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On the mixed finite element approximation of wave problems. Application to shallow water flows

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Abstract

The purpose of this paper is to present a finite element approximation of the scalar hyperbolic wave equation written in mix form, that is, introducing an auxiliary vector field to transform the problem into a first order problem in space and time. We explain why the standard Galerkin method is inappropriate to solve this problem, and propose as alternative a stabilized finite element method that can be cast in the variational multiscale framework. The formulation is extended also to the modified Boussinesq equations as a model for waves in shallow water flows.

1 Introduction

There are several mathematical models for flows in shallow domains. However, a feature they have in common is the mathematical structure of the coupling between the water elevation and the velocity, the unknowns of the problem. This coupling is already present in the simplest setting, modeling linear gravity waves in shallow domains, and is also present in more complex models, such as the Saint-Venant or the Boussinesq equations.

In this work we present a finite element approximation of the modified Boussinesq equations introduced in [5]. Our main concern is the development of a formulation allowing to use *equal* interpolation for the water elevation and the velocity. In general, this is not possible, not even for the linear problem which we use to explain the source on instability of the classical Galerkin method.

Our formulation is based on the variational multiscale approach in the format introduced in [3, 4]. The basic idea is to split the unknowns into a *resolvable* component, which can be reproduced by the discretization method (in our case finite elements) and the remainder, which we will call *sub-grid scale* or *subscale*. Rather than solving exactly for the latter, the formulation results from a closed form approximation for the subscales, which is designed in order to capture their *effect* on the discrete finite element solution. This leads to a formulation that allows the use of equal velocity-depth interpolations. We prove analytically this fact in a particular case, only aiming to explain the stabilization mechanism introduced by the approximation of the subscales.

2 Problem statement

2.1 Initial and boundary value problem

Let us consider the motion of a fluid in a shallow domain whose horizontal projection is $\Omega \subset \mathbb{R}^2$ and whose depth, measured when the fluid is at rest from a horizontal free surface to the bottom of the domain, is $H(\boldsymbol{x})$, $\boldsymbol{x} = (x_1, x_2) \in \Omega$. The vertical coordinate is taken $x_3 = 0$ at the free surface, so that $x_3 = -H(\boldsymbol{x})$ is the equation for the bathymetry. Let $\eta(\boldsymbol{x}, t)$ be the free surface elevation of the fluid in motion and $\boldsymbol{u}(\boldsymbol{x}, t)$ the velocity measured at $x_3 = \beta H$, with the parameter β given, and with $t \in [0, T]$, the time interval of analysis.

Let a be the amplitude and λ the wavelength of a characteristic mode of a wave propagating in the domain of analysis. Let also H_0 be a characteristic depth of this domain, and define the dimensionless numbers $\varepsilon := a/H_0$, $\mu := H_0/\lambda$. The Boussinesq wave theory is obtained by expanding the equations of motion for an inviscid incompressible fluid in terms of ε and μ, and retaining only terms of order up to $\mathcal{O}(\varepsilon)$ and $\mathcal{O}(\mu^2)$, so that it requires $\varepsilon << 1$, $\mu << 1$ and $\varepsilon/\mu^2 = \mathcal{O}(1)$.

The modified Boussinesq equations presented in [5] can be written as

$$\partial_t \eta + \nabla \cdot (H\boldsymbol{u}) + \varepsilon \nabla \cdot (\eta \boldsymbol{u}) + \mu^2 \nabla \cdot \boldsymbol{J}_\eta = 0, \tag{1}$$

$$\partial_t \boldsymbol{u} + g \nabla \eta + \varepsilon \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \mu^2 \boldsymbol{J}_u = \boldsymbol{0}, \tag{2}$$

where g is the magnitude of the gravity acceleration and we have introduced the auxiliary fields $\boldsymbol{J}_\eta := C_1 H^3 \boldsymbol{E} + C_3 H^2 \boldsymbol{E}^H$, $\boldsymbol{J}_u := C_2 H^2 \partial_t \boldsymbol{E} + \beta H \partial_t \boldsymbol{E}^H$, $\boldsymbol{E} := \nabla D$, $D := \nabla \cdot \boldsymbol{u}$, $\boldsymbol{E}^H := \nabla D^H$, $D^H := \nabla \cdot (H\boldsymbol{u})$, where C_i, $i = 1, 2, 3$, are constants defined in terms of β (see [5]).

The boundary conditions to be considered are of two types:

- Inflow boundary, $\Gamma_{\rm I}$. The elevation is known, so that $\eta = \bar{\eta}$ on $\Gamma_{\rm I}$, where the overbar denotes given boundary conditions. The velocity $\bar{\boldsymbol{u}}$ depends on the elevation $\bar{\eta}$.
- Reflecting boundary, $\Gamma_{\rm R}$. In this case, the normal component of the velocity must be zero. It can be shown that this implies that the normal component of $\boldsymbol{J}_\eta$ must vanish [6], so that $\boldsymbol{n} \cdot \boldsymbol{u} = 0$ and, $\boldsymbol{n} \cdot \boldsymbol{J}_\eta = 0$ on $\Gamma_{\rm R}$.

Finally, initially conditions of the form $\eta(\boldsymbol{x}, 0) = \eta^0(\boldsymbol{x})$ and $\boldsymbol{u}(x, 0) = \boldsymbol{u}^0(\boldsymbol{x})$ have to be appended to the problem.

2.2 Variational problem

Let $\xi(\boldsymbol{x})$ and $\boldsymbol{v}(\boldsymbol{x})$ be the elevation and velocity test functions, respectively, belonging to the appropriate functional spaces (see below for the functional setting in the linear case).

To account for the boundary conditions described, ξ must vanish on $\Gamma_{\rm I}$ and the normal component of $\boldsymbol{v}$ must vanish on $\Gamma_{\rm R}$.

Multiplying (1) by ξ and (2) by $\boldsymbol{v}$ and integrating by parts, one gets

$$\int_\Omega \xi \partial_t \eta \, {\rm d}\boldsymbol{x} - \int_\Omega \nabla\xi \cdot (H\boldsymbol{u}) \, {\rm d}\boldsymbol{x} - \varepsilon \int_\Omega \nabla\xi \cdot (\eta\boldsymbol{u}) \, {\rm d}\boldsymbol{x} - \mu^2 \int_\Omega \nabla\xi \cdot \boldsymbol{J}_\eta \, {\rm d}\boldsymbol{x} = 0, \tag{3}$$

$$\int_\Omega \boldsymbol{v} \cdot \partial_t \boldsymbol{u} \, {\rm d}\boldsymbol{x} + g \int_\Omega \boldsymbol{v} \cdot \nabla\eta \, {\rm d}\boldsymbol{x} + \varepsilon \int_\Omega \boldsymbol{v} \cdot (\boldsymbol{u} \cdot \nabla\boldsymbol{u}) \, {\rm d}\boldsymbol{x} + \mu^2 \int_\Omega \boldsymbol{v} \cdot \boldsymbol{J}_u \, {\rm d}\boldsymbol{x} = 0, \tag{4}$$

which must hold for all test functions ξ and $\boldsymbol{v}$.

The auxiliary fields $\boldsymbol{J}_\eta$ and $\boldsymbol{J}_u$ involve second derivatives of the velocity. To cope with them, there are basically two options, either to project directly $\boldsymbol{E}$ and $\boldsymbol{E}^H$ or to project first D and D^H.

3 Space discretization using the Galerkin method

3.1 Galerkin method

Let $\{\Omega^e\}$ be a finite element partition of the domain Ω, with $e = 1, ..., n_{\rm el}$, of size $h = \max_e h^e$, $h^e = {\rm diam}(\Omega^e)$. Let also V_h be a finite element space constructed from this partition using *continuous* Lagrangian interpolation within each element domain. Clearly, this space is a subspace of the space where the continuous unknowns (elevation and velocity components) must be defined. We intend to use equal interpolation for both, and therefore the problem consists in seeking $\eta_h(\cdot, t) \in V_h$ and $\boldsymbol{u}_h(\cdot, t) \in V_h^2$ satisfying the adequate boundary conditions and solution of the finite dimensional time evolution problem

$$\int_\Omega \xi_h \partial_t \eta_h \, {\rm d}\boldsymbol{x} - \int_\Omega \nabla\xi_h \cdot (H\boldsymbol{u}_h) \, {\rm d}\boldsymbol{x} - \varepsilon \int_\Omega \nabla\xi_h \cdot (\eta_h\boldsymbol{u}_h) \, {\rm d}\boldsymbol{x} - \mu^2 \int_\Omega \nabla\xi_h \cdot \boldsymbol{J}_{\eta,h} \, {\rm d}\boldsymbol{x} = 0, \tag{5}$$

$$\int_\Omega \boldsymbol{v}_h \cdot \partial_t \boldsymbol{u}_h \, {\rm d}\boldsymbol{x} + g \int_\Omega \boldsymbol{v}_h \cdot \nabla\eta_h \, {\rm d}\boldsymbol{x} + \varepsilon \int_\Omega \boldsymbol{v}_h \cdot (\boldsymbol{u}_h \cdot \nabla\boldsymbol{u}_h) \, {\rm d}\boldsymbol{x} + \mu^2 \int_\Omega \boldsymbol{v}_h \cdot \boldsymbol{J}_{u,h} \, {\rm d}\boldsymbol{x} = 0, \tag{6}$$

which must hold for all test functions $\xi_h \in V_h$ and $\boldsymbol{v}_h \in V_h^2$ satisfying the corresponding homogeneous boundary conditions. Initial conditions have to be appended to this initial value problem.

The standard Galerkin finite element approximation to problem (3)-(4) is (5)-(6). It is the main goal of this work to show that it is unstable and to devise a modification to enhance its stability properties.

3.2 Mathematical framework for the linear non-dispersive problem

Let us redefine $\eta \leftarrow \eta - \bar{\eta}$, so that the boundary conditions are homogeneous: $\eta = 0$ on $\Gamma_{\rm I}$ and $\boldsymbol{n} \cdot \boldsymbol{u} = 0$ on $\Gamma_{\rm R}$. If $\varepsilon = 0$ (linear problem) and $\mu^2 = 0$ (non-dispersive problem), the equations to be solved for H constant are

$$\partial_t \eta + H \nabla \cdot \boldsymbol{u} = f_\eta, \tag{7}$$

$$\partial_t \boldsymbol{u} + g \nabla\eta = \boldsymbol{f}_u, \tag{8}$$

for appropriate f_η and $\boldsymbol{f}_u$ that depend on the boundary value $\bar{\eta}$. Problem (7)-(8) can be re-written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\eta\\ \boldsymbol{u}\end{bmatrix} + \begin{bmatrix}0 & H\nabla\cdot(\cdot)\\ g\nabla(\cdot) & \mathbf{0}\end{bmatrix}\begin{bmatrix}\eta\\ \boldsymbol{u}\end{bmatrix} = \begin{bmatrix}f_\eta\\ \boldsymbol{f}_u\end{bmatrix}$$

This problem is well posed for $\eta \in V_\eta$ and $\boldsymbol{u} \in \boldsymbol{V}_u$, where

$$\begin{aligned} V_\eta &= \left\{\xi \in H^1(\Omega) \mid \xi = 0 \text{ on } \Gamma_\mathrm{I}\right\},\\ \boldsymbol{V}_u &= \{\boldsymbol{v} \in H(\mathrm{div}, \Omega) \mid \boldsymbol{n}\cdot\boldsymbol{v} = 0 \text{ on } \Gamma_\mathrm{R}\}. \end{aligned}$$

The usual notation is used in these expressions.

If we define the spaces $V = V_\eta \times \boldsymbol{V}_u$, $L = L^2(\Omega) \times \boldsymbol{L}^2(\Omega)$, problem (7)-(8) can be cast into the following abstract framework: find $u \in C^0([0,T];V) \cap C^1([0,T];L)$ such that

$$\frac{\mathrm{d}u}{\mathrm{d}t} + Au = f, \tag{9}$$

where $A : V \longrightarrow L$ is defined by

$$u = \begin{bmatrix}\eta\\ \boldsymbol{u}\end{bmatrix} \mapsto Au = \begin{bmatrix}H\nabla\cdot\boldsymbol{u}\\ g\nabla\eta\end{bmatrix}.$$

To simplify the notation, let us take $g = 1$, $H = 1$ and assume all the variables are dimensionless.

Let $(\cdot,\cdot)$ be the inner product in L. The norm is a space X is denoted by $\|\cdot\|_X$. Problem (9) is equivalent to find $u \in C^0([0,T];V) \cap C^1([0,T];L)$ such that

$$\left(\frac{\mathrm{d}u}{\mathrm{d}t}, v\right) + (Au, v) = (f, v) \qquad \forall v \in L. \tag{10}$$

The well-posedness of the problem relies on the following properties:

1. A is monotone: $(Au, u) \geq 0$.

2. A is maximal. In the case A is monotone and L reflexive, as in our case, this is implied by

$$\|Au\|_L \geq c_1\|u\|_V - c_2\|u\|_L \quad \forall u \in V.$$

In our case, monotonicity and maximality of A are trivially checked. Since $Av \in L$ for all $v \in V$, we have

$$\begin{aligned} (Au, u) &= \int_\Omega \eta\nabla\cdot\boldsymbol{u}\,\mathrm{d}\boldsymbol{x} + \int_\Omega \nabla\eta\cdot\boldsymbol{u}\,\mathrm{d}\boldsymbol{x} = 0,\\ \|Au\|_L^2 &= \int_\Omega |\nabla\cdot\boldsymbol{u}|^2\,\mathrm{d}\boldsymbol{x} + \int_\Omega |\nabla\eta|^2\,\mathrm{d}\boldsymbol{x} = \|u\|_V^2 - \|u\|_L^2. \end{aligned}$$

If these two conditions hold, Hille-Yosida theorem guarantees that there exists a unique solution to the problem that can be bounded as follows (see, for example, [2]):

$$\sup_{t\in[0,T]} \|u\|_L \leq c\left(\|u_0\|_L + T \sup_{t\in[0,T]} \|f\|_L\right) \tag{11}$$

$$\sup_{t\in[0,T]} \left\|\frac{\mathrm{d}u}{\mathrm{d}t}\right\|_L \leq c\left(\|u_0\|_V + T \sup_{t\in[0,T]} \left\|\frac{\mathrm{d}f}{\mathrm{d}t}\right\|_L\right) \tag{12}$$

$$\sup_{t\in[0,T]} \|u\|_V \leq c\left(\|u_0\|_V + T \sup_{t\in[0,T]} \left\|\frac{\mathrm{d}f}{\mathrm{d}t}\right\|_L\right) \tag{13}$$

Let us indicate how to prove these bounds. Bound (11) is obtained by taking $v = u(\cdot, t)$ in (10), using the monotonicity of A and integrating from $t = 0$ to an arbitrary t'. Bound (12) follows using a similar argument to the equation differentiated with respect to t. The important point is bound (13). It follows taking $v = Au$ in the variational equation,

$$\|Au\|_L^2 = (Au, Au) = (f, Au) - \left(\frac{\mathrm{d}u}{\mathrm{d}t}, Au\right) \leq \|f\|_L^2 + \left\|\frac{\mathrm{d}u}{\mathrm{d}t}\right\|_L^2 + \frac{1}{2}\|Au\|_L^2,$$

and then using the maximality of A and bound (12).

The discrete problem using the Galerkin method is: find $u \in C^1([0,T]; V_h)$ such that

$$\left(\frac{\mathrm{d}u_h}{\mathrm{d}t}, v_h\right) + (Au_h, v_h) = (f, v_h) \qquad \forall v_h \in V_h.$$

Note that the test function and trial solution spaces are the same.

Bounds (11) and (12) can be proved for the discrete problem exactly as for the continuous one. However, *bound (13) does not hold for the discrete problem.* The reason is quite simple: *Au_h cannot be taken as test function*, since for $u_h \in V_h$, $Au_h \notin V_h$. Observe also that from the numerical point of view, (13) is what *prevents oscillations*, since in our case it gives control on the divergence of the velocity and the gradient of the elevation.

As a conclusion, the standard Galerkin method may yield oscillations.

4 Stabilized finite element method

4.1 Application to the linearized non-dispersive problem

Let us consider first the non-dispersive problem and linearized using a constant velocity field $\boldsymbol{u}_0$, so that the differential equations of the problem are

$$\partial_t \eta + H\nabla \cdot \boldsymbol{u} + \varepsilon \boldsymbol{u}_0 \cdot \nabla \eta = f_\eta, \tag{14}$$

$$\partial_t \boldsymbol{u} + g\nabla \eta + \varepsilon \boldsymbol{u}_0 \cdot \nabla \boldsymbol{u} = \boldsymbol{f}_u. \tag{15}$$

We will first present the stabilized method we propose for this problem and give a stability estimate. This stabilized method is based on a decomposition of the unknowns η and $\boldsymbol{u}$ into their finite element component and a subscale, that is to say, $\eta = \eta_h + \eta'$ and $\boldsymbol{u} = \boldsymbol{u}_h + \boldsymbol{u}'$. The next step is to give a closed form expression for the subscales in terms of the finite element components. Using heuristic arguments, whose motivation will be omitted, it can be shown that the subscales can be approximated by

$$\eta' = \tau P'(\partial_t \eta_h + H\nabla\cdot\boldsymbol{u}_h + \varepsilon\boldsymbol{u}_0\cdot\nabla\eta_h - f_\eta), \tag{16}$$

$$\boldsymbol{u}' = \tau P'(\partial_t \boldsymbol{u}_h + g\nabla\eta_h + \varepsilon\boldsymbol{u}_0\cdot\nabla\boldsymbol{u}_h - \boldsymbol{f}_u), \tag{17}$$

where P' is a projection onto the space of subscales chosen and τ is a parameter whose expression is given by

$$\tau = \frac{h}{C_1\varepsilon|\boldsymbol{u}_0| + C_2\sqrt{gH}}, \tag{18}$$

C_1 and C_2 being algorithmic constants. See [1] for details of the derivation.

Inserting expressions (16) and (17) into the variational formulation of the problem and taking into account that for H and $\boldsymbol{u}_0$ constant the stabilization parameter τ will be the same for all the elements of the finite element mesh, the stabilized formulation we propose is:

$$\begin{aligned} 0 = {} & \frac{g}{H}(\partial_t\eta_h, \xi_h) - g(\boldsymbol{u}_h, \nabla\xi_h) - \frac{g}{H}(\varepsilon\boldsymbol{u}_0\eta_h, \nabla\xi_h) - \frac{g}{H}(f_\eta, \xi_h) \\ & + (\partial_t\boldsymbol{u}_h, \boldsymbol{v}_h) + g(\nabla\eta_h, \boldsymbol{v}_h) + (\varepsilon\boldsymbol{u}_0\cdot\nabla\boldsymbol{u}_h, \boldsymbol{v}_h) - (\boldsymbol{f}_u, \boldsymbol{v}_h) \\ & + \tau\frac{g}{H}(P'(\partial_t\eta_h + H\nabla\cdot\boldsymbol{u}_h + \varepsilon\boldsymbol{u}_0\cdot\nabla\eta_h - f_\eta), H\nabla\cdot\boldsymbol{v}_h + \varepsilon\boldsymbol{u}_0\cdot\nabla\xi_h) \\ & + \tau(P'(\partial_t\boldsymbol{u}_h + g\nabla\eta_h + \varepsilon\boldsymbol{u}_0\cdot\nabla\boldsymbol{u}_h - \boldsymbol{f}_u), g\nabla\xi_h + \varepsilon\boldsymbol{u}_0\cdot\nabla\boldsymbol{v}_h), \end{aligned} \tag{19}$$

which must hold for all test functions ξ_h and $\boldsymbol{v}_h$ in the appropriate spaces. The terms in the first two rows of this variational equation correspond to the Galerkin contribution, whereas those multiplied by τ should provide stabilization. We will see next that they indeed provide additional stability.

Let us consider the case $P' = P_h^\perp$, that is, the space of subscales is orthogonal to the finite element space. We assume that the velocity and elevation are interpolated using equal continuous functions. We will obtain in this situation a stability estimate for the finite element unknown $\boldsymbol{u}_h(\boldsymbol{x},t)$, $\eta_h(\boldsymbol{x},t)$, solution of the semidiscrete problem (continuous in time). For that purpose, it is enough to consider the case without forcing terms in (19). The problem to be considered is then

$$\begin{aligned} 0 = {} & \frac{g}{H}(\partial_t\eta_h, \xi_h) - g(\boldsymbol{u}_h, \nabla\xi_h) - \frac{g}{H}(\varepsilon\boldsymbol{u}_0\eta_h, \nabla\xi_h) \\ & + (\partial_t\boldsymbol{u}_h, \boldsymbol{v}_h) + g(\nabla\eta_h, \boldsymbol{v}_h) + (\varepsilon\boldsymbol{u}_0\cdot\nabla\boldsymbol{u}_h, \boldsymbol{v}_h) \\ & + \tau\frac{g}{H}(P_h^\perp(H\nabla\cdot\boldsymbol{u}_h + \varepsilon\boldsymbol{u}_0\cdot\nabla\eta_h), H\nabla\cdot\boldsymbol{v}_h + \varepsilon\boldsymbol{u}_0\cdot\nabla\xi_h) \\ & + \tau(P_h^\perp(g\nabla\eta_h + \varepsilon\boldsymbol{u}_0\cdot\nabla\boldsymbol{u}_h), g\nabla\xi_h + \varepsilon\boldsymbol{u}_0\cdot\nabla\boldsymbol{v}_h). \end{aligned} \tag{20}$$

If at each time t we take $\xi_h = \eta_h$, $\boldsymbol{v}_h = \boldsymbol{u}_h$ it is found that

$$\begin{aligned} & \frac{1}{2}\frac{g}{H}\frac{\mathrm{d}}{\mathrm{d}t}\|\eta_h\|^2 + \frac{g}{H}\tau\|P_h^\perp(H\nabla\cdot\boldsymbol{u}_h + \varepsilon\boldsymbol{u}_0\cdot\nabla\eta_h)\|^2 \\ & \quad + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{u}_h\|^2 + \tau\|P_h^\perp(g\nabla\eta_h + \varepsilon\boldsymbol{u}_0\cdot\nabla\boldsymbol{u}_h)\|^2 \leq 0. \end{aligned}$$

Here and in what follows, we use the abbreviation $\|\cdot\| \equiv \|\cdot\|_{L^2(\Omega)}$. Integrating from $t=0$ to any time t' one gets

$$\frac{1}{2}\frac{g}{H}\|\eta_h(t')\|^2 + \frac{1}{2}\|\boldsymbol{u}_h(t')\|^2 + \frac{g}{H}\tau\int_0^{t'} \|P_h^\perp(H\nabla\cdot\boldsymbol{u}_h(t) + \varepsilon\boldsymbol{u}_0\cdot\nabla\eta_h(t))\|^2 \mathrm{d}t$$
$$+ \tau\int_0^{t'} \|P_h^\perp(g\nabla\eta_h(t) + \varepsilon\boldsymbol{u}_0\cdot\nabla\boldsymbol{u}_h(t))\|^2 \mathrm{d}t$$
$$\leq \frac{1}{2}\frac{g}{H}\|\eta_h(0)\|^2 + \frac{1}{2}\|\boldsymbol{u}_h(0)\|^2. \tag{21}$$

If now we differentiate (20) with respect to time and at each time t we take $\xi_h = \partial_t\eta_h$, $\boldsymbol{v}_h = \partial_t\boldsymbol{u}_h$ and after this we integrate from $t=0$ to any time t' we get

$$\frac{1}{2}\frac{g}{H}\|\partial_t\eta_h(t')\|^2 + \frac{1}{2}\|\partial_t\boldsymbol{u}_h(t')\|^2$$
$$+ \frac{g}{H}\tau\int_0^{t'} \|P_h^\perp(H\nabla\cdot\partial_t\boldsymbol{u}_h(t) + \varepsilon\boldsymbol{u}_0\cdot\nabla\partial_t\eta_h(t))\|^2 \mathrm{d}t$$
$$+ \tau\int_0^{t'} \|P_h^\perp(g\nabla\partial_t\eta_h(t) + \varepsilon\boldsymbol{u}_0\cdot\nabla\partial_t\boldsymbol{u}_h(t))\|^2 \mathrm{d}t$$
$$\leq \frac{1}{2}\frac{g}{H}\|\partial_t\eta_h(0)\|^2 + \frac{1}{2}\|\partial_t\boldsymbol{u}_h(0)\|^2. \tag{22}$$

We assume now that the family of finite element meshes is quasi-uniform, so that there exists a constant C_{inv} such that

$$\|\nabla v_h\| \leq \frac{C_{\text{inv}}}{h}\|v_h\| \tag{23}$$

for all constants in the finite element spaces, either of velocity or of elevation.

Evaluating (20) at $t=0$, taking $\xi_h = \partial_t\eta_h(0)$, $\boldsymbol{v}_h = \partial_t\boldsymbol{u}_h(0)$, using Young's inequality $ab \leq \frac{1}{2\alpha}a^2 + \frac{\alpha}{2}b^2$ for all constants $\alpha > 0$, and using the inverse estimate (23) and the expression of τ (18), we obtain that there exists a constant C for which

$$\frac{g}{H}\|\partial_t\eta_h(0)\|^2 + \|\partial_t\boldsymbol{u}_h(0)\|^2$$
$$\leq C\left(\frac{g}{H}\|H\nabla\cdot\boldsymbol{u}_h(0)\|^2 + \|g\nabla\eta_h(0)\|^2 + \frac{g}{H}\|\varepsilon\boldsymbol{u}_0\cdot\nabla\eta_h(0)\|^2 + \|\varepsilon\boldsymbol{u}_0\cdot\nabla\boldsymbol{u}_h(0)\|^2\right). \tag{24}$$

Here and in what follows, C will denote a generic constant, not necessarily the same in different appearances.

To obtain stability for $\nabla\cdot\boldsymbol{u}_h$ and $\nabla\eta_h$ it is seen from (21) that we need to bound only their component in the appropriate finite element space. For that, we can take as tests functions in (20)

$$\xi_h = \tau P_h(\bar{\xi}_h), \quad \bar{\xi}_h = H\nabla\cdot\boldsymbol{u}_h + \varepsilon\boldsymbol{u}_0\cdot\nabla\eta_h,$$
$$\boldsymbol{v}_h = \tau P_h(\bar{\boldsymbol{v}}_h), \quad \bar{\boldsymbol{v}}_h = g\nabla\eta_h + \varepsilon\boldsymbol{u}_0\cdot\nabla\boldsymbol{u}_h,$$

which yields

$$\frac{g}{H}\tau\|P_h(\bar{\xi}_h)\|^2 + \tau\|P_h(\bar{\boldsymbol{v}}_h)\|^2$$
$$\leq C\left(\frac{g}{H}\tau\|\partial_t\eta_h\|^2 + \tau\|\partial_t\boldsymbol{u}_h\|^2 + \frac{g}{H}\tau\|P_h^\perp(\bar{\xi}_h)\|^2 + \tau\|P_h^\perp(\bar{\boldsymbol{v}}_h)\|^2\right),$$

which combined with (21), (22) and (24) yields the stability estimates we were looking for:

$$\frac{g}{H}\max_{0<s<t}\|\eta_h(s)\|^2 + \max_{0<s<t}\|\boldsymbol{u}_h(s)\|^2 \leq \frac{g}{H}\|\eta_h^0\|^2 + \|\boldsymbol{u}_h^0\|^2. \tag{25}$$

$$\begin{aligned}&\frac{g}{H}\tau\int_0^t\|\bar{\xi}_h(s)\|^2\mathrm{d}s + \tau\int_0^t\|\bar{\boldsymbol{v}}_h(s)\|^2\mathrm{d}s\\ &\quad\leq C\left(\frac{g}{H}\|\eta_h^0\|^2 + \|\boldsymbol{u}_h^0\|^2 + \frac{g}{H}\tau\|H\nabla\cdot\boldsymbol{u}_h^0\|^2 t + \tau\|g\nabla\eta_h^0\|^2 t\right.\\ &\qquad\left.+\frac{g}{H}\tau\|\varepsilon\boldsymbol{u}_0\cdot\nabla\eta_h^0\|^2 t + \tau\|\varepsilon\boldsymbol{u}_0\cdot\nabla\boldsymbol{u}_h^0\|^2 t\right).\end{aligned} \tag{26}$$

From the numerical point of view, estimate (25), which bounds the $C^0(0,T;L^2(\Omega))$ norm of the unknowns, is weaker that (26), since in this last case we have some control on the divergence of the velocity, the gradient of the elevation and the convective term in both equations (14) and (15).

4.2 Stabilized finite element method for the general problem

Extending the ideas of the previous subsection to the Boussinesq equations, the stabilized finite element formulation we propose in this case is

$$\begin{aligned}&\frac{g}{H_0}(\partial_t\eta_h,\xi_h) - \frac{g}{H_0}(H\boldsymbol{u}_h,\nabla\xi_h) - \frac{g}{H_0}\varepsilon(\eta_h\boldsymbol{u}_h,\nabla\xi_h) - \frac{g}{H_0}\mu^2(\boldsymbol{J}_{\eta,h},\nabla\xi_h)\\ &\quad+(\partial_t\boldsymbol{u}_h,\boldsymbol{v}_h) + g(\nabla\eta_h,\boldsymbol{v}_h) + \varepsilon(\boldsymbol{u}_h\cdot\nabla\boldsymbol{u}_h,\boldsymbol{v}_h) + \mu^2(\boldsymbol{J}_{u,h},\boldsymbol{v}_h)\\ &\quad+\frac{g}{H_0}\sum_{e=1}^{n_{\mathrm{el}}}\tau^e\left\langle P'(\partial_t\eta_h + \nabla\cdot(H\boldsymbol{u}_h) + \varepsilon\nabla\cdot(\eta_h\boldsymbol{u}_h) + \mu^2\nabla\cdot\boldsymbol{J}_{\eta,h})\right.,\\ &\qquad\qquad\left.\nabla\cdot(H\boldsymbol{v}_h) + \varepsilon\nabla\cdot(\xi_h\boldsymbol{u}_h)\right\rangle_{\Omega^e}\\ &\quad+\sum_{e=1}^{n_{\mathrm{el}}}\tau^e\left\langle P'(\partial_t\boldsymbol{u}_h + g\nabla\eta_h + \varepsilon\boldsymbol{u}_h\cdot\nabla\boldsymbol{u}_h + \mu^2\boldsymbol{J}_{u,h}), g\nabla\xi_h + \varepsilon\boldsymbol{u}_h\cdot\nabla\boldsymbol{v}_h\right\rangle_{\Omega^e} = 0,\end{aligned} \tag{27}$$

which must hold for all test functions ξ_h and $\boldsymbol{v}_h$. Here, H_0 is a characteristic depth only needed to scale the equations. This stabilized finite element formulation has proven to be stable and accurate in numerical simulations.

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