$ is compact on the space $E = \mathbb{R}^3$.

Now we can show that it is possible to associate a bifurcation diagram for the special case

$$f(x_i) = -C_f \ \ , \ \ \mathcal{G}(-10-\epsilon) + C > 0 \ \ \textit{and} \ \ \frac{\mathcal{G}(-10+\epsilon) + C}{\mathcal{G}(-10-\epsilon) + C} \leq \frac{S_2M}{S_1m}. \tag{21}$$

Theorem 2 *If we denote by* Σ *the set of pairs* $(Q, \mathbf{y}) \in \mathbb{R}^+ \times \mathbb{R}^3$*, where* $\mathbf{y}$ *verifies* $(\mathbf{P}_Q^\infty)$ *then* Σ *contains an unbounded connected component containing the point* $(0, \mathcal{G}^{-1}(-\mathbf{C}))$*.*

Proof. *Σ has an unbounded component*: We claim that the following result, due to Rabinowitz [16], can be applied to our case:

"Let E a Banach space. If $F : I\!R \times E \to E$ is compact and $F(0,u) \equiv 0$, then Σ contains a pair of unbounded components C^+ and C^- in $I\!R^+ \times E$, $I\!R^- \times E$ respectively and $C^+ \cap C^- = \{(0,0)\}$".

In order to do that, we consider the translation of $\mathbf{y}$ given by $\mathbf{z} := \mathbf{y} - \mathcal{G}^{-1}(-\mathbf{C})$. Obviously, v is a solution of $(\mathbf{P}_Q^\infty)$ with $\hat{\mathcal{G}}(\sigma) = \mathcal{G}(\sigma + \mathcal{G}^{-1}(-C)) + C$ and $\hat{\beta}(\sigma) = \beta(\sigma + \mathcal{G}^{-1}(-C))$. We define $\hat{\Sigma}$ in an analogous way to Σ. Let $E = \mathbb{R}^3$ and define $\mathbf{F}(\mathbf{z}) := (\mathbf{A} + \mathbf{v}\mathcal{G})^{-1}(\mathbf{u}Q\mathbf{S}(\cdot)\beta(z) + \mathbf{f})$ which is compact on the space $E = \mathbb{R}^3$. On the other hand, if $Q = 0$, problem $(\mathbf{P}_Q^\infty)$ has a unique solution $v = 0$, so $F(0,0) = 0$. In conclusion, $\hat{\Sigma}$ contains two unbounded components $\hat{C}^+$ and $\hat{C}^-$ on $I\!R^+ \times \mathbb{R}^3$ and $I\!R^- \times \mathbb{R}^3$ respectively and $\hat{C}^+ \cap \hat{C}^- = \{(0,0)\}$. Since Σ is a translation of $\hat{\Sigma}$ then Σ contains two unbounded components C^+ and C^- on $I\!R^+ \times \mathbb{R}^3$ and $I\!R^- \times \mathbb{R}^3$ respectively and $C^+ \cap C^- = \{(0, \mathcal{G}^{-1}(-C))\}$. Since $Q \geq 0$ in the studied model, we are interested in C^+. In order to establish the behaviour of C^+, we also recall that for every $q > 0$ there exists a constant $L = L(q)$ such that if $0 \leq Q \leq q$ then every solution $\mathbf{y}_Q$ of $(\mathbf{P}_Q^\infty)$ verifies $\|\mathbf{y}_Q\|_\infty \leq L(q)$. Since the principal component is unbounded its projection over the Q-axis is $[0,\infty)$. On the other hand, if Q is large enough $(\mathbf{P}_Q^\infty)$ has a unique solution $\mathbf{y}_Q$ and this solution is greater than $\mathcal{G}^{-1}(QS_0M - C)$. Since $\text{lím}_{|s|\to\infty} |\mathcal{G}(s)| = +\infty$, then the unbounded branch C^+ containing $(0, \mathcal{G}^{-1}(-\mathbf{C}))$ should go to (∞, ∞).

Remark 1. In the continuous problem it is well known that there are many other solutions that do not belong to the branch C^+ (see [8]). In some special cases (for instance, the zero-dimensional model: $k = 0$ and constant coefficients) it is possible to characterize the different parts of the brach correspoponding to stable (and unstable) solutions.

Remark 2. It is easy to show that under symmetry conditions on $S(x)$ and $f(x)$ the branch C^+ is formed by symmetry stationary solutions $(\mathbf{y})_1 = (\mathbf{y})_3$.

Remark 3. It is not difficult to make a similar study about a branch of solutions when Q is fixed and the emmisivity u is taken as a variable parameter.

3. Connecting stationary solutions by means of controls

We consider the problem of transferring the system from a stationary state to another one (when $Q = Q_0$ is fixed) but now by means of suitable choices of the controls $u(t)$ and $v(t)$. In fact, we shall consider here only the case of a single control $v(t)$ and when both solutions are in the same connected component (the branch C^+). For the sake of simplicity, we shall consider the connection between an arbitrary (possibly unstable) symmetric state $(\mathbf{y}^0, v^0 Q_0)$ to a final stable symmetric one $(\mathbf{y}^f, v^f Q_0)$, both in the principal branch C^+. The case when $v(t)$ is fixed and the only control is $u(t)$ follows the same arguments. Finally, the case of two controls $u(t)$ and $v(t)$ is even easier. We start with the uniform diffusion case $\mathbf{A} = \mathbf{A}_N$ with Neumann boundary conditions

Theorem 3. *i) Assume $\mathbf{A} = \mathbf{A}_N$, $u(t) \equiv 1$ and that the controls $u(t), v(t)$ act globally in space $((l_1, l_2) = (-1, 1))$. Let $(\mathbf{y}^f, Q_0 v^f)$ be a stable symmetric stationary solution in the branch C^+. Then, for any other symmetric state $(\mathbf{y}^0, v^0 Q_0)$ in C^+ there exists a time*

$T > 0$ and a piece-wise continuous control $v \in L^\infty(0,T)$ with $v(0) = v^0$ and $v(T) = v^f$ such that the solution $\mathbf{y}(t)$ of the problem $(\mathbf{P}_{Q_0})$ with initial datum $\mathbf{y}^0$ verifies that $\mathbf{y}(T) = \mathbf{y}^f$.
ii) In the case of a localized control $v(t)$ in $((l_1, l_2) \subsetneq (-1,1))$ the same conclusion holds when, in addition, $(\mathbf{y}^0, v^0 Q_0)$ and $(\mathbf{y}^f, v^f Q_0)$ are closed enough.

Proof. We divide the proof of i) into two different steps. In the first step, given an small $\epsilon > 0$ we connect $(\mathbf{y}^0, v^0 Q_0)$ with a point $(\mathbf{y}^f, Q_0 v^f)$ by means of the branch of stationary solutions C^+ and so, by means of a parametrization $(\mathbf{y}^*(\tau), Q(\tau))$ with $Q(\tau) = (1-\tau)v^0 Q_0 + \tau v^f Q_0$ for $\tau \in [0,1]$. Obviously, this orbit does not need to be a solution of $(\mathbf{P}_{Q_0})$ but, given $\varepsilon > 0$, we can construct the function $[0, 1/\varepsilon] \to \mathbb{R}^3 \times \mathbb{R}$ given by $(\mathbf{y}^\varepsilon(t), v^\varepsilon(t)) = ((\mathbf{y}^*(\varepsilon t), Q(\varepsilon t))$ which is almost a solution since

$$\left\| \dot{\mathbf{y}}(t) = \mathbf{f}(\mathbf{y}(t), 1, v(t), Q) \right\| = O(\varepsilon). \tag{22}$$

Then, since $(\mathbf{y}^f, v^f Q_0)$ is stable we can assume that $\mathbf{y}^\varepsilon(T_\varepsilon)$ (with $T_\varepsilon = 1/\varepsilon$) is near $\mathbf{y}^f$. The second step consists in connecting $\mathbf{y}^\varepsilon(T_\varepsilon)$ with $\mathbf{y}^f$ by means of a control $\widehat{v}(t)$ for $t \in [T_\varepsilon, T]$, for some $T > T_\varepsilon$. This can be done thanks to well-known results (see, e.g. [12], [17]) since the Kalman's condition for the linearized equation near $(\mathbf{y}^f, 1, v^f Q_0)$ holds. Note that due to the symmetry assumption we can reduce the system $(\mathbf{P}_{Q_0})$ to a system of only two equations leading to a linearization

$$\dot{\mathbf{y}}(t) = \mathbf{C}\mathbf{y}(t) + \mathbf{B}u(t) \tag{23}$$

where $\mathbf{C}= \nabla_{\mathbf{y}} \mathbf{f}(\mathbf{y}^f, 1, v^f Q_0)$ and $\mathbf{B} = \nabla_v \mathbf{f}(\mathbf{y}^f, 1, v^f Q_0)$, and the Kalman's condition holds

$$Range(\mathbf{B}, \mathbf{CB}) = 2 \tag{24}$$

ii) For a localized control $v(t)$ appearing only in the second equation of $(\mathbf{P}_{Q_0})$ the argument of connecting branch of stationary solutions C^+ may fail but at least we can apply the local controllability results for nonliear equations since the Kalman's condition holds.

Remark 4. It is a courious fact that, in the case of the original 3-system $(\mathbf{P}_{Q_0})$, the necessary and sufficient condition in order to have the Kalman's condition for the linearized equation allows to see that there are other solutions (not necessarely symmetric) which does not satisfy it.

We end this section with the consideration of the degenerate case $\mathbf{A} = \mathbf{A}_D$. As indicated before, now the first and third equations of $(\mathbf{P}_{Q_0})$ are uncoupled and so the problem (neither its linearizations) can be locally controllable. Nevertheless, we can state some result on a relaxed notion of controllability given in terms of the reachability set:

Theorem 4. *i) Assume $\mathbf{A} = \mathbf{A}_D$, $u(t) \equiv 1$ and that the controls $u(t), v(t)$ act globally in space $((l_1, l_2) = (-1,1))$. Let $(\mathbf{y}^f, Q_0 v^f)$ be a stable symmetric stationary solution in the branch C^+. Then, for any other symmetric state $(\mathbf{y}^0, v^0 Q_0)$ in C^+ and for any $\varepsilon > 0$ there exists a time $T^\epsilon > 0$ and a piece-wise continuous control $v \in L^\infty(0, T^\epsilon)$ with $v(0) = v^0$ and $v(T^\epsilon) = v^f$ such that the solution $\mathbf{y}(t)$ of the problem $(\mathbf{P}_{Q_0})$ with initial datum $\mathbf{y}^0$ verifies that $\left\|\mathbf{y}(T^\epsilon) - \mathbf{y}^f\right\| \leq \varepsilon$.*
ii) In the case of a localized control $v(t)$ in $((l_1, l_2) \subsetneq (-1,1))$ the same conclusion holds when, in addition, $(\mathbf{y}^0, v^0 Q_0)$ and $(\mathbf{y}^f, v^f Q_0)$ are closed enough.

Proof. It is enough to apply the arguments of the proof of Theorem 3 replacing the local controllability condition for $(\mathbf{P}_{Q_0})$ by the fact that the reachability set is open since the Lie bracket condition is satisfied (see [17]).

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