The non–degenerate center problem in certain families of planar differential systems.

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Key words: center problem, commutator, isochronous center, time–reversible.

Abstract

This work concerns the non–degenerated center problem in certain families of differential systems in $\mathbb{R}^2$. We study the existence of uniformly isochronous centers and the form of their commutators. We also classify all centers of the family of the BiLiénard systems of degree five.

1   Introduction

We consider analytic systems of differential equations in the real plane of the form:

$$\dot{x} = -y + P(x,y), \quad \dot{y} = x + Q(x,y), \tag{1}$$

where $P(x,y)$ and $Q(x,y)$ are real analytic functions in a neighborhood of the origin without constant nor linear terms.

2   Uniformly Isochronous Centers

In this section, we consider the family of polynomial differential systems of the form:

$$\dot{x} = -y + xR(x,y), \quad \dot{y} = x + yR(x,y), \tag{2}$$

with $R(x,y) = \sum_{i=1}^{n} R_i(x,y)$ where R_i is a homogeneous polynomial of degree i. The centers of these systems are called *uniformly isochronous centers*. The origin, when it is a center, is called uniformly isochronous center of system (2) because in polar coordinates (2) takes the form: $\dot{r} = F(r,\theta)$, $\dot{\theta} = 1$, see [7, 8].

The center problem for this type of systems has been studied by several authors. The case in which $R(x,y)$ is a homogeneous polynomial of degree i has been studied in [7] where the following result is given:

Theorem 1. *Let $R(x,y)$ be a homogeneous polynomial. Then the origin is an isochronous center of system* (2) *if, and only if, one of the following conditions holds:*

  (i) *system* (2) *has even degree;*

  (ii) *system* (2) *has odd degree $n = 2m + 1$, and*

$$\sum_{\ell=0}^{2m} r_\ell \int_0^{2\pi} (\cos\varphi)^{2m-\ell} (\sin\varphi)^\ell d\varphi = 0,$$

where $R(x,y) = \sum_{\ell=0}^{n-1} r_\ell\, x^{n-1-\ell} y^\ell$.

In the nonhomogeneous class, the first case that has been studied corresponds to the systems with $R(x,y) = R_1 + R_2$, see [6]. In [13] the authors study the case when $R(x,y) = R_1 + R_3$. In [4] the authors study the case when $R(x,y) = R_1 + R_2 + R_3$ with $R_1^2 + R_2^2 + R_3^2 \neq 0$. The case $R(x,y) = R_1 + R_2 + R_3 + R_4$ with $R_4 \neq 0$ and only one R_i not equal to zero, for $i = 1, 2, 3$, is studied in [1]. Systems of type (2) with $R(x,y) = R_2 + R_4$ and $R(x,y) = R_2 + R_6$ have been studied in [20] and [21], respectively.

In all the presented cases up to now, the families of centers are time–reversible, i.e, the centers are symmetrical with respect to a straight line passing through the origin. Therefore, the centers are invariant under the next transformation (modulo a rotation): $(x, y, t) \to (x, -y, -t)$ or $(x, y, t) \to (-x, y, -t)$.

Hence the natural question is, are all the centers of family (2) time–reversible? It is known that the answer to this question is negative. As we will see in the next section, there exist non–reversible centers in the family of the uniformly isochronous systems.

2.1 Isochronous centers and commutators

A more geometric approach to differential systems of equations in the real plane gives the notion of planar vector field.

Definition 2. *The vector field associated to the differential system* (1) *is $\mathcal{X} = (-y + P(x,y))\partial/\partial x + (x + Q(x,y))\partial/\partial y$.*

The following definition gives the notion of commutator of a vector field.

Definition 3. *Two vector fields $\mathcal{X}$ and $\mathcal{Y}$ commute if their Lie bracket is null, that is $[\mathcal{X}, \mathcal{Y}] = D\mathcal{X}.\mathcal{Y} - D\mathcal{Y}.\mathcal{X} \equiv 0$.*

The following theorem shows the relation between the isochronous center problem and the existence of transversal commutator, see for instance [18].

Theorem 4. *A center at the origin of system* (1)*, with associated vector field $\mathcal{X}$, is isochronous if, and only if, there exists an analytic vector field $\mathcal{Y}$ such that $[\mathcal{X}, \mathcal{Y}] \equiv 0$ and $\mathcal{X}$ and $\mathcal{Y}$ are transversal in a punctured neighborhood of the origin.*

Furthermore, we have the following result proved in [17], see also [12].

Theorem 5. *The origin of system* (1)*, with associated vector field* $\mathcal{X}$*, is an isochronous center if, and only if, there exists an analytic vector field of the form* $\mathcal{Y} = (x+o(x,y))\partial/\partial x + (y + o(x,y))\partial/\partial y$ *such that* $[\mathcal{X}, \mathcal{Y}] \equiv 0$.

Hence, the isochronous center problem is equivalent to find a transversal commutator in a punctured neighborhood of the origin. There are only a few families of polynomial differential systems in which a complete classification of the isochronous centers is known, and almost all of them have a polynomial commutator, see for instance [5, 10, 15, 16, 18, 19]. Moreover, several works are devoted to find polynomial commutators for different families of polynomial differential systems, see [10, 18]. However, there exist polynomial differential systems without any polynomial commutator. The first example of a polynomial isochronous center without any polynomial commutator was found by Devlin in [9]. This example is a quartic system, with homogeneous nonlinear part, where an isochronous center at the origin and others two non–isochronous centers coexist. The example is:

$$\dot{x} = -y - x^4 - 4x^2y^2 + y^4, \quad \dot{y} = x - 4x^3y. \tag{3}$$

The non–existence of a polynomial commutator is based upon the following theorem:

Theorem 6. *Let us consider a polynomial system* (1) *with an analytic commutator defined in an open set* $\mathcal{U}$. *If there exists a center in* $\mathcal{U}$, *it is an isochronous center.*

Proof. Let $\mathcal{Y}$ the analytic commutator defined in $\mathcal{U}$ and $\mathcal{X}$ the vector field associated to the polynomial system. $\mathcal{X}$ and $\mathcal{Y}$ are not transversal in the set $V = 0$ with $V := \mathcal{X} \wedge \mathcal{Y}$. Moreover, $V(x,y)$ is inverse integrating factor. Hence, $V = 0$ is an invariant curve of the polynomial system (1). Let (x_0, y_0) a center of $\mathcal{X}$ in $\mathcal{U}$. Since (x_0, y_0) is a singular point of $\mathcal{X}$, then, $V(x_0, y_0) = 0$. Since (x_0, y_0) is a center in $\mathcal{U}$, (x_0, y_0) is an isolated zero of V. Therefore, there exists a punctured neighborhood of (x_0, y_0) where the fields $\mathcal{X}$ and $\mathcal{Y}$ are transversal and by Theorem 4, (x_0, y_0) is an isochronous center. Hence, Devlin example cannot have a polynomial commutator. □

We note that, in particular, every center of a system with a polynomial commutator is an isochronous center. Nevertheless, it can exist a focus singular point in $\mathcal{U}$ as the following example shows $\dot{x} = -y + x(x^2+y^2), \dot{y} = x + y(x^2+y^2)$, which has a focus at the origin and the following polynomial and transversal (in a punctured neighborhood of the origin) commutator: $\dot{x} = x(x^2+y^2), \dot{y} = y(x^2+y^2)$.

3 Uniformly isochronous centers and commutators

We have seen that the isochronous center problem is equivalent to find commutators transversal in a punctured neighborhood of the singular point. In [2], the characterization of the polynomial commutators of uniformly isochronous centers is studied. Using these results, the authors classify the centers of the families with $R(x,y) = R_1 + R_n$ and the case with $R(x,y) = R_2 + R_{2n}$ with $n \in \mathbb{N}$, see [2, 3]. These results exhibit the usefulness of commutators in the classification of uniformly isochronous centers. Moreover, in all the families of uniformly isochronous centers studied in [2], the authors find that either

they are time–reversible or they have a polynomial commutator. Consequently, one asks whether this is the general rule. In [14], an example with a uniformly isochronous center at the origin which is no time–reversible and has no polynomial commutator is found. The example is the following:

$$
\begin{aligned}
\dot{x} &= -y + x\,(y^3 - 3xy^2 + 2x^2y)\,(1 + x^2 + y^2), \\
\dot{y} &= x + y\,(y^3 - 3xy^2 + 2x^2y)\,(1 + x^2 + y^2).
\end{aligned}
\tag{4}
$$

In addition, this system is shown to commute with

$$
\begin{aligned}
\dot{x} &= x\,(x^2 + y^2)\,\sqrt{x^2 + y^2}\,(1 + x^2 + y^2), \\
\dot{y} &= y\,(x^2 + y^2)\,\sqrt{x^2 + y^2}\,(1 + x^2 + y^2).
\end{aligned}
\tag{5}
$$

We note that this commutator has not radial linear part, so Theorem 5 does not apply. Moreover, although the vector fields are transversal in a punctured neighborhood of the origin, the commutator is not analytic, so we cannot use Theorem 4. Hence, this commutator does not ensure the existence of a uniformly isochronous center at the origin.

In the next subsection we want to continue the study of the existence of polynomial and analytic commutators for uniformly isochronous centers started in [2, 3].

3.1 Commutators of polynomial systems

First, we present a result that gives the degree of a polynomial commutator, provided that this polynomial commutator exists, see [3].

Theorem 7. *If system* (2) *has a polynomial commutator, then this polynomial commutator is of the form:*

$$
\dot{x} = xK(x, y), \quad \dot{y} = yK(x, y),
$$

where K is a polynomial of the same degree as R.

In fact, the bound of the degree of the commutator is the same for systems of the form:

$$
\dot{x} = -y + P_2 + P_3 + \cdots + xR_n, \quad \dot{y} = x + Q_2 + Q_3 + \cdots + yR_n,
$$

where P_i and Q_i are homogeneous polynomials of degree i. This type of systems are called *infinity degenerated systems*.

Consequently, the problem to detect the existence or not of a polynomial commutator reduce to a computation problem because the degree is fixed given a polynomial system. Moreover the form of these polynomial commutators is studied in [2, 3]. More specifically, in [3] are proved the following theorems:

Theorem 8. *System* (2), *with $R_1 = R_2 = \ldots = R_{j-1} = 0$ and $R_j \neq 0$ has a polynomial commutator with radial lineal part if, and only if, there are α_ℓ, β_ℓ homogeneous polynomials of order ℓ ($\ell \leq j$, ℓ divides to j) verifying $x\partial_y\beta_\ell - y\partial_x\beta_\ell = \ell\alpha_\ell$ such that the system reads for:*

$$
\dot{x} = -y + x\alpha_\ell \sum_{k=j/\ell-1}^{r-1} a_k\beta_\ell^k, \quad \dot{y} = x + y\alpha_\ell \sum_{k=j/\ell-1}^{r-1} a_k\beta_\ell^k,
\tag{6}
$$

with a_k arbitrary real numbers and $r = [(n-1)/\ell]$. The commutator is given by

$$\dot{x} = x + x \sum_{k=j/\ell-1}^{r-1} a_k \beta_\ell^{k+1}, \quad \dot{y} = y + y \sum_{k=j/\ell-1}^{r-1} a_k \beta_\ell^{k+1}.$$

By Theorem 5, system (6) has a uniformly isochronous center at the origin because the commutator is polynomial and has radial lineal part.

Theorem 9. *System* (2), *where* $R(x,y) = \sum_{j=0}^{n-1} R_j(x,y)$ *with* $R_j(x,y)$ *homogeneous polynomial of degree* j, *has a polynomial commutator with null linear part if, and only if, it is of the form:*

$$\begin{aligned} \dot{x} &= -y + xP_{2\ell}(x,y)\sum_{j=0}^m a_j(x^2+y^2)^j, \\ \dot{y} &= x + yP_{2\ell}(x,y)\sum_{j=0}^m a_j(x^2+y^2)^j, \end{aligned} \tag{7}$$

with $P_{2\ell}(x,y)$ *homogeneous polynomial of degree* 2ℓ, $\ell \geq 0$, *and* a_j *arbitrary real numbers. In this case, the commutator is given by*

$$\dot{x} = x\sum_{j=0}^m a_j(x^2+y^2)^{j+\ell}, \quad \dot{y} = y\sum_{j=0}^m a_j(x^2+y^2)^{j+\ell}. \tag{8}$$

The following theorem gives the conditions to have a uniformly isochronous center at the origin for system (7). Moreover, when the system has a uniformly isochronous center, by Theorem 5, it also admits an analytic commutator with radial linear part. Therefore, the natural question is how to find it. The following theorem also gives the analytic commutator with radial linear part of the family of systems (7), when it has a uniformly isochronous center at the origin.

Theorem 10. *System* (7) *with the condition* $\int_0^{2\pi} P_{2\ell}(\cos\varphi, \sin\varphi)d\varphi = 0$ *has a uniformly isochronous center at the origin. Moreover, it has an analytic commutator with radial linear part of the form:*

$$\begin{aligned} \dot{x} &= x(x^2+y^2)^\ell \sum_{j=0}^m a_j(x^2+y^2)^j H^{-1}, \\ \dot{y} &= y(x^2+y^2)^\ell \sum_{j=0}^m a_j(x^2+y^2)^j H^{-1}, \end{aligned} \tag{9}$$

where H *is an analytic first integral of the form* $H = (x^2+y^2)^{\ell+k}/(1+h(x,y))$ *and* a_k *is the first non null coefficient of the* $\sum_{j=0}^m a_j(x^2+y^2)^j$ *in* (7).

Proof. Taking polar coordinates, system (7) reads for

$$\dot{r} = r^{2\ell+1}\sum_{j=0}^m a_j r^{2j} P_{2\ell}(\cos\varphi, \sin\varphi), \quad \dot{\varphi} = 1.$$

In the case $\int_0^{2\pi} P_{2\ell}(\cos\varphi, \sin\varphi)d\varphi = 0$, system (7) has an analytic first integral that in polar coordinates takes the form

$$H = \frac{-2a_k r^{2(\ell+k)}(\ell+k)}{1 + \alpha r^2 + \beta r^4 + \cdots + 2a_k r^{2(\ell+k)}(\ell+k)\int P_{2\ell}(\cos\varphi, \sin\varphi)d\varphi}.$$

Hence, system (7) has a uniformly isochronous center at the origin. Using this analytic first integral we can construct the analytic commutator (9) with radial linear part. $\quad\square$

We remark that the vector fields associated to system (7) and the commutator (8) are always transversal in a punctured neighborhood of the origin. We note that if the commutator (8) is analytic in a neighborhood of the origin (that is, ℓ is a natural number) then, using Theorem 4, the origin of system (7) is a uniformly isochronous center.

In short, there exist uniformly isochronous centers without polynomial commutators but they always have an analytic one. For instance, the Volokitin example (4) is a polynomial system of degree 6, which has no polynomial commutator, see [14]. Using Theorem 10, system (4) commutes with

$$
\begin{aligned}
\dot{x} &= x(x^2+y^2)^{\frac{3}{2}}(1+x^2+y^2)H^{-1}, \\
\dot{y} &= y(x^2+y^2)^{\frac{3}{2}}(1+x^2+y^2)H^{-1},
\end{aligned}
\tag{10}
$$

where H is the first integral

$$
H = \frac{3(x^2+y^2)^{3/2}}{4(-1+3x^2+4x^3+3y^2+6xy^2+3(x^2+y^2)^{\frac{3}{2}}\arctan(\sqrt{x^2+y^2}))}.
$$

System (10) provides an analytic commutator with radial lineal part for system (4).

In summary, the polynomial systems which have a polynomial commutator and a uniformly isochronous center at the origin are determined by Theorem 8 or Theorem 10. Moreover, there exist uniformly isochronous centers of polynomial systems which are only characterized by the existence of an analytic commutator of the form $\mathcal{Y} = (x + o(x,y))\partial/\partial x + (y + o(x,y))\partial/\partial y$, see Theorem 5.

3.2 Commutators of analytic systems

The following theorem is given in [3] and establishes the form of an analytic commutator for an analytic system:

Theorem 11. *If the analytic system: $\dot{x} = -y + x\,R(x,y)$, $\dot{y} = x + y\,R(x,y)$, with $R(0,0) = 0$, has a center at the origin, then there exists an analytic commutator of the form: $\dot{x} = x + x\,K(x,y)$, $\dot{y} = y + y\,K(x,y)$, with K an analytic function around the origin with $K(0,0) = 0$.*

From Theorem 11 the commutativity condition reduces to the following partial differential equation:

$$
x\left(\frac{\partial K}{\partial y} - \frac{\partial H}{\partial x} + H\frac{\partial K}{\partial x} - K\frac{\partial H}{\partial x}\right) + y\left(-\frac{\partial K}{\partial x} - \frac{\partial H}{\partial y} + H\frac{\partial K}{\partial y} - K\frac{\partial H}{\partial y}\right) = 0
\tag{11}
$$

A straightforward generalization of Theorem 8 for analytic systems is the following:

Proposition 12. *Consider the system*

$$
\dot{x} = -y + xa(x,y)g(b(x,y)), \quad \dot{y} = x + ya(x,y)g(b(x,y))
\tag{12}
$$

with $a(x,y) = (x\partial_y b(x,y) - y\partial_x b(x,y))/\ell$, g an arbitrary analytic function and $b(x,y)$ a homogeneous polynomial of degree ℓ with $\ell \neq 0$. This system has a commutator of the form:

$$
\dot{x} = x + xb(x,y)g(b(x,y)), \quad \dot{y} = y + yb(x,y)g(b(x,y)),
\tag{13}
$$

and it has an isochronous center at the origin.

Proof. Since $b(x, y)$ is a homogeneous polynomial of degree ℓ, we have

$$x\frac{\partial b(x, y)}{\partial x} + y\frac{\partial b(x, y)}{\partial y} = \ell b(x, y), \tag{14}$$

and taking into account the derivatives of (14), the Lie bracket of the vector fields associated to systems (12) and (13) is null and the claim follows. $\square$

A straightforward generalization of Theorem 9 for analytic systems is the following:

Proposition 13. *The analytic system:*

$$\dot{x} = -y + xf(x^2 + y^2)g(x, y), \quad \dot{y} = x + yf(x^2 + y^2)g(x, y)$$

where $f(x^2 + y^2)$ is an analytic function and $g(x, y)$ is a homogeneous polynomial of degree 2ℓ, has a commutator of the form:

$$\dot{x} = x(x^2 + y^2)^\ell f(x^2 + y^2), \quad \dot{y} = y(x^2 + y^2)^\ell f(x^2 + y^2)$$

Proof. The Lie bracket of the vector field associated to the systems gives:

$$[\mathcal{X}, \mathcal{Y}] = f(x^2 + y^2)\left(x\frac{\partial g(x, y)}{\partial x} + y\frac{\partial g(x, y)}{\partial y} - 2\ell g(x, y)\right)\mathcal{Y}$$

Finally, since $g(x, y)$ is a homogeneous polynomial of degree 2ℓ, the claim follows. $\square$

4 BiLiénard equation

In this section we study another family of polynomial systems which corresponds:

$$\dot{x} = -y + F(x), \quad \dot{y} = x + G(y),$$

where $F(x)$ and $G(y)$ are polynomials without constant neither linear terms. These systems are called *BiLiénard systems*, see [11]. In this case, the center problem has been studied with $F(x)$ and $G(y)$ polynomials until fourth degree and all the centers are time–reversible, see [13]. Furthermore, there are families of centers for $F(x)$ and $G(y)$ of arbitrary degree, see [11]. In the following theorem we classify all centers in which $F(x)$ and $G(y)$ are polynomials of degree five.

Theorem 14. *Consider the system:*

$$\dot{x} = -y + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5, \quad \dot{y} = x + b_2y^2 + b_3y^3 + b_4y^4 + b_5y^5, \tag{15}$$

where a_i and b_i are real numbers. All centers at the origin of system (15) are time–reversible.

Acknowledgements

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