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Semilinear order property and infinite games

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À minha esposa, Lina

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RESUMEN DE LA TESIS

Título: “Semilinear order property and infinite games”

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En este trabajo se analiza la determinación de juegos de Lipschitz y Wadge, junto con la propiedad de ordenación semilineal, estrechamente relacionada con estos juegos, en el contexto de la Aritmética de segundo orden y el programa de la Matemática inversa (Reverse Mathematics). En primer lugar, se obtienen pruebas directas, formalizables en la Aritmética de segundo orden, de la determinación de los juegos de Lipschitz y Wadge para los primeros niveles de la Jerarquía de diferencias de Haussdorf. A continuación, se determinan los axiomas de existencia suficientes para la formalización de dichas pruebas dentro de los subsistemas clásicos de la Aritmética de segundo orden ($\mathbf{RCA}_0$, $\mathbf{WKL}_0$, $\mathbf{ACA}_0$, $\mathbf{ATR}_0$ y Π^1_1–$\mathbf{CA}_0$). Finalmente, en algunos casos se muestra que dichos axiomas de existencia son óptimos, probando que resultan ser equivalentes (sobre un subsistema débil adecuado, como $\mathbf{RCA}_0$ o $\mathbf{ACA}_0$) a las correspondientes formalizaciones de los principios de determinación o de ordenación semilineal. Los principales resultados obtenidos son los siguientes:

Teorema A. Sobre $\mathbf{RCA}_0$ son equivalentes:

(1) $\mathbf{ACA}_0$.
(2) $(\Sigma^0_1)_2$–$\mathbf{Det}^*_L$
(el principio de determinación para juegos de Lipschitz entre subconjuntos del espacio de Cantor que son diferencia de dos cerrados).
(3) $(\Sigma^0_1)_2$–$\mathbf{SLO}^*_L$
(la propiedad de ordenación semilineal de la reducibilidad Lipschitz entre subconjuntos del espacio de Cantor que son diferencia de dos cerrados).

Teorema B. Sobre $\mathbf{RCA}_0$ son equivalentes:

(1) $\mathbf{ATR}_0$.
(2) $(\Sigma^0_1 \cup \Pi^0_1)$–$\mathbf{Det}_L$
(el principio de determinación para juegos de Lipschitz entre subconjuntos abiertos o cerrados del espacio de Baire).

Teorema C. Sobre $\mathbf{ACA}_0$ son equivalentes:

(1) $\mathbf{ATR}_0$.
(2) Δ^0_1–$\mathbf{Det}_L$
(el principio de determinación para juegos de Lipschitz entre subconjuntos del espacio de Baire que son simultáneamente abiertos y cerrados).
(3) Δ^0_1–$\mathbf{SLO}_L$
(la propiedad de ordenación semilineal de la reducibilidad Lipschitz entre subconjuntos del espacio de Baire simultáneamente abiertos y cerrados).

Contents

Introduction

In this thesis we analyze the determinacy of the Lipschitz and Wadge games, as well as the tightly related semilinear ordering principle, in two different settings. Firstly, we work in the setting of descriptive set theory and, secondly, in the setting of second order arithmetic. Very roughly, the main goals of the present thesis are:

- *to give direct proofs of the determinacy of Lipschitz and Wadge games for the first levels of the Wadge hierarchy;*
- *to formalize these proofs in the setting of second order arithmetic in order to calibrate the strength of Lipschitz and Wadge determinacy in terms of the (set existence) axioms needed to prove them, as a new contribution to the research program of Reverse Mathematics; and*
- *to examine the relation between the semilinear ordering principle and the determinacy of Lipschitz and Wadge games in the formal context of second order arithmetic, and to search for the axioms needed to prove the equivalence between these principles.*

Lipschitz and Wadge games were first introduced in the late 1960's by W. W. Wadge as a tool for studying the *complexity of subsets of real numbers* (Wadge had already obtained most of his results on this topic by the end of 1972 [WWW72], but published them only much later in his Ph. D. thesis [WWW83].) Given subsets of real numbers A and B, A is said to be *Wadge reducible* to B if there is a real continuous function F such that

$$A = F^{-1}(B).$$

Intuitively, the existence of $F : \mathbb{R} \to \mathbb{R}$ such that $x \in A$ if and only if $F(x) \in B$ for every $x \in \mathbb{R}$ means that the problem of verifying membership in A can be reduced to the problem of verifying membership in B and therefore we can regard A as being simpler or at most as complex as B. It is natural to regard this reducibility relation as an order relation. This explains the usual notation $A \leq_W B$, meaning that A is Wadge reducible to B by a continuous function. If the function which witnesses the reduction of A to B is, in addition, Lipschitz, then the notation becomes $A \leq_L B$ and A is said to be *Lipschitz reducible* to B. The equivalence classes of the equivalence relations induced by $\leq_W$ and $\leq_L$ are called Wadge and Lipschitz degrees, respectively, and the preorders $\leq_W$ and $\leq_L$ induce a partial order $\leq$ on them.

As it is common practice in descriptive set theory, Wadge did not study the reducibility relations $\leq_W$ and $\leq_L$ for the real line $\mathbb{R}$ itself. Instead, he considered the *Baire space* ω^ω, i.e. the space of all ω–sequences of natural numbers endowed with the product topology inherited from the discrete topology on ω. The reason for this is that the use of $\mathbb{R}$ leads to minor but annoying difficulties. Seen as a topological space, $\mathbb{R}$ is not homeomorphic to any of its powers, and the standard decimal representation of the elements of $\mathbb{R}$ is badly behaved in a sense that real numbers very closed to each other (e.g., 1 and $0.999 \cdots 9$) can have completely different expansions. These difficulties are related to the presence of the rational numbers among the reals, and can be avoided simply by replacing $\mathbb{R}$ by the Baire space. The Baire space which is well–known to be homeomorphic to the irrationals $\mathbb{I}$. Another natural choice is to study the reducibility relation in the *Cantor space* 2^ω, i.e., the space of all ω–sequences of 0's and 1's endowed with the product topology inherited from the discrete topology on $\{0, 1\}$. Cantor space is compact and Baire space is not. But, despite this difference, both spaces are zero–dimensional Polish spaces satisfying that elements which are very close together will have in common a large initial segment of their respective representations.

Wadge showed that if we restrict our attention to the Baire or Cantor space, the reducibility relations $\leq_W$ and $\leq_L$ can be naturally studied in terms of the so–called *Wadge and Lipschitz games*. These games are variants of the traditional Gale–Stewart games used in descriptive set theory.

Given $A \subseteq \omega^\omega$ (resp., 2^ω), the Gale–Stewart game for A, in symbols $G(A)$, is the game on ω (resp., $\{0, 1\}$) where player I (male) and player II (female) alternatively play natural numbers x_i and y_i and player I wins just in case $\langle x_0, y_0, x_1, y_1, x_1, y_2, \dots \rangle \in A$.

Player I	x_0		x_1		x_2		$\dots$
Player II		y_0		y_1		y_2	$\dots$

In contrast, in Lipschitz and Wadge games each player builds his/her own ω–sequence and the winning condition for the game now depends on two sets. Namely, given $A, B \subseteq \omega^\omega$ (resp., 2^ω), the Lipschitz game for A and B, in symbols $G_L(A, B)$, is the game on ω (resp., $\{0, 1\}$) where player I and player II alternatively play natural numbers x_i and y_i and player II wins just in case $\langle x_0, x_1, x_2, \dots \rangle \in A \Leftrightarrow \langle y_0, y_1, y_2, \dots \rangle \in B$.

Player I	x_0		x_1		x_2		$\dots$
Player II		y_0		y_1		y_2	$\dots$

The Wadge game for A and B, in symbols $G_W(A, B)$, is the variant of $G_L(A, B)$ where player II is allowed to pass (i.e., not to play) at any round, but she must play infinitely often otherwise she loses. The payoff is as before: player II wins if, and only if, $\langle x_0, x_1, x_2, \dots \rangle \in A \Leftrightarrow \langle y_0, y_1, y_2, \dots \rangle \in B$.

Player I	x_0		x_1	$\dots$	x_k		x_{k+1}	$\dots$	x_l		$\dots$	
Player II		**p**		**p** $\cdots$ **p**		y_0		**p** $\cdots$ **p**		y_1		$\dots$

It is worth noting that despite the differences in their definitions Lipschitz and Wadge games can be viewed as particular Gale–Stewart games with appropriate payoff sets.

For each of the above mentioned games, a strategy for player I (resp. player II) is a function from finite sequences of natural numbers to natural numbers which assigns to each partial run of the game player's I (resp. player II) next move. A strategy is said to be a *winning strategy* for a player if he or she wins the game as long as he or she plays according the strategy, no matter how his or her opponent plays. Thus, saying that player I or II wins a game means to say that he or she has a winning strategy in that game.

The key idea to relate Wadge reducibility and infinite games is the following simple but crucial result, which is known as *Wadge's lemma.*

- (Wadge, 1972) Let $A, B \subseteq \omega^\omega$ (or 2^ω). Then:

  1. Player II wins the game $G_L(A, B)$ iff $A \leq_L B$.
  2. Player II wins the game $G_W(A, B)$ iff $A \leq_W B$.
  3. If player I wins $G_L(A, B)$ or $G_W(A, B)$ then $B \leq_L A^c$.

Where X^c stands for the complement of X. Two person infinite games admit no draws and only one of the players can win the game. But the fact that one of them has a winning strategy is something that must be verified: a two person infinite game is said to be *determined* if one of the players has a winning strategy. The assertion that all Gale–Stewart games in the Baire space are determined is the content of the axiom of determinacy (**AD**) from set theory:

$$\mathbf{AD} = \forall A \subseteq \omega^\omega,\ G(A) \text{ is determined.}$$

It is well known that making use of the axiom of choice (**AC**), one can construct $A \subseteq \omega^\omega$ with $G(A)$ non–determined. Thus, **AD** contradicts full **AC**. However, it is compatible with a weaker form of choice: the axiom of dependent choices (**DC**).

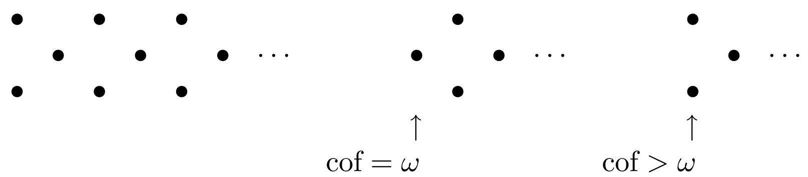
It is important to note that Wadge was not concerned with the question of proving the determinacy of Wadge and Lipschitz games. Instead, he assumed **AD** as a working hypothesis and then obtained Wadge and Lipschitz determinacy as an immediate consequence. In particular, putting together determinacy and Wadge's lemma, he derived the following somewhat surprising comparability property for sets of *reals*:

$$\mathbf{SLO}_L = \forall A, B \subseteq \omega^\omega,\ A \leq_L B \vee B^c \leq_L A$$

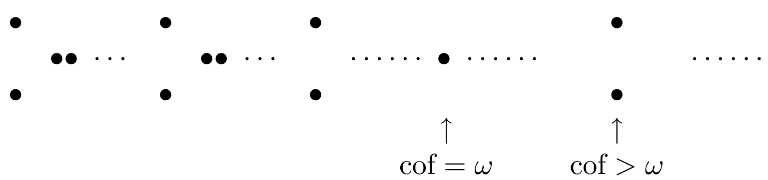
$$\mathbf{SLO}_W = \forall A, B \subseteq \omega^\omega,\ A \leq_W B \vee B^c \leq_W A$$

That is to say, assuming **AD** the collections of the Wadge and Lipschitz degrees are "almost" linearly ordered. (If we identify the degree of a set with that of its complement, the hierarchy becomes a strict linear order.) The notation **SLO** stands for *Semi–Linear Ordering* and the subscript indicates whether we consider Wadge or Lipschitz reductions.

Wadge soon realized the relevance of the semilinear ordering principle **SLO** and, starting from this principle, in [WWW83] he extensively studied the structure of Wadge degrees in the Baire space, particularly of those Wadge degrees formed by Borel sets. His methodology was based on a *topological analysis* of Wadge and Lipschitz games by using the fundamental result due to Hausdorff and Kuratowski that $\mathbf{\Delta}^0_{\alpha+1}$ sets can be generated by taking (possibly transfinite) differences of $\mathbf{\Pi}^0_\alpha$ sets (the so–called *Hausdorff Difference Hierarchy* for $\mathbf{\Pi}^0_\alpha$ sets). Very roughly, given Borel sets A and B, if we view them as members of the difference hierarchy, then the underlying topological structure gives us enough information for constructing a winning strategy in the game $G_{L/W}(A, B)$. Remarkably, he was able to prove that Borel Wadge degrees are well–founded and to calculate the exact ordinal length of such hierarchy. After this work, many descriptive set theorists further developed the so–called Wadge theory, including, among others, Donald A. Martin, D. Monk, J. Steel, R. Van Wesep and A. Andretta ([Mar73], [Stee77], [VWsp77], [VWsp78], [AA03], [AA04].) From these works, a detailed picture of the hierarchies of *all* (not necessarily Borel) Wadge and Lipschitz degrees emerged, showing two hierarchies finer than the Hausdorff difference hierarchy and much finer than the Borel hierarchy. In fact, working in $\mathbf{ZF} + \mathbf{DC} + \mathbf{AD}$, both the collection of *all* Wadge degrees and the collection of *all* Lipschitz degrees are well–founded. Moreover, the hierarchy of Wadge degrees looks like an infinite ladder where pairs of non–self–dual Wadge degrees (i.e., degrees formed by sets incomparable with their complements) alternate with self–dual Wadge degrees (i.e., degrees formed by sets Wadge equivalent to their complements), and where the levels of countable cofinality are occupied by self-dual Wadge degrees and the levels of uncountable cofinality are occupied by non-self-dual Wadge degrees:



As to Lipschitz degrees, it was shown that pairs of non–self–dual Wadge degrees correspond to pairs of non–self–dual Lipschitz degrees; whereas each self-dual Wadge degree is the union of a block of ω_1 (ω, if we consider the Cantor space) consecutive self–dual Lipschitz degrees. Thus, the hierarchy of Lipschitz degrees looks like follows:



Interestingly, a good portion of these results can be recovered without assuming **AD**. This is due to the prominent result of Donald Martin that

- (Martin, 1975) $\mathbf{ZF} + \mathbf{DC}$ proves determinacy for all Borel sets.

In particular, it follows that $\mathbf{ZF} + \mathbf{DC}$ proves the determinacy of Wadge and Lipschitz games for Borel sets (and hence the semilinear ordering principle for Borel sets too). This is because Wadge or Lipschitz games for Borel sets can be reexpressed as Borel Gale–Stewart games. However, the proof of Martin's result is well–known to require the assumption of strong existence axioms. H. Friedman [Frd71] famously proved that Borel determinacy is not provable in **ZFC** (**ZF** plus the axiom of choice) without the power set axiom. Indeed, he showed that $\aleph_1$ many iterations of the power set are needed to prove it. He also proved that

- (Friedman, 1971) Second order arithmetic $\mathbf{Z}_2$ *cannot* prove that all Borel Gale–Stewart games are determined.

More recently, in [MS12], A. Montalbán and R. A. Shore established the precise bounds for the amount of determinacy provable in second order arithmetic by showing that Gale–Stewart determinacy for all finite Boolean combinations of $\mathbf{\Pi}^0_3$ sets is already not provable within $\mathbf{Z}_2$. In contrast, by a result of A. Louveau and J. Saint Raymond [LSR87],

- (A. Louveau and Saint Raymond, 1987) Second order arithmetic $\mathbf{Z}_2$ *does* prove that all Borel Wadge and Lipschitz games are determined.

Thus, although we can recover Borel Wadge and Lipschitz determinacy within $\mathbf{ZF} + \mathbf{DC}$ from Martin's notorious result, we would be using a principle strictly more powerful than needed. In addition, the exact strength of Borel Wadge and Lipschitz determinacy is not known. Although provable in full $\mathbf{Z}_2$, it has not been known any *natural* subsystem of $\mathbf{Z}_2$ which suffices for proving Borel Wadge and Lipschitz determinacy. A natural approach to attack this problem then arises:

- *to give direct proofs of Wadge and Lipschitz determinacy and of the semilinear ordering principle for Borel sets that allow us to calibrate the exact strength of those principles in terms of subsystems of second order arithmetic.*

The present thesis can be seen as a first step towards an answer to this question. The material in this thesis naturally divides into two parts. In the first part, we work in $\mathbf{ZF} + \mathbf{DC}$ and show that Wadge's topological analysis of games can be reinterpreted in order to give direct proofs of Wadge and Lipschitz determinacy for the first levels of the Wadge hierarchy. In the second part, and this is the bulk of the present work, we formalize previous arguments inside second order arithmetic (so far no explicit formalization of Wadge and Lipschitz games in second order arithmetic was available in the literature) and obtain a number of results on the strength of such principles in terms of classical fragments of $\mathbf{Z}_2$.

In what follows, we briefly describe the tools we have used and the main results that we have obtained in both parts.

PART I: Wadge/Lipschitz determinacy in descriptive set theory.

As previously mentioned, in [WWW83] Wadge's goal was not the study of determinacy of games. Instead, he was interested in proving the existence of complete (i.e. maximal relatively to $\leq_W$) sets for a given Wadge degree. Nonetheless, in Chapter 2 we show that the analysis he developed, after its reinterpretation in a more combinatorial manner, can be used to prove the determinacy of both Lipschitz and Wadge games over sets of real numbers located in given levels of the Wadge hierarchy. Namely, we present direct, topological proofs of Lipschitz and Wadge determinacy (and hence also of the semilinear ordering principle) in both the Cantor and Baire spaces for the following classes (that correspond to the first five levels of the Wadge hierarchy or, alternatively, to the first three levels of Hausdorff's hierarchy of differences):

$$
\underbrace{\overbrace{\begin{array}{c}\{\emptyset\}\\ \\ \{X^\omega\}\end{array}}^{\mathbf{Df}_0}}_{\breve{\mathbf{Df}}_0}
\qquad
\underbrace{\overbrace{\begin{array}{cc} & \mathbf{\Pi}^0_1-\mathbf{\Sigma}^0_1\\ \mathbf{\Delta}^0_1-\{\emptyset,X^\omega\} & \\ & \mathbf{\Sigma}^0_1-\mathbf{\Pi}^0_1\end{array}}^{\mathbf{Df}_1=\mathbf{\Pi}^0_1}}_{\breve{\mathbf{Df}}_1=\mathbf{\Sigma}^0_1}
\qquad
\underbrace{\overbrace{\begin{array}{cc} & \mathbf{Df}_2-\breve{\mathbf{Df}}_2\\ (\mathbf{Df}_2\cap\breve{\mathbf{Df}}_2)-\left(\mathbf{\Pi}^0_1\cup\mathbf{\Sigma}^0_1\right) & \\ & \breve{\mathbf{Df}}_2-\mathbf{Df}_2\end{array}}^{\mathbf{Df}_2}}_{\breve{\mathbf{Df}}_2}
$$

where

1. $\mathbf{\Pi}^0_1$ = closed sets
2. $\mathbf{\Sigma}^0_1$ = open sets
3. $\mathbf{\Delta}^0_1$ = clopen (i.e. both closed and open) sets
4. $\mathbf{Df}_2$ = differences of closed sets.
5. $\breve{\mathbf{Df}}_2$ = complements of differences of closed sets.

The proof ideas are based on the fact that a closed set in the Baire or Cantor space can be characterized as the *set of paths* of a pruned tree. This point of departure is important in two aspects. Firstly it allows us to develop the arguments in a more combinatorial way, and secondly it can be recovered almost without restrictions in second order arithmetic.

Let us see how this is done according to the type of the set in Cantor and Baire spaces. Firstly, we consider clopen sets. In both spaces clopen sets can be characterized using well-founded trees, i.e. trees that have no path. In the Cantor space, these trees are finite, and to each clopen set we can associate a natural number: the maximal length of a sequence in the tree characterizing it. In the context of Lipschitz or Wadge games, this natural number stands for the maximum number of moves a player can make preserving the possibility to decide whether he or she remains within or leaves the tree. In the Baire space, trees corresponding to closed sets can be infinitely splitting. Thus, to each well-founded tree describing a clopen set we associate a rank, which is a countable ordinal. Now the possibility to decide whether a player remains within or leaves the tree stays open as long as the rank of the sequence that he or she is enumerating is different from zero. Given two clopen sets A and B, comparison of the associated natural numbers or countable ordinals gives us natural criteria to decide which player will win the corresponding Lipschitz or Wadge game, and to built a winning strategy.

Secondly, a set can be closed and not open, which implies that its boundary is non-empty. This means that the corresponding tree characterizing it has a path with extensions outside the tree for every initial sequence of the path. So at each stage of a enumeration of such a path one can always decide to stay in the tree or leave it forever. In terms of games, this means that if player II plays using this *special path* then she will always win the game whenever player I plays in a closed set. Using this idea, we can show that Lipschitz and Wadge games for closed sets are determined.

Thirdly, we have to examine differences of closed sets. These sets can be viewed as differences of sets of paths of pruned trees. The study of these trees is underpinned by a previous topological analysis of differences of closed sets in terms of Hausdorff's residues. The crucial step here consists in identifying the condition under which the complement of a difference of closed sets is itself a difference of sets as well. This allows us to distinguish clearly the types of differences we have to deal with, particularly those involving well-founded trees and those not involving well-founded trees. To the former we can apply the methods described above with the appropriate adaptations. The others have to be examined accordingly to their particular structure.

The following table summarizes the main results we have obtained. The first column shows the classes to which the payoff sets A and B belong. The two columns on the right refer the lemmas and propositions which prove determinacy for the corresponding Lipschitz and Wadge games, resp., in Cantor space and in Baire space.

(A, B)	Cantor	Baire
$\mathbf{\Delta}^0_1$	Lemma 2.2	Lemma 2.20
$\mathbf{\Pi}^0_1$	Lemma 2.5	Lemma 2.22
$\mathbf{\Sigma}^0_1 \cup \mathbf{\Pi}^0_1$	Lemma 2.8	Lemma 2.24
$\mathbf{Df}_2 \cap \breve{\mathbf{Df}}_2$	Lemma 2.10	Lemma 2.26
$\mathbf{Df}_2$	Lemma 2.13	Lemma 2.28
$\mathbf{Df}_2 \cup \breve{\mathbf{Df}}_2$	Theorem 2.18	Theorem 2.31

The results referred in the above table are set up in Sections 2 and 3 of Chapter 2. We have only written in detail the proofs for the Lipschitz case. Wadge determinacy, which is usually easier to prove, can be obtained from our arguments for the Lipschitz case after suitable modifications. In fact, it is not hard to see that a winning strategy for player II in a Lipschitz game yields automatically a winning strategy for player II in the corresponding Wadge game. Nonetheless, this does not hold for player I. There are cases where player I wins the Lipschitz game, while player II has a winning strategy in the corresponding Wadge game.

PART II: Wadge/Lipschitz determinacy in second order arithmetic.

Reverse mathematics is the program of discovering which set existence axioms are needed to prove known mathematical theorems. Research in this field has shown that in the setting of second order arithmetic almost all theorems of ordinary, non-set-theoretic mathematics fall into a small number of equivalence classes with respect to provable equivalence over a weak base theory, the so called "big five". These five subsystems of second order arithmetic are $\mathbf{RCA}_0$, $\mathbf{WKL}_0$, $\mathbf{ACA}_0$, $\mathbf{ATR}_0$, and Π^1_1-$\mathbf{CA}_0$ and they are distinguished from one another by their increasing stronger set existence axioms: recursive comprehension, weak König's lemma, arithmetical comprehension, arithmetical transfinite recursion and Π^1_1-comprehension, respectively. The subsystem $\mathbf{RCA}_0$ is the weakest and the ideal one to be used as base theory. The common procedure in reverse mathematics consists in showing that a given mathematical theorem taken over a weak base theory (ideally $\mathbf{RCA}_0$) is equivalent to the principal set existence axiom of one of the subsystems $\mathbf{WKL}_0$, $\mathbf{ACA}_0$, $\mathbf{ATR}_0$, or Π^1_1-$\mathbf{CA}_0$ (for more information see the classical monograph [Smp99]).

The reverse mathematics of the determinacy of Gale-Stewart games has been thoroughly investigated by J. R. Steel, K. Tanaka, M. O. MedSalem, and T. Nemoto ([Stee77], [Tan90], [NMT07], [N09a], and [N09b]). After Friedman had shown that $\mathbf{Z}_2$ *cannot* prove that all Borel Gale–Stewart games are determined, and shortly after he had laid the foundations of reverse mathematics [Frd75], Steel established in his Ph. D. thesis the exact strength of clopen and open determinacy in the Baire space (ch. I, B, of [Stee77]; see also Theorem V.8.7 in [Smp99]):

- (J. R. Steel, 1977) The following assertions are pairwise equivalent over $\mathbf{RCA}_0$:
    1. $\mathbf{ATR}_0$.
    2. Δ^0_1 (clopen) Gale–Stewart determinacy in the Baire space.
    3. Σ^0_1 (open) Gale–Stewart determinacy in the Baire space.

Historically, this is one of the first results in the field of reverse mathematics. Several years later K. Tanaka obtained a further result([Tan90]):

- (K. Tanaka, 1990) The following assertions are pairwise equivalent over $\mathbf{RCA}_0$:
    1. Π^1_1-$\mathbf{CA}_0$.
    2. $\Sigma^0_1 \wedge \Pi^0_1$ (intersections of open and closed sets) Gale–Stewart determinacy in the Baire space.

More recently, subsystems $\mathbf{WKL}_0$ and $\mathbf{ACA}_0$ have been also fully characterized in terms of Gale-Stewart determinacy, but this time considering infinite games in the Cantor space. (Let us observe that above level Δ^0_3 the strength of determinacy in Cantor and Baire space coincide, but for levels below Δ^0_3 determinacy in Cantor space is strictly weaker than determinacy in Baire space.) In what follows, **Det** and **Det*** stand for Gale-Stewart determinacy in Baire and in Cantor space, respectively.

- (T. Nemoto, M. MedSalem, K. Tanaka, 2007) The following assertions are pairwise equivalent over $\mathbf{RCA}_0$:
  1. $\mathbf{WKL}_0$.
  2. Δ^0_1-$\mathbf{Det}^*$.
  3. Σ^0_1-$\mathbf{Det}^*$.

- (T. Nemoto, M. MedSalem, K. Tanaka, 2007) The following assertions are pairwise equivalent over $\mathbf{RCA}_0$:
  1. $\mathbf{ACA}_0$.
  2. $\Sigma^0_1 \wedge \Pi^0_1$-$\mathbf{Det}^*$.

Many other related results have been obtained (see Section 1 of Chapter 3 for a comprehensive account of the more relevant ones), providing us with a detailed picture of the reverse mathematics of Gale-Stewart determinacy. We can say that currently we have a very complete knowledge of the strength of Gale-Stewart determinacy, in both Cantor and Baire spaces, from level $\boldsymbol{\Delta}^0_1$ to a little further beyond level $\boldsymbol{\Delta}^0_3$. In addition, a recent result of A. Montalbán and R.A. Shore shows that this is precisely where the bound for Gale-Stewart determinacy in second order arithmetic stays ([MS12]):

- (A. Montalbán and R.A. Shore, 2012)
  1. For each natural number k, $\mathbf{Z}_2$ proves determinacy for Boolean combinations of k many $\boldsymbol{\Pi}^0_3$ sets.
  2. $\mathbf{Z}_2$ does not prove determinacy for *all* finite Boolean combinations of Π^0_3 sets. In particular, $\mathbf{Z}_2$ does not prove Δ^0_4-$\mathbf{Det}$.

Thus, we not only have a fairly detailed information about the strength of Gale-Stewart determinacy in terms of subsystems of second order arithmetic, but also know the exact amount of such determinacy provable in $\mathbf{Z}_2$.

In contrast, the situation for Lipschitz and Wadge games is completely different. It is known that Lipschitz games are determined for all Borel sets in $\mathbf{Z}_2$ (a result of A. Louveau and J. Saint Raymond already mentioned). But it is not known any informative characterization of which portion of $\mathbf{Z}_2$ suffices; and, somewhat surprisingly, there is no detailed analysis of the strength of Lipschitz or Wadge determinacy in terms of subsystems of second order arithmetic. In fact, to the best of our knowledge, it does not even exist an explicit formalization of Lipschitz or Wadge games in the language of second order arithmetic in the literature. Also, despite its relevance in the study of Lipschitz and Wadge degrees, the reverse mathematics of the semilinear ordering principle has not been investigated either.

This is certainly a notable gap in our understanding of the reverse mathematics of infinite games in descriptive set theory, and the present thesis can be seen as a first step in order to fill this gap.

First of all, it should be noted that it is possible to recover some information on the strength of Lipschitz and Wadge determinacy from the known results on Gale-Stewart determinacy. This is because every Lipschitz and Wadge game can be effectively (checkable in the theory $\mathbf{RCA}_0$) reduced to a Gale-Stewart game. However, in the formal context of second order arithmetic, this reduction is done at the price of a significant increase of the payoff set complexity. Roughly speaking, this reduction allows one to infer Lipschitz determinacy for Γ sets from Gale-Stewart determinacy for $\Gamma \wedge (\neg\Gamma)$ sets, and Wadge determinacy for Γ sets from Gale-Stewart determinacy for $\max\{\Pi^0_2, \Gamma \wedge (\neg\Gamma)\}$ sets. This provides us with upper bounds on the strength of Lipschitz and Wadge determinacy for Γ sets. But these bounds needn't be, however, optimal.

This fact justifies the methodology used in this work: to give explicit formalizations of Lipschitz and Wadge games in second order arithmetic, and to formalize *direct* proofs of Lipschitz and Wadge determinacy and the semilinear ordering principle in order to obtain finer results on the strength of these principles.

The formalization of Lipschitz and Wadge games in the language of $\mathbf{Z}_2$ will be accomplished in Chapter 3. We formalize not only the determinacy axioms, but also the semilinear ordering axioms, whose reverse mathematics had not been studied so far. As a result, we introduce a number of new principles in the language of second order arithmetic:

Γ-$\mathbf{Det}^*_{L/W}$: Lipschitz/Wadge determinacy for Γ sets in the Cantor space.

Γ-$\mathbf{Det}_{L/W}$: Lipschitz/Wadge determinacy for Γ sets in the Baire space.

Γ-$\mathbf{SLO}^*_{L/W}$: semilinear ordering principle for Γ sets in the Cantor space.

Γ-$\mathbf{SLO}_{L/W}$: semilinear ordering principle for Γ sets in the Baire space.

Sometimes we also need to consider Lipschitz or Wadge games in which player I's and player II's payoff sets belong to different classes Γ_1 and Γ_2. We will write (Γ_1, Γ_2)-$\mathbf{Det}^*_{L/W}$, (Γ_1, Γ_2)-$\mathbf{Det}_{L/W}$ and so on to denote the corresponding theories.

Let us observe that, in order to formalize the semilinear ordering principle, we do not use the original definition in terms of reducibility by continuous functions. Instead, we use the simpler equivalent definition in terms of winnings strategies given in Wadge's lemma. But this is unessential because we also show in Chapter 3 that Wadge's lemma itself is provable in our base theory $\mathbf{RCA}_0$ (a result of independent interest).

We have studied the strength of these principles both in Cantor space (Chapter 4) and Baire space (Chapter 5). The following table summarizes the main results that we have obtained. Each determinacy principle has been shown to be provable within the subsystem of $\mathbf{Z}_2$ indicated in the table.

Subsystem	Cantor	Baire
$\mathbf{RCA}_0$	Δ^0_1-$\mathbf{Det}^*_W$	
$\mathbf{WKL}_0$	Δ^0_1-$\mathbf{Det}^*_L$, (Δ^0_1, Σ^0_1)-$\mathbf{Det}^*_{L/W}$, (Σ^0_1, Δ^0_1)-$\mathbf{Det}^*_L$	
$\mathbf{ACA}_0$	$(\Sigma^0_1 \cup \Pi^0_1)$-$\mathbf{Det}^*_{L/W}$, $\left(\Sigma^0_1\right)_2$-$\mathbf{Det}^*_{L/W}$	Δ^0_1-$\mathbf{Det}_W$, $(\Sigma^0_1 \cup \Pi^0_1)$-$\mathbf{Det}_W$
$\mathbf{ATR}_0$		Δ^0_1-$\mathbf{Det}_L$, $(\Sigma^0_1 \cup \Pi^0_1) - \mathbf{Det}_L$
Π^1_1-$\mathbf{CA}_0$		$\left(\Sigma^0_1\right)_2 \cup \neg \left(\Sigma^0_1\right)_2$-$\mathbf{Det}_{L/W}$

Where $\left(\Sigma^0_1\right)_2$ stands for the second level of the hierarchy of differences and amounts to $\Sigma^0_1 \wedge \Pi^0_1$.

To each one of these results we can add the corresponding Γ-$\mathbf{SLO}_{L/W}$ or Γ-$\mathbf{SLO}^*_{L/W}$ principle, since it can be proved in $\mathbf{RCA}_0$ that the semilinear ordering principle for each class of sets Γ is a consequence of determinacy for the games whose payoff sets belong to that class Γ.

Of special interest are the results that $\mathbf{ACA}_0$ proves $\left(\Sigma^0_1\right)_2$-$\mathbf{Det}^*_{L/W}$, that $\mathbf{ATR}_0$ proves Σ^0_1-$\mathbf{Det}_{L/W}$ and that Π^1_1-$\mathbf{CA}_0$ proves $\left(\Sigma^0_1\right)_2$-$\mathbf{Det}_{L/W}$. These are salient examples of novel results that cannot be derived from previously known theorems on Gale-Stewart determinacy.

The following natural step is to look for reversals for Lipschitz and Wadge determinacy and for semilinear ordering principle in second order arithmetic. Since Lipschitz and Wadge determinacy is formally weaker than classical Gale-Stewart determinacy, any reversal of this kind is a new result and of great interest. We have been able to obtain two reversals, one for $\mathbf{ACA}_0$ and the other for $\mathbf{ATR}_0$.

Firstly, we have obtained a reversal for $\mathbf{ACA}_0$ in Cantor space. If we add to our result the equivalence $\mathbf{ACA}_0 \equiv \left(\Sigma^0_1\right)_2$-$\mathbf{Det}^*$ proved by T. Nemoto, M. MedSalem, and K. Tanaka ([NMT07]), we have obtained that:

- (Theorem 4.41) The following assertions are pairwise equivalent over $\mathbf{RCA}_0$:

    1. $\mathbf{ACA}_0$.
    2. $\left(\Sigma^0_1\right)_2$-$\mathbf{Det}^*_L$.
    3. $\left(\Sigma^0_1\right)_2$-$\mathbf{SLO}^*_L$.

4. $\left(\Sigma_1^0\right)_2$-**Det***.

This means that within the base system $\mathbf{RCA}_0$ Gale-Stewart determinacy, Lipschitz determinacy, and Lipschitz semilinear ordering principle for $\left(\Sigma_1^0\right)_2$ sets in Cantor space are equivalent principles. Furthermore they all are equivalent to arithmetical comprehension, $\mathbf{ACA}_0$.

Secondly, we have obtained a reversal for $\mathbf{ATR}_0$ in Baire space. Again, taking into account the already known result of T. Nemoto, M. MedSalem, and K. Tanaka ([NMT07]) that $\mathbf{ATR}_0 \equiv \Delta_1^0$-**Det** $\equiv \Sigma_1^0$-**Det**, we have all together:

- (Theorem 5.28) The following assertions are pairwise equivalent over $\mathbf{RCA}_0$:

  1. $\mathbf{ATR}_0$.
  2. $(\Sigma_1^0 \cup \Pi_1^0)$-$\mathbf{Det}_L$.
  3. Σ_1^0-$\mathbf{Det}_L$.
  4. (Δ_1^0, Π_1^0)-$\mathbf{Det}_L$.
  5. Δ_1^0-**Det**.
  6. Σ_1^0-**Det**.

As an intermediate step towards the above result, we have also obtained a second reversal for $\mathbf{ATR}_0$, but this time over the stronger base theory $\mathbf{ACA}_0$. Nevertheless, this second result has the advantage that it is also a reversal for the semilinear ordering principle.

- (Theorem 5.21)The following assertions are pairwise equivalent over $\mathbf{ACA}_0$:

  1. $\mathbf{ATR}_0$.
  2. Δ_1^0-$\mathbf{Det}_L$.
  3. Δ_1^0-$\mathbf{SLO}_L$.

Comparing the recent result of A. Montalbán and R. A. Shore to the earlier result of A. Louveau and Saint Raymond, it is natural to think that there must exist a huge difference between Lipschitz and Gale-Stewart determinacy. Lipschitz determinacy is much weaker than Gale-Stewart determinacy, since full Borel Lipschitz determinacy can be proved in $\mathbf{Z}_2$, whereas Δ_4^0 Gale-Stewart determinacy is already not provable in $\mathbf{Z}_2$. However, the two reversals we have just mentioned suggest that this difference does not reveal itself at the lower stages of the hierarchy of sets. In fact, we have proven that Gale-Stewart and Lipschitz determinacy in Cantor space are equivalent (over $\mathbf{RCA}_0$) when restricted to differences of closed sets. Similarly, we have proven that Gale-Stewart and Lipschitz determinacy in the Baire space are equivalent (over $\mathbf{ACA}_0$) when restricted to clopen or closed sets.

Although we have not obtained a reversal for $\mathbf{WKL_0}$, it seems plausible to aim at proving an equivalence between $\mathbf{WKL_0}$ and Σ^0_1-$\mathbf{Det}^*_L$. However, we have only been able to obtain Σ^0_1-$\mathbf{Det}^*_{L/W}$ from $\mathbf{WKL_0}$ plus an assertion stating that every closed set is either true closed (i.e. closed and not open) or clopen. Whether this dichotomy property is provable within $\mathbf{WKL_0}$ is left pending. In addition, we have considered several other natural assertions which imply this dichotomy property. Interestingly, all these assertions have turned out to be equivalent to $\mathbf{ACA_0}$. Thus, as a by-product, this investigation has yielded up several reversals for $\mathbf{ACA_0}$. We collect them in the next theorem. The respective proofs can be found in Chapter 4.

- (Theorem 4.41, Proposition 4.30, Proposition 4.7) The following assertions are pairwise equivalent over $\mathbf{RCA_0}$:

  1. $\mathbf{ACA_0}$.
  2. $\left(\Sigma^0_1\right)_2$-$\mathbf{Det}^*_L$.
  3. $\left(\Sigma^0_1\right)_2$-$\mathbf{SLO}^*_L$.
  4. Every binary tree can be pruned.
  5. Weak König Lemma for Σ^0_1 trees.
  6. Weak Radó selection lemma.
  7. The scheme of $\Sigma^0_1 \wedge \Pi^0_1$ choice.

In Baire space, $(\Sigma^0_1 \cup \Pi^0_1)$-$\mathbf{Det}_W$ can be proved in $\mathbf{ACA_0}$ and $\left(\Sigma^0_1\right)_2 \cup \neg\left(\Sigma^0_1\right)_2$–$\mathbf{Det}_L$ can be obtained from Π^1_1-$\mathbf{CA_0}$, but we do not know if they are strong enough to obtain a reversal for $\mathbf{ACA_0}$ and Π^1_1-$\mathbf{CA_0}$, respectively. Thus, a reversal for Π^1_1-$\mathbf{CA_0}$ in terms of Lipschitz determinacy is also left pending.

The results obtained in the thesis are far from exhausting the subject of calibrating the strength of Lipschitz and Wadge determinacy, and of Lipschitz and Wadge semilinear ordering principles, in terms of subsystems of second order arithmetic. Nonetheless we think that they represent an interesting first step towards a better understanding of the subject.

We conclude this introduction by giving a brief summary of the structure of the thesis. The work is divided into the present Introduction and Chapters 1, 2, 3, 4, 5, and 6.

In Chapter 1 we present necessary background material concerning Baire and Cantor spaces, infinite games (classical Gale-Stewart, and Lipschitz and Wadge games), the principle of semilinear order, and the basic definitions and results of descriptive set theory and of second order arithmetic. We also introduce the basic tool of trees and rank functions defined on well-founded trees, which will be needed in the analysis developed in the following chapters.

In Chapter 2 we give direct proofs of the determinacy of Lipschitz and Wadge games for the first five levels of the Wadge hierarchy, fulfilling this way the first goal of the thesis.

In Chapter 3 we formalize Lipschitz and Wadge games in the language of second order arithmetic. Underpinned by this formalization we prove several basic facts within $\mathbf{RCA}_0$ concerning the general relations between Lipschitz and Wadge determinacy, and Lipschitz and Wadge semilinear ordering principles. This chapter also contains our proof of Wadge's lemma within $\mathbf{RCA}_0$.

Chapter 4 investigates the reverse mathematics of Lipschitz and Wadge determinacy in the Cantor space. We show that our analysis of games in the Cantor space can be carried out within the subsystems $\mathbf{WKL}_0$ and $\mathbf{ACA}_0$. This chapter also contains one of the main results of the thesis. Namely, we show that within the base system $\mathbf{RCA}_0$, Lipschitz semilinear ordering principle and Lipschitz determinacy in Cantor space are as strong as the subsystem of second order arithmetic $\mathbf{ACA}_0$. This fulfills the second and third goals of the thesis for differences of closed sets in Cantor space.

Chapter 5 is devoted to Lipschitz and Wadge determinacy in Baire space. We show that our analysis of the games in the Baire space can be carried out within the subsystems $\mathbf{ATR}_0$ and Π^1_1-$\mathbf{CA}_0$. This chapter contains the second main result of the thesis. Here we show that Lipschitz determinacy for closed sets and $\mathbf{ATR}_0$ are equivalent over base system $\mathbf{RCA}_0$, fulfilling in this way the second goal of the thesis for closed sets in the Baire space.

Finally, Chapter 6 contains some concluding remarks and some open problems.

Chapter 1

Preliminaries

Throughout this thesis the analysis of Lipschitz and Wadge games will be held within one of the two main topological spaces of Descriptive Set Theory, the Cantor space or the Baire space. Thus in the present chapter we begin by characterizing the topology of both Baire and Cantor spaces and by pointing out their connection with other topological spaces. It will become clear that these spaces are salient examples of Polish spaces. Additional information on the concepts and the proofs of the assertions stated in the first section of this chapter can be found in [Sri98].

In Section 2 we succinctly describe the main classifications of subsets of Polish spaces used in Descriptive Set Theory, the Borel and the projective hierarchies. The first levels of the Hausdorff hierarchy will be also discussed in detail in later sections.

In Section 3 we define the concept of an infinite nullsum two person game with perfect memory and perfect information in its classical form, the Gale-Stewart games. The assumption that in all these games one of the players has a strategy which allows him to win the game whatever his opponent does is the content of the so called axiom of determinacy from Descriptive Set Theory. The detailed proofs of the results mentioned in this and in the previous section can be found in the classical work of A. S. Kechris [Kec95].

In Section 4 we introduce Lipschitz and Wadge games. We define the Lipschitz and Wadge reducibility relations and degrees, and state the well-known Wadge's Lemma.

We detail the structure of Lipschitz and Wadge hierarchies of degrees in Section 5. The main features of these hierarchies were obtained by Wadge assuming the axiom of determinacy. They are contained in the clauses of Theorem 1.10 at the end of Section 5.

Section 6 is devoted to the principle of semilinear order. This principle was introduced by Wadge as an immediate consequence of the axiom of determinacy. Gradually it became clear that the principle of semilinear order shares with the axiom of determinacy many of its consequences and currently there is even the conjecture that in an appropriate realm of set theory these assertions are equivalent. Further information on the subject of this and the previous two sections can be found in [WWW83] and [AA03].

In Section 7 we explain the relation between Lipschitz hierarchy and classical Hausdorff´s hierarchy of differences.

In Section 8 we introduce some basic notation and define the combinatorial tools we will need in Chapter 2. This includes trees and rank functions defined on trees. Proofs of the propositions formulated in this section can be found in [Kec95].

In Section 9 we introduce the language of second order arithmetic and describe some of the subsystems of second order arithmetic that will be used in later chapters. More detailed information concerning the subsystems of second order arithmetic described in this section is contained in the classical book of S. G. Simpson [Smp99].

In Sections 1 to 8 we will be using the language of set theory and our meta-theory will be **ZF**. Whenever we need an additional principle we will mention it explicitly.

1.1 Baire and Cantor spaces

Let us introduce some notation concerning sequences. For any nonempty set X and $n \in \omega$, we denote by X^n the set of sequences $s = \langle s(0), ..., s(n-1) \rangle$ of *length* n from X. The length of a finite sequence s is denoted by $|s|$. If $|s| = 0$ then s is the *empty sequence*, i.e. $|s| = \langle \rangle$. If $k \leq n = |s|$, then we let

$$s\,[k] = \langle s(0), ..., s(k-1) \rangle \,.$$

Obviously $s\,[0] = \langle \rangle$. Let

$$X^{<\omega} = \bigcup_{n \in \omega} X^n,$$

i.e. $X^{<\omega}$ is the *set of all finite sequences of elements of* X. If $s, t \in X^{<\omega}$, we say that s is an *initial segment of* t (or t is an *extension of* s), if $s = t[k]$ for some $k \leq |t|$. We denote this by $s \subseteq t$. Clearly $\langle \rangle \subseteq t$ for any t. We abbreviate $s \subseteq t \wedge s \neq t$ by $s \subset t$. Moreover, for any $s, t \in X^{<\omega}$ with $|s| = k$ and $|t| = l$, the *concatenation of* s *and* t is the sequence

$$s * t = \langle s(0), ..., s(k-1), t(0), ..., t(l-1) \rangle \,.$$

Let X^ω denote the *set of all infinite sequences of elements of* X. For any infinite sequence $x \in X^\omega$ and all $k \in \omega$ we also define

$$x\,[k] = \langle x(0), ..., x(k-1) \rangle$$

Obviously $x[0] = \langle \rangle$. If $s \in X^{<\omega}$ and $x \in X^\omega$ we say that s is an *initial segment of* x (or x is an *extension of* s), if $s = x[|s|]$. In this case we write $s \subset x$ or sometimes $s \subseteq x$. Clearly $\langle \rangle \subset x$ for any $x \in X^\omega$.

Baire and Cantor spaces are topological spaces. Let X be either $\{0,1\}$ or ω. For each $s \in X^{<\omega}$ we define the set N_s of all infinite sequences from X^ω with the common initial segment s putting

$$N_s = \{x \in X^\omega : s \subseteq x\}.$$

The set $\{N_s : s \in X^{<\omega}\}$ is a basis for a topology on X^ω. Endowed with this topology the space $\{0,1\}^\omega$ or 2^ω is called *Cantor space* and the space ω^ω is called *Baire space*. It is also usual to write $\mathcal{N}$ to denote the Baire space.

Observe that for any $s, t \in X^{<\omega}$

$$N_s \cap N_t \in \{\emptyset, N_s, N_t\}$$

and that each open set N_s is also closed. Since they are simultaneously open and closed these sets are called *clopen*.

Next we mention some important properties of the topology of Cantor and Baire spaces.

Lemma 1.1 *Let X be either $\{0,1\}$ or ω. Then:*

1. *If X is endowed with the discrete topology, i.e. every subset of X is open, then the above mentioned topology of X^ω coincides with the product topology $\prod_{i\in\omega} X_i$, where $X_i = X$ for every $i \in \omega$.*
2. *The topology of X^ω is induced by the metric $d : X^\omega \times X^\omega \to \mathbb{Q}$ defined for all $x, y \in X^\omega$ by*
$$d(x;y) = \begin{cases} 2^{-\min\{k\in\omega : x(k)\neq y(k)\}-1}, & \text{if } x \neq y \\ 0, & \text{otherwise} \end{cases}$$
3. *The space X^ω is complete, i.e. every Cauchy sequence in X^ω converges to an element of X^ω.*
4. *The space X^ω is 0-dimensional, i.e. X^ω has a basis consisting of clopen sets.*
5. *The space X^ω is second countable, i.e. X^ω has a countable basis.*
6. *The space X^ω is separable, i.e. X^ω contains a countable dense subset.*
7. *The space X^ω is totally disconnected, i.e. the empty set and one-point sets are the only subsets that cannot be represented as the union of two or more disjoint nonempty open subsets.*
8. *The space X^ω is homeomorphic to the product of any finite number of copies of itself.*

Moreover, the Cantor space is *compact* while the Baire space is not. The open cover $\{N_{\langle n\rangle} : n \in \omega\}$ has no finite subcover.

A *Polish space* is a topological space which is separable and completely metrizable. Polish spaces are a very large family of topological spaces. However, topological problems in general Polish spaces can be transferred to the Baire space ω^ω or the Cantor space 2^ω as it becomes clear from the following two theorems ([Sri98], section 2.6):

Theorem 1.2 *Every compact metric space is a continuous image of 2^ω.*

Theorem 1.3 *Every Polish space is a continuous image of ω^ω.*

The Baire space is homeomorphic to the set of irrational numbers when they are given the subspace topology inherited from the real line (a homeomorphism can be constructed using continued fractions). In Descriptive Set Theory the set of real numbers is commonly identified with the Baire space.

1.2 Borel and projective hierarchies

The Borel and the projective hierarchies are classifications of classes of sets, called *pointclasses*, in semilinear orders of increasing complexity. These pointclasses are called *boldface* classes because they are closed under continuous preimages. The hierarchies can be defined for each Polish space.

Let X be a Polish space. Then the class $\mathbb{B}(X)$ of the Borel sets is the smallest collection of subsets of X that contains all open sets and is closed under complement and countable unions. This definition of the Borel sets is the simplest, but to put them in a hierarchy of boldface pointclasses we give a more constructive definition by simultaneous transfinite recursion:

$$\begin{aligned}
&\mathbf{\Sigma}^0_1(X) = \{G \subseteq X : G \text{ is open}\},\\
&\mathbf{\Pi}^0_1(X) = \{F \subseteq X : F \text{ is closed}\},\\
&\mathbf{\Sigma}^0_\alpha(X) = \{\textstyle\bigcup_{n\in\omega} A_n : A_n \in \mathbf{\Pi}^0_{\beta_n}(X), 0 < \beta_n < \alpha\}, \text{ for } 1 < \alpha,\\
&\mathbf{\Pi}^0_\alpha(X) = \{\textstyle\bigcap_{n\in\omega} A_n : A_n \in \mathbf{\Sigma}^0_{\beta_n}(X), 0 < \beta_n < \alpha\}, \text{ for } 1 < \alpha,\\
&\mathbf{\Delta}^0_\alpha(X) = \mathbf{\Sigma}^0_\alpha(X) \cap \mathbf{\Pi}^0_\alpha(X), \text{ for } 1 \leq \alpha.
\end{aligned}$$

The slightly complex third and fourth clauses in the definition are needed only at limit ordinals; for successor ordinals we could simply say that

$$\begin{aligned}
\mathbf{\Sigma}^0_{\alpha+1}(X) &= \{\textstyle\bigcup_{n\in\omega} A_n : A_n \in \mathbf{\Pi}^0_\alpha(X)\},\\
\mathbf{\Pi}^0_{\alpha+1}(X) &= \{\textstyle\bigcap_{n\in\omega} A_n : A_n \in \mathbf{\Sigma}^0_\alpha(X)\}.
\end{aligned}$$

If X is a countable Polish space, then all subsets of X and their complements are countable unions of singletons, so, for all $A \subseteq X$, $A \in \mathbf{\Delta}^0_2(X)$. The Borel sets become interesting when X is an uncountable Polish space. In this case there are subsets of X that are not Borel and the Borel sets form a hierarchy which does not collapse.

Remark 1.4 *Since* $\mathbb{R} - \mathbb{I}rr = \mathbb{Q} = \bigcup_{x\in\mathbb{Q}}\{x\}$ *is a countable union of closed sets and* $\mathbb{I}rr \cong \mathcal{N}$ *the classes of Borel sets* $\mathbb{B}(\mathbb{R})$, $\mathbb{B}(\mathbb{I}rr)$, *and* $\mathbb{B}(\mathcal{N})$ *are isomorphic for* $\alpha > 2$.

Proposition 1.5 *Let* X *be an uncountable Polish space. Then for every* $1 < \alpha < \omega_1$,

$$\mathbf{\Sigma}^0_\alpha(X) \neq \mathbf{\Pi}^0_\alpha(X),\ \mathbf{\Delta}^0_\alpha(X) \subsetneqq \mathbf{\Sigma}^0_\alpha(X) \subsetneqq \mathbf{\Delta}^0_{\alpha+1}(X) \quad \textit{and} \quad \mathbf{\Delta}^0_\alpha(X) \subsetneqq \mathbf{\Pi}^0_\alpha(X) \subsetneqq \mathbf{\Delta}^0_{\alpha+1}(X).$$

Assuming the axiom of countable choices, $\mathbf{AC}_\omega$ it can be proved that in the above recursive way we only need ω_1 steps to produce all Borel sets.

Proposition 1.6 ($\mathbf{AC}_\omega$) *Let* X *be metrizable and infinite. Then:*

$$\mathbb{B}(X) = \bigcup_{0<\alpha<\omega_1} \mathbf{\Sigma}^0_\alpha(X)$$

Thus, this is the usual picture of the Borel hierarchy for uncountable Polish spaces, where the arrows represent strict inclusions:

$$\underbrace{\begin{array}{cccccccccccc} & & \mathbf{\Sigma}^0_1 & & & & \mathbf{\Sigma}^0_2 & & & & \mathbf{\Sigma}^0_3 & \\ & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow \\ \mathbf{\Delta}^0_1 & & & & \mathbf{\Delta}^0_2 & & & & \mathbf{\Delta}^0_3 & & & \\ & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow \\ & & \mathbf{\Pi}^0_1 & & & & \mathbf{\Pi}^0_2 & & & & \mathbf{\Pi}^0_3 & \end{array}}_{\omega_1} \quad \cdots = \mathbb{B}$$

There are projections (continuous images) of Borel sets which are not Borel sets. The hierarchy of projective sets, which can be also defined for each Polish space, is based on this fact.

Let X and Y be topological spaces and $B \subseteq X \times Y$. Then the set

$$p(B) = \{x \in X : \exists\, y \in Y \ \ (x, y) \in B\},$$

is said to be the *projection* of B on X.

Let X be a Polish space. A subset $A \subseteq X$ is called *analytic* if there is a Borel set $B \in \mathbb{B}(X \times X)$ such that $A = p(B)$, and is called *coanalytic* if its complement A^c is analytic.

As we did with the Borel sets we define the projective sets by simultaneous recursion. For each $n \in \omega$, with $n > 0$, we define

$$\begin{array}{l} \mathbf{\Sigma}^1_1(X) = \{A \subseteq X : A \text{ is analytic}\}, \\ \mathbf{\Pi}^1_n(X) = \{A \subseteq X : A^c \in \mathbf{\Sigma}^1_n(X)\}, \\ \mathbf{\Sigma}^1_{n+1}(X) = \{A \subseteq X : A = p(B) \text{ and } B \in \mathbf{\Pi}^1_n(X \times X)\}, \\ \mathbf{\Delta}^1_n(X) = \mathbf{\Sigma}^1_n(X) \cap \mathbf{\Pi}^1_n(X). \end{array}$$

Proposition 1.7 ($\mathbf{AC}_\omega$) *Let X be an uncountable Polish space. Then for every $n \geq 1$,*

$$\mathbf{\Sigma}^1_n(X) \neq \mathbf{\Pi}^1_n(X),\ \mathbf{\Delta}^1_n(X) \subsetneqq \mathbf{\Sigma}^1_n(X) \subsetneqq \mathbf{\Delta}^1_{n+1}(X) \quad \textit{and} \quad \mathbf{\Delta}^1_n(X) \subsetneqq \mathbf{\Pi}^1_n(X) \subsetneqq \mathbf{\Delta}^1_{n+1}(X).$$

For uncountable Polish spaces the class of the analytic sets is strictly larger than the class of Borel sets. To see this it suffices to prove that $\mathbf{\Delta}^1_1(X) = \mathbb{B}(X)$. This result is called the Suslin theorem.

Theorem 1.8 ($\mathbf{AC}_\omega$) *Let X be an uncountable Polish space. Then for all $A \subseteq X$,*

A is Borel iff A is analytic and coanalytic.

In particular, $\mathbb{B}(X) = \mathbf{\Delta}^1_1(X)$.

The usual picture of the projective hierarchy for uncountable Polish spaces, where the arrows represent strict inclusions, is the following:

$$\underbrace{\begin{array}{ccccccccccccc}
 & & \boldsymbol{\Sigma}^1_1 & & & & \boldsymbol{\Sigma}^1_2 & & & & \boldsymbol{\Sigma}^1_3 & & \\
 & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & \\
\mathbb{B} = \boldsymbol{\Delta}^1_1 & & & & \boldsymbol{\Delta}^1_2 & & & & \boldsymbol{\Delta}^1_3 & & & & \cdots \\
 & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & \\
 & & \boldsymbol{\Pi}^1_1 & & & & \boldsymbol{\Pi}^1_2 & & & & \boldsymbol{\Pi}^1_3 & &
\end{array}}_{\omega}$$

Assuming the full axiom of choice it is provable that all Borel sets and all analytic sets have the three *regularity properties*. Namely, in any Polish space X if $A \subseteq X$ is a Borel or an analytic set:

(-) A is *Lebesgue measurable*, i.e. there are Borel sets B and B' such that $(A \setminus B) \cup (B \setminus A) \subseteq B'$ and $m(B') = 0$ where m stands for a Lebesgue measure.

(-) A has the *perfect set property*, i.e. A is countable or else has a perfect subset (a nonempty closed set without isolated points).

(-) A has the *property of Baire*, i.e. for some open set G, $(A \setminus G) \cup (G \setminus A)$ is a union of countably many nowhere dense sets (sets whose closure has an empty interior).

This is the best result of this kind that can be proved within **ZFC**. In fact, there are models of **ZFC** (to wit, $V = L$) where the regularity properties fail for $\boldsymbol{\Delta}^1_2$ sets. In addition, the perfect set property is already independent of **ZFC** for $\boldsymbol{\Pi}^1_1$ sets. (See T. Jech's book "Set Theory" (2nd Ed.), Springer, pp. 527-529 for further information.)

1.3 Gale-Stewart games and the axiom of determinacy

Let X be either $\{0, 1\}$ or ω, and let $A \subseteq X^\omega$. The *Gale-Stewart game* for A, denoted $G(A)$, is defined as follows: Two players, say player I (male) and player II (female), alternately choose natural numbers $f(i)$ and $g(i)$, respectively, in X.

I	$f(0)$		$f(1)$		$f(2)$		$\ldots$
II		$g(0)$		$g(1)$		$g(2)$	$\ldots$

After ω turns, player I has produced a sequence $f : \omega \to X$ of elements of X, and player II has produced a sequence $g : \omega \to X$ of elements of X. The resulting play is the sequence $f \otimes g : \omega \to X$ given by $f \otimes g = \langle f(0), g(0), f(1), g(1), \ldots \rangle$, which collects the alternating moves of players I and II.

Player I wins a play of the game $G(A)$ if and only if

$$f \otimes g \in A.$$

Otherwise player II wins that play.

Gale-Stewart games are infinite nullsum two person games with perfect memory and perfect information. Infinite because players play infinite times; nullsum because only one of the players can win (and there is no ties); two person because there are only two players; with perfect memory because both players know and do not forget all previous moves; and with perfect information because both players are able to calculate all possible moves starting at any given position.

A strategy for player I in the game $G(A)$ is a function assigning an element of X to every sequence from X of even length. (We think of this as telling player I which move to make at any finite stage on the game.) Similarly, a strategy for player II in the game $G(A)$ is a function assigning an element of X to every sequence from X of odd length.

A strategy for player I is called winning if player I wins the game as long as he plays following it, no matter what his opponent plays. Similarly, a strategy for player II is called winning if player II wins the game as long as she plays following it, no matter what her opponent plays.

A game $G(A)$ (or the set A) is said to be determined if either player I or player II has a winning strategy in $G(A)$. The full axiom of choice, **AC**, implies that all Borel games, i.e. games $G(A)$ where A is a Borel set, are determined. However, to obtain the determinacy of slightly more complex sets, for example analytic sets, we already need to assume other type of axioms, namely axioms stipulating the existence of large cardinals.

The axiom of determinacy **AD** (introduced by Jan Mycielski and Hugo Steinhaus in 1962) is the principle stating that all subsets of real numbers are determined. Thus,

$$\mathbf{AD} = \text{For all } A \subseteq \omega^\omega, \text{ the game } G(A) \text{ is determined.}$$

In presence of **AD** no uncountable subset of ω^ω is well-ordered and so **AD** contradicts **AC**. However **AD** has other interesting consequences. Namely, the axiom of countable choices, $\mathbf{AC}_\omega(\omega^\omega)$, is a consequence of **AD** and the above mentioned three regularity properties (Lebesgue measurability, the perfect set property and the property of Baire) are also derivable from **AD** for every subset of the reals without restrictions to specific pointclasses.

1.4 Lipschitz and Wadge games, and Wadge determinacy

Let X be either $\{0,1\}$ or ω, and let $A, B \subseteq X^\omega$. We will use f's and g's to refer to members of A and B, respectively. In other words, f and g stand for functions from ω to X.

The *Lipschitz game* for A and B, denoted $G_L(A,B)$, is the infinite nullsum two person game on X with perfect information and perfect memory where player I (male) and player II (female) take turns and play natural numbers $f(i)$ and $g(i)$, respectively. Player I plays first:

I	$f(0)$		$f(1)$		$f(2)$		$\ldots$	
II		$g(0)$		$g(1)$		$g(2)$		$\ldots$

Player II wins the game $G_L(A,B)$ if

$$f \in A \text{ iff } g \in B,$$

If the equivalence does not hold, player I wins.

Next we define the concepts of a strategy for player I and for player II. Let us denote by $\mathrm{Seq}_{\mathrm{even}}$ the subset of $X^{<\omega}$ containing all finite sequences of even length, and by $\mathrm{Seq}_{\mathrm{odd}}$ the subset of $X^{<\omega}$ containing all finite sequences of odd length. Then a *strategy for player* I in a Lipschitz game $G_L(A,B)$ is a function assigning a member of X to every finite sequence of even length. Analogously, a *strategy for player* II in Lipschitz game $G_L(A,B)$ is a function assigning a member of X to every finite sequence of odd length. That is to say, a strategy for player I in the game $G_L(A,B)$ is a function $\mathcal{E}_{\mathrm{I}} : \mathrm{Seq}_{\mathrm{even}} \to X$ and a strategy for player II in the game $G_L(A,B)$ is a function $\mathcal{E}_{\mathrm{II}} : \mathrm{Seq}_{\mathrm{odd}} \to X$.

Thus if player I is pitting a strategy $\mathcal{E}_{\mathrm{I}} : \mathrm{Seq}_{\mathrm{even}} \to X$ against the moves produced by player II we have

I	$f(0) = \mathcal{E}_{\mathrm{I}}(\langle\rangle)$		$f(1) = \mathcal{E}_{\mathrm{I}}(\langle f(0), g(0)\rangle)$		$f(2) = \mathcal{E}_{\mathrm{I}}(\langle f(0), g(0), f(1), g(1)\rangle)$	$\ldots$
II		$g(0)$		$g(1)$		$\ldots$

for any $g \in X^{\omega}$. Then $\mathcal{E}_{\mathrm{I}} \otimes g$ stands for the complete play of $G_L(A,B)$ where we distinguish the play of player I from the play of player II putting

$$f(n) = (\mathcal{E}_{\mathrm{I}} \otimes g)(2n) \quad \text{and} \quad g(n) = (\mathcal{E}_{\mathrm{I}} \otimes g)(2n+1),$$

respectively, for each $n \in \omega$. In order to keep the reference to the strategy that player I is using we will denote the play of player I by $\mathcal{E}_{\mathrm{I}} \otimes^{\mathrm{I}} g$ and the play of player II by $\mathcal{E}_{\mathrm{I}} \otimes^{\mathrm{II}} g$. Thus for every $n \in \omega$, we have

$$f(n) = (\mathcal{E}_{\mathrm{I}} \otimes^{\mathrm{I}} g)(n) \quad \text{and} \quad g(n) = (\mathcal{E}_{\mathrm{I}} \otimes^{\mathrm{II}} g)(n).$$

An important part of the proofs of determinacy in the forthcoming chapters is the definition of a strategy for player I. To do that we start by specifying $\mathcal{E}_{\mathrm{I}}(\langle\rangle)$. Then we take an arbitrary partial play $s \otimes t$ of even length and explain how to obtain the value of the strategy $\mathcal{E}_{\mathrm{I}} : \mathrm{Seq}_{\mathrm{even}} \to X$ for that argument.

Now we explain what a partial play is. Let $s, t \in X^{<\omega}$ and suppose player I is using strategy $\mathcal{E}_{\mathrm{I}}$. Then $s \otimes t$ is a *partial play of length* $2j$ where $|s| = |t| = j$ if

$$t = \langle (t)_0, (t)_1, ..., (t)_{j-1} \rangle$$

for any $(t)_0, (t)_1, ..., (t)_{j-1} \in X$ and

$$s = \langle (s)_0, (s)_1, ..., (s)_{j-1} \rangle$$

where $(s)_0 = \mathcal{E}_\mathrm{I}(\langle\rangle)$, $(s)_1 = \mathcal{E}_\mathrm{I}(\langle (s)_0, (t)_0 \rangle)$, ..., and

$$(s)_{j-1} = \mathcal{E}_\mathrm{I}(\langle (s)_0, (t)_0, (s)_1, (t)_1, ..., (s)_{j-2}, (t)_{j-2} \rangle).$$

On the other hand, if player II is using strategy $\mathcal{E}_\mathrm{II} : \mathrm{Seq}_\mathrm{odd} \to X$, for any $f \in X^\omega$ we have

$$\begin{array}{c|lll} \mathbf{I} & f(0) & f(1) & \cdots \\ \hline \mathbf{II} & g(0) = \mathcal{E}_\mathrm{II}(\langle f(0) \rangle) & g(1) = \mathcal{E}_\mathrm{II}(\langle f(0), g(0), f(1) \rangle) & \cdots \end{array}$$

The complete play of $G_L(A, B)$ is denoted $f \otimes \mathcal{E}_\mathrm{II}$. Again, we distinguish the play f of player I, $f(n) = (f \otimes \mathcal{E}_\mathrm{II})(2n)$, from the play of player II, $g(n) = (f \otimes \mathcal{E}_\mathrm{II})(2n+1)$. In order to keep the reference to the strategy player II is using we will denote the play of player I by $f \otimes^\mathrm{I} \mathcal{E}_\mathrm{II}$ and the play of player II by $f \otimes^\mathrm{II} \mathcal{E}_\mathrm{II}$. Thus for every $n \in \omega$,

$$f(n) = (f \otimes^\mathrm{I} \mathcal{E}_\mathrm{II})(n) \ \text{ and } g(n) = (f \otimes^\mathrm{II} \mathcal{E}_\mathrm{II})(n).$$

In order to define a strategy for player II we will take an arbitrary partial play $s \otimes t$ of odd length and explain how to obtain the value of the strategy $\mathcal{E}_\mathrm{II} : \mathrm{Seq}_\mathrm{odd} \to X$ for that partial play. Let $s, t \in X^{<\omega}$ and suppose player II is using strategy $\mathcal{E}_\mathrm{II}$. Then $s \otimes t$ is *a partial play of length* $2j+1$ where $|s| = j+1$ and $|t| = j$ if

$$s = \langle (s)_0, (s)_1, ..., (s)_j \rangle$$

for any $(s)_0, (s)_1, ..., (s)_j \in \{0, 1\}$ and

$$t = \langle (t)_0, (t)_1, ..., (t)_{j-1} \rangle$$

with $(t)_0 = \mathcal{E}_\mathrm{II}(\langle (s)_0 \rangle)$, $(t)_1 = \mathcal{E}_\mathrm{II}(\langle (s)_0, (t)_0, (s)_1 \rangle)$, ..., and

$$(t)_{j-1} = \mathcal{E}_\mathrm{II}(\langle (s)_0, (t)_0, (s)_1, ..., (t)_{j-2}, (s)_{j-1} \rangle).$$

A strategy for player I, $\mathcal{E}_\mathrm{I}$, is called a *winning strategy for player* I if player I wins the game as long as he plays following it, no matter what his opponent plays. Thus, $\mathcal{E}_\mathrm{I}$ is a winning strategy for player I in $G_L(A, B)$ if for every $g \in X^\omega$,

$$\mathcal{E}_\mathrm{I} \otimes^\mathrm{I} g \in A^c \ \text{ iff } g \in B,$$

where A^c stands for the complement of A in X^ω, i.e. $A^c = X^\omega - A$. Similarly, a strategy for player II, $\mathcal{E}_\mathrm{II}$, is called a *winning strategy for player* II if player II wins the game as long as she plays following it, no matter what player I plays. Thus, $\mathcal{E}_\mathrm{II}$ is a winning strategy for player II in $G_L(A, B)$ if for every $f \in X^\omega$,

$$f \in A \ \text{ iff } \ f \otimes^\mathrm{II} \mathcal{E}_\mathrm{II} \in B.$$

Finally, we say that a Lipschitz game $G_L(A, B)$ is *determined* if either player I or player II has a winning strategy in $G_L(A, B)$.

The *Wadge game* for A and B, denoted $G_W(A,B)$, is the variant of the Lipschitz game where player II is allowed to pass at any round, but she must play infinitely often otherwise she loses:

I	$f(0)$		$f(1)$		$\ldots$	$f(k)$			$f(k+1)$		$\ldots$	$f(l)$		$\ldots$
II		$\mathbf{p}$		$\mathbf{p}\cdots\mathbf{p}$			$g(0)$	$\mathbf{p}$		$\mathbf{p}\cdots\mathbf{p}$			$g(1)$	$\ldots$

The $\mathbf{p}$'s denote the moves in which player II passes, i.e. does not play. Thus the moves of player II in a Wadge game are the terms of a sequence $g' \in (X \cup \{\mathbf{p}\})^\omega$. The final play of player II, however, is the sequence $g \in X^\omega$ such that for every $n \in \omega$, $g(n) = g'(k_n)$, where

$$
\begin{aligned}
k_0 &= \mu i \ \left[g'(i) \neq \mathbf{p}\right] \\
k_{n+1} &= \mu i \ \left[k_n < i \wedge g'(i) \neq \mathbf{p}\right]
\end{aligned}
$$

So the final sequence built by player II, g, disregards all $\mathbf{p}$'s. As before player II wins the game $G_W(A,B)$ if

$$f \in A \ \text{ iff } \ g \in B,$$

otherwise player I wins. The notions of a strategy and a winning strategy for a Wadge game are defined similarly.

The principle stating that all Lipschitz games are determined is denoted $\mathbf{AD}_L$, i.e.

$$\mathbf{AD}_L = \ \text{For all } A, B \subseteq \omega^\omega \text{, the game } G_L(A,B) \text{ is determined.}$$

Analogously we denote by $\mathbf{AD}_W$ the assertion stating that all Wadge games are determined, i.e.

$$\mathbf{AD}_W = \ \text{For all } A, B \subseteq \omega^\omega \text{, the game } G_W(A,B) \text{ is determined.}$$

Both $\mathbf{AD}_L$ and $\mathbf{AD}_W$ are consequences of the more general $\mathbf{AD}$. There is the conjecture, probably due to R. Soloway, that assuming $\mathbf{ZF}$ plus the constructibility axiom $\mathbf{V} = \mathbf{L}(\mathbb{R})$ we have

$$\mathbf{AD} \Longleftrightarrow \mathbf{AD}_W.$$

However, there is no known canonical way to reduce an arbitrary Gale-Stewart game to a Wadge game and an indirect proof was never obtained, so the problem remains open to this day (see [AA03], p. 165).

Lipschitz and Wadge games provide a substantial help in studying the reducibility relation based on the topological notion of inverse image by a continuous function. A function $F : X^\omega \to X^\omega$ is said to be *continuous* if for all $f \in X^\omega$

$$\forall n \in \omega \exists m \in \omega \ F(N_{f[m]}) \subseteq N_{F(f)[n]}.$$

For $A, B \subseteq X^\omega$, we say that A is *Wadge reducible* to B, denoted $A \leq_W B$, if there is a continuous function $F : X^\omega \longrightarrow X^\omega$ such that

$$f \in A \text{ iff } F(f) \in B$$

for all $x \in X^\omega$. It is not hard to see that the relation $\leq_W$ is a preorder, i.e. reflexive and transitive on the class of subsets of X^ω. $A <_W B$ stands for $A \leq_W B$ and $B \nleq_W A$.

It is immediate from the definition that $A \leq_W B$ iff $A^c \leq_W B^c$. A subset A is called *Wadge self-dual* if $A \leq_W A^c$ and is called *Wadge non-self-dual* otherwise. $A \equiv_W B$ means $A \leq_W B$ and $B \leq_W A$. Note that $A \leq_W A^c$ implies $A \equiv_W A^c$, so that A being self-dual actually means $A \equiv_W A^c$. The *Wadge degree* of A is the equivalence class $[A]_W = \{B : B \equiv_W A\}$, and the *relation* $\preceq_W$ defined on these equivalence classes is a partial order induced by the preorder $\leq_W$. The above definitions can be extended to the Wadge degrees. $[A]_W \prec_W [B]_W$ stands for $[A]_W \preceq_W [B]_W$ and $[B]_W \npreceq_W [A]_W$. $[A]_W$ is the dual of $[A^c]_W$, and if $[A]_W \preceq_W [A^c]_W$ the degree $[A]_W$ is called *Wadge self-dual degree.* Note that if $[A]_W$ is self-dual, we actually have $[A]_W = [A^c]_W$. On the other hand if $[A]_W \npreceq_W [A^c]_W$ then $[A]_W \neq [A^c]_W$ and $\{[A]_W, [A^c]_W\}$ form a pair of Wadge degrees called a *Wadge non-self-dual pair.*

A function $F : X^\omega \longrightarrow X^\omega$ is said to be *Lipschitz* if there is a constant $c \leq 1$ such that for all $f, g \in X^\omega$

$$d(F(f), F(g)) \leq c \cdot d(f, g).$$

If for $A, B \subseteq X^\omega$ there exists a Lipschitz function $F : X^\omega \longrightarrow X^\omega$ such that

$$f \in A \text{ iff } F(f) \in B$$

then we say that A is *Lipschitz reducible* to B and denote this by writing $A \leq_L B$. The notions $A <_L B$, $A \equiv_L B$, $[A]_L$, and $\preceq_L$ are defined similarly using $\leq_L$ instead of $\leq_W$. Obviously, $A \leq_L B$ implies $A \leq_W B$.

The basic tool in the study of Lipschitz and Wadge degrees is the definition of the relations $\leq_L$ and $\leq_W$ in terms of Lipschitz and Wadge games, respectively. Continuous functions from X^ω to X^ω can be determined by monotone functions from $X^{<\omega}$ to $X^{<\omega}$ which, in turn, can be identified with strategies for Lipschitz/Wadge games. This is the content of Wadge's lemma ([WWW83], Ch. I, B). See also Section 3.6 of Chapter 3 of the present thesis, where a proof of Wadge's lemma within Second Order Arithmetic is given.

Lemma 1.9 (Wadge's lemma) *Let X be $\{0, 1\}$ or ω. Let $A, B \subseteq X^\omega$. Then:*

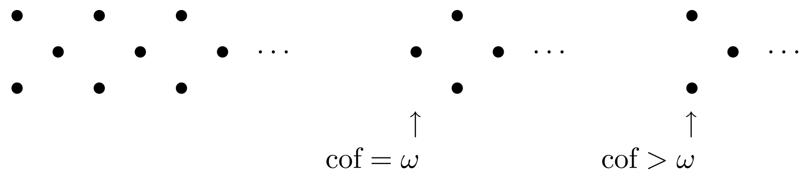
1. *Player* II *wins the game $G_L(A, B)$ iff $A \leq_L B$.*

2. *Player* II *wins the game $G_W(A, B)$ iff $A \leq_W B$.*

3. *If player* I *wins $G_L(A, B)$ or $G_W(A, B)$ then $B \leq_L A^c$.*

1.5 Lipschitz and Wadge hierarchies

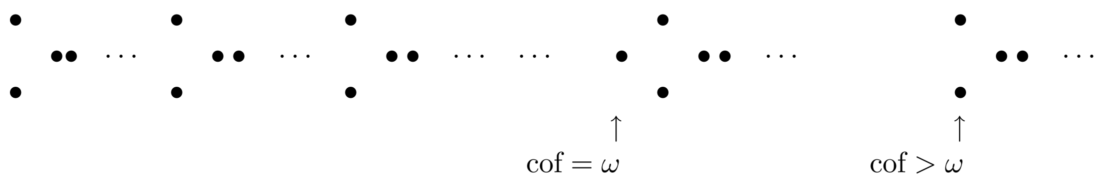
In his Ph.D. thesis [WWW83], Wadge deduced the fundamental properties of the Wadge degrees assuming the determinacy of the Wadge games over the real numbers. He showed that Wadge degrees build a hierarchy of growing complexity, similar to the Borel hierarchy, that looks like a ladder:



Let X be either $\{0,1\}$ or ω. The first degree (level 0) is the non-self-dual pair $\{[\emptyset]_W, [X^\omega]_W\}$ where $[\emptyset]_W = \{\emptyset\}$ and $[X^\omega]_W = \{X^\omega\}$. For all $A \subseteq X^\omega$ such that $A \neq \emptyset$ and $A \neq X^\omega$ we have that $[\emptyset]_W \prec_W [A]_W$ and $[X^\omega]_W \prec_W [A]_W$. A *successor Wadge degree* $[A]_W$ is a degree such that there is a degree $[B]_W \prec_W [A]_W$, but there is no other degree in between. In this case $[A]_W$ is said to be the immediate successor of $[B]_W$. The immediate successor of $[\emptyset]_W$ and $[X^\omega]_W$ is a self-dual degree (level 1) which entails all clopen sets except $\emptyset$ and X^ω. The next level is occupied by a non-self-dual pair formed from the class of the closed sets which are not open and the class of the open sets which are not closed. Along the hierarchy non-self-dual and self-dual degrees continue to alternate.

A degree that is not a successor degree and is neither $[\emptyset]_W$ nor $[X^\omega]_W$ is called a *limit Wadge degree*. A limit Wadge degree is of *countable cofinality* if it is the least upper bound of a countable sequence of smaller Wadge degrees. A limit degree which is not of countable cofinality is said to be of *uncountable cofinality*. Wadge showed that at limit levels of countable cofinality there is a self-dual Wadge degree, and at all other limit levels there is a non-self-dual Wadge degree.

Similar definitions and results hold for the Lipschitz degrees. However, there is a difference. Lipschitz hierarchy is even finer than the Wadge hierarchy since immediately above a non-self-dual Lipschitz pair there is a strictly increasing sequence of ω_1 (ω in the Cantor space) consecutive self-dual Lipschitz degrees.



The relation between the hierarchies of Lipschitz and Wadge is obtained as a consequence of a result of J. Steel and Van Wesep (see [VWsp78] and [Stee77]). The non-self-dual Wadge pairs correspond to the non-self-dual Lipschitz pairs and each self-dual Wadge degree is exactly the union of all consecutive self-dual Lipschitz degrees contained in one of the sequences of ω_1 (ω in the Cantor space) consecutive self-dual Lipschitz degrees which appear in the Lipschitz hierarchy.

Assuming **AD** and the axiom of dependent choice, **DC**, which is independent of **AD**, we have in the Baire space:

Theorem 1.10 *Assume* $\mathbf{AD} + \mathbf{DC}$. *Then:*

1. *$\prec_W$ and $\prec_L$ are well-founded relations,*
2. *immediately above a non-self-dual Wadge (Lipschitz) pair there is a single self-dual Wadge degree,*
3. *immediately above a non-self-dual Lipschitz (Wadge) pair there is a strictly increasing sequence of ω_1 consecutive self-dual Lipschitz degrees,*
4. *immediately above a single self-dual Wadge degree there is a non-self-dual Wadge (Lipschitz) pair,*
5. *immediately above a strictly increasing sequence of ω_1 consecutive self-dual Lipschitz degrees there is a non-self-dual Lipschitz (Wadge) pair,*
6. *at limit levels of countable cofinality there is a self-dual Wadge (resp. Lipschitz) degree, and at all other limit levels there is a non-self-dual Wadge (resp. Lipschitz) pair.*

1.6 Semilinear order principle

Let X be either $\{0,1\}$ or ω. The Wadge semilinear order principle, $\mathbf{SLO}_W$, is the assertion stating that for all $A, B \subseteq X^\omega$, $A \leq_W B$ or $B^c \leq_W A$ holds. Thus,

$$\mathbf{SLO}_W = \text{ For all } A, B \subseteq X^\omega,\ A \leq_W B \vee B^c \leq_W A.$$

Similarly, $\mathbf{SLO}_L$ denotes the Lipschitz semilinear order principle, i.e.

$$\mathbf{SLO}_L = \text{ For all } A, B \subseteq X^\omega,\ A \leq_L B \vee B^c \leq_L A.$$

As a consequence of Wadge´s lemma we obtain that the relation between Wadge and Lipschitz determinacy axioms and Wadge and Lipschitz semilinear principles is the following:

$$\begin{array}{ccc} \mathbf{AD}_L & \Longrightarrow & \mathbf{SLO}_L \\ & & \Downarrow \\ \mathbf{AD}_W & \Longrightarrow & \mathbf{SLO}_W \end{array}$$

Recently A. Andretta ([AA03] and [AA04]) has proved that all these principles are actually equivalent, so we know now that

$$\mathbf{SLO}_W \Longleftrightarrow \mathbf{AD}_W \Longleftrightarrow \mathbf{AD}_L \Longleftrightarrow \mathbf{SLO}_L.$$

This result was obtained within $\mathbf{ZF}+\mathbf{BP}+\mathbf{DC}$, i.e. within Zermelo-Fraenckel set theory without the full axiom of choice but with two assumptions, the property of Baire and the axiom of dependent choice.

The principle $\mathbf{SLO}_W$ shares with $\mathbf{AD}$ important consequences. For example, $\mathbf{SLO}_W$ implies the perfect set property, a result which was proved by Wadge ([WWW83], Ch. II, C). This means that $\mathbf{SLO}_W$ contradicts the axiom of choice. However, $\mathbf{SLO}_W$ does not exclude all kinds of choice. The weaker principle of countable choices, $\mathbf{AC}_\omega$, is a consequence of $\mathbf{SLO}_W$ ([AA03], pp. 176-178).

A. Andretta also proved that in Theorem 1.10 **AD** can be replaced by $\mathbf{SLO}_W + \mathbf{BP}$, and that many properties of Wadge and Lipschitz hierarchies can be obtained assuming only $\mathbf{SLO}_W$.

L. Harrington [Har78] has shown that $\mathbf{SLO}_W$ and **AD**, restricted to $\mathbf{\Pi}^1_1$ sets, are equivalent. G. Hjorth [Hjr96] proved the equivalence of both principles when they are restricted to $\mathbf{\Pi}^1_2$ sets. Thus, since the proofs of A. Andretta are "local" ([AA03], pp. 164), for $n = 1, 2$ we have

$$\mathbf{AD}\left(\mathbf{\Pi}^1_n\right) \Longleftrightarrow \mathbf{SLO}_W\left(\mathbf{\Pi}^1_n\right) \Longleftrightarrow \mathbf{AD}_W\left(\mathbf{\Pi}^1_n\right) \Longleftrightarrow \mathbf{AD}_L\left(\mathbf{\Pi}^1_n\right) \Longleftrightarrow \mathbf{SLO}_L\left(\mathbf{\Pi}^1_n\right).$$

A generalization to higher projective pointclasses, as we mentioned in section 1.4, remains an open problem.

1.7 Hausdorff differences and Lipschitz games

Hausdorff introduced a sequence of classes of differences of sets over the ordinals. Since we are only interested in the initial terms of this sequence, we define only finite differences. Let $n \in \omega$ and $S = (F_0, F_1, .., F_n)$ be a sequence of closed subsets of X^ω, i.e. $F_i \in \mathbf{\Pi}^0_1(X^\omega)$ for all $0 \le i \le n$. Then we define

$$D_n(S) = \bigcup_{i<n} (F_{2i} - F_{2i+1}).$$

For example, for $n = 0, 1, 2, 3$ we have

$$\begin{aligned} D_0(S) &= \emptyset \\ D_1(S) &= F_0 - F_1 \\ D_2(S) &= (F_0 - F_1) \cup F_2 \\ D_3(S) &= (F_0 - F_1) \cup (F_2 - F_3) \end{aligned}$$

Now we can define classes of differences of closed sets by

$$\mathbf{Df}_n = \left\{ D_n(S) : S \text{ is monotone nonincreasing in } \mathbf{\Pi}^0_1(X^\omega) \wedge\ F_n = \emptyset \right\}$$

Again, for $n = 0, 1, 2, 3$ and any subset $A \subseteq X^\omega$

$$\begin{aligned} &A \in \mathbf{Df}_0 \ \text{ iff } \ A = \emptyset \\ &A \in \mathbf{Df}_1 \ \text{ iff } \ A \in \mathbf{\Pi}^0_1(X^\omega) \\ &A \in \mathbf{Df}_2 \ \text{ iff } \ (A = F_0 - F_1 \wedge F_0 \supseteq F_1) \\ &A \in \mathbf{Df}_3 \ \text{ iff } \ (A = F_0 - (F_1 - F_2) \wedge F_0 \supseteq F_1 \supseteq F_2) \end{aligned}$$

where $F_0, F_1, F_2 \in \mathbf{\Pi}^0_1(X^\omega)$.

We introduce also for each $n \in \omega$ the class $\breve{\mathbf{Df}}_n$ of the complements of the sets in $\mathbf{Df}_n$. Thus for each $A \subseteq X^\omega$ we have

$$A \in \mathbf{Df}_n \ \text{ iff } \ A = F_0 - (F_1 - (\cdots - (F_{n-1} - F_n) \cdots))$$

and

$$A \in \breve{\mathbf{Df}}_n \ \text{ iff } \ A^c = F_0 - (F_1 - (\cdots - (F_{n-1} - F_n) \cdots))$$

where F_0, F_1, ..., F_{n-1}, $F_n \in \mathbf{\Pi}^0_1(X^\omega)$ and $F_0 \supseteq F_1 \supseteq \cdots \supseteq F_{n-1} \supseteq F_n = \emptyset$.

As we mentioned above, the definitions can be extended to the ordinals. Using a classical result of Hausdorff stating that

$$\mathbf{\Delta}^0_2(X^\omega) = \sum_{\alpha < \omega_1} \mathbf{Df}_\alpha$$

([Kec95], p. 176), Wadge carried out a very detailed analysis of Wadge degrees below $\mathbf{\Delta}^0_2$. He found the classes which occupy each degree noting that for each $i \in \omega$, $\mathbf{\Sigma}^0_i$ (resp., $\mathbf{\Pi}^0_i$) is an initial segment of $\leq_W$, i.e. if A is $\mathbf{\Sigma}^0_i$ (resp., $\mathbf{\Pi}^0_i$) and $B \leq_W A$, then B is $\mathbf{\Sigma}^0_i$ (resp., $\mathbf{\Pi}^0_i$). Moreover the sets in $\mathbf{\Sigma}^0_i - \mathbf{\Pi}^0_i$ (resp., $\mathbf{\Pi}^0_i - \mathbf{\Sigma}^0_i$) are $\mathbf{\Sigma}^0_i$-complete (resp., $\mathbf{\Pi}^0_i$-complete), i.e. if A is $\mathbf{\Sigma}^0_i - \mathbf{\Pi}^0_i$ (resp., $\mathbf{\Pi}^0_i - \mathbf{\Sigma}^0_i$) and B is $\mathbf{\Sigma}^0_i$ (resp., $\mathbf{\Pi}^0_i$), then $B \leq_W A$.

In the following table we show the first five levels of the hierarchy of Wadge we are interested in. They correspond to the first three levels of Hausdorff´s hierarchy of differences.

$$
\begin{array}{ccccc}
\mathbf{Df}_0 & \multicolumn{2}{c}{\mathbf{Df}_1 = \mathbf{\Pi}^0_1} & \multicolumn{2}{c}{\mathbf{Df}_2} \\
0 & 1 & 2 & 3 & 4 \\
\{\emptyset\} & & \mathbf{\Pi}^0_1 - \mathbf{\Sigma}^0_1 & & \mathbf{Df}_2 - \breve{\mathbf{Df}}_2 \\
 & \mathbf{\Delta}^0_1 - \{\emptyset, X^\omega\} & & (\mathbf{Df}_2 \cap \breve{\mathbf{Df}}_2) - (\mathbf{\Pi}^0_1 \cup \mathbf{\Sigma}^0_1) & \\
\{X^\omega\} & & \mathbf{\Sigma}^0_1 - \mathbf{\Pi}^0_1 & & \breve{\mathbf{Df}}_2 - \mathbf{Df}_2 \\
0^- & & 2^- & & 4^- \\
\breve{\mathbf{Df}}_0 & \multicolumn{2}{c}{\breve{\mathbf{Df}}_1 = \mathbf{\Sigma}^0_1} & \multicolumn{2}{c}{\breve{\mathbf{Df}}_2}
\end{array}
$$

Note that, according to (3) of Theorem 1.10, self-dual Wadge degrees 1 and 3 correspond to ω_1 (ω in the Cantor space) self-dual Lipschitz degrees, while the non-self-dual pairs 0, 0^-, 2, 2^-, and 4, 4^-, correspond to non-self-dual pairs of Lipschitz degrees.

In the following chapter we shall show that the Lipschitz game $G_L(A, B)$ and the Wadge game $G_W(A, B)$ are determined in both Cantor and Baire spaces when A and B belong to any of the Lipschitz degrees we referred above.

1.8 Well-founded trees and ranks

A subset $T \subseteq X^{<\omega}$ is called a *tree* on X if T is closed under initial segments, i.e.

$$\forall s, t \in X^{<\omega} \, (s \in T \wedge t \subset s \to t \in T).$$

In particular, $\langle\rangle \in T$ if T is nonempty. We call the elements of T the *nodes* of T. T is said to be *infinite* if the set of nodes of T is infinite. If $S \subseteq X^{<\omega}$ is a tree on X and $S \subseteq T$, then S is called a subtree of T.

Let T be a tree on a set X. Then:

(-) A node $s \in T$ is called *terminal* if it has no proper extension in T, i.e. if

$$\forall a \in X \ \left(s * \langle a \rangle \notin T\right).$$

(-) T is said to be *pruned* if it has no terminal nodes.

(-) T is said to be *finitely splitting* if for every node $s \in T$, $\{a \in X \, : s * \langle a \rangle \in T\}$ is finite. Otherwise T is called *infinitely splitting*.

(-) $f \in X^{\omega}$ is called a *path* or an *infinite branch* of T if

$$\forall n \in \omega \ \left(f\,[n] \in T\right).$$

(-) The *body* of T is written as $[T]$ and is the set of all paths of T, i.e.

$$[T] = \{f \in X^{\omega} : \forall n \in \omega \ f\,[n] \in T\}.$$

(-) T is called a *well-founded* tree if $[T] = \emptyset$, i.e. if it has no path. Otherwise we say that T is *ill-founded*.

It is not hard to see that every tree can be pruned, i.e. that for every tree $T \subseteq X^{<\omega}$ there exists a pruned subtree $S \subseteq T$ such that $[S] = [T]$. In fact,

$$S = \{t \in X^{<\omega} \, : t \in T \wedge \exists f \in X^{\omega} \, (f \in [T] \wedge t \subset f)\}.$$

Of course, if T has no infinite path, then $S = \emptyset$. In set theory the assertion that every nonempty pruned tree $S \subseteq X^{<\omega}$ has a path is equivalent to the axiom of dependent choices, $\mathbf{DC}(X^{<\omega})$ (see Proposition 1.12 of [AA01]).

König´s Lemma will be often referred to in later chapters. Its proof requires only the axiom of countable choice, $\mathbf{AC}_{\omega}$.

Proposition 1.11 (König Lemma) *If $T \subseteq X^{<\omega}$ is an infinite, finitely splitting tree, then T has a path, i.e. is ill-founded.*

Corollary 1.12 *Every infinite binary tree $T \subseteq 2^{<\omega}$ has a path, i.e. is ill-founded.*

As we mentioned in Section 1, X^ω can be viewed as the topological space of all ω-sequences of elements of X endowed with the product topology inherited by the discrete topology on ω. The following characterization allows us to classify closed sets by the combinatorial properties of the trees which define them:

Proposition 1.13 *A subset F of X^ω is closed iff it is the body of a tree T on X, i.e. $F = [T]$.*

If T is a well-founded tree on a set X, then there is a unique function $\rho_T : X^{<\omega} \longrightarrow \mathrm{Ord}$ defined recursively by

$$\begin{aligned}\rho_T(t) &= \sup\{\rho_T(s) + 1 : s \in T \wedge t \subset s\}\\ &= \sup\{\rho_T(t * \langle n\rangle) + 1 : t * \langle n\rangle \in T\},\end{aligned}$$

for each $t \in T$. For all sequences $t \in X^{<\omega}$ such that $t \notin T$ we put $\rho_T(t) = 0$. If $t \in T$ and t is terminal we also have $\rho_T(t) = \sup\emptyset = 0$.

The *rank* of a well-founded tree T is defined to be the ordinal

$$\rho(T) = \sup\{\rho_T(t) + 1 : t \in T\}.$$

Thus if T is nonempty, $\rho(T) = \rho_T(\langle\rangle) + 1$.

The notation introduced in this section will be also used in the setting of second order arithmetic with the necessary adjustments, especially as regards the interpretation of symbols.

1.9 Second order arithmetic

In this section, we recall some basic definitions and facts about second order arithmetic. The *language* L_2 of second order arithmetic is a two-sorted language with *number variables* x, y, z, ..., which are intended to range over the set ω of all natural numbers, and *set variables* X, Y, Z, ..., which are intended to range over subsets of ω, i.e. over $\wp(\omega)$. The set variables can also be viewed as variables intended to range over the $\{0,1\}-$valued functions, that is, the characteristic functions of sets of natural numbers. We have also the *constant number symbols* 0 and 1, *binary function symbols* $+$ and $\cdot$, and *binary relation symbols* $=$, $<$, all of them on ω. Additionally, we have the *membership relation symbol* $\in$, which is intended to range over $\omega\times\ \wp(\omega)$.

Terms and formulas are defined as usual. *Numerical terms* are number variables, the constant symbols 0 and 1, and $t_1 + t_2$ and $t_1 \cdot t_2$ whenever t_1 and t_2 are numerical terms. If

t_1 and t_2 are numerical terms and X is any set variable, then $t_1 = t_2$, $t_1 < t_2$, and $t_1 \in X$ are *atomic formulas*. If X and Y are set variables, expressions $X = Y$ and $X \subseteq Y$ are to be seen as a shorthand for $\forall x\ (x \in X \leftrightarrow x \in Y)$ and $\forall x\ (x \in X \to x \in Y)$, respectively.

The *axioms of second order arithmetic* consist of the universal closure of the following L_2-formulas:

(i) discrete order semi-ring axioms for $(\omega, +, \cdot, 0, 1, <)$,

(ii) induction axiom:

$$(0 \in X \wedge \forall x(x \in X \to x+1 \in X)) \to \forall x\ x \in X,$$

(iii) comprehension scheme:

$$\exists X \forall x\,(x \in X \leftrightarrow \varphi(x))\,,$$

where $\varphi(x)$ is any formula of L_2 in which X does not occur freely. The formula $\varphi(x)$ may contain free variables in addition to x. These free variables may be referred as parameters of this instance of the comprehension scheme. Parameters will be also allowed in the rest of axiom schemes we shall introduce in this section.

The *formal system of second order arithmetic*, shortly $\mathbf{Z}_2$, consists of the axioms of second order arithmetic, together with all formulas of L_2 which are deducible from the axioms by means of the usual logical axioms and rules of inference. In order to introduce formal subsystems of second order arithmetic we need a classification of the formulas attending to their quantifier complexity.

Let x be a variable, let t be a numerical term not containing x, and let φ be a formula of L_2. We use the following abbreviations:

$$\begin{array}{lll} \forall x < t\ \varphi & \text{abbreviates} & \forall x(x < t \to \varphi), \\ \exists x < t\ \varphi & \text{abbreviates} & \exists x(x < t \wedge \varphi), \end{array}$$

and

$$\begin{array}{lll} \forall x \leq t\ \varphi & \text{abbreviates} & \forall x < t+1\ \varphi, \\ \exists x \leq t\ \varphi & \text{abbreviates} & \exists x < t+1\ \varphi. \end{array}$$

A formula of L_2 is said to be *arithmetical* if it contains no set quantifiers. Now we classify the *arithmetical formulas*, whose class is denoted Π^0_∞. Let φ be a formula of L_2:

(-) φ is a *bounded quantifier formula* (shortly Σ^0_0 or Π^0_0) if it is built up from atomic formulas by using connectives and bounded number quantifiers $\forall x < t$ and $\exists x < t$, where t does not contain x. Note that a bounded quantifier formula may contain free set or number variables, also called parameters,

(-) φ is a Σ^0_1 *formula* if it is of the form $\exists x\ \theta$, where x is a number variable and θ is a bounded quantifier formula.

(-) φ is a Π^0_1 *formula* if it is of the form $\forall x\ \theta$, where x is a number variable and θ is a bounded quantifier formula.

In general, for $0 < k \in \omega$:

(-) φ is said to be a Σ^0_k *formula* if it is of the form

$$\exists x_1 \forall x_2 \exists x_3 \cdots Q x_k\ \theta,$$

with $Q = \exists$, if k is odd, and $Q = \forall$, if k is even, and where x_1, x_2, ..., x_k are number variables and θ is a bounded quantifier formula.

(-) φ is said to be a Π^0_k *formula* if it is of the form

$$\forall x_1 \exists x_2 \forall x_3 \cdots Q x_k\ \theta,$$

with $Q = \forall$, if k is odd, and $Q = \exists$, if k is even, and where x_1, x_2, ..., x_k are number variables and θ is a bounded quantifier formula.

Similarly we can set up a classification of formulas whose quantifiers range over set variables. We denote the class of these formulas by Π^1_∞. Let φ be a formula of L_2:

(-) φ is a Σ^1_1 *formula* if it is of the form $\exists X\ \theta$, where X is a set variable and θ is an arithmetical formula.

(-) φ is a Π^1_1 *formula* if it is of the form $\forall X\ \theta$, where X is a set variable and θ is an arithmetical formula.

In general, for $0 < k \in \omega$:

(-) φ is said to be a Σ^1_k *formula* if it is of the form

$$\exists X_1 \forall X_2 \exists X_3 \cdots Q X_k\ \theta,$$

with $Q = \exists$, if k is odd, and $Q = \forall$, if k is even, and where X_1, X_2, ..., X_k are number variables and θ is an arithmetical formula.

(-) φ is said to be a Π^1_k *formula* if it is of the form

$$\forall X_1 \exists X_2 \forall X_3 \cdots Q X_k\ \theta,$$

with $Q = \forall$, if k is odd, and $Q = \exists$, if k is even, and where X_1, X_2, ..., X_k are number variables and θ is an arithmetical formula.

Let $i \in \{0,1\}$ and $k \in \omega$. Clearly any Σ^i_k formula is logically equivalent to the negation of a Π^i_k formula, and vice versa. Moreover, up to logical equivalence of formulas, we have $\Sigma^i_k \cup \Pi^i_k \subseteq \Sigma^i_{k+1} \cap \Pi^i_{k+1}$.

We loosely say that a formula is Σ^i_k (respectively Π^i_k) if it is equivalent over a base theory to a Σ^i_k (respectively Π^i_k) formula.

Using the above classification, we define schemata of comprehension and induction as follows.

Definition 1.14 *Let Γ be Π or Σ. Then for $i \in \{0,1\}$:*

1. *The scheme of Γ^i_k comprehension, denoted by Γ^i_k-**CA**, consists of all axioms of the form*
$$\exists X \forall x\,(x \in X \leftrightarrow \varphi(x))\,,$$
where φ is Γ^i_k, and X does not occur freely in $\varphi(x)$.
2. *The scheme of bounded Γ^i_k comprehension consists of all axioms of the form*
$$\forall n \exists X \forall x\,(x \in X \leftrightarrow (x < n \wedge \varphi(x)))\,,$$
where φ is Γ^i_k, and X does not occur freely in $\varphi(x)$.
3. *The scheme of Δ^i_k comprehension, denoted by Δ^i_k-**CA**, consists of all axioms of the form*
$$\forall x\,(\varphi(x) \leftrightarrow \psi(x)) \rightarrow \exists X \forall x\,(x \in X \leftrightarrow \varphi(x))\,,$$
where φ is Σ^i_k, $\psi(x)$ is Π^i_k, and X does not occur freely in $\varphi(x)$.
4. *The scheme of Γ^i_k induction, denoted by Γ^i_k-**IND**, consists of all axioms of the form*
$$\varphi(0) \wedge \forall x(\varphi(x) \rightarrow \varphi(x+1)) \rightarrow \forall x \varphi(x),$$
where φ is Γ^i_k.
5. *The scheme of Γ^i_k axiom of choice, denoted Γ^i_k-**AC**, consists of all axioms of the form*
$$\forall x \exists X \varphi(x, X) \rightarrow \exists Y \forall x \varphi(x, (Y)_x),$$
where φ is Γ^i_k and $(Y)_x = \{i : (i,x) \in Y\}$. Note that we are using the paring function $(i,j) = (i+j)^2 + i$.
6. *The scheme of strong Γ^i_k collection consists of all axioms of the form*
$$\forall x \exists y \forall i < x(\exists j\ \varphi(i,j) \rightarrow \exists j < y\ \varphi(i,j)),$$
where φ is Γ^i_k and y does not occur freely in φ.

A *subsystem of second order arithmetic* is a formal system in the language L_2 each of whose axioms is a theorem of second order arithmetic. In order to define a subsystem of second order arithmetic, we have to specify the axioms of the system by referring some formulas of L_2. The axioms are then taken to be the universal closures of those formulas.

Now we define the subsystem of second order arithmetic *Recursive Comprehension axiom with restricted induction*, $\mathbf{RCA}_0$. The subscript 0 denotes restricted induction. This means that the system does not include the *full second order induction scheme*, but a restricted version of it.

Definition 1.15 $\mathbf{RCA}_0$ *is the formal system in the language of* L_2 *which consists of*

1. *the discrete order semi-ring axioms for* $(\omega, +, \cdot, 0, 1, <)$,
2. Δ^0_1-$\mathbf{CA}$,
3. Σ^0_1-$\mathbf{IND}$.

Within $\mathbf{RCA}_0$ the *set of natural numbers* is defined as the unique set X such that $\forall x\ (x \in X)$, and it is denoted $\mathbb{N}$. Within $\mathbf{RCA}_0$ we can encode *finite sets* and *finite sequences* as single natural numbers. The *set of all codes of finite sequences* exists by Σ^0_0-$\mathbf{CA}$, and is denoted $\mathbb{N}^{<\mathbb{N}}$. Similarly the *set of all codes of finite sequences of* 0´*s and* 1´*s* exists and is denoted $2^{<\mathbb{N}}$. We continue to use the symbology introduced in the previous section to deal with members of $\mathbb{N}^{<\mathbb{N}}$ and of $2^{<\mathbb{N}}$, bearing in mind that the symbols now represent codes. We will also continue to use the terminology concerning trees introduced in section 1.8. These expressions will then denote the corresponding concepts codified in second order arithmetic (for a detailed definition see [Smp99], pp. 20 and 121.)

Within $\mathbf{RCA}_0$ the number systems $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$, are developed in a way very similar to the Dedekind-Cauchy construction. We will refer this fact in more detail in the last section of Chapter 3.

Within $\mathbf{RCA}_0$, we define a numerical paring function by letting

$$(m, n) = (m + n)^2 + m.$$

Using Δ^0_1-$\mathbf{CA}$, we can prove that for all sets X, $Y \subseteq \mathbb{N}$, there exists a set $X \times Y \subseteq \mathbb{N}$ consisting of all (m, n) such that $m \in X$ and $n \in Y$. Then a function $f : X \to Y$ is defined to be a set $f \subseteq X \times Y$ such that for all $m \in X$ there is exactly one $n \in Y$ such that $(m, n) \in f$. For $m \in X$, $f(m)$ is defined to be the unique n such that $(m, n) \in f$. We can also deal with functions of any arity $k \geq 1$ by using the pairing function. The usual properties of such functions can be proved in $\mathbf{RCA}_0$. In particular, $\mathbf{RCA}_0$ proves that functions are closed under composition, primitive recursion, and minimization. Let us observe that the language of second order arithmetic does not formally contain any function variables. However, one can naturally express the fact that "G is the graph of a function $f : \mathbb{N} \to X$" by using a Π^0_2 formula. Thus, we can freely use variables ranging over functions in our language. In addition, within $\mathbf{RCA}_0$ we have ([Smp99], p. 72):

Proposition 1.16 $\mathbf{RCA}_0$ *proves:*

1. *Bounded* Σ^0_1 *comprehension.*
2. Π^0_1-$\mathbf{IND}$.
3. *Strong* Σ^0_1 *collection.*

Proposition 1.17 *For each* $k \in \omega$ *such that* $k > 0$, $\mathbf{RCA}_0$ *proves:*

1. Σ^0_k-$\mathbf{IND} \leftrightarrow \Pi^0_k$-$\mathbf{IND}$.
2. Σ^0_k-$\mathbf{IND} \leftrightarrow$ *bounded* Σ^0_k *comprehension.*
3. Σ^0_k-$\mathbf{IND} \leftrightarrow$ *strong* Σ^0_k *collection.*

The other subsystems of second order arithmetic are defined by adding some existence axioms to $\mathbf{RCA}_0$. In order to define the next subsystem we need the *Weak König's Lemma*, which states that:

For all tree $T \subseteq 2^{<\mathbb{N}}$, if T is infinite, then T has a path.

Now we can define the subsystem *Weak König's Lemma with restricted induction,* $\mathbf{WKL}_0$.

Definition 1.18 $\mathbf{WKL}_0$ *is the formal system in the language of* L_2 *which consists of*

1. *the axioms of* $\mathbf{RCA}_0$,
2. *Weak König's Lemma.*

Subsystems $\mathbf{RCA}_0$ and $\mathbf{WKL}_0$ are close in proof theoretic strength. In fact, both share the same first order part (viz. Σ_1-induction $I\Sigma_1$) and by a well known theorem of Harrington, $\mathbf{WKL}_0$ is conservative over $\mathbf{RCA}_0$ for Π^1_1-sentences. However, $\mathbf{WKL}_0$ is much stronger than $\mathbf{RCA}_0$ from the viewpoint of mathematical practice. In fact, $\mathbf{WKL}_0$ is strong enough to prove many well known nonconstructive theorems which are not provable in $\mathbf{RCA}_0$.

The subsystem *Arithmetical Comprehension with restricted induction*, $\mathbf{ACA}_0$, is defined as follows.

Definition 1.19 $\mathbf{ACA}_0$ *is the formal system in the language of* L_2 *which consists of*

1. *the discrete order semi-ring axioms for* $(\omega, +, \cdot, 0, 1, <)$,

2. *the induction axiom:*

$$(0 \in X \land \forall x(x \in X \rightarrow x+1 \in X)) \rightarrow \forall x\ x \in X,$$

3. *the arithmetical comprehension scheme,* Π^0_∞-**CA**:

$$\exists X \forall x \left(x \in X \leftrightarrow \varphi(x)\right),$$

where $\varphi(x)$ *is any arithmetical formula of* L_2 *in which* X *does not occur freely.*

As a consequence of the induction axiom and arithmetical comprehension scheme we have in $\mathbf{ACA}_0$ the *arithmetical induction scheme*, Π^0_∞-**IND**, i.e.

$$\varphi(0) \land \forall x(\varphi(x) \rightarrow \varphi(x+1)) \rightarrow \forall x \varphi(x),$$

restricted to arithmetical formulas of L_2. In fact, the first order part of $\mathbf{ACA}_0$ is, precisely, Peano arithmetic **PA**.

The following proposition will be useful (see [Smp99], Lemma III.1.3 for a proof).

Proposition 1.20 *The following are equivalent over* $\mathbf{RCA}_0$*:*

1. $\mathbf{ACA}_0$.

2. Σ^0_1-**CA**.

The next subsystem of second order arithmetic we shall refer is a bit more elaborate to describe and we will postpone a formal definition to Chapter 5. Informally, *Arithmetical Transfinite Recursion with restricted induction*, shortly $\mathbf{ATR}_0$, is a system stronger than $\mathbf{ACA}_0$, where we can not only prove assertions by transfinite induction, which is already possible within $\mathbf{ACA}_0$, but also make definitions by transfinite recursion. This latter feature is not available in $\mathbf{ACA}_0$ and it is responsible for the possibility of proving within $\mathbf{ATR}_0$ that the countable ordinals form a linear ordering. This fact will be used in Chapter 5 to prove results which depend on countable ordinals.

The system $\mathbf{ATR}_0$ consists of $\mathbf{ACA}_0$ plus a set existence axiom known as *Arithmetical Transfinite Recursion*. Intuitively we can describe this axiom as follows. Let $\theta(n, X)$ be an arithmetical formula with a free number variable n and a free set variable X. We may view this formula as an arithmetical operator $\Theta : \wp(\mathbb{N}) \rightarrow \wp(\mathbb{N})$ defined by

$$\Theta(X) = \{n \in \mathbb{N} : \theta(n, X)\}\ .$$

Let α stand for a countable well ordering, i.e. a relation on $\mathbb{N}$ such that for all $x, y, z \in \alpha$

$$(x < y \wedge y < z) \rightarrow x < z$$

and

$$x = y \vee x < y \vee y < x,$$

and there is no sequence $\{x_i : i \in \mathbb{N}\} \subseteq \alpha$ with $x_{i+1} < x_i$ for all $i \in \mathbb{N}$. Now consider the set $Y \subseteq \mathbb{N} \times \alpha$ obtained by iterating the operator Θ along α, i.e.

$$Y = \{(n, 0) : n \in \Theta(\emptyset)\} \cup \{(n, 1) : n \in \Theta(\Theta(\emptyset))\} \cup \cdots$$

Thus, *Arithmetical Transfinite Recursion* is the axiom scheme asserting that the set Y exists, for every arithmetical operator Θ and every countable well ordering α.

Definition 1.21 $\mathbf{ATR}_0$ *is the formal system in the language of* L_2 *which consists of*

1. $\mathbf{ACA}_0$,
2. *Arithmetical Transfinite Recursion.*

The following result will be useful in Chapter 5.

Proposition 1.22 $\mathbf{ATR}_0$ *proves the* Σ^1_1 *axiom of choice, i.e. the scheme*

$$\forall x \exists X (\varphi(x, X) \rightarrow \exists Y \forall x \varphi(x, (Y)_x)),$$

where $\varphi(x)$ *is* Σ^1_1 *and* $(Y)_x = \{i : (i, x) \in Y)\}$.

Finally, we introduce the subsystem of second order arithmetic Π^1_1-*Comprehension with restricted induction*, shortly Π^1_1-$\mathbf{CA}_0$. This subsystem of $\mathbf{Z}_2$ is defined as follows:

Definition 1.23 Π^1_1-$\mathbf{CA}_0$ *is the formal system in the language of* L_2 *which consists of*

1. *the discrete order semi-ring axioms for* $(\omega, +, \cdot, 0, 1, <)$*;*
2. *induction axiom:*

$$(0 \in X \wedge \forall x(x \in X \rightarrow x + 1 \in X)) \rightarrow \forall x \; x \in X \;\; ;$$

3. *comprehension scheme,* Π^1_1-$\mathbf{CA}$*:*

$$\exists X \forall x \, (x \in X \leftrightarrow \varphi(x)) \, ,$$

where $\varphi(x)$ *is a* Π^1_1 *formula of* L_2 *in which* X *does not occur freely.*

Let us note that subsystems Π^1_1-$\mathbf{CA}_0$ and Σ^1_1-$\mathbf{CA}_0$ are equivalent, i.e. they have the same theorems (see [Smp99], p. 16).

There is a relation of strict inclusion between all the subsystems that we described:

$$\mathbf{RCA}_0 \subset \mathbf{WKL}_0 \subset \mathbf{ACA}_0 \subset \mathbf{ATR}_0 \subset \Pi^1_1\text{-}\mathbf{CA}_0$$

(see [Smp99], pp. 35 and 39).

The five subsystems of second order arithmetic, $\mathbf{RCA}_0$, $\mathbf{WKL}_0$, $\mathbf{ACA}_0$, $\mathbf{ATR}_0$, and Π^1_1-$\mathbf{CA}_0$, also known as the "big five", encompass almost all ordinary mathematics in the sense that almost all mathematical results are equivalent to results that can be proved in at least one of them. In addition, a huge amount of theorems of ordinary mathematics have turned out to be either provable in $\mathbf{RCA}_0$ or exactly equivalent over $\mathbf{RCA}_0$ to one of the remaining "big five". This is the main theme in Reverse Mathematics. Further information on these systems can be found in Simpson's book [Smp99].

Finally, let us note that it is immediate that

$$\mathbf{Z}_2 = \bigcup_{k\in\omega} \Pi^1_k\text{-}\mathbf{CA}_0.$$

Thus, full second order arithmetic is sometimes also denoted by Π^1_∞-$\mathbf{CA}_0$.

Chapter 2

Topological analysis of Lipschitz and Wadge games

In this chapter we fulfill the first goal of the thesis. Namely, we give direct proofs of the determinacy of Lipschitz and Wadge games for the first levels of the Wadge hierarchy. This will be done in Sections 2 and 3. The proofs take advantage of Wadge´s topological analysis of $\mathbf{\Delta}^0_2$ sets developed in [WWW83]. Let us recall that Wadge's goal was not to prove the determinacy of Wadge games but to discover the structure of the class of degrees. Here we show that, however, it is possible to adapt Wadge's ideas in order to give direct proofs of the determinacy of both Lipschitz and Wadge games. This is particularly interesting because in Chapters 4 and 5 we will formalize these proofs for the study of Lipschitz and Wadge determinacy in second order arithmetic.

The chapter is divided into three sections. In Section 1 we examine the first levels of Hausdorff's hierarchy of differences below $\mathbf{\Delta}^0_2$. Relying on the work of Hausdorff, Wadge showed that for any $\mathbf{\Delta}^0_2$ sets A and B, A is reducible to B iff B has at least as many nonempty residues and adjoins as does A, i.e. for any countable ordinal α,

$$\mathrm{Rs}_\alpha(A) \neq \emptyset \text{ implies } \mathrm{Rs}_\alpha(B) \neq \emptyset, \text{ and } \mathrm{Rs}_\alpha(A^c) \neq \emptyset \text{ implies } \mathrm{Rs}_\alpha(B^c) \neq \emptyset,$$

where Rs_α stands for the α-th residue or adjoin. In this section we describe Hausdorff residues and their relation to differences of closed sets. This description allied to the representation of closed sets in terms of trees creates the conditions to develop more combinatorial arguments on which the proofs of Sections 2 and 3 can be built.

In Section 2 we work in Cantor space and prove the determinacy of Lipschitz and Wadge games up to differences of closed sets and their complements, i.e. for subsets of the Cantor space which occupy degrees corresponding to the first five levels of Wadge hierarchy.

In Section 3 we also prove the determinacy of Lipschitz and Wadge games up to differences of closed sets and their complements but, in this section, we work in the Baire space.

Finally, let us note that in this chapter we work in the setting of Zermelo-Fraenckel set theory and, as is customary in descriptive set theory, our metatheory will be $\mathbf{ZF}+\mathbf{DC}$. As before we use boldface notation $\mathbf{\Delta}^0_1$, $\mathbf{\Sigma}^0_1$, $\mathbf{\Pi}^0_1$, and $\mathbf{\Sigma}^0_1 \cup \mathbf{\Pi}^0_1$ to denote, respectively, the pointclass of clopen sets, the pointclass of open sets, the pointclass of closed sets, and the pointclass of sets which are open or closed.

2.1 Hausdorff residues and adjoins

We follow section C of Chapter I of Wadge thesis [WWW83] as well as Chapter VII of Hausdorff's book [H57]. In order to deal with differences of sets Hausdorff introduced the notion of a *residue* and the related notion of an *adjoin*. Here we will introduce these concepts in a slightly different way and we will define them as Wadge did in his thesis. Actually we will not use explicitly the concept of adjoin.

We will use some common notation. Let X be either $\{0,1\}$ or ω and $A, B \subseteq X^\omega$. Then:

1. A^c denotes the *complement* of A in X^ω, i.e. $A^c = X^\omega - A$.
2. $\overline{A}$ denotes the *closure* of A, i.e. $\overline{A} = \cap\{F \in \mathbf{\Pi}^0_1 : A \subseteq F\}$.
3. $\mathrm{int}(A)$ denotes the *interior* of A, i.e. $\mathrm{int}(A) = \cup\{G \in \mathbf{\Sigma}^0_1 : G \subseteq A\}$.
4. ∂A denotes the *boundary* of A, i.e. $\partial A = \left(\overline{A} \cap \overline{A^c}\right)$.
5. $\overline{A}^B$ denotes the *closure* of A in B, i.e. $\overline{A}^B = \overline{A} \cap B$.
6. $\mathrm{int}_B(A)$ denotes the *interior* of A in B, i.e.
$$\mathrm{int}_B(A) = \cup\{G \cap B : G \in \mathbf{\Sigma}^0_1,\ G \cap B \subseteq A\}.$$
7. $\partial_B A$ denotes the *boundary* of A in B, i.e. $\partial_B A = \overline{A}^B \cap \overline{B - A}^B$.

Now we define the Hausdorff sequence of finite residues. Let A be any subset of X^ω. Then we define the sequence of *residues* of A, $\{\mathrm{Rs}_n(A) : n \in \omega\}$, recursively by putting

$$\mathrm{Rs}_0(A) = A,$$
$$\mathrm{Rs}_{n+1}(A) = \overline{\mathrm{Rs}_n(A^c)} \cap A.$$

In Hausdorff's terminology $\mathrm{Rs}_1(A^c) = \overline{A} \cap A^c$ is the first adjoin of A and $\mathrm{Rs}_2(A^c) = \overline{\mathrm{Rs}_1(A)} \cap A^c = \overline{\overline{A^c} \cap A} \cap A^c$ is the first residue of A^c. According to Hausdorff's theorem differences of closed sets can be characterize by residues and adjoins. We will only need the finite version of this result ([H57]; see also [WWW83], p. 87):

Proposition 2.1 *For all $A \subseteq X^\omega$ and all $n \in \omega$,*

$$\mathrm{Rs}_n(A) = \emptyset \quad \textit{iff} \quad A \in \begin{cases} \mathbf{Df}_n & \textit{if } n \textit{ is even} \\ \breve{\mathbf{Df}}_n & \textit{if } n \textit{ is odd} \end{cases}$$

For our proofs of determinacy it will be crucial to know the structure of the residue of a set taking into account to which class of differences its complement belongs. Let X be either $\{0,1\}$ or ω and $A \subseteq X^\omega$. We highlight some useful facts:

1. A is a closed set if and only if the first adjoin of A is empty, i.e. $\mathrm{Rs}_1(A^c) = \overline{A} \cap A^c = A \cap A^c = \emptyset$. Note that for every closed set A, $\partial A \subseteq A$.

2. If $A \in \mathbf{\Pi}^0_1 - \mathbf{\Sigma}^0_1$ then $\partial A \subseteq A$ and
$$\mathrm{Rs}_1(A) = \overline{A^c} \cap A = \overline{A^c} \cap \overline{A} = \partial A \neq \emptyset.$$
On the other hand a closed set A is clopen if and only if $\mathrm{Rs}_1(A) = \partial A = \emptyset$.

The next two facts are immediate consequences of Hausdorff's result (see also [WWW83], p. 80):

3. $A \in \mathbf{Df}_2$ if and only if $\mathrm{Rs}_2(A) = \emptyset$.

4. $A, A^c \in \mathbf{Df}_2$ if and only if $\mathrm{Rs}_2(A) = \mathrm{Rs}_2(A^c) = \emptyset$.

Now we search for a necessary and sufficient condition for $A^c \in \mathbf{Df}_2$ when we *already* know that $A \in \mathbf{Df}_2$. This fact will play an important role in the proofs of Lipschitz and Wadge determinacy, so we examine it in some detail.

5. If $A \in \mathbf{Df}_2$, i.e. if $A = F_0 - F_1$ for some $F_0, F_1 \in \mathbf{\Pi}^0_1$ with $F_1 \subseteq F_0$, then $A^c \in \mathbf{Df}_2$ if and only if
$$\partial_{F_0} F_1 \cap \overline{\partial F_0 - F_1} = \emptyset.$$
Since by Hausdorff's theorem $A^c \in \mathbf{Df}_2$ if and only if $\mathrm{Rs}_2(A^c) = \emptyset$, it is enough to show that $\mathrm{Rs}_2(A^c) = \partial_{F_0} F_1 \cap \overline{\partial F_0 - F_1}$. Bearing in mind that $A = F_0 \cap F_1^c$ and thus $A^c = F_0^c \cup F_1$, we get
$$\begin{aligned}
\mathrm{Rs}_2(A^c) &= \overline{\overline{A^c} \cap A} \cap A^c \\
&= \overline{\overline{A^c} \cap F_0 \cap F_1^c} \cap (F_0^c \cup F_1) \\
&= \left(\overline{\overline{A^c} \cap F_0 \cap F_1^c} \cap F_0^c\right) \cup \left(\overline{\overline{A^c} \cap F_0 \cap F_1^c} \cap F_1\right)
\end{aligned}$$
Now, let us notice that
$$\begin{aligned}
\overline{A^c} \cap F_0 \cap F_1^c &= \overline{F_0^c \cup F_1} \cap F_0 \cap F_1^c \\
&= (\overline{F_0^c} \cup F_1) \cap F_0 \cap F_1^c \\
&= \overline{F_0^c} \cap F_0 \cap F_1^c \\
&= \partial F_0 - F_1
\end{aligned}$$
Then
$$\begin{aligned}
\mathrm{Rs}(A^c) &= \left(\overline{\partial F_0 - F_1} \cap F_0^c\right) \cup \left(\overline{\partial F_0 - F_1} \cap F_1\right) \\
&= \overline{\partial F_0 - F_1} \cap F_1
\end{aligned}$$

since, $\overline{\partial F_0 - F_1} \cap F_0^c \subseteq F_0 \cap F_0^c = \emptyset$. On the other hand, F_1 is a closed set in the subspace F_0, so $F_1 = \partial_{F_0} F_1 \cup \mathrm{int}_{F_0}(F_1)$.

Let us prove that $\overline{\partial F_0 - F_1} \cap \mathrm{int}_{F_0}(F_1) = \emptyset$. If $f \in \mathrm{int}_{F_0}(F_1)$ then there exists an open neighborhood G of f such that $G \cap F_0 \subseteq F_1$. If in addition, $f \in \overline{\partial F_0 - F_1}$ then there exists a sequence $(f_n)_{n\in\omega}$ such that $\lim f_n = f$ and for all $n \in \omega$, $f_n \in \partial F_0 - F_1$. Then there exists $k \in \omega$ such that for all $n \geq k$, $f_n \in G \cap F_0 \subseteq F_1$. Thus $f_n \in F_1$ for every $n \geq k$ and this is a contradiction.

We can now conclude our argument as follows:

$$\begin{aligned}
\mathrm{Rs}_2(A^c) &= \overline{\partial F_0 - F_1} \cap F_1 \\
&= \overline{\partial F_0 - F_1} \cap (\partial_{F_0} F_1 \cup \mathrm{int}_{F_0}(F_1)) \\
&= (\overline{\partial F_0 - F_1} \cap \delta_{F_0} F_1) \cup (\overline{\partial F_0 - F_1} \cap \mathrm{int}_{F_0}(F_1)) \\
&= \overline{\partial F_0 - F_1} \cap \partial_{F_0} F_1
\end{aligned}$$

as required.

We will make use of the above topological concepts in order to prove several Lipschitz and Wadge determinacy results in the next two sections. Thus, it will be crucial to express these notions in the language of second order arithmetic in order to formalize such proofs. To do that, the key idea is to re-express these notions by means of paths of appropriate trees. Below we give some examples.

Let $T, S \subseteq X^{<\omega}$ be trees with $T \subseteq S$. The boundary of T is the following set

$$\delta(T) = \{t \in T : \exists t' (t \subset t' \wedge t' \notin T)\}$$

The boundary of T in S is the set

$$\delta_S T = \{t \in T : \exists t' (t \subset t' \wedge t' \in S - T)\}$$

We recall that $[T]$ is the set of all infinite branches of T, i.e.

$$[T] = \{f \in X^\omega : \forall n \in \omega\, f[n] \in T\}.$$

Now let us notice that:

1. $\delta(T)$ and $\delta_S T$ are also trees.
2. $[\delta(T)]$ is, precisely, the boundary of the set $[T]$, i.e.
$$\partial[T] = [\delta(T)] = \{f \in [T] : \forall k\, \exists s\, (f[k] \subseteq s \wedge s \notin T)\}.$$
3. $[\delta_S T]$ is, precisely, the boundary of $[T]$ in $[S]$, i.e.
$$\partial_{[S]}[T] = [\delta_S T] = \{f \in [T] : \forall k\, \exists s\, (f[k] \subseteq s \wedge s \in S - T)\}$$

Also note that, in general,

$$[\delta_S T] \neq [\delta(T)] \cap [S] = [\delta(T) \cap S]$$

2.2 Determinacy of Lipschitz and Wadge games in Cantor space

In this section we prove Lipschitz and Wadge determinacy in the Cantor space for the first five levels of Wadge hierarchy (see Section 7 of the previous chapter). For $A, B \subseteq 2^\omega$ we denote by $G^*_L(A, B)$ the Lipschitz game in Cantor space for A and B, and by $G^*_W(A, B)$ the corresponding Wadge game. We will use the notation and definitions introduced in Section 4 of the previous chapter with adequate modifications. We will denote by $2^{<\omega}$ the set of finite sequences from $\{0, 1\}$, by $\mathrm{Seq}^*_{\mathrm{even}}$ the subset of $2^{<\omega}$ containing all finite sequences of even length, and by $\mathrm{Seq}^*_{\mathrm{odd}}$ the subset of $2^{<\omega}$ containing all finite sequences of odd length.

The arguments that we present in this chapter will be later formalized in second order arithmetic. This will be done in Chapter 4 and to make that work easier we will here set up the winning strategies needed in the proofs of Lipschitz determinacy in a very detailed form.

The following lemma settles the issue of Lipschitz and Wadge determinacy in Cantor space for the first two levels of the Wadge hierarchy.

Lemma 2.2 *Let A and B be clopen sets in Cantor space.*

1. *$G^*_L(A, B)$ is determined.*
2. *$G^*_W(A, B)$ is determined.*

Proof. We shall assume that A and B are both different from $\emptyset$ and 2^ω. We can check easily that the result holds in these cases.
Since $A, B \in \mathbf{\Delta}^0_1$, there exist pruned binary trees S, S', T and T' such that

$$A = [S], \quad A^c = [S'], \quad B = [T], \text{ and } B^c = [T'].$$

Since $[S \cap S'] = [S] \cap [S'] = \emptyset$, by König's lemma, $S \cap S'$ is a finite tree. Let us define $l_1 = \max\{|s| : s \in S \cap S'\}$. In a similar way, we define $l_2 = \max\{|t| : t \in T \cap T'\}$.

(1): We must show $G^*_L([S], [T])$ is determined. We distinguish two cases:
<u>Case 1</u>: $l_1 \leq l_2$.
Then player II has a winning strategy in the game $G^*_L([S], [T])$. Indeed, let $t_0 \in T \cap T'$ such that $|t_0| = l_2$ (such an element exists by definition of l_2). Observe that, since T and T' are pruned trees and due to the maximal character of l_2, we have

$$\forall t\, (t_0 * \langle 0 \rangle \subseteq t \to t \in T) \wedge \forall t\, (t_0 * \langle 1 \rangle \subseteq t \to t \in T')$$

or

$$\forall t\, (t_0 * \langle 0 \rangle \subseteq t \to t \in T') \wedge \forall t\, (t_0 * \langle 1 \rangle \subseteq t \to t \in T).$$

Using that $l_1 \leq l_2$ we also get that for each s_0 with $|s_0| \geq l_2$,

$$\forall s\, (s_0 * \langle 0 \rangle \subseteq s \to s \in S) \wedge \forall s\, (s_0 * \langle 1 \rangle \subseteq s \to s \in S')$$

or

$$\forall s\,(s_0 * \langle 0\rangle \subseteq s \to s \in S') \wedge \forall s\,(s_0 * \langle 1\rangle \subseteq s \to s \in S).$$

Having this in mind, a strategy $\mathcal{E}_{\mathrm{II}}$: $\mathrm{Seq}^*_{\mathrm{odd}} \to \{0,1\}$ for player II can be defined as follows. For all $s,t \in 2^{<\omega}$ with $|s| = j+1$ and $|t| = j$, where $j \in \omega$, we define

$$\mathcal{E}_{\mathrm{II}}(s \otimes t) = \begin{cases} (t_0)_j & \text{if } j+1 < l_2 \\ \min\{k : t_0 * \langle k\rangle \in T\} & \text{if } s \in S \wedge j+1 = l_2 \\ \min\{k : t_0 * \langle k\rangle \in T'\} & \text{if } s \in S' \wedge j+1 = l_2 \\ 0 & \text{otherwise} \end{cases}$$

It is not hard to see that $\mathcal{E}_{\mathrm{II}}$ is a winning strategy for player II. Indeed, according to $\mathcal{E}_{\mathrm{II}}$ player II enumerates t_0 no matter what player I does. At the time player II reaches $(t_0)_{l_2-1}$ player I has already irrevocably committed himself to enumerate an element of either A or A^c since $l_1 \leq l_2$. Player II, however, can still decide to enumerate an element of either B or B^c. Now if player I is enumerating an element of A, and he will do it forever, player II chooses k and enumerates an element of B, also forever. If player I is enumerating an element of A^c, then player II chooses k' and enumerates an element of B^c. Thus, for every $f \in 2^\omega$ we have $f \in A$ iff $f \otimes^{\mathrm{II}} \mathcal{E}_{\mathrm{II}} \in B$ and hence $\mathcal{E}_{\mathrm{II}}$ is a winning strategy for player II.

Case 2: $l_2 < l_1$.
Then player I has a winning strategy in the game $G^*_L([S],[T])$. Indeed, let $s_0 \in S \cap S'$ such that $|s_0| = l_1$. Then

$$\forall s\,(s_0 * \langle 0\rangle \subseteq s \to s \in S) \wedge \forall s\,(s_0 * \langle 1\rangle \subseteq s \to s \in S')$$

or

$$\forall s\,(s_0 * \langle 0\rangle \subseteq s \to s \in S') \wedge \forall s\,(s_0 * \langle 1\rangle \subseteq s \to s \in S).$$

In addition, for each finite sequence t_0 with $|t_0| \geq l_1 - 1$,

$$\forall t\,(t_0 * \langle 0\rangle \subseteq t \to t \in T) \wedge \forall t\,(t_0 * \langle 1\rangle \subseteq t \to t \in T')$$

or

$$\forall t\,(t_0 * \langle 0\rangle \subseteq t \to t \in T') \wedge \forall t\,(t_0 * \langle 1\rangle \subseteq t \to t \in T).$$

Proceeding as before we can define a strategy $\mathcal{E}_{\mathrm{I}}$: $\mathrm{Seq}^*_{\mathrm{even}} \to \{0,1\}$ for player I. First we put $\mathcal{E}_{\mathrm{I}}(\langle\rangle) = (s_0)_0$. Now for all $s,t \in 2^{<\omega}$ with $|s| = |t| = j$ and $j \geq 1$, we define

$$\mathcal{E}_{\mathrm{I}}(s \otimes t) = \begin{cases} (s_0)_j & \text{if } j < l_1 \\ \min\{k : s_0 * \langle k\rangle \in S'\} & \text{if } t \in T \wedge j = l_1 \\ \min\{k : s_0 * \langle k\rangle \in S\} & \text{if } t \in T' \wedge j = l_1 \\ 0 & \text{otherwise} \end{cases}$$

To see that $\mathcal{E}_{\mathrm{I}}$ is a winning strategy for player I let us observe the following. Player I enumerates s_0 no matter what player II plays. After player I reaches $(s_0)_{l_1-1}$ player II must decide to continue in either B or B^c, if she has not already decided. Now if player II decides to commit herself to B, the next move of player I is to choose k' and to play forever outside A, i.e. in A^c. If, on the contrary, player II decides to stay in B^c for the rest of the game, then in the next move player I chooses k and remains forever inside A. In both cases, for every $g \in 2^\omega$, $g \in B$ iff $\mathcal{E}_{\mathrm{I}} \otimes g \in A^c$. Thus $\mathcal{E}_{\mathrm{I}}$ is a winning strategy for player I.

(2): We distinguish the same two cases. Concerning case 1, a winning strategy for player II in a Lipschitz game yields automatically a winning strategy for player II in the corresponding Wadge game. On the other hand, it is not hard to see that if player II is allowed to pass, she will also win the game in case 2. In fact, player II can keep passing until player I commits himself to remaining either in A or outside A forever. In conclusion, the Wadge game $G^*_W(A, B)$ is determined and now it is player II who always has a winning strategy.

This completes the proof of the lemma. ■

Remark 2.3

1. *In the previous proof we used pruned trees and König's lemma. In set theory this is not a concern since every tree can be pruned, i.e. contains a pruned subtree with the same set of paths, and the König's lemma is a consequence of* $\mathbf{AC}_\omega$*. In second order arithmetic, however, it will be crucial to derive the fact that every tree corresponding to a clopen set can be pruned from König's lemma itself. This will enable us to prove the result within* $\mathbf{WKL}_0$*.*

2. *The above proof of determinacy of games* $G^*_W(A, B)$*, where* A *and* B *are clopen sets, implies that clopen sets different from* $\emptyset$ *and* 2^ω *form a Wadge degree. Similarly we get that each clopen set different from* $\emptyset$ *and* 2^ω *is* $\mathbf{\Delta}^0_1$*-complete for Wadge reducibility.*

The following notation will be useful in the proofs of this and the next sections. Let X be $\{0, 1\}$ or ω and $s \in X^{<\omega}$. Then the tree $s * X^{<\omega}$ of the initial segments and extensions of s is defined by

$$s * X^{<\omega} = \{t \in X^{<\omega} : t \subseteq s \vee s \subseteq t\}.$$

Also we define

$$s * X^\omega = \{f \in X^\omega : s \subset f\}.$$

Observe that $[s * X^{<\omega}] = s * X^\omega$.

Corollary 2.4 *There is a sequence of Lipschitz degrees* $\{[A_i]_L : i \in \omega\}$ *in Cantor space such that:*

1. *for each* $i \in \omega$*,* A_i *is a clopen set different from* $\emptyset$ *and* 2^ω*;*

2. *for each* $i \in \omega$*,* $[A_i]_L \prec_L [A_{i+1}]_L$*; and*

3. *for each clopen set B different from $\emptyset$ and 2^ω there exists $i \in \omega$ such that $[B]_L = [A_i]_L$.*

Proof. It follows from Lemma 2.2 putting $A_i = 0^{(i+1)} * 2^\omega$. ■

Lemma 2.5 *Let A and B be closed sets in Cantor space.*

1. *$G^*_L(A, B)$ is determined.*
2. *$G^*_W(A, B)$ is determined.*

Proof. (1): It is enough to show that if $S, T \subseteq 2^{<\omega}$ are pruned binary trees then the Lipschitz game $G^*_L([S], [T])$ is determined. We distinguish three cases:
Case 1: $\partial[T] = [\delta(T)] \neq \emptyset$.
Then there is $g \in [T]$ such that $\forall k \, \exists t \, (g[k] \subseteq t \wedge t \notin T)$ and a winning strategy for player II, $\mathcal{E}_{\mathrm{II}}$, can be defined as follows. For all $s, t \in 2^{<\omega}$ with $|s| = j + 1$ and $|t| = j$, we put

$$\mathcal{E}_{\mathrm{II}}(s \otimes t) = \begin{cases} g(j) & \text{if } s \in S \\ \min\{k : t * \langle k \rangle \notin T\} & \text{if } s \notin S \wedge (t * \langle 0 \rangle \notin T \vee t * \langle 1 \rangle \notin T) \\ g(j) & \text{if } s \notin S \wedge (t * \langle 0 \rangle \in T \wedge t * \langle 1 \rangle \in T) \end{cases}$$

To see that $\mathcal{E}_{\mathrm{II}}$ is a winning strategy for player II it suffices to observe that following strategy $\mathcal{E}_{\mathrm{II}}$ player II enumerates g as long as player I is enumerating an element of $[S]$, and that she can always use a branch of g to leave $[T]$ if player I decides to leave $[S]$. Thus, for every $f \in 2^\omega$ we have $f \in A$ iff $f \otimes^{\mathrm{II}} \mathcal{E}_{\mathrm{II}} \in B$ and hence $\mathcal{E}_{\mathrm{II}}$ is a winning strategy for player II.

Case 2: $[\delta(T)] = \emptyset$ and $\partial[S] = [\delta(S)] \neq \emptyset$.
Then there exists $f \in [S]$ such that $\forall k \, \exists s \, (f[k] \subseteq s \wedge s \notin S)$. Note that $\delta(T)$ is a binary tree with no path (since $[\delta(T)] = \emptyset$). Hence, $\delta(T)$ must be finite and, as a consequence, there exists $l = \max(\{|t| : \; t \in \delta(T)\}$. A winning strategy for player I, $\mathcal{E}_{\mathrm{I}}$, can be defined as follows. Let $\mathcal{E}_{\mathrm{I}}(\langle\rangle) = f(0)$ and for all $s, t \in 2^{<\omega}$ with $|s| = |t| = j \geq 1$,

$$\mathcal{E}_{\mathrm{I}}(s \otimes t) = \begin{cases} f(j) & \text{if } t \notin T \vee (t \in T \wedge j \leq l) \\ \min\{k : t * \langle k \rangle \notin S\} & \text{if } t \in T \wedge j > l \wedge (s * \langle 0 \rangle \notin S \vee s * \langle 1 \rangle \notin S) \\ f(j) & \text{if } t \in T \wedge j > l \wedge (s * \langle 0 \rangle \in S \wedge s * \langle 1 \rangle \in S) \end{cases}$$

To see that $\mathcal{E}_{\mathrm{I}}$ is a winning strategy for player I let us observe the following. Player I starts by enumerating f and continues enumerating f until he reaches $f(l)$, no matter what player II plays. At this point player II must decide to continue in either B or B^c, if she has not already decided. Now if player II decides to commit herself to B, player I continues to enumerate f; otherwise he leaves A as soon as he can. Hence, for every $g \in 2^\omega$, $g \in B$ iff $\mathcal{E}_{\mathrm{I}} \otimes g \in A^c$ and $\mathcal{E}_{\mathrm{I}}$ is a winning strategy for player I.

Case 3: $[\delta(T)] = \emptyset$ and $[\delta(S)] = \emptyset$.
Then, as in Case 2, $\delta(T)$ is a finite tree and we can define $l = \max(\{|t'| : t' \in T\}$. From the definition of l we immediately get that $[T]$ is $\mathbf{\Delta}^0_1$:

$$g \in [T] \leftrightarrow \forall u\,(g[u] \in T) \leftrightarrow g[l] \in [T].$$

A similar reasoning can be applied to $[S]$ and, therefore, $G_L([S],[T])$ is determined since $[S]$ and $[T]$ are clopen (see Lemma 2.2).

(2): For Wadge games, it is clear that player II has a winning strategy in cases 1 and 3 above (as usual, we are assuming that $[S]$ and $[T]$ are different from $\emptyset$ and 2^ω). As for case 2, it is player I who has a winning strategy. Indeed, player I can enumerate $f \in \partial[S]$ as long as player II is passing or enumerating some $t \in \delta(T)$. Since she cannot do this forever, player I wins the game.

This completes the proof of the lemma. ■

Remark 2.6

1. *Cases 2 and 3 are obtained by applying König's lemma to a finite tree defined by an existential condition. Since this possibility is not available within* $\mathbf{WKL}_0$*, we will be forced to derive this result within a subsystem of second order arithmetic stronger than* $\mathbf{WKL}_0$*, namely within* $\mathbf{ACA}_0$*.*

   In addition, we have assumed that the trees characterizing closed sets A and B are pruned. This fact can also be used in second order arithmetic. In fact, in Chapter 4 we will show that the principle asserting that every tree can be pruned is equivalent to $\mathbf{ACA}_0$ *over* $\mathbf{RCA}_0$*.*

2. *It follows from the previous proof that* $\mathbf{\Pi}^0_1 - \mathbf{\Sigma}^0_1$ *form both a Lipschitz and a Wadge degree. Also, it follows that every closed set with a nonempty boundary is* $\mathbf{\Pi}^0_1$*-complete for both Lipschitz and Wadge reducibility.*

Corollary 2.7 *There exists a closed and not open set of the Cantor space, B, such that for every closed and not open set of the Cantor space, A, we have* $[A]_L =_L [B]$

Proof. It follows from Lemma 2.5 putting $B = \{\vec{0}\} \cup \bigcup_k 0^{(k)} * \langle 1 \rangle * \langle 0 \rangle * 2^\omega$. ■

Lemma 2.8 *Let A and B be subsets of the Cantor space such that* $A, B \in \mathbf{\Sigma}^0_1 \cup \mathbf{\Pi}^0_1$*.*

1. $G^*_L(A,B)$ *is determined.*

2. $G^*_W(A,B)$ *is determined.*

Proof. The case $A, B \in \mathbf{\Pi}^0_1$ is just Lemma 2.5. Now bearing in mind that the strategies for a game $G^*_L(A,B)$ are also strategies for the dual game $G^*_L(A^c, B^c)$ we obtain that $G^*_L(A,B)$ is determined when $A, B \in \mathbf{\Sigma}^0_1$. Analogously we obtain that $G^*_W(A,B)$ is determined when $A, B \in \mathbf{\Sigma}^0_1$.

(1): Let us prove that $G^*_L(A,B)$ is determined for $A \in \mathbf{\Sigma}^0_1$ and $B \in \mathbf{\Pi}^0_1$. (The case where $B \in \mathbf{\Sigma}^0_1$ and $A \in \mathbf{\Pi}^0_1$ follows from this one by duality.)
It is enough to show that if $S, T \subseteq 2^{<\omega}$ are pruned binary trees, then the Lipschitz game $G^*_L([S]^c, [T])$ is determined. We distinguish two cases:

Case 1: $\partial[S] = [\delta(S)] \neq \emptyset$.
Then there exists $f \in [S]$ such that $\forall k\, \exists s\, (f[k] \subseteq s \wedge s \notin S)$.
A winning strategy $\mathcal{E}_{\mathrm{I}}$, for player I, can be defined as follows. Let $\mathcal{E}_{\mathrm{I}}(\langle\rangle) = f(0)$ and for all $s, t \in 2^{<\omega}$ with $|s| = |t| = j \geq 1$, we put

$$\mathcal{E}_{\mathrm{I}}(s \otimes t) = \begin{cases} f(j) & \text{if } t \in T \\ \min\{k : t * \langle k\rangle \notin S\} & \text{if } t \notin T \wedge (s * \langle 0\rangle \notin S \vee s * \langle 1\rangle \notin S) \\ f(j) & \text{if } t \notin T \wedge (s * \langle 0\rangle \in S \wedge s * \langle 1\rangle \in S) \end{cases}$$

Case 2: $\partial[S] = [\delta(S)] = \emptyset$.
Then $\delta(S)$ is a finite binary tree. Let $l = \max(\{|s| : \ s \in \delta(S)\}$. We have

$$\forall f\, (f \in [S] \leftrightarrow f[l+1] \in S)$$

and, so, $[S]$ is clopen and $G^*_L([S]^c, [T])$ is determined by Lemma 2.5.

(2): Again it suffices to show that $G^*_W([S]^c, [T])$ is determined for $S, T \subseteq 2^{<\omega}$ pruned binary trees. We distinguish the same two cases. Although in Wadge games player II has an extra resource (she can pass a finite number of times), in case 1 player I can still use essentially the same strategy to win the game. Case 2 is completely analogous to the corresponding case for Lipschitz games.

This completes the proof of the lemma. ∎

Remark 2.9

1. *The above arguments also yield that* $\mathbf{\Pi}^0_1 - \mathbf{\Sigma}^0_1$ *and* $\mathbf{\Sigma}^0_1 - \mathbf{\Pi}^0_1$ *form a pair of incomparable dual Lipschitz/Wadge degrees.*

2. *Being a non-self-dual pair, it is natural to obtain that it is player I who has a winning strategy when his opponent plays in a set which belongs to the other non-self-dual degree or to degrees below.*

In the next lemmas the topological analysis of Section 1, especially the useful fact number 5 there, will be used profusely. The first lemma deals with sets in the third level of Wadge hierarchy.

Lemma 2.10 *Let A and B be subsets of the Cantor space such that $A, B, A^c, B^c \in \mathbf{Df}_2$.*

1. *$G_L^*(A, B)$ is determined.*
2. *$G_W^*(A, B)$ is determined.*

Proof. Without loss of generality, we can assume that there exist pruned binary trees S_0, S_1, T_0, T_1 such that

1. $S_1 \subseteq S_0$ and $T_1 \subseteq T_0$.
2. $A = [S_0] - [S_1]$, and $B = [T_0] - [T_1]$.

Since $A^c \in \mathbf{Df}_2$,

$$\mathrm{Rs}_2(A^c) = \partial_{[S_0]}[S_1] \cap \overline{\partial[S_0] - [S_1]} = [\delta_{S_0} S_1] \cap \overline{[\delta(S_0)] - [S_1]} = \emptyset.$$

Observe that $\mathrm{Rs}_2(A^c)$ is a closed set and so $\mathrm{Rs}_2(A^c) = [\delta(S_0, S_1)]$ for some tree $\delta(S_0, S_1)$, namely

$$\delta(S_0, S_1) = \{s \in S_1 : \exists f_1, f_2\, (f_1 \in [\delta_{S_0} S_1] \wedge f_2 \in [\delta(S_0)] - [S_1] \wedge s \subset f_1 \wedge s \subset f_2)\}.$$

By König's lemma, $\delta(S_0, S_1)$ must be finite. Similarly,

$$\mathrm{Rs}_2(B^c) = \partial_{[T_0]}[T_1] \cap \overline{\partial[T_0] - [T_1]} = [\delta_{T_0} T_1] \cap \overline{[\delta(T_0)] - [T_1]} = \emptyset$$

and hence $\delta(T_0, T_1)$ is also a finite tree defined putting

$$\delta(T_0, T_1) = \{t \in T_1 : \exists g_1, g_2\, (g_1 \in [\delta_{T_0} T_1] \wedge g_2 \in [\delta(T_0)] - [T_1] \wedge t \subset g_1 \wedge t \subset g_2)\}.$$

Let $a = \max\{|s| : s \in \delta(S_1, S_1)\}$ and $b = \max\{|s| : s \in \delta(T_0, T_1)\}$.

(1): To prove that $G_L^*(A, B)$ is determined we distinguish three main cases with several subcases.
<u>Case 1</u>: $[\delta_{T_0} T_1] \neq \emptyset$, $[\delta(T_0)] - [T_1] \neq \emptyset$, and $[\delta_{S_0} S_1] \neq \emptyset$, $[\delta(S_0)] - [S_1] \neq \emptyset$.
We distinguish two subcases:

1. If $a \leq b$, then player II has a winning strategy, $\mathcal{E}_{\mathrm{II}}$.

   Let $t_b \in \delta(T_0, T_1)$ be such that $|t_b| = b$, i.e. t_b is a sequence of $\delta(T_0, T_1)$ of maximal length. Let us fix $g_1, g_2 \in 2^\omega$ such that $t_b \subseteq g_1$, $g_1 \in [\delta_{T_0} T_1]$, $t_b \subseteq g_2$, and $g_2 \in [\delta(T_0)] - [T_1]$. Also recall that $(t_b)_j = t_b(j)$ and $s\,[k]$ denotes the finite sequence $\langle s(0), ..., s(k-1)\rangle$. We can define a winning strategy for player II as follows. For all $s, t \in 2^{<\omega}$ with $|s| = j + 1$ and $|t| = j$, we put

$$
\mathcal{E}_{\mathrm{II}}(s\otimes t) = \begin{cases}
(t_b)_j & \text{if } j < b \\
g_1(j) & \text{if } j \geq b \wedge s[b+1] \notin S_0 \\
g_2(j) & \text{if } j \geq b \wedge s[b+1] \in S_0 - S_1 \wedge s \in S_0 \\
g_2(j) & \text{if } j \geq b \wedge s[b+1] \in S_0 - S_1 \wedge s \notin S_0 \wedge t * \langle 0 \rangle \in T_0 \wedge t * \langle 1 \rangle \in T_0 \\
k & \text{if } j \geq b \wedge s[b+1] \in S_0 - S_1 \wedge s \notin S_0 \wedge (t * \langle 0 \rangle \notin T_0 \vee t * \langle 1 \rangle \notin T_0) \\
 & \text{and } k = \min\{i : t * \langle i \rangle \notin T_0\} \\
g_1(j) & \text{if } j \geq b \wedge s[b+1] \in S_1 \wedge \exists f\, (f \in [\delta_{S_0} S_1] \wedge s[b+1] \subset f) \wedge \\
 & (s \in S_1 \vee s \notin S_0) \\
g_1(j) & \text{if } j \geq b \wedge s[b+1] \in S_1 \wedge \exists f\, (f \in [\delta_{S_0} S_1] \wedge s[b+1] \subset f) \wedge \\
 & s \in S_0 - S_1 \wedge \exists s'(s \subseteq s' \wedge s' \notin S_0) \\
g_1(j) & \text{if } j \geq b \wedge s[b+1] \in S_1 \wedge \exists f\, (f \in [\delta_{S_0} S_1] \wedge s[b+1] \subset f) \wedge \\
 & s \in S_0 - S_1 \wedge \ \forall s'(s \subset s' \rightarrow s' \in S_0) \wedge \\
 & (t * \langle 0 \rangle \notin T_0 - T_1 \wedge t * \langle 1 \rangle \notin T_0 - T_1) \\
k & \text{if } j \geq b \wedge s[b+1] \in S_1 \wedge \exists f\, (f \in [\delta_{S_0} S_1] \wedge s[b+1] \subset f) \wedge \\
 & s \in S_0 - S_1 \wedge \ \forall s'(s \subset s' \rightarrow s' \in S_0) \wedge \\
 & (t * \langle 0 \rangle \in T_0 - T_1 \vee t * \langle 1 \rangle \in T_0 - T_1) \\
 & \text{and } k = \min\{i : t * \langle i \rangle \in T_0 - T_1\} \\
g_2(j) & \text{if } j \geq b \wedge s[b+1] \in S_1 \wedge \neg\exists f\, (f \in [\delta_{S_0} S_1] \wedge s[b+1] \subset f) \wedge \\
 & (s \in \delta_{S_0} S_1 \vee s \in S_0 - S_1) \\
g_2(j) & \text{if } j \geq b \wedge s[b+1] \in S_1 \wedge \neg\exists f\, (f \in [\delta_{S_0} S_1] \wedge s[b+1] \subset f) \wedge \\
 & s \notin \delta_{S_0} S_1 \wedge (s \in S_1 \vee s \notin S_0) \wedge t * \langle 0 \rangle \in T_0 \wedge t * \langle 1 \rangle \in T_0 \\
k & \text{if } j \geq b \wedge s[b+1] \in S_1 \wedge \neg\exists f\, (f \in [\delta_{S_0} S_1] \wedge s[b+1] \subset f) \wedge \\
 & s \notin \delta_{S_0} S_1 \wedge (s \in S_1 \vee s \notin S_0) \wedge (t * \langle 0 \rangle \notin T_0 \vee t * \langle 1 \rangle \notin T_0) \\
 & \text{and } k = \min\{i : t * \langle i \rangle \notin T_0\}
\end{cases}
$$

That is to say, player II plays using t_b during the first b rounds. Then, she plays according to one of the following cases:

Case 1A: At the $(b+1)$-th round, player I has played outside S_0. Then, player I will continue outside S_0 forever (S_0 is a tree) and player II can simply play using g_1 (recall that $g_1 \notin [T_0] - [T_1]$ and $t_b \subset g_1$).

Case 1B: At the $(b+1)$-th round, player I has played in $S_0 - S_1$. Then:

- As long as player I continues playing inside S_0, player II will play using g_2. Thus, if player I continues playing inside S_0 forever (and so his resulting play is a point in $[S_0] - [S_1]$), player II has also produced a point in $[T_0] - [T_1]$.
- Assume that at some round $c > b+1$, player I plays outside S_0 (and hence he will continue outside S_0 forever). Since player II has been playing using g_2 so far and g_2 belongs to $[\delta(T_0)]$, there must be some round $d \geq c$ in which player II can play outside T_0. Then, player II continues playing using g_2 waiting for such a round d to appear; and at round d, she plays outside T_0 (and hence she will continue outside T_0 forever).

Case 1C: At the $(b+1)$-th round, player I has played in S_1 and $s[b+1]$ has some extensions in $[\delta_{S_0} S_1]$.

- As long as player I continues playing inside S_1, player II will play using g_1. Thus, if player I continues playing inside S_1 forever (and so his resulting play is a point not in $[S_0] - [S_1]$), player II has also produced a point not in $[T_0] - [T_1]$.

Now assume that at some round $c > b+1$, player I plays outside S_1. Then:

- As long as player I plays inside $\delta(S_0)$, player II will continue using g_1. But observe that player I cannot continue playing in $\delta(S_0)$ forever. For otherwise $s[b+1]$ would be in $\delta(S_0, S_1)$, which is impossible since $a \leq b$. So, after a finite number of steps one of the following two cases will hold.
- If player I plays outside S_0 (and so his resulting play will be a point not in $[S_0] - [S_1]$), player II will simply continue playing using g_1.
- If player I plays in $S_0 - \delta(S_0)$, then he will be forced to play in $S_0 - S_1$ forever and thus his resulting play will be a point in $[S_0] - [S_1]$. But so far player II has been playing using g_1, which is a point of $[\delta_{T_0} T_1]$. Consequently, there must be some round d in which player II can play in $T_0 - T_1$. Then player II continues playing using g_1 waiting for such a round to appear; and at round d, she plays in $T_0 - T_1$ and will continue playing in $T_0 - T_1$ forever (this is possible because T_0 is a pruned tree).

Case 1D: At the $(b+1)$-th round, player I has played in S_1 and $s\,[b+1]$ has no extension in $[\delta_{S_0}S_1]$.

- As long as player I plays in $\delta_{S_0}S_1$, player II will play using g_2. But observe that player I cannot continue playing in $\delta_{S_0}S_1$ forever, for we are assuming that $s\,[b+1]$ has no extension in $[\delta_{S_0}S_1]$. So, after a finite number of steps player I will necessarily play outside $\delta_{S_0}S_1$. Then:

- As long as player I plays in $S_0 - S_1$, player II will continue using g_2.

- If at some round player I has played s either in S_1 or outside S_0, then player I is forced to play outside $[S_0] - [S_1]$ since we are assuming that $s \notin \delta(S_0, S_1)$. But so far player II has been playing using g_2, which is a point of $[\delta(T_0)] - [T_1]$. Consequently, there must be some round d in which player II can play outside T_0. Then player II continues playing using g_2 waiting for such a round to appear; and at round d, she plays outside T_0 and hence will play outside T_0 forever.

This proves that $\mathcal{E}_{\mathrm{II}}$ is a winning strategy for player II.

2. If $a > b$, then player I has a winning strategy.

Let $s_a \in \delta(S_0, S_1)$ be such that $|s_a| = a$, i.e. let s_a be a sequence of $\delta(S_0, S_1)$ of maximal length. Let us fix $f_1, f_2 \in 2^\omega$ such that $s_a \subseteq f_1$, $f_1 \in [\delta_{S_0}S_1]$, $s_a \subseteq f_2$, and $f_2 \in [\delta(S_0)] - [S_1]$. Then player I wins the game with the following strategy. Let $\mathcal{E}_{\mathrm{I}}(\langle\rangle) = (s_a)_0$ and for all $s, t \in 2^{<\omega}$ with $|s| = |t| = j \geq 1$, put

$$\mathcal{E}_{\mathrm{I}}(s \otimes t) = \begin{cases} (s_a)_j & \text{if } j < a \\ f_2(j) & \text{if } j \geq a \wedge t[a] \notin T_0 \\ f_1(j) & \text{if } j \geq a \wedge t[a] \in T_0 - T_1 \wedge t \in T_0 \\ f_1(j) & \text{if } j \geq a \wedge t[a] \in T_0 - T_1 \wedge t \notin T_0 \wedge s * \langle 0 \rangle \notin S_0 - S_1 \wedge s * \langle 1 \rangle \notin S_0 - S_1 \\ k & \text{if } j \geq a \wedge t[a] \in T_0 - T_1 \wedge t \notin T_0 \wedge (s * \langle 0 \rangle \in S_0 - S_1 \vee s * \langle 1 \rangle \in S_0 - S_1) \\ & \text{and } k = \min\{i : s * \langle i \rangle \in S_0 - S_1\} \\ f_2(j) & \text{if } j \geq a \wedge t[a] \in T_1 \wedge \exists g\, (g \in [\delta_{T_0} T_1] \wedge t[a] \subset g) \wedge (t \in T_1 \vee t \notin T_0) \\ f_2(j) & \text{if } j \geq a \wedge t[a] \in T_1 \wedge \exists g\, (g \in [\delta_{T_0} T_1] \wedge t[a] \subset g) \wedge \\ & t \in T_0 - T_1 \wedge \exists t' (t \subseteq t' \wedge t' \notin T_0) \\ f_2(j) & \text{if } j \geq a \wedge t[a] \in T_1 \wedge \exists g\, (g \in [\delta_{T_0} T_1] \wedge t[a] \subset g) \wedge \\ & t \in T_0 - T_1 \wedge \ \forall t' (t \subset t' \to t' \in T_0) \wedge \\ & (s * \langle 0 \rangle \in S_0 - S_1 \wedge s * \langle 1 \rangle \in S_0 - S_1) \\ k & \text{if } j \geq a \wedge t[a] \in T_1 \wedge \exists g\, (g \in [\delta_{T_0} T_1] \wedge t[a] \subset g) \wedge \\ & t \in T_0 - T_1 \wedge \ \forall t' (t \subset t' \to t' \in T_0) \wedge \\ & (s * \langle 0 \rangle \notin S_0 - S_1 \vee s * \langle 1 \rangle \notin S_0 - S_1) \\ & \text{and } k = \min\{i : s * \langle i \rangle \in S_0 - S_1\} \\ f_1(j) & \text{if } j \geq a \wedge t[a] \in T_1 \wedge \neg \exists g\, (g \in [\delta_{T_0} T_1] \wedge t[a] \subset g) \wedge \\ & (t \in \delta_{T_0} T_1 \vee t \in T_0 - T_1) \\ f_1(j) & \text{if } j \geq a \wedge t[a] \in T_1 \wedge \neg \exists g\, (g \in [\delta_{T_0} T_1] \wedge t[a] \subset g) \wedge \\ & t \notin \delta_{T_0} T_1 \wedge (t \in T_1 \vee t \notin T_0) \wedge s * \langle 0 \rangle \notin S_0 - S_1 \wedge s * \langle 1 \rangle \notin S_0 - S_1 \\ k & \text{if } j \geq a \wedge t[a] \in T_1 \wedge \neg \exists g\, (g \in [\delta_{T_0} T_1] \wedge t[a] \subset g) \wedge \\ & t \notin \delta_{T_0} T_1 \wedge (t \in T_1 \vee t \notin T_0) \wedge (s * \langle 0 \rangle \in S_0 - S_1 \vee s * \langle 1 \rangle \in S_0 - S_1) \\ & \text{and } k = \min\{i : t * \langle i \rangle \in S_0 - S_1\} \end{cases}$$

That is to say, player I plays using s_a until round a. Then, he plays according to one of the following cases:

Case 2A: At the a-th round, player II has played outside T_0. Then, player II will continue outside T_0 forever (T_0 is a tree) and player I can simply play using f_2 (recall that $f_2 \in [S_0] - [S_1]$ and $s_a \subset f_2$).

Case 2B: At the a-th round, player II has played in $T_0 - T_1$. Then:

- As long as player II continues playing inside T_0, player I will play using f_1. Thus, if player II continues playing inside T_0 forever (and so her resulting play is a point in $[T_0] - [T_1]$, player I has produced a point in $[S_1]$.
- Assume that at some round $c > a$, player II plays outside T_0 (and hence she will continue outside T_0 forever). Since player I has been playing using f_1 so far and f_1 belongs to $[\delta_{S_0} S_1]$, there must be some round $d \geq c$ in which player I can enter $S_0 - S_1$. Then, player I continues playing using f_1 waiting for such a round d to appear; and at round d, he enters $S_0 - S_1$ (and hence he will continue inside $S_0 - S_1$ forever).

Case 2C: At the a-th round, player II has played in T_1 and $t\,[a]$ has some extensions in $[\delta_{T_0} T_1]$.

- As long as player II continues playing inside T_1, player I will play using f_2. Thus, if player II continues playing inside T_1 forever (and so his resulting play is a point not in $[T_0] - [T_1]$, player I has produced a point in $[S_0] - [S_1]$.

Now assume that at some round $c > a$, player II plays outside T_1. Then:

- As long as player II plays inside $\delta\,(T_0)$, player I will continue using f_2. But observe that player II cannot continue playing in $\delta\,(T_0)$ forever. For otherwise $t\,[a]$ would be in $\delta(T_0, T_1)$, which is impossible since $a > b$. So, after a finite number of steps one of the following two cases will hold.
- If player II plays outside T_0 (and so her resulting play will be a point not in $[T_0] - [T_1]$), player I will simply continue playing using f_2.
- If player II plays in $T_0 - \delta\,(T_0)$, then she will be forced to play in $T_0 - T_1$ forever and thus her resulting play will be a point in $[T_0] - [T_1]$. But so far player I has been playing using f_2, which is a point of $[\delta(S_0)] - [S_1]$. Consequently, there must be some round d in which player I can play not in $S_0 - S_1$. Then player I continues playing using f_2 waiting for such a round to appear; and at round d, he plays outside $S_0 - S_1$ and will continue playing outside $S_0 - S_1$ forever (this can be done because $[S_0]$ is a pruned tree).

Case 2D: At the a-th round, player II has played in T_1 and $t\,[a]$ has no extension in $[\delta_{T_0} T_1]$.

- As long as player II plays in $\delta_{T_0}T_1$, player I will play using f_1. But observe that player II cannot continue playing in $\delta_{T_0}T_1$ forever, for we are assuming that $t\,[a]$ has no extension in $[\delta_{T_0}T_1]$. So, after a finite number of steps player II will necessarily play outside $\delta_{T_0}T_1$. Then:
- As long as player II plays in $T_0 - T_1$, player I will continue using f_1.
- If at some round player II has played t either in T_1 or outside T_0, then player II is forced to play outside $[T_0]-[T_1]$ since we are assuming that $t \notin \delta(T_0, T_1)$. But so far player I has been playing using f_1, which is a point of $[\delta_{S_0}S_1]$. Consequently, there must be some round d in which player I can play in $S_0 - S_1$. Then player I continues playing using f_1 waiting for such a round to appear; and at round d, he plays in $S_0 - S_1$ and hence will play outside $S_0 - S_1$ forever.

Hence $\mathcal{E}_{\mathrm{I}}$ is a winning strategy for player I.

In the following discussion, we shall say that (T_0, T_1) (and similarly (S_0, S_1)) are in a *degenerated position* if $[\delta_{T_0}T_1] = \emptyset$ or $[\delta(T_0)] - [T_1] = \emptyset$.
Observe that if (T_0, T_1) are in a degenerated position then $[T_0] - [T_1]$ must be an open or closed set. Indeed, if $[\delta(T_0)] - [T_1] = \emptyset$ then $[\delta(T_0)] \subseteq [T_1]$ and thus,

$$[T_0] - [T_1] = ([\delta(T_0)] \cup \mathrm{int}([T_0])) - [T_1] = \mathrm{int}([T_0]) - [T_1]$$

is an open in Cantor space. If $[\delta_{T_0}T_1] = \emptyset$ then $[T_1]$ is open in $[T_0]$ and $[T_0] - [T_1]$ is closed in $[T_0]$ (and therefore also in Cantor space).

Case 2: One (and only one) of (T_0, T_1) or (S_0, S_1) are in a degenerated position.
If player I plays in a degenerated position, then A is closed or open and player II has a winning strategy (essentially, player II plays simulating the strategy described in Lemma 2.2 (case 1)):

- If A is closed then, since (T_0, T_1) is not in a degenerate position there exists $g_2 \in [\delta(T_0)] - [T_1]$ and player II can win the game using g_2.

- If A is an open set then, since (T_0, T_1) is not in a degenerate position there exists $g_1 \in [\delta_{T_0}T_1]$ and player II can win the game using g_1.

In a similar way it can be proved that if player II plays in a degenerated position then player I has a winning strategy:

- If B is open then, since (S_0, S_1) is not in a degenerate position there exists $f_2 \in [\delta(S_0)] - [S_1]$ and player I can win the game using f_2.

- If B is an closed set then, since (S_0, S_1) is not in a degenerate position there exists $f_1 \in [\delta_{S_0}S_1]$ and player I can win the game using f_1.

Case 3: (T_0, T_1) and (S_0, S_1) are in a degenerated position.
Recall that in these degenerated cases $[T_0] - [T_1]$ and $[S_0] - [S_1]$ are closed or open sets, so, the corresponding game is determined by Lemma 2.8.

(2): Taking into account the former cases and the fact that a winning strategy for player II in a Lipschitz game yields a winning strategy for player II in the corresponding Wadge game, it remains to examine the second part of case 1 and the second part of case 2. In the second part of case 1, since player II is now allowed to pass, she can wait until the moves of player I form a sequence of length a. Then according to the decision of player I of continuing to play in $[S_1]$, in A or outside A, player II plays in $[T_1]$, in B or outside B. Following this strategy player II eventually wins the game. In the second part of case 2, however, the allowance to pass is not enough for player II to win the game and it is easy to check that player I still has an winning strategy in $G_W^*(A, B)$.

This completes the proof of the lemma. ∎

Remark 2.11 *As a consequence of the latter lemma, sets $A \in \mathbf{Df}_2$, which are neither open nor closed, and whose complements also belong to $\mathbf{Df}_2$, form a Wadge degree.*

Corollary 2.12 *There is a sequence of Lipschitz degrees $\{[A_i]_L : i \in \omega\}$ in Cantor space such that:*

1. *for each $i \in \omega$, A_i and A_i^c are differences of closed sets which are neither open nor closed;*
2. *for each $i \in \omega$, $[A_i]_L \prec_L [A_{i+1}]_L$; and*
3. *for each set B such that B and B^c are differences of closed sets and B is neither open nor closed, there exists $i \in \omega$ such that $[B]_L = [A_i]_L$.*

Proof. Putting

$$A_i = 0^{(i+1)} * (\langle 0\rangle * (\{\vec{0}\} \cup \bigcup_k 0^{(k)} * \langle 1\rangle * \langle 0\rangle * 2^\omega) \cup (\langle 1\rangle * (\bigcup_k 0^{(k)} * \langle 1\rangle * \langle 0\rangle * 2^\omega)))$$

we observe that A_i is the difference of two closed sets $[F_0]$ and $[F_1]$ such that $0^{(i+1)} * \vec{0}$, $0^{(i+1)} * \langle 1\rangle * \vec{0} \in [F_0]$ and $[F_0] = \{\langle 1\rangle * \vec{0}\}$, and that A_i is neither open (since $\vec{0} \in A_i$) nor closed (since $\langle 1\rangle * \vec{0} \in \partial A_i \not\subseteq A_i$). The result follows from case 1 of the previous lemma. ∎

Now we examine the sets occupying the fourth level of Wadge hierarchy.

Lemma 2.13 *Let A and B be subsets of the Cantor space such that $A, B \in \mathbf{Df}_2$.*

1. *$G_L^*(A, B)$ is determined.*
2. *$G_W^*(A, B)$ is determined.*

Proof. Without loss of generality, we can assume that there exist pruned binary trees S_0, S_1, T_0, T_1 such that

1. $S_1 \subseteq S_0$ and $T_1 \subseteq T_0$.

2. $A = [S_0] - [S_1]$, and $B = [T_0] - [T_1]$.

(1): We distinguish three cases.
<u>Case 1</u>: $\mathrm{Rs}_2(B^c) = [\delta_{T_0} T_1] \cap \overline{[\delta(T_0)] - [T_1]} \neq \emptyset$,
Let $g_1 \in [\delta_{T_0} T_1] \cap \overline{[\delta(T_0)] - [T_1]}$. Then, player II wins using the following strategy. (This time we only give an informal description of the winning strategy; a detailed definition can be obtained using the ideas in the proof of Lemma 2.10.)

1. If player I plays $s \in S_1$ with $|s| = j + 1$ then player II plays $g_1(j)$.

2. If player I plays $s \notin S_0$, with $|s| = j + 1$, and player II's last move is $t = g_1 \lceil j \rceil$, then player II plays $g_1(j)$.

3. Suppose player I plays $s \in S_0 - S_1$ with $|s| = j + 1$ and $s \lceil j \rceil \in S_1$. Let $t \in T_1$ be player II's last move. Then there exists $g_2 \in [\delta(T_0)] - [T_1]$ such that $t \subseteq g_2$ (recall that $g_1 \in \overline{[\delta(T_0)] - [T_1]}$). Let us fix g_2 in advance. Then player II plays $g_2(j)$.

4. If player I continues playing $s \in S_0 - S_1$, then player II uses g_2 as in previous case.

5. Finally, if player I plays $s \notin S_0$ and $t \in T_0 - T_1$ is player II's last move, then player II plays the least $i \leq 1$ such that $t * \langle i \rangle \notin T_0$, if there exists such an element i. If there is no such i, player II plays $g_2(j)$ waiting for such an i to appear (sooner or later it will appear since $g_1 \in [\delta(T_0)]$).

6. If $s \notin S_0$ and $t \notin T_0$ then player II plays 0.

<u>Case 2</u>: $\mathrm{Rs}_2(B^c) = \emptyset$ but $\mathrm{Rs}_2(A^c) = [\delta_{S_0} S_1] \cap \overline{[\delta(S_0)] - [S_1]} \neq \emptyset$.
Since $[\delta_{T_0} T_1] \cap \overline{[\delta(T_0)] - [T_1]} = [\delta(T_0, T_1)]$, by König's lemma $\delta(T_0, T_1)$ is finite. Let $b = \max\{|t| : t \in \delta(T_0, T_1)\}$ and $f_1 \in [\delta_{S_0} S_1] \cap \overline{[\delta(S_0)] - [S_1]}$.
Then player I wins the game with the following strategy:
Player I starts by playing f_1 (that is, $f_1(0)$, $f_1(1), \ldots$ and so on) until round $b+1$ (included). Let t_0 denote the sequence of player II's moves until round $b+1$. We distinguish several cases:

1. $t_0 * 2^{<\omega} \subseteq T_1$ or $t_0 \notin T_0$.

   Since $f_1 \in \overline{[\delta(S_0)] - [S_1]}$, there exists $f_2 \in [\delta(S_0)] - [S_1]$ such that $f_1[b+1] \subseteq f_2$ and player I wins the game by playing f_2.

2. There exists $g_1 \in [\delta_{T_0}T_1]$ such that $t_0 \subseteq g_1$.

   Then there exists $f_2 \in [\delta(S_0)] - [S_1]$ such that $f_1[b+1] \subseteq f_2$ and player I plays using f_2, as long as player II plays in T_1. If eventually player II produces $t_0' \notin T_1$, then we distinguish two cases:

   (-) If $t_0' \notin T_0$, player I continues playing using f_2.

   (-) Suppose $t_0' \in T_0 - T_1$. Then, player I plays using f_2 but now eventually player II must produce t_0'' such that $t_0' \subseteq t_0''$ and $t_0'' * 2^{<\omega} \subseteq T_0$. Then, player I plays outside S_0 (note that he can eventually do such a move since $f_2 \in [\delta(S_0)]$).

3. $t_0 \in T_0 - T_1$.

   Then player I plays using f_1. If eventually player II plays outside T_0 then player I can choose $f_2 \in [\delta(S_0)] - [S_1]$ (as in the previous case) and can win the game by playing according to f_2.

Case 3: $\mathrm{Rs}_2(A^c) = \emptyset$ and $\mathrm{Rs}_2(B^c) = \emptyset$.
Then $G_L^*(A, B)$ is determined by Lemma 2.10.

(2): Considering the former cases and taking into account that a winning strategy for player II in a Lipschitz game yields a winning strategy for player II in the corresponding Wadge game, it suffices to examine case 2. It is not hard to see that the winning strategy for player I described in the proof also works when player II is allowed to pass a finite number of times. Thus player I wins in this case also in the Wadge game $G_W^*(A, B)$.

This completes the proof of the lemma. ■

Remark 2.14 *In Chapter 4 we will prove the result of reverse mathematics stating that in second order arithmetic both Lipschitz determinacy and $\mathbf{SLO}_L^*$ for sets in $\mathbf{Df}_2$ are equivalent to $\mathbf{ACA}_0$.*

Corollary 2.15 *There exists a set $B \in \mathbf{Df}_2 - \breve{\mathbf{Df}}_2$ of the Cantor space such that for every $A \in \mathbf{Df}_2 - \breve{\mathbf{Df}}_2$ of the Cantor space A, $[A]_L =_L [B]$.*

Proof. It follows from case 1 of the previous lemma putting $B = \{\vec{0}\} \cup \bigcup_k 0^{(k)} * \langle 1 \rangle * \langle 0 \rangle * B'$, with

$$B' = \langle 0 \rangle * (\langle 0 \rangle * (\{\vec{0}\} \cup \bigcup_k 0^{(k)} * \langle 1 \rangle * \langle 0 \rangle * 2^\omega) \cup (\langle 1 \rangle * (\bigcup_k 0^{(k)} * \langle 1 \rangle * \langle 0 \rangle * 2^\omega))).$$

■

Lemma 2.16 *Let A and B be subsets of the Cantor space such that $A^c, B \in \mathbf{Df}_2$.*

1. *$G_L^*(A, B)$ is determined.*
2. *$G_W^*(A, B)$ is determined.*

Proof. The proof is similar to that of Lemma 2.13 and we omit it. ■

Remark 2.17 *From the above arguments it is not hard to see that both $\mathbf{Df}_2 - \breve{\mathbf{Df}}_2$ and $\breve{\mathbf{Df}}_2 - \mathbf{Df}_2$ are initial classes and form a pair of incomparable dual Lipschitz and Wadge degrees.*

We are ready for the main result of this section, which summarizes Lemmas 2.10, 2.13, and 2.16.

Theorem 2.18 *Let A and B be subsets of the Cantor space such that $A, B \in \mathbf{Df}_2 \cup \breve{\mathbf{Df}}_2$.*

1. *$G_L^*(A, B)$ is determined.*
2. *$G_W^*(A, B)$ is determined.*

Proof. Follows from the previous lemmas. ■

Corollary 2.19 *$\mathbf{SLO}_L^*$ and $\mathbf{SLO}_W^*$ hold for subsets A and B of the Cantor space such that $A, B \in \mathbf{Df}_2 \cup \breve{\mathbf{Df}}_2$.*

Proof. The Lipschitz and Wadge semilinear order principles are consequences of the determinacy of Lipschitz and Wadge games, respectively. Moreover, the implications are local, i.e. the determinacy of $G_L^*(A, B)$ for $A, B \in \mathbf{\Delta}_1^0$ implies $\mathbf{SLO}_L^*$ for $A, B \in \mathbf{\Delta}_1^0$; the determinacy of $G_L^*(A, B)$ for $A, B \in \mathbf{\Pi}_1^0$ implies $\mathbf{SLO}_L^*$ for $A, B \in \mathbf{\Pi}_1^0$, and so on. ■

We have proved the determinacy of Lipschitz and Wadge games for sets in the first five levels of Wadge hierarchy. The proofs were based heavily on the analysis of the first residue of a set. Thus it is natural to expect that proofs of determinacy concerning further finite levels could be obtained in the same way. Of course, it would be interesting to obtain a proof by induction for all finite levels and to extend the procedure for all countable ordinals.

2.3 Determinacy of Lipschitz and Wadge games in Baire space

In this section we show how our previous arguments can be adapted to prove determinacy of Lipschitz and Wadge games in Baire space for degrees included in the first five levels of the Wadge hierarchy. The key notion is that of a well–founded tree, and the ordinal rank associated to each such a tree. Well–founded trees and ordinal ranks play here the role of finite trees and tree heights, which were used in the previous section.

For $A, B \subseteq \omega^\omega$ we denote by $G_L(A, B)$ the Lipschitz game in Baire space and by $G_W(A, B)$ the Wadge game in Baire space. Notation and definitions related to these concepts were introduced in Section 4 of Chapter 1. The difference between the games studied in this section and the games studied in the previous section is that the moves of players I and II are not restricted to the set $\{0,1\}$. Players I and II can now play any natural number.

As before we start by examining the clopen sets.

Lemma 2.20 *Let A and B be clopen sets in Baire space.*

1. *$G_L(A,B)$ is determined.*
2. *$G_W(A,B)$ is determined.*

Proof. We shall assume that A and B are both different from $\emptyset$ and ω^ω. We can check easily that the result holds in these cases.
Since $A, B \in \mathbf{\Delta}^0_1$, there exist pruned trees S, S', T and T' such that

$$A = [S], \quad A^c = [S'], \quad B = [T], \text{ and } B^c = [T'].$$

Since $[S \cap S'] = [S] \cap [S'] = \emptyset$, $S_0 = S \cap S'$ is a well–founded tree. Let α be the rank of S_0, i.e. $\alpha = \rho(S_0)$. In a similar way, we define $\beta = \rho(T_0)$, where $T_0 = T \cap T'$.

(1): We distinguish two cases:
<u>Case 1</u>: $\alpha \leq \beta$.
Then player II has a winning strategy in the game $G_L(A,B)$. Let us observe that, since T and T' are pruned trees if $t_0 \notin T_0 = T \cap T'$ then

$$\forall t_1\, (t_0 \subset t_1 \to t_1 \in T) \vee \forall t_2\, (t_0 \subset t_2 \to t_2 \in T').$$

Similarly if $s_0 \notin S_0 = S \cap S'$ then

$$\forall s_1\, (s_0 \subset s_1 \to s_1 \in S) \vee \forall s_2\, (s_0 \subset s_2 \to s_2 \in S').$$

As a consequence a strategy $\mathcal{E}_{\mathrm{II}} :\ \mathrm{Seq}_{\mathrm{odd}} \to \omega$ for player II can be defined as follows. For all $s, t \in \omega^{<\omega}$ with $|s| = j+1$ and $|t| = j$, we define

$$\mathcal{E}_{\mathrm{II}}(s \otimes t) = \begin{cases} \min\{k :\ t * \langle k \rangle \in T_0 \wedge \rho_{S_0}(s) \leq \rho_{T_0}(t * \langle k \rangle)\} & \text{if } \ s \in S_0 \\ \min\{k :\ \exists t_1 \in T\, (t * \langle k \rangle \subseteq t_1)\} & \text{if } \ s \in S - S_0 \wedge \\ & t \in T_0 \wedge \rho_{T_0}(t) \neq 0 \\ \min\{k :\ \exists t_2 \in T'\, (t * \langle k \rangle \subseteq t_2)\} & \text{if } \ s \in S' - S_0 \wedge \\ & t \in T_0 \wedge \rho_{T_0}(t) \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

Function $\mathcal{E}_{\mathrm{II}}$ is a winning strategy for player II. Indeed, according to $\mathcal{E}_{\mathrm{II}}$ player II enumerates an element of T_0 no matter what player I plays. When it finally happens that

$\rho_{T_0}(t) = 0$ player I has irrevocably committed himself to enumerate an element of either A or A^c since $\rho(S_0) \leq \rho(T_0)$. Now if player I is enumerating an element of A, player II chooses to go into B, and if player I is enumerating an element of A^c, then player II decides to enter the set B^c. Thus, for every $f \in \omega^\omega$ we have $f \in A$ iff $f \otimes^{\mathrm{II}} \mathcal{E}_{\mathrm{II}} \in B$ and hence $\mathcal{E}_{\mathrm{II}}$ is a winning strategy for player II.

Case 2: $\beta < \alpha$.
Then player I has a winning strategy in the game $G_L(A, B)$.
Player I's first move is

$$\mathcal{E}_{\mathrm{I}}(\langle\rangle) = \min\{k : \langle k \rangle \in S_0 \wedge \rho_{S_0}(\langle k \rangle) \geq \beta\}.$$

Now for all $s, t \in \omega^{<\omega}$ with $|t| = j \geq 1$, we define:

$$\mathcal{E}_{\mathrm{I}}(s \otimes t) = \begin{cases} \min\{k : s * \langle k \rangle \in S_0 \wedge \rho_{T_0}(t) \leq \rho_{S_0}(s * \langle k \rangle)\} & \text{if } t \in T_0 \\ \min\{k : \exists s_2 \in S' \, (s * \langle k \rangle \subseteq s_2)\} & \text{if } t \in T - T_0 \wedge \\ & s \in S_0 \wedge \rho_{S_0}(s) \neq 0 \\ \min\{k : \exists s_1 \in S \, (s * \langle k \rangle \subseteq s_1)\} & \text{if } t \in T' - T_0 \wedge \\ & s \in S_0 \wedge \rho_{S_0}(s) \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

To see that $\mathcal{E}_{\mathrm{I}}$ is a winning strategy for player I let us observe the following. Player I starts to enumerate an element of S_0 no matter what player II plays. When the sequence s that player I is enumerating satisfies $\rho_{S_0}(s) = 0$, player II must decide to continue in either B or B^c, if she has not already decided it. Now if player II decides to commit herself to B, the next move of player I is to choose k such that $s * \langle k \rangle \subseteq s_2$ for some $s_2 \in S'$ and to play forever outside A, i.e. in A^c. If, on the contrary, player II decides to stay in B^c for the rest of the game, then in the next move player I chooses k such that $s * \langle k \rangle \subseteq s_1$ for some $s_1 \in S$ and remains forever inside A. In both cases, for every $g \in \omega^\omega$, $g \in B$ iff $\mathcal{E}_{\mathrm{I}} \otimes g \in A^c$. Thus $\mathcal{E}_{\mathrm{I}}$ is a winning strategy for player I.

(2): We distinguish the same two cases. Concerning case 1, a winning strategy for player II in a Lipschitz game yields automatically a winning strategy for player II in the corresponding Wadge game. Hence this case follows from case 1 for Lipschitz games. On the other hand, it is not hard to see that if player II is allowed to pass, she would also win the game in case 2. In fact, player II can keep passing until player I commits himself to remaining either in A or outside A forever. In conclusion, the Wadge game $G_W(A, B)$, where A and B are clopen sets of the Baire space, is determined.

This completes the proof of the lemma. ■

Remark 2.21

1. *In the proof we used pruned trees and the rank of a tree. We can prove in set theory that every tree can be pruned, i.e. contains a pruned subtree with the same*

set of paths, and that every well-founded tree has a rank. In second order arithmetic, however, as we will prove in Chapter 5, both facts are available only in $\mathbf{ATR}_0$ *and in stronger subsystems of second order arithmetic. This will enable us to prove the result within* $\mathbf{ATR}_0$*.*

2. *Since the rank of any* ω *tree of the Baire space is a countable ordinal the lemma implies that there exists a sequence* $\{[A_\alpha]_L : \alpha \in \omega_1\}$ *of Lipschitz degrees in the Baire space such that:*

   (a) *for each* $\alpha \in \omega_1$*,* A_α *is a clopen set different from* $\emptyset$ *and* ω^ω*;*

   (b) *for each* $\alpha \in \omega_1$*,* $[A_\alpha]_L \prec_L [A_\beta]_L$*, where* $\beta \in \omega_1$ *is a successor or a limit ordinal above* α*;*

   (c) *for each clopen set* B *different from* $\emptyset$ *and* ω^ω *there exists* $\alpha \in \omega_1$ *such that* $[B]_L = [A_\alpha]_L$*.*

3. *The determinacy of the Wadge games* $G_W(A, B)$*, where* A *and* B *are clopen sets of the Baire space, implies that clopen sets different from* $\emptyset$ *and* ω^ω *form a Wadge degree. Additionally, using Wadge's lemma we obtain that the clopen sets form a initial class, i.e. for all* A *and* B*, if* A *is clopen and II wins* $G(B, A)$*, then* B *is clopen. Similarly we obtain that every clopen set is* $\mathbf{\Delta}^0_1$*-complete, i.e. given a clopen set* A*, for all clopen* B *player II wins* $G(B, A)$*.*

Lemma 2.22 *Let* A *and* B *be closed sets in the Baire space.*

1. $G_L(A, B)$ *is determined.*

2. $G_W(A, B)$ *is determined.*

Proof. It is enough to show that if $S, T \subseteq \omega^{<\omega}$ are pruned trees then the Lipschitz game $G_L([S], [T])$ is determined.

(1): We distinguish three cases:
Case 1: $[\delta(T)] \neq \emptyset$.
Then there exists $g \in [T]$ such that $\forall k\, \exists t\, (g[k] \subseteq t \wedge t \notin T)$. Let us see that there exists a winning strategy for player II, $\mathcal{E}_{\mathrm{II}}$ defined as follows:
For all $s, t \in \omega^{<\omega}$ with $|s| = j + 1$ and $|t| = j$,

$$\mathcal{E}_{\mathrm{II}}(s \otimes t) = \begin{cases} g(j) & \text{if } s \in S \\ \min\{k : t * \langle k \rangle \notin T\} & \text{if } s \notin S \wedge \exists t \ (t * \langle k \rangle \notin T) \\ g(j) & \text{if } s \notin S \wedge \forall k \, (t * \langle k \rangle \in T) \end{cases}$$

To see that $\mathcal{E}_{\mathrm{II}}$ is a winning strategy for player II it suffices to observe that following strategy $\mathcal{E}_{\mathrm{II}}$ player II enumerates g as long as player I is enumerating an element of $[S]$ and that she can always use a branch of g to leave $[T]$ if player I decides to leave $[S]$.

Thus, for every $f \in \omega^\omega$ we have $f \in A$ iff $f \otimes^{\mathrm{II}} \mathcal{E}_{\mathrm{II}} \in B$ and hence $\mathcal{E}_{\mathrm{II}}$ is a winning strategy for player II.

Case 2: $[\delta(T)] = \emptyset$ and $[\delta(S)] \neq \emptyset$.
Then there exists $f \in [S]$ such that $\forall k\, \exists s\, (f[k] \subseteq s \wedge s \notin S)$.
Let us define $T' = \{t' \in T : \exists t\, (t' \subseteq t \wedge t \notin T)\}$. Then T' is a tree with no infinite branch (since Case 1 fails) and a winning strategy for player I, $\mathcal{E}_{\mathrm{I}}$ can be defined as follows:
Let $\mathcal{E}_{\mathrm{I}}(\langle\rangle) = f(0)$ and for all $s, t \in \omega^{<\omega}$ with $|s| = |t| = j \geq 1$,

$$\mathcal{E}_{\mathrm{I}}(s \otimes t) = \begin{cases} f(j) & \text{if } \ t \notin T \vee t \in T' \\ \min\{k : \ s * \langle k\rangle \notin S\} & \text{if } \ t \in T - T' \wedge \exists k\, (s * \langle k\rangle \notin S) \\ f(j) & \text{if } \ t \in T - T' \wedge \forall k\, (s * \langle k\rangle \notin S) \end{cases}$$

Since T' is well–founded player II must eventually play outside T'. Let us denote by t the sequence of player II's movements. Then $t \notin T'$ means that

$$\forall t'\, (t \subset t' \to t' \notin T) \vee \forall t'\, (t \subset t' \to t' \in T)$$

Thus, according to $\mathcal{E}_{\mathrm{I}}$ Player I starts enumerating f and continues enumerating f as long as player II plays inside T'. For player II leaving T' amounts to decide whether she ends the game in B or in B^c. So if player II decides to commit herself to B, player I continues to enumerate f, otherwise he leaves A as soon as he can. Hence, for every $g \in \omega^\omega$, $g \in B$ iff $\mathcal{E}_{\mathrm{I}} \otimes g \in A^c$ and $\mathcal{E}_{\mathrm{I}}$ is a winning strategy for player I.

Case 3: $[\delta(T)] = \emptyset$ and $[\delta(S)] = \emptyset$.
Then, $[T]$ and $[S]$ are clopen and therefore $G_L([S],[T])$ is determined by Lemma 2.20.

(2): It is not hard to see that if we consider Wadge games, instead of Lipschitz games, player II would have a winning strategy in every single case considered above except in case 2. In this case player I can enumerate f while player II is passing or enumerating a t'. Since she cannot do this forever, player I wins the game.

This completes the proof of the lemma. ■

Remark 2.23

1. *In Chapter 5 we prove the reverse mathematics result stating that the Lipschitz determinacy of closed sets in Baire space is equivalent to the subsystem of second order arithmetic* $\mathbf{ATR}_0$.
2. *As a consequence of the above lemma and of Wadge's lemma we obtain that the closed sets form a initial class, and that every closed set with a nonempty boundary is* $\mathbf{\Pi}^0_1$*-complete.*

In some of the following proofs we will define winning strategies without detailing the reasons why they are winning.

Lemma 2.24 *Let A and B be subsets of the Baire space such that $A, B \in \mathbf{\Sigma}^0_1 \cup \mathbf{\Pi}^0_1$.*

1. *$G_L(A, B)$ is determined.*
2. *$G_W(A, B)$ is determined.*

Proof. The case A, $B \in \mathbf{\Pi}^0_1$ is just Lemma 2.22. Bearing in mind that the strategies for a game $G_L(A, B)$ are also strategies for the dual game $G_L(A^c, B^c)$ we obtain from Lemma 2.22 that $G_L(A, B)$ is determined when A, $B \in \mathbf{\Sigma}^0_1$. Analogously we obtain that $G_W(A, B)$ is determined when A, $B \in \mathbf{\Sigma}^0_1$.

(1): Let us prove that $G_L(A, B)$ is determined for $A \in \mathbf{\Sigma}^0_1$ and $B \in \mathbf{\Pi}^0_1$ (the remaining case follows from this one by duality).
It is enough to show that if $S, T \subseteq \omega^{<\omega}$ are pruned trees then the Lipschitz game $G_L([S]^c, [T])$ is determined. We distinguish two cases:

Case 1: $[\delta(S)] \neq \emptyset$.
Then there exists $f \in [S]$ such that $\forall k\, \exists s\, (f[k] \subseteq s \wedge s \notin S)$.
A winning strategy $\mathcal{E}_{\rm I}$, for player I, can be defined as follows:
Let $\mathcal{E}_{\rm I}(\langle\rangle) = g(0)$ and for all $s, t \in \omega^{<\omega}$ with $|s| = |t| = j \geq 1$,

$$\mathcal{E}_{\rm I}(s \otimes t) = \begin{cases} f(j) & \text{if } t \in T \\ \min\{k : s * \langle k\rangle \notin S\} & \text{if } t \notin T \wedge \exists k\, (s * \langle k\rangle \notin S) \\ f(j) & \text{if } t \notin T \wedge \forall k\, (s * \langle k\rangle \in S) \end{cases}$$

Case 2: $[\delta(S)] = \emptyset$.
Then $[S]^c$ is closed and $G_L([S]^c, [T])$ is determined by Lemma 2.22.

(2): We distinguish the same two cases.
Observe that in case 1 player I can still use the same winning strategy to win $G_W(A, B)$ with $A \in \mathbf{\Sigma}^0_1$ and $B \in \mathbf{\Pi}^0_1$. Case 2 is analogous to the corresponding case for Lipschitz games. Finally, the determinacy of $G_W(A, B)$ for $B \in \mathbf{\Sigma}^0_1$ and $A \in \mathbf{\Pi}^0_1$ follows from the previous cases by duality.

This completes the proof of the proposition. ■

Remark 2.25 *The above arguments yield also that $\mathbf{\Pi}^0_1 - \mathbf{\Sigma}^0_1$ and $\mathbf{\Sigma}^0_1 - \mathbf{\Pi}^0_1$ are initial classes and form a pair of incomparable dual Lipschitz and Wadge degrees.*

As it was the case in the former section the topological analysis of Section 1 will play a crucial role in the next lemmas.

Lemma 2.26 *Let A and B be subsets of the Baire space such that $A, B, A^c, B^c \in \mathbf{Df}_2$.*

1. *$G_L(A, B)$ is determined.*

2. $G_W(A, B)$ is determined.

Proof. Without loss of generality, we can assume that there exists pruned trees S_0, S_1, T_0, T_1 such that

1. $S_1 \subseteq S_0$ and $T_1 \subseteq T_0$.

2. $A = [S_0] - [S_1]$, and $B = [T_0] - [T_1]$.

Since $A^c \in \mathbf{Df}_2$, we know that $\mathrm{Rs}_2(A^c) = [\delta_{S_0} S_1] \cap \overline{[\delta(S_0)] - [S_1]} = \emptyset$. Observe that $\mathrm{Rs}_2(A^c)$ is a closed set, so $\mathrm{Rs}_2(A^c) = [\delta(S_0, S_1)]$ for some tree $\delta(S_0, S_1)$, namely

$$\delta(S_0, S_1) = \{s \in S_1 : \exists f_1, f_2\, (f_1 \in [\delta_{S_0} S_1] \wedge f_2 \in [\delta(S_0)] - [S_1] \wedge s \subset f_1 \wedge s \subset f_2)\}.$$

By hypothesis $\delta(S_0, S_1)$ must be well–founded. Similarly, $\mathrm{Rs}_2(B^c) = [\delta_{T_0} T_1] \cap \overline{[\delta(T_0)] - [T_1]} = \emptyset$ and, therefore, $\delta(T_0, T_1)$ is also a well–founded tree. Let $\alpha = \rho\,(\delta(S_0, S_1))$ and $\beta = \rho\,(\delta(T_0, T_1))$.

(1): To prove that $G_L(A, B)$ is determined we distinguish three main cases with several subcases.
Case 1: $[\delta_{T_0} T_1] \neq \emptyset$, $[\delta(T_0)] - [T_1] \neq \emptyset$, and $[\delta_{S_0} S_1] \neq \emptyset$, $[\delta(S_0)] - [S_1] \neq \emptyset$.
We distinguish two subcases:

1. If $\alpha \leq \beta$, then player II has a winning strategy, $\mathcal{E}_{\mathrm{II}}$.

   For all $s, t \in \omega^{<\omega}$ with $|s| = j + 1$ and $|t| = j$ if $s \in \delta(S_0, S_1)$ with $\alpha_j = \rho_{\delta(S_0,S_1)}(s)$, then
   $$\mathcal{E}_{\mathrm{II}}(s \otimes t) = \min\{k : t * \langle k \rangle \in \delta(T_0, T_1) \wedge \rho_{\delta(T_0,T_1)}(t * \langle k \rangle)) \geq \alpha_j\}.$$

   If at some point player I plays outside $\delta(S_0, S_1)$ (this must be eventually the case, since $\delta(S_0, S_1)$ is well–founded) and player II is still playing inside $\delta(T_0, T_1)$, with $\rho_{\delta(T_0,T_1)}(t) > 0$, then
   $$\mathcal{E}_{\mathrm{II}}(s \otimes t) = \min\{k : t * \langle k \rangle \in \delta(T_0, T_1)\}$$

   If $\rho_{\delta(T_0,T_1)}(t) = 0$, or $t \notin \delta(T_0, T_1)$, then player II decides her move depending on the position, s, of player I, as follows:

   For each $u \in \delta(T_0, T_1)$ such that $\rho_{\delta(T_0,T_1)}(u) = 0$, let us fix $g_1^u \in [\delta_{T_0} T_1]$ with $u \subseteq g_1^u$ and $g_2^u \in [\delta(T_0)] - [T_1]$ with $u \subseteq g_2^u$ (here we are using $\mathbf{AC}_\omega$). Let us define $b_t = \max\{k \leq |t| : t\,[k] \in \delta(T_0, T_1)\}$. Then $\rho_{\delta(T_0,T_1)}(t\,[b_t]) = 0$ and taking $t' = t\,[b_t]$, we have at our disposal functions $g_1^{t'}$ and $g_2^{t'}$. Then the strategy of player II continues as follows:

$$
\mathcal{E}_{\mathrm{II}}(s \otimes t) = \begin{cases}
g_1^{t'}(j) & \text{if } j \geq b_t \wedge s[b_t+1] \notin S_0 \\
g_2^{t'}(j) & \text{if } j \geq b_t \wedge s[b_t+1] \in S_0 - S_1 \wedge s \in S_0 \\
g_2^{t'}(j) & \text{if } j \geq b_t \wedge s[b_t+1] \in S_0 - S_1 \wedge s \notin S_0 \wedge \forall k\,(t * \langle k \rangle \in T_0) \\
k & \text{if } j \geq b_t \wedge s[b_t+1] \in S_0 - S_1 \wedge s \notin S_0 \wedge \exists k\,(t * \langle k \rangle \notin T_0) \\
 & \text{and } k = \min\{i : t * \langle i \rangle \notin T_0\} \\
g_1^{t'}(j) & \text{if } j \geq b_t \wedge s[b_t+1] \in S_1 \wedge \exists f\,(f \in [\delta_{S_0} S_1] \wedge s[b_t+1] \subset f) \wedge \\
 & (s \in S_1 \vee s \notin S_0) \\
g_1^{t'}(j) & \text{if } j \geq b_t \wedge s[b_t+1] \in S_1 \wedge \exists f\,(f \in [\delta_{S_0} S_1] \wedge s[b_t+1] \subset f) \wedge \\
 & s \in S_0 - S_1 \wedge \exists s'(s \subseteq s' \wedge s' \notin S_0) \\
g_1^{t'}(j) & \text{if } j \geq b_t \wedge s[b_t+1] \in S_1 \wedge \exists f\,(f \in [\delta_{S_0} S_1] \wedge s[b_t+1] \subset f) \wedge \\
 & s \in S_0 - S_1 \wedge \ \forall s'(s \subset s' \to s' \in S_0) \wedge \forall k\,(t * \langle k \rangle \notin T_0 - T_1) \\
k & \text{if } j \geq b_t \wedge s[b_t+1] \in S_1 \wedge \exists f\,(f \in [\delta_{S_0} S_1] \wedge s[b_t+1] \subset f) \wedge \\
 & s \in S_0 - S_1 \wedge \ \forall s'(s \subset s' \to s' \in S_0) \wedge \exists k\,(t * \langle k \rangle \in T_0 - T_1) \\
 & \text{and } k = \min\{i : t * \langle i \rangle \in T_0 - T_1\} \\
g_2^{t'}(j) & \text{if } j \geq b_t \wedge s[b_t+1] \in S_1 \wedge \neg\exists f\,(f \in [\delta_{S_0} S_1] \wedge s[b_t+1] \subset f) \wedge \\
 & (s \in \delta_{S_0} S_1 \vee s \in S_0 - S_1) \\
g_2^{t'}(j) & \text{if } j \geq b_t \wedge s[b_t+1] \in S_1 \wedge \neg\exists f\,(f \in [\delta_{S_0} S_1] \wedge s[b_t+1] \subset f) \wedge \\
 & s \notin \delta_{S_0} S_1 \wedge (s \in S_1 \vee s \notin S_0) \wedge \forall k\,(t * \langle k \rangle \in T_0) \\
k & \text{if } j \geq b_t \wedge s[b_t+1] \in S_1 \wedge \neg\exists f\,(f \in [\delta_{S_0} S_1] \wedge s[b_t+1] \subset f) \wedge \\
 & s \notin \delta_{S_0} S_1 \wedge (s \in S_1 \vee s \notin S_0) \wedge \exists k\,(t * \langle k \rangle \notin T_0) \\
 & \text{and } k = \min\{i : t * \langle i \rangle \notin T_0\}
\end{cases}
$$

That is to say, player II plays using t' during the first b_t rounds. Then, she plays according to cases 1A′, 1B′, 1C′, and 1D′, which can be described as in cases 1A, 1B, 1C, and 1D of Lemma 2.10 by replacing b, g_1, and g_2 by b_t, $g_1^{t'}$, and $g_2^{t'}$, respectively.

2. If $\alpha > \beta$, then player I has a winning strategy.

First, $\mathcal{E}_{\mathrm{I}}(\langle\rangle) = \min\{k : \langle k\rangle \in \delta(S_0, S_1) \wedge \rho_{\delta(S_0,S_1)}(\langle k\rangle) \geq \beta\}$.

For all $s, t \in \omega^{<\omega}$ with $|s| = |t| = j \geq 1$, if player II plays $t \in \delta(T_0, T_1)$ with $\beta_j = \rho_{\delta(T_0,T_1)}(t)$ then

$$\mathcal{E}_{\mathrm{I}}(s \otimes t) = \min\{k : \; s * \langle k\rangle \in \delta(S_0, S_1) \wedge \rho_{\delta(S_0,S_1)}(s * \langle k\rangle) \geq \beta_j\}$$

If at some point player II plays outside $\delta(T_0, T_1)$ (this must eventually be the case, since $\delta(T_0, T_1)$ is well–founded) and player I is still playing inside $\delta(S_0, S_1)$, with $\rho_{\delta(S_0,S_1)}(s) > 0$, then

$$\mathcal{E}_{\mathrm{I}}(s_1 \otimes s_2) = \min\{k : \; s * \langle k\rangle \in \delta(S_0, S_1)\}.$$

If $\rho_{\delta(S_0,S_1)}(s) = 0$, or $s \notin \delta(S_0, S_1)$, then player I decides his move depending on the position t of II, as follows:

For each $u \in \delta(S_0, S_1)$ such that $\rho_\delta(S_0, S_1)(u) = 0$, let us fix $f_1^u \in [\delta_{S_0} S_1]$ with $u \subseteq f_1^u$ and $f_2^u \in [\delta(S_0)] - [S_1]$ such that $u \subseteq f_2^u$. Let $a_s = \max\{k \leq |s| : s\,[k] \in \delta(S_0, S_1)\}$. Then, taking $s' = s\,[a_s]$, the strategy of player I continues as follows:

$$\mathcal{E}_{\mathrm{I}}(s \otimes t) = \begin{cases} f_2^{s'}(j) & \text{if } j \geq a_s \wedge t[a_s] \notin T_0 \\ f_1^{s'}(j) & \text{if } j \geq a_s \wedge t[a_s] \in T_0 - T_1 \wedge t \in T_0 \\ f_1^{s'}(j) & \text{if } j \geq a_s \wedge t[a_s] \in T_0 - T_1 \wedge t \notin T_0 \wedge \forall k\, (s * \langle k \rangle \notin S_0 - S_1) \\ k & \text{if } j \geq a_s \wedge t[a_s] \in T_0 - T_1 \wedge t \notin T_0 \wedge \exists k\, (s * \langle k \rangle \in S_0 - S_1) \\ & \text{and } k = \min\{i : s * \langle i \rangle \in S_0 - S_1\} \\ f_2^{s'}(j) & \text{if } j \geq a_s \wedge t[a_s] \in T_1 \wedge \exists g\, (g \in [\delta_{T_0} T_1] \wedge t[a_s] \subset g) \wedge \\ & (t \in T_1 \vee t \notin T_0) \\ f_2^{s'}(j) & \text{if } j \geq a_s \wedge t[a_s] \in T_1 \wedge \exists g\, (g \in [\delta_{T_0} T_1] \wedge t[a_s] \subset g) \wedge \\ & t \in T_0 - T_1 \wedge \exists t'(t \subseteq t' \wedge t' \notin T_0) \\ f_2^{s'}(j) & \text{if } j \geq a_s \wedge t[a_s] \in T_1 \wedge \exists g\, (g \in [\delta_{T_0} T_1] \wedge t[a_s] \subset g) \wedge \\ & t \in T_0 - T_1 \wedge \ \forall t'(t \subset t' \to t' \in T_0) \wedge \forall k\, (s * \langle k \rangle \in S_0 - S_1) \\ k & \text{if } j \geq a_s \wedge t[a_s] \in T_1 \wedge \exists g\, (g \in [\delta_{T_0} T_1] \wedge t[a_s] \subset g) \wedge \\ & t \in T_0 - T_1 \wedge \ \forall t'(t \subset t' \to t' \in T_0) \wedge \exists k\, (s * \langle k \rangle \notin S_0 - S_1) \\ & \text{and } k = \min\{i : s * \langle i \rangle \in S_0 - S_1\} \\ f_1^{s'}(j) & \text{if } j \geq a_s \wedge t[a_s] \in T_1 \wedge \neg\exists g\, (g \in [\delta_{T_0} T_1] \wedge t[a_s] \subset g) \wedge \\ & (t \in \delta_{T_0} T_1 \vee t \in T_0 - T_1) \\ f_1^{s'}(j) & \text{if } j \geq a_s \wedge t[a_s] \in T_1 \wedge \neg\exists g\, (g \in [\delta_{T_0} T_1] \wedge t[a_s] \subset g) \wedge \\ & t \notin \delta_{T_0} T_1 \wedge (t \in T_1 \vee t \notin T_0) \wedge \forall k\, (s * \langle k \rangle \notin S_0 - S_1) \\ k & \text{if } j \geq a_s \wedge t[a_s] \in T_1 \wedge \neg\exists g\, (g \in [\delta_{T_0} T_1] \wedge t[a_s] \subset g) \wedge \\ & t \notin \delta_{T_0} T_1 \wedge (t \in T_1 \vee t \notin T_0) \wedge \exists k\, (s * \langle k \rangle \in S_0 - S_1) \\ & \text{and } k = \min\{i : t * \langle i \rangle \in S_0 - S_1\} \end{cases}$$

That is to say, player I plays using s' during the first a_s rounds. Then, he plays according to cases 2A′, 2B′, 2C′, and 2D′, which can be described as in cases 2A, 2B, 2C, and 2D of Lemma 2.10 by replacing a, f_1, and f_2 by a_s, $f_1^{s'}$, and $f_2^{s'}$, respectively.

Recall that we say that (T_0, T_1) (and similarly (S_0, S_1)) are in a *degenerated position* if $[\delta_{T_0} T_1] = \emptyset$ or $[\delta(T_0)] - [T_1] = \emptyset$.
Let us observe again that if (T_0, T_1) are in a degenerated position then $[T_0] - [T_1]$ must be an open or closed set.

Case 2: One (and only one) of (T_0, T_1) or (S_0, S_1) are in a degenerated position.
If player I plays in a degenerated position, then A is closed or open and player II has a winning strategy (essentially, player II plays simulating the strategy described in Lemma 2.22 (case 1)). The subcases have been described in Case 2 of Lemma 2.10.

Case 3: (T_0, T_1) and (S_0, S_1) are in a degenerated position.
In these degenerated cases $[T_0] - [T_1]$ and $[S_0] - [S_1]$ are closed or open sets, so, the corresponding game is determined by Lemma 2.24.

(2): Taking into account the former cases and the fact that a winning strategy for player II in a Lipschitz game yields a winning strategy for player II in the corresponding Wadge game, it remains to examine the second part of case 1 and the second part of case 2. In the second part of case 1, since player II is now allowed to pass, she can wait until player I's moves form a sequence s' of length $\rho_\delta(S_0, S_1)(s') = 0$. Then according to the decision of player I of continuing to play in $[S_1]$, in A or outside A, player II plays in $[T_1]$, in B or outside B. Following this strategy player II eventually wins the game. In the second part of case 2 the allowance to pass is not enough for player II to win the game and player I still has an winning strategy in $G_W(A, B)$.

This completes the proof of the proposition. ■

Remark 2.27

1. *The lemma concerns sets in the third level of Wadge hierarchy. Similarly to lemma 2.20, we can derive the existence of a ω_1-sequence of Lipschitz degrees in the Baire space. Since the rank of any ω tree of the Baire space is a countable ordinal there exists a sequence $\{[A_\alpha]_L : \alpha \in \omega_1\}$ of Lipschitz degrees in the Baire space such that:*
   (a) *for each $\alpha \in \omega_1$, A_α is a difference of closed sets and is neither open nor closed;*
   (b) *for each $\alpha \in \omega_1$, $[A_\alpha]_L \prec_L [A_\beta]_L$, where $\beta \in \omega_1$ is a successor or a limit ordinal above α;*
   (c) *for each set B which is a difference of closed sets and is neither open nor closed there exists $\alpha \in \omega_1$ such that $[B]_L = [A_\alpha]_L$.*
2. *As a consequence of the above lemma sets $A \in \mathbf{Df}_2$, which are neither open nor closed, and whose complements also belong to $\mathbf{Df}_2$, form a Wadge degree.*

Now we move on to the fourth level of Wadge hierarchy.

Lemma 2.28 *Let A and B be subsets of the Baire space such that $A, B \in \mathbf{Df}_2$.*

1. *$G_L(A, B)$ is determined.*
2. *$G_W(A, B)$ is determined.*

Proof. Without loss of generality, we can assume that there exist pruned trees S_0, S_1, T_0, T_1 such that

1. $S_1 \subseteq S_0$ and $T_1 \subseteq T_0$.
2. $A = [S_0] - [S_1]$, and $B = [T_0] - [T_1]$.

Now we distinguish the same cases as in the proof of Lemma 2.13 and reason accordingly. Equipped with Lemma 2.26, the proof is similar to that of Lemma 2.13 and we omit it. ■

Lemma 2.29 *Let A and B be subsets of the Baire space such that $A^c, B \in \mathbf{Df}_2$.*

1. *$G_L(A, B)$ is determined.*
2. *$G_W(A, B)$ is determined.*

Proof. The proof is similar to that of Lemma 2.13 and we omit it. ■

Remark 2.30 *From the above arguments it is not hard to see that both $\mathbf{Df}_2 - \breve{\mathbf{Df}}_2$ and $\breve{\mathbf{Df}}_2 - \mathbf{Df}_2$ are initial classes and form a pair of incomparable dual Lipschitz and Wadge degrees.*

We are ready for the main result of this section, which summarizes lemmas 2.26, 2.28, and 2.29.

Theorem 2.31 *Let A and B be subsets of the Baire space such that $A, B \in \mathbf{Df}_2 \cup \breve{\mathbf{Df}}_2$.*

1. *$G_L(A, B)$ is determined.*
2. *$G_W(A, B)$ is determined.*

Proof. Follows from the previous lemmas. ■

Corollary 2.32 $\mathbf{SLO}_L$ *and* $\mathbf{SLO}_W$ *hold for subsets A and B of the Baire space such that $A, B \in \mathbf{Df}_2 \cup \breve{\mathbf{Df}}_2$.*

Proof. The Lipschitz and Wadge semilinear order principle are consequences of the determinacy of Lipschitz and Wadge games, respectively. Again the implications are local in the sense of Corollary 2.19 ■

Remark 2.33 *In Chapter 5 we will derive this result within subsystem* Π^1_1–$\mathbf{CA}_0$ *of second order arithmetic.*

We have proved Lipschitz determinacy only for sets of both Cantor and Baire spaces which occupy degrees corresponding to the first five levels of Wadge hierarchy.

It is likely that the argument can be extended to prove Lipschitz and Wadge determinacy for all finite differences of closed sets (and of open sets) or even to countable ordinal differences of closed sets (and open sets).

Chapter 3

Infinite games in Second Order Arithmetic

This chapter is devoted to the formalization of infinite nullsum two person games with perfect information in the language of second order arithmetic. Originally the subject was studied in the realm of Zermelo-Fraenckel set theory. However, around 1975 H. Friedman realized that the standard theorems of classical (countable) mathematics require only the basic axioms for arithmetic and set existence axioms just for subsets of $\mathbb{N}$. This motivated him to transfer the work of calibrating the strength of classical mathematical theorems in terms of set existence axioms from the set theoretic realm to the setting of second order arithmetic and its subsystems. One year later John R. Steel showed that the determinacy of $\mathbf{\Sigma}^0_1$ games is equivalent over $\mathbf{RCA}_0$ to $\mathbf{ATR}_0$, and many results on the reverse mathematics of Gale-Stewart games have been obtained since then.

In Section 1 we survey the most relevant results in this area as well as we describe the usual formalization of Gale-Stewart games in second order arithmetic. This formalization is well known (see [Smp99], V.8) and we only refer it for the sake of completion. We will consider games played in both the Baire and the Cantor space. In terms of formalization, the difference between Gale-Stewart games played in the Baire space or in the Cantor space reduces itself to the interpretation of the winning set. Nevertheless, when we examine the strength of such principles for sets at levels below Δ^0_3, we find substantial differences as can be seen in the end of Section 1.

In Sections 2 and 3 we formalize Lipschitz and Wadge games in the language of second order arithmetic. We state the principles of Lipschitz and Wadge determinacy and the semilinear order property, and set up the respective formalized theories. To the best of our knowledge, there was no explicit formalization of these games in second order arithmetic in the literature and we do it for the first time.

In Section 4 we state some basic facts concerning Lipschitz and Wadge determinacy, semilinear order principle, and the relations between these concepts. All these basic facts can be proved in our base theory $\mathbf{RCA}_0$ and will be used in the latter chapters sometimes without explicit reference.

In Section 5 we study the relation between Gale-Stewart and Lipschitz/Wadge games. There is no obvious way of reducing an arbitrary Gale-Stewart game to a Lipschitz or Wadge game. However, as we show in Section 5, every Lipschitz and every Wadge game can be canonically reduced to a Gale-Stewart game, and this reduction can be easily verified within $\mathbf{RCA}_0$. Still, this reduction is obtained at the price of increasing the payoff set complexity. This is something that cannot be disregarded when the goal is to calibrate the strength of determinacy in terms of subsystems of second order arithmetic.

Finally, in Section 6 we show that Wadge's Lemma can be proved within $\mathbf{RCA}_0$. This fact justifies our formalization of the semilinear order principle by means of Lipschitz/Wadge games, instead of having used the original definition with continuous reductions.

From now on, we always work in the language of second order arithmetic and our base theory will be $\mathbf{RCA}_0$.

3.1 Gale-Stewart games

Let $X \subseteq \mathbb{N}$ be nonempty. For a given formula $\varphi(f)$ with a distinguished function variable $f \in X^{\mathbb{N}}$, a game in the space $X^{\mathbb{N}}$ is defined as follows: Two players, say player I (male) and player II (female), alternately choose an element x in X to form $f \in X^{\mathbb{N}}$ which is called the resulting play. Player I plays first.

Player I	x_0		x_1		x_2		$\dots$
Player II		y_0		y_1		y_2	$\dots$

After ω turns, player I has produced a sequence of elements of X, $\langle x_0, x_1, \dots \rangle$, and player II has produced a sequence of elements of X, $\langle y_0, y_1, \dots \rangle$. The resulting play is the function $f \in X^{\mathbb{N}}$ given by $\langle x_0, y_0, x_1, y_1, \dots \rangle$. Player I wins if and only if $\varphi(f)$ holds. Otherwise Player II wins that play. We denote by $G^X(\varphi)$ the Gale-Stewart game in the space $X^{\mathbb{N}}$ defined by the formula $\varphi(f)$.

A strategy for player I in the game $G^X(\varphi)$ is a function assigning an element of X to every sequence from X of even length. A strategy for player II in the game $G^X(\varphi)$ is a function assigning an element of X to every sequence from X of odd length. That is to say, if we define

$$\mathrm{Seq}^X = \{s \in \mathrm{Seq} : s \text{ is a sequence of elements from } X\}$$

$$\mathrm{Seq}^X_{\mathrm{even}} = \{s \in \mathrm{Seq}^X : |s| \text{ is even}\}$$

$$\mathrm{Seq}^X_{\mathrm{odd}} = \{s \in \mathrm{Seq}^X : |s| \text{ is odd}\}$$

then a strategy for player I in the game $G^X(\varphi)$ is a function $\sigma_{\mathrm{I}} : \mathrm{Seq}^X_{\mathrm{even}} \to X$ and a strategy for player II in the game $G^X(\varphi)$ is a function $\sigma_{\mathrm{II}} : \mathrm{Seq}^X_{\mathrm{odd}} \to X$.

If players I and II follow strategies σ_{I} and σ_{II}, respectively, the resulting play is uniquely determined and denoted by $\sigma_{\mathrm{I}} \otimes \sigma_{\mathrm{II}}$. In fact, $\sigma_{\mathrm{I}} \otimes \sigma_{\mathrm{II}}$ is the function $h : \mathbb{N} \to X$ defined by the recursive equations

$$h(2k) = \sigma_{\mathrm{I}}(h[2k])$$
$$h(2k+1) = \sigma_{\mathrm{II}}(h[2k+1]).$$

Recall that $h[i]$ denotes the finite sequence $\langle h(0),\, h(1) \ldots, h(i-1)\rangle$.

A strategy for player I, σ_{I}, is winning if player I wins the game as long as he plays following it, no matter what his opponent plays. Thus, the fact that σ_{I} is a winning strategy for player I in $G^X(\varphi)$ can be formalized as follows

$$\forall \sigma_{\mathrm{II}}\, \varphi(\sigma_{\mathrm{I}} \otimes \sigma_{\mathrm{II}}),$$

where σ_{II} ranges over strategies for player II. Similarly, a strategy for player II, σ_{II}, is winning if player II wins the game as long as she plays following it, no matter what her opponent plays. Thus, the fact that σ_{II} is a winning strategy for player II in $G^X(\varphi)$ can be formalized as follows

$$\forall \sigma_{\mathrm{I}}\, \neg\varphi(\sigma_{\mathrm{I}} \otimes \sigma_{\mathrm{II}}),$$

where σ_{I} ranges over strategies for player I.

A game $G^X(\varphi)$ is determined if either player I or player II has a winning strategy. Hence, the determinacy of the game $G^X(\varphi)$ can be expressed by the axiom

$$Det^X(\varphi) \equiv \exists \sigma_{\mathrm{I}}\, \forall \sigma_{\mathrm{II}}\, \varphi(\sigma_{\mathrm{I}} \otimes \sigma_{\mathrm{II}}) \,\vee\, \exists \sigma_{\mathrm{II}}\, \forall \sigma_{\mathrm{I}}\, \neg\varphi(\sigma_{\mathrm{I}} \otimes \sigma_{\mathrm{II}}),$$

where σ_{I} and σ_{II} range over strategies for player I and strategies for player II, respectively.

We are now in a position to give the axiomatizations of the Gale-Stewart determinacy principles that have been considered in the literature (see, e.g., Section V.8 of [Smp99].)

Definition 3.1 (Gale–Stewart Determinacy theories) *Let Γ be a class of formulas with a distinguished function variable $f \in X^{\mathbb{N}}$.*

1. *The scheme of Γ-determinacy in $X^{\mathbb{N}}$, Γ-$\mathbf{Det}^X$, is given by the axiom scheme*

$$Det^X(\varphi)$$

where $\varphi(f)$ is in Γ.

2. *The scheme of Δ^0_n-determinacy in $X^{\mathbb{N}}$, Δ^0_n-$\mathbf{Det}^X$, is given by the axiom scheme*

$$\forall f \in X^{\mathbb{N}}(\varphi(f) \leftrightarrow \psi(f)) \,\to\, Det^X(\varphi)$$

where $\varphi(f)$ is in Σ^0_n and $\psi(f)$ is in Π^0_n.

3. *If $X = \mathbb{N}$ (determinacy in the Baire space), we will omit the superscript X and write Γ-$\mathbf{Det}$ and Δ^0_n-$\mathbf{Det}$, respectively. If $X = \{0,1\}$ (determinacy in the Cantor space), we will write Γ-$\mathbf{Det}^*$ and Δ^0_n-$\mathbf{Det}^*$, respectively.*

Remark 3.2

1. *In all the theories we will consider, formulas in the corresponding axiom schemes are allowed to contain number and set parameters, although in most cases we will not write them explicitly.*

2. *For the sake of notational simplicity, if the set X is clear from the context, we will omit the superscript X in our notations $G^X(\varphi)$, Seq^X, Seq^X_{even}, Seq^X_{odd}, $Det^X(\varphi)$. In particular, this will be the case when working in the Cantor space ($X = \{0,1\}$) and in the Baire space ($X = \mathbb{N}$).*

3. *Recall that the language of second order arithmetic does not formally contain any function variables. However, one can naturally express the fact that "G is the graph of a function $f : \mathbb{N} \to X$" by using a Π^0_2 formula. The price to pay is a possible increase of the quantifier complexity of the formulas involved. This point could cause problems in a system without full arithmetical comprehension. However, we will only consider base theories below* $\mathbf{ACA}_0$ *when working in the Cantor space and, as usual, if $X = \{0,1\}$ then we can regard f as a set variable simply by identifying $f(n) = 0$ and $f(n) = 1$ with $n \notin f$ and $n \in f$, respectively.*

From now on, we consider the class of formulas Γ to be one of the following:

- Formulas Σ^0_n/Π^0_n in the arithmetic hierarchy,
- Formulas Δ^0_n as defined above,
- Formulas $(\Sigma^0_n)_k$, with $k \in \omega$ and $k > 0$, corresponding to k-th level of the difference hierarchy on Σ^0_n sets. These formula classes are defined as follows. For $k = 1$, $(\Sigma^0_n)_1 = \Sigma^0_n$. For $k > 1$, $\varphi \in (\Sigma^0_n)_k$ iff φ can be written as $\varphi_1 \wedge \varphi_2$, where $\neg\varphi_1 \in (\Sigma^0_n)_{k-1}$ and $\varphi_2 \in \Sigma^0_n$.

In particular, these classes of formulas are well known to satisfy nice closure properties that will be used in the sequel sometimes without explicit mention.

Recall that given a class of formulas Γ, $\neg\Gamma$ denotes the class of formulas given by $\{\neg\varphi : \varphi \in \Gamma\}$. It is well known that determinacy for Γ games and determinacy for $\neg\Gamma$ games are equivalent principles. This fact easily formalizes in $\mathbf{RCA}_0$.

Lemma 3.3 (Lemma 3.6.3 of [N09b]) *It is provable in* $\mathbf{RCA}_0$ *that* Γ-$\mathbf{Det}^X$ *and* $(\neg\Gamma)$-$\mathbf{Det}^X$ *are equivalent.*

Proof. By symmetry, it suffices to show that Γ-$\mathbf{Det}^X$ implies $(\neg\Gamma)$-$\mathbf{Det}^X$. Assume Γ-determinacy in $X^{\mathbb{N}}$. Consider $\varphi(f) \in \neg\Gamma$. We must show that $Det^X(\varphi)$ holds. To this end, consider $\theta(f) \equiv \neg\varphi(f')$, where f' denotes the function defined from f by putting $f'(n) = f(n+1)$. Since $\theta(f)$ is equivalent in $\mathbf{RCA}_0$ to a formula in Γ, by applying Γ-$\mathbf{Det}^X$ we obtain that

$$\exists\sigma_{\mathrm{I}}\,\forall\sigma_{\mathrm{II}}\,\neg\varphi((\sigma_{\mathrm{I}}\otimes\sigma_{\mathrm{II}})') \;\vee\; \exists\sigma_{\mathrm{II}}\,\forall\sigma_{\mathrm{I}}\,\varphi((\sigma_{\mathrm{I}}\otimes\sigma_{\mathrm{II}})')$$

Case 1: $\exists\sigma_{\mathrm{I}}\,\forall\sigma_{\mathrm{II}}\,\neg\varphi((\sigma_{\mathrm{I}}\otimes\sigma_{\mathrm{II}})')$.
Then, we can use σ_{I} in order to construct a winning strategy for player II in the game $G^X(\varphi)$. In fact, let $a=\sigma_I(\langle\rangle)$. Put

$$\tau_{\mathrm{II}}(\langle x_0\rangle)=\sigma_{\mathrm{I}}(\langle a,x_0\rangle)$$

$$\tau_{\mathrm{II}}(\langle x_0,y_0,\ldots,x_{k+1}\rangle)=\sigma_{\mathrm{I}}(\langle a,x_0,y_0,\ldots,x_k,y_k\rangle)$$

It is easy to see that τ_{II} is a winning strategy for player II in $G^X(\varphi)$.

Case 2: $\exists\sigma_{II}\,\forall\sigma_I\,\varphi((\sigma_I\otimes\sigma_{II})')$.
Then, we can use σ_{II} in order to construct a winning strategy for player I in the game $G^X(\varphi)$. Namely, pick $b\in X$ and define a strategy τ_{I} as follows

$$\tau_{\mathrm{I}}(\langle\rangle)=\sigma_{\mathrm{II}}(\langle b\rangle)$$

$$\tau_{\mathrm{I}}(\langle x_0,y_0,\ldots,x_k,y_k\rangle)=\sigma_{\mathrm{II}}(\langle b,x_0,y_0,\ldots,x_k,y_k\rangle)$$

It is easy to see that τ_{I} is a winning strategy for player I in $G^X(\varphi)$.

This completes the proof of the lemma. ■

As a consequence, the following hierarchy of determinacy principles of increasing strength emerges.

Baire space:

$$\Delta^0_1\text{-}\mathbf{Det}\subseteq \begin{array}{c}\Sigma^0_1\text{-}\mathbf{Det}\\ |||\\ \Pi^0_1\text{-}\mathbf{Det}\end{array}\subseteq(\Sigma^0_1)_2\text{-}\mathbf{Det}\subseteq(\Sigma^0_1)_3\text{-}\mathbf{Det}\subseteq\cdots\subseteq(\Sigma^0_1)_\omega\text{-}\mathbf{Det}\subseteq\Delta^0_2\text{-}\mathbf{Det}\subseteq\ldots$$

Cantor space:

$$\Delta^0_1\text{-}\mathbf{Det}^*\subseteq \begin{array}{c}\Sigma^0_1\text{-}\mathbf{Det}^*\\ |||\\ \Pi^0_1\text{-}\mathbf{Det}^*\end{array}\subseteq(\Sigma^0_1)_2\text{-}\mathbf{Det}^*\subseteq(\Sigma^0_1)_3\text{-}\mathbf{Det}^*\subseteq\cdots\subseteq(\Sigma^0_1)_\omega\text{-}\mathbf{Det}^*\subseteq\Delta^0_2\text{-}\mathbf{Det}^*\subseteq\ldots$$

where $(\Sigma^0_n)_\omega$-**Det** denotes the principle of determinacy for the class of all finite Boolean combinations $(\Sigma^0_n)_\omega$ sets, as defined in [MS12].

It is easy to check that games in the Cantor space can be reduced to games in the Baire space, without altering the payoff complexity of the game. Actually, the following 'folklore' result holds.

Lemma 3.4 *It is provable in* $\mathbf{RCA}_0$ *that* Γ-**Det** *implies* Γ-**Det***.

Proof. Assume Γ-**Det**. We must show that Γ-**Det*** holds too. Consider $\varphi(f)$ in Γ and define $\theta(f)\equiv\varphi(f')$, where f' denotes the function defined from f by putting $f'(n)=0$ if

$f(n) = 0$ and $f'(n) = 1$ if $f(n) \neq 0$. Since $\theta(f)$ is equivalent in $\mathbf{RCA}_0$ to a formula in Γ, by applying Γ-**Det** we obtain that

$$\exists \sigma_{\mathrm{I}}\, \forall \sigma_{\mathrm{II}}\, \varphi((\sigma_{\mathrm{I}} \otimes \sigma_{\mathrm{II}})') \,\vee\, \exists \sigma_{\mathrm{II}}\, \forall \sigma_{\mathrm{I}}\, \neg\varphi((\sigma_{\mathrm{I}} \otimes \sigma_{\mathrm{II}})')$$

where σ_{I} and σ_{II} range over strategies for the game $G^{\mathbb{N}}(\theta)$.
Case 1: $\exists \sigma_{\mathrm{I}}\, \forall \sigma_{\mathrm{II}}\, \varphi((\sigma_{\mathrm{I}} \otimes \sigma_{\mathrm{II}})')$.
Then, we can use σ_{I} in order to construct a winning strategy for player I in the game $G^{\{0,1\}}(\varphi)$. Namely, we define a strategy τ_{I} as follows. Given a sequence s, let $\sigma_{\mathrm{I}} * s$ denote the sequence of length $2 \cdot |s|$ corresponding to the run of the game in which player I plays following σ_{I} and player II plays s. Put

$$\tau_{\mathrm{I}}(\langle x_0, y_0, \dots x_k, y_k\rangle) = \begin{cases} 0 & \text{if } \sigma_{\mathrm{I}}(\sigma_{\mathrm{I}} * \langle y_0, \dots, y_k\rangle) = 0 \\ 1 & \text{if } \sigma_{\mathrm{I}}(\sigma_{\mathrm{I}} * \langle y_0, \dots, y_k\rangle) \neq 0 \end{cases}$$

Case 2: $\exists \sigma_{\mathrm{II}}\, \forall \sigma_{\mathrm{I}}\, \neg\varphi((\sigma_{\mathrm{I}} \otimes \sigma_{\mathrm{II}})')$.
The proof is similar to the one of Case 1 and we omit it. ■

As to the opposite direction, it is also well known that games in the Baire space can be effectively (checkable in $\mathbf{RCA}_0$) reduced to games in the Cantor space, but now at the price of increasing the payoff complexity Γ (see, e.g., [MS12], p. 229). If Γ is at least at the level of Δ^0_3, this complexity increase has no effect and Γ determinacy in one space is equivalent to Γ determinacy in the other. In contrast, at levels below Δ^0_3, the strengths of Γ determinacy in the Cantor space and Γ determinacy in the Baire space are known to differ significantly (see the table below).

The precise bounds for the amount of Gale-Stewart determinacy provable in second order arithmetic were obtained by Montalbán and Shore in [MS12]. We state the result for the Baire space, but observe that the same bounds apply equally well to the Cantor space, for they lie above Δ^0_3 determinacy.

Proposition 3.5 ([MS12])

1. *For each $k \in \omega$, $\mathbf{Z}_2$ proves $(\Sigma^0_3)_k$-**Det**.*
2. *$\mathbf{Z}_2$ does not prove $(\Sigma^0_3)_\omega$-**Det**. In particular, $\mathbf{Z}_2$ does not prove Δ^0_4-**Det**.*

The strength of determinacy for levels below Δ^0_4 has been calibrated in terms of the common subsystems of second order arithmetic, both for games in the Cantor space and for games in the Baire space. The following table summarizes the main results obtained (cf. [Smp99], and [NMT07]).

The left most column contains subsystems of second order arithmetic from weaker to stronger. The center column and the right most column contain classes of games in Cantor space and Baire space, respectively. In the table a subsystem of second order arithmetic and types of determinacy included in the same row are pairwise equivalent within $\mathbf{RCA}_0$. Subsystem Π^1_1-$\mathbf{TR}_0$ is the system $\mathbf{RCA}_0$ plus Π^1_1 transfinite recursion. On the other hand, for each $k \in \omega$, the axiom scheme $\left[\Sigma^1_1\right]^k$-$\mathbf{ID}_0$ represents the iteration of inductive definitions with k operators (see [Smp99] and [MT07] for precise definitions).

Within $\mathbf{RCA}_0$		Determinacy in $2^{\mathbb{N}}$	Determinacy in $\mathbb{N}^{\mathbb{N}}$
$\mathbf{WKL}_0$	$\Leftrightarrow$	Δ^0_1, Σ^0_1	
$\mathbf{ACA}_0$	$\Leftrightarrow$	$(\Sigma^0_1)_2$	
$\mathbf{ATR}_0$	$\Leftrightarrow$	Δ^0_2, Σ^0_2	Δ^0_1, Σ^0_1
$\Pi^1_1 - \mathbf{CA}_0$	$\Leftrightarrow$		$(\Sigma^0_1)_2$
$\Pi^1_1 - \mathbf{TR}_0$	$\Leftrightarrow$		Δ^0_2
$\Sigma^1_1 - \mathbf{ID}_0$	$\Leftrightarrow$	$(\Sigma^0_2)_2$	Σ^0_2
$\vdots$		$\vdots$	$\vdots$
$\left[\Sigma^1_1\right]^k - \mathbf{ID}_0$	$\Leftrightarrow$	$(\Sigma^0_2)_{k+1}$	$(\Sigma^0_2)_k$
$\vdots$	$\Leftrightarrow$	$\vdots$	$\vdots$

From the table it is clear that within $\mathbf{RCA}_0$ we have $(\Sigma^0_2)_k$-$\mathbf{Det}^* \leftrightarrow (\Sigma^0_2)_{k-1}$-$\mathbf{Det}$ for $1 < k < \omega$. It can also be proved that Δ^0_3-$\mathbf{Det}^* \leftrightarrow \Delta^0_3$-$\mathbf{Det}$ within $\mathbf{RCA}_0$.

As it was previously mentioned, second order arithmetic $\mathbf{Z}_2$ does not prove Δ^0_4-$\mathbf{Det}$, not even $(\Sigma^0_3)_\omega$-$\mathbf{Det}$. Nevertheless, since $\mathbf{Z}_2$ proves $(\Sigma^0_3)_k$-$\mathbf{Det}$, for each $k \in \omega$, it makes sense to ask in the spirit of reverse mathematics whether there exist results in the other direction. I.e. whether $(\Sigma^0_3)_k$-$\mathbf{Det}$ can be proved to be equivalent to $\mathbf{Z}_2$. This is, however, not the case. M. O. MedSalem and K. Tanaka have proved (see [MT07]) that Borel determinacy, Δ^1_1-$\mathbf{Det}$, does not imply even Δ^1_2-$\mathbf{CA}_0$, let alone $\mathbf{Z}_2$.

3.2 Lipschitz games

Let $X \subseteq \mathbb{N}$ be nonempty. For given formulas $A(f)$ and $B(g)$ with distinguished function variables $f, g \in X^{\mathbb{N}}$, respectively, a Lipschitz game in the space $X^{\mathbb{N}}$ is defined as follows: Two players, say player I (male) and player II (female), alternately choose an element x in X to form two functions $f, g \in X^{\mathbb{N}}$. Player I plays first.

Player I	x_0		x_1		x_2		$\dots$
Player II		y_0		y_1		y_2	$\dots$

After ω turns, player I has produced a sequence of elements of X, $\langle x_0, x_1, \dots \rangle$, and player II has produced a sequence of elements of X, $\langle y_0, y_1, \dots \rangle$. The resulting play for player I is the function $f \in X^{\mathbb{N}}$ given by $\langle x_0, x_1, \dots \rangle$. The resulting play for player II is the function $g \in X^{\mathbb{N}}$ given by $\langle y_0, y_1, \dots \rangle$. Player I wins if and only if $\neg(A(f) \leftrightarrow B(g))$. Player II wins if and only if $A(f) \leftrightarrow B(g)$. We denote by $G_L^X(A, B)$ the Lipschitz game in the space $X^{\mathbb{N}}$ defined by the formulas $A(f)$, $B(g)$.

A strategy for player I in the game $G_L^X(A, B)$ is a function assigning an element of X to every sequence from X of even length. A strategy for player II in the game $G_L^X(A, B)$ is a function assigning an element of X to every sequence from X of odd length. That is to say, a strategy for player I in the game $G_L^X(A, B)$ is a function $\sigma_{\mathrm{I}} : \mathrm{Seq}^X_{\mathrm{even}} \to X$ and a strategy for player II in the game $G_L^X(A, B)$ is a function $\sigma_{\mathrm{II}} : \mathrm{Seq}^X_{\mathrm{odd}} \to X$.

If players I and II follow strategies σ_{I} and σ_{II}, respectively, the resulting plays are uniquely determined. We will write $\sigma_{\mathrm{I}} \otimes_L^{\mathrm{I}} \sigma_{\mathrm{II}}$ to denote player I's resulting play and write $\sigma_{\mathrm{I}} \otimes_L^{\mathrm{II}} \sigma_{\mathrm{II}}$ to denote player II's resulting play. Notice that

$$\sigma_{\mathrm{I}} \otimes_L^{\mathrm{I}} \sigma_{\mathrm{II}}(n) = \sigma_{\mathrm{I}} \otimes \sigma_{\mathrm{II}}(2n)$$

$$\sigma_{\mathrm{I}} \otimes_L^{\mathrm{II}} \sigma_{\mathrm{II}}(n) = \sigma_{\mathrm{I}} \otimes \sigma_{\mathrm{II}}(2n+1).$$

A strategy for player I, σ_{I}, is winning if player I wins the game as long as he plays following it, no matter what his opponent plays. Thus, the fact that σ_{I} is a winning strategy for player I in $G_L^X(A, B)$ can be formalized as follows

$$\forall \sigma_{\mathrm{II}}\, \neg(A(\sigma_{\mathrm{I}} \otimes_L^{\mathrm{I}} \sigma_{\mathrm{II}}) \leftrightarrow B(\sigma_{\mathrm{I}} \otimes_L^{\mathrm{II}} \sigma_{\mathrm{II}})).$$

where σ_{II} ranges over strategies for player II. Similarly, a strategy for player II, σ_{II}, is winning if player II wins the game as long as she plays following it, no matter what her opponent plays. Thus, the fact that σ_{II} is a winning strategy for player II in $G_L^X(A, B)$ can be formalized as follows

$$\forall \sigma_{\mathrm{I}}\, (A(\sigma_{\mathrm{I}} \otimes_L^{\mathrm{I}} \sigma_{\mathrm{II}}) \leftrightarrow B(\sigma_{\mathrm{I}} \otimes_L^{\mathrm{II}} \sigma_{\mathrm{II}})),$$

where σ_{I} ranges over strategies for player I.

Hence, the determinacy of the game $G_L^X(A, B)$, denoted $Det_L^X(A, B)$, can be expressed by the axiom

$$\exists \sigma_{\mathrm{I}}\, \forall \sigma_{\mathrm{II}}\, \neg(A(\sigma_{\mathrm{I}} \otimes_L^{\mathrm{I}} \sigma_{\mathrm{II}}) \leftrightarrow B(\sigma_{\mathrm{I}} \otimes_L^{\mathrm{II}} \sigma_{\mathrm{II}})) \vee \exists \sigma_{\mathrm{II}}\, \forall \sigma_{\mathrm{I}}\, (A(\sigma_{\mathrm{I}} \otimes_L^{I} \sigma_{\mathrm{II}}) \leftrightarrow B(\sigma_{\mathrm{I}} \otimes_L^{\mathrm{II}} \sigma_{\mathrm{II}})),$$

where σ_{I} and σ_{II} range over strategies for player I and strategies for player II, respectively.

We are now in a position to give the precise axiomatizations of the theories we will be interested in.

Definition 3.6 (Lipschitz Determinacy theories) *Let Γ_1 and Γ_2 be classes of formulas with distinguished function variables $f, g \in X^{\mathbb{N}}$, respectively.*

1. *The scheme of* (Γ_1, Γ_2)*-Lipschitz determinacy in* $X^{\mathbb{N}}$, (Γ_1, Γ_2)-$\mathbf{Det}_L^X$, *is given by the axiom scheme*

$$Det_L^X(A, B)$$

where $A(f)$ *is in* Γ_1 *and* $B(g)$ *is in* Γ_2.

For simplicity, if $\Gamma_1 = \Gamma_2 = \Gamma$, *we will write* Γ-$\mathbf{Det}_L^X$ *instead of* (Γ, Γ)-$\mathbf{Det}_L^X$.

2. *The scheme of* (Δ_n^0, Δ_m^0)*-Lipschitz determinacy in* $X^{\mathbb{N}}$, (Δ_n^0, Δ_m^0)-$\mathbf{Det}_L^X$, *is given by the axiom scheme*

$$\forall f \in X^{\mathbb{N}}(A(f) \leftrightarrow C(f)) \wedge \forall g \in X^{\mathbb{N}}(B(g) \leftrightarrow D(g)) \ \rightarrow \ Det_L^X(A, B)$$

where $A(f)$ *is in* Σ_n^0, $C(f)$ *is in* Π_n^0, $B(g)$ *is in* Σ_m^0 *and* $D(g)$ *is in* Π_m^0.

For simplicity, if $n = m$, *we will write* Δ_n^0-$\mathbf{Det}_L^X$ *instead of* (Δ_n^0, Δ_n^0)-$\mathbf{Det}_L^X$.

3. *The scheme of* (Γ, Δ_n^0)*-Lipschitz determinacy in* $X^{\mathbb{N}}$, (Γ, Δ_n^0)-$\mathbf{Det}_L^X$, *is given by the axiom scheme*

$$\forall g \in X^{\mathbb{N}}(B(g) \leftrightarrow D(g)) \ \rightarrow \ Det_L^X(A, B)$$

where $A(f)$ *is in* Γ, $B(g)$ *is in* Σ_n^0 *and* $D(g)$ *is in* Π_n^0.

4. *The scheme of* (Δ_n^0, Γ)*-Lipschitz determinacy in* $X^{\mathbb{N}}$, (Δ_n^0, Γ)-$\mathbf{Det}_L^X$, *is given by the axiom scheme*

$$\forall f \in X^{\mathbb{N}}(A(f) \leftrightarrow C(f)) \ \rightarrow \ Det_L^X(A, B)$$

where $A(f)$ *is in* Σ_n^0, $C(f)$ *is in* Π_n^0 *and* $B(g)$ *is in* Γ.

5. *If* $X = \mathbb{N}$ *(determinacy in the Baire space), we will omit the superscript* X *and write* (Γ_1, Γ_2)-$\mathbf{Det}_L$, (Δ_n^0, Δ_m^0)-$\mathbf{Det}_L$, *and so on. If* $X = \{0, 1\}$ *(determinacy in the Cantor space), we will write* (Γ_1, Γ_2)-$\mathbf{Det}_L^*$, (Δ_n^0, Δ_m^0)-$\mathbf{Det}_L^*$, *and so on.*

In a similar vein, one can naturally formalize in the language of second order arithmetic the semilinear ordering principle for the Lipschitz reducibility relation. First of all, recall that a set $A \subseteq X^{\mathbb{N}}$ is Lipschitz reducible to a set $B \subseteq X^{\mathbb{N}}$, written $A \leq_L B$, if and only if player II has a winning strategy in the game $G_L^X(A, B)$. Hence, given two formulas $A(f)$ and $B(g)$, the fact that $A \leq_L B$ can be formalized by the following axiom

$$Red_L^X(A, B) \equiv \exists \sigma_{\mathrm{II}} \, \forall \sigma_{\mathrm{I}} \, (A(\sigma_{\mathrm{I}} \otimes_L^{\mathrm{I}} \sigma_{\mathrm{II}}) \leftrightarrow B(\sigma_{\mathrm{I}} \otimes_L^{\mathrm{II}} \sigma_{\mathrm{II}})),$$

where σ_{I} and σ_{II} range over strategies for player I and strategies for player II, respectively.

The semilinear ordering principle SLO_L^X says that given $A, B \subseteq X^{\mathbb{N}}$, either $A \leq_L B$ or $\neg B \leq_L A$. Thus, the following axiomatizations appear naturally.

Definition 3.7 (Lipschitz SLO theories) *Let* Γ_1 *and* Γ_2 *be classes of formulas with distinguished function variables* $f, g \in X^{\mathbb{N}}$, *respectively.*

1. *The scheme of* (Γ_1, Γ_2)*-Lipschitz semilinear ordering principle in* $X^{\mathbb{N}}$*,* (Γ_1, Γ_2)-$\mathbf{SLO}_L^X$*, is given by the axiom scheme*

$$Red_L^X(A, B) \vee Red_L^X(\neg B, A)$$

where $A(f)$ *is in* Γ_1 *and* $B(g)$ *is in* Γ_2*.*

For simplicity, if $\Gamma_1 = \Gamma_2 = \Gamma$*, we will write* Γ-$\mathbf{SLO}_L^X$ *instead of* (Γ, Γ)-$\mathbf{SLO}_L^X$*.*

2. *The scheme of* (Δ^0_n, Δ^0_m)*-Lipschitz semilinear ordering principle in* $X^{\mathbb{N}}$*,* (Δ^0_n, Δ^0_m)-$\mathbf{SLO}_L^X$*, is given by the axiom scheme*

$$\forall f \in X^{\mathbb{N}}(A(f) \leftrightarrow C(f)) \wedge \forall g \in X^{\mathbb{N}}(B(g) \leftrightarrow D(g)) \rightarrow Red_L^X(A, B) \vee Red_L^X(\neg B, A)$$

where $A(f)$ *is in* Σ^0_n*,* $C(f)$ *is in* Π^0_n*,* $B(g)$ *is in* Σ^0_m *and* $D(g)$ *is in* Π^0_m*.*

For simplicity, if $n = m$*, we will write* Δ^0_n-$\mathbf{SLO}_L^X$ *instead of* (Δ^0_n, Δ^0_n)-$\mathbf{SLO}_L^X$*.*

3. *The scheme of* (Γ, Δ^0_n)*-Lipschitz semilinear ordering principle in* $X^{\mathbb{N}}$*,* (Γ, Δ^0_n)-$\mathbf{SLO}_L^X$*, is given by the axiom scheme*

$$\forall g \in X^{\mathbb{N}}(B(g) \leftrightarrow D(g)) \rightarrow Red_L^X(A, B) \vee Red_L^X(\neg B, A)$$

where $A(f)$ *is in* Γ*,* $B(g)$ *is in* Σ^0_n *and* $D(g)$ *is in* Π^0_n*.*

4. *The scheme of* (Δ^0_n, Γ)*-Lipschitz semilinear ordering principle in* $X^{\mathbb{N}}$*,* (Δ^0_n, Γ)-$\mathbf{SLO}_L^X$*, is given by the axiom scheme*

$$\forall f \in X^{\mathbb{N}}(A(f) \leftrightarrow C(f)) \rightarrow Red_L^X(A, B) \vee Red_L^X(\neg B, A)$$

where $A(f)$ *is in* Σ^0_n*,* $C(f)$ *is in* Π^0_n *and* $B(g)$ *is in* Γ*.*

5. *If* $X = \mathbb{N}$ *(determinacy in the Baire space), we will omit the superscript* X *and write* (Γ_1, Γ_2)-$\mathbf{SLO}_L$*,* (Δ^0_n, Δ^0_m)-$\mathbf{SLO}_L$*, and so on. If* $X = \{0, 1\}$ *(determinacy in the Cantor space), we will write* (Γ_1, Γ_2)-$\mathbf{SLO}^*_L$*,* (Δ^0_n, Δ^0_m)-$\mathbf{SLO}^*_L$*, and so on.*

3.3 Wadge games

Wadge games are variants of Lipschitz games in which player II is allowed to pass (i.e., not to play) at any round, but she must play infinitely often otherwise she loses.

Let $X \subseteq \mathbb{N}$ be nonempty. For given formulas $A(f)$ and $B(g)$ with distinguished function variables $f, g \in X^{\mathbb{N}}$, respectively, a Wadge game in the space $X^{\mathbb{N}}$ is defined as follows: Two players, say player I (male) and player II (female), alternately play to form two functions $f, g \in X^{\mathbb{N}}$. Player I plays first and in each of his turns he must choose an element x in X. In each of her turns player II either chooses an element x in X or has the option to pass (we will write `p` to denote this action.) But Player II has to play infinitely often otherwise she loses.

Player I	x_0		x_1		x_2		x_3		x_4		$\dots$
Player II		$\mathtt{p}$		$\mathtt{p}$		y_0		$\mathtt{p}$		y_1	$\dots$

After ω turns, player I has produced a sequence of elements of X, $\langle x_0, x_1, x_2 \dots \rangle$, and player II has produced a sequence, $\langle \mathtt{p}, \dots, \mathtt{p}, y_0, \mathtt{p}, \dots, \mathtt{p}, y_1, \mathtt{p}, \dots, \mathtt{p}, y_2, \dots \rangle$. The resulting play for player I is the function $f \in X^{\mathbb{N}}$ given by $\langle x_0, x_1, x_2 \dots \rangle$. The resulting play for player II is the function $g \in X^{\mathbb{N}}$ given by $\langle y_0, y_1, y_2 \dots \rangle$. Player I wins if and only if $\neg(A(f) \leftrightarrow B(g))$. Player II wins if and only if $A(f) \leftrightarrow B(g)$. We denote by $G_W^X(A, B)$ the Wadge game in the space $X^{\mathbb{N}}$ defined by the formulas $A(f), B(g)$.

In order to formalize the fact that player II can pass, we will identity passing with picking the number zero. Accordingly, we will consider that if one of the two players chooses number $n+1$ in the formalized Wadge game, he or she has actually chosen number n in the *real world* game. As a consequence, player I and player II would play in different spaces, for player I is not allowed to choose the number zero. To avoid this, we opt for allowing player I to pick number zero as well and we consider that for player I, both choosing 0 and 1 mean choosing 0 in the *real world* game. Thus, as an example, a (formalized) play of a Wadge game of the form

$$2, 3, 0, 0, 1, 1, 4, 0 \dots$$

corresponds to the following *real world* play

Player I	1		0		0		3		$\dots$
Player II		2		$\mathtt{p}$		0		$\mathtt{p}$	$\dots$

(Notice that the formalization of a play of a Wadge game is not unique. The previous one could also be formalized as $2, 3, 1, 0, 1, 1, 4, 0 \dots$ or as $2, 3, 0, 0, 0, 1, 4, 0 \dots$)

We are now in a position to give our definitions. Let X^+ be the set defined as $\{0\} \cup \{i + 1 : i \in X\}$. A strategy for player I in the game $G_W^X(A, B)$ is a function assigning an element of X^+ to every sequence from X^+ of even length. A strategy for player II in the game $G_W^X(A, B)$ is a function assigning an element of X^+ to every sequence from X^+ of odd length. That is to say, a strategy for player I in the game $G_L^X(A, B)$ is a function $\sigma_{\mathrm{I}} : \mathrm{Seq}_{\mathrm{even}}^{X^+} \to X^+$ and a strategy for player II in the game $G_W^X(A, B)$ is a function $\sigma_{\mathrm{II}} : \mathrm{Seq}_{\mathrm{odd}}^{X^+} \to X^+$.

Given strategies σ_{I} and σ_{II}, the restriction that player II must play infinitely often can be expressed by the Π_2^0 formula

$$Inf(\sigma_{\mathrm{I}}, \sigma_{\mathrm{II}}) \equiv \forall n \, \exists k > n \; \sigma_{\mathrm{I}} \otimes \sigma_{\mathrm{II}}(2k + 1) \neq 0$$

(Recall that $\sigma_{\mathrm{I}} \otimes \sigma_{\mathrm{II}}$ denotes the resulting play for a Gale-Stewart game previously defined.)

If players I and II follow strategies σ_{I} and σ_{II}, respectively, and $Inf(\sigma_{\mathrm{I}}, \sigma_{\mathrm{II}})$ holds, the resulting plays are uniquely determined. We will write $\sigma_{\mathrm{I}} \otimes_W^{\mathrm{I}} \sigma_{\mathrm{II}}$ to denote player I's resulting play and write $\sigma_{\mathrm{I}} \otimes^{\mathrm{II}}_{W} \sigma_{\mathrm{II}}$ to denote player II's resulting play. Note that both $\sigma_{\mathrm{I}} \otimes^{\mathrm{I}}_{W} \sigma_{\mathrm{II}}$ and $\sigma_{\mathrm{I}} \otimes^{\mathrm{I}}_{W} \sigma_{\mathrm{II}}$ will be functions from $\mathbb{N}$ into X. Actually, we define

$$\sigma_{\mathrm{I}} \otimes^{\mathrm{I}}_{W} \sigma_{\mathrm{II}}(n) = (\sigma_{\mathrm{I}} \otimes \sigma_{\mathrm{II}}(2n)) \dot{-} 1$$

$$\begin{aligned} move(0) &= \mu i\,[(\sigma_{\mathrm{I}} \otimes \sigma_{\mathrm{II}})(2i+1) \neq 0] \\ move(n+1) &= \mu i\,[i > move(n) \wedge (\sigma_{\mathrm{I}} \otimes \sigma_{\mathrm{II}})(2i+1) \neq 0] \end{aligned}$$

$$\sigma_{\mathrm{I}} \otimes^{\mathrm{II}}_{W} \sigma_{\mathrm{II}}(n) = (\sigma_{\mathrm{I}} \otimes \sigma_{\mathrm{II}}(2move(n)+1)) - 1$$

where $\dot{-}$ denotes the modified subtraction function given by $a \dot{-} b = \max(0, a-b)$.

A strategy for player I, σ_{I}, is winning if player I wins the game as long as he plays following it, no matter what his opponent plays. Thus, the fact that σ_{I} is a winning strategy for player I in $G^X_W(A,B)$ can be formalized as follows

$$\forall \sigma_{\mathrm{II}}\ (Inf(\sigma_{\mathrm{I}}, \sigma_{\mathrm{II}}) \rightarrow \neg(A(\sigma_{\mathrm{I}} \otimes^{\mathrm{I}}_{W} \sigma_{\mathrm{II}}) \leftrightarrow B(\sigma_{\mathrm{I}} \otimes^{\mathrm{II}}_{W} \sigma_{\mathrm{II}}))),$$

where σ_{II} ranges over strategies for player II in the game $G^X_W(A,B)$. Similarly, a strategy for player II, σ_{II}, is winning if player II wins the game as long as she plays following it, no matter what her opponent plays. Thus, the fact that σ_{II} is a winning strategy for player II in $G^X_W(A,B)$ can be formalized as follows

$$\forall \sigma_{\mathrm{I}}\ (Inf(\sigma_{\mathrm{I}}, \sigma_{\mathrm{II}}) \wedge (A(\sigma_{\mathrm{I}} \otimes^{\mathrm{I}}_{W} \sigma_{\mathrm{II}}) \leftrightarrow B(\sigma_{\mathrm{I}} \otimes^{\mathrm{II}}_{W} \sigma_{\mathrm{II}}))),$$

where σ_{I} ranges over strategies for player I in the game $G^X_W(A,B)$.

Hence, the determinacy of the game $G^X_W(A,B)$, denoted $Det^X_W(A,B)$, can be expressed by the axiom

$$\begin{aligned} &\exists \sigma_{\mathrm{I}}\, \forall \sigma_{\mathrm{II}}\ \big[Inf(\sigma_{\mathrm{I}}, \sigma_{\mathrm{II}}) \rightarrow \neg(A(\sigma_{\mathrm{I}} \otimes^{\mathrm{I}}_{W} \sigma_{\mathrm{II}}) \leftrightarrow B(\sigma_{\mathrm{I}} \otimes^{\mathrm{II}}_{W} \sigma_{\mathrm{II}}))\big] \\ &\qquad \vee\ \exists \sigma_{\mathrm{II}}\, \forall \sigma_{\mathrm{I}}\ \big[Inf(\sigma_{\mathrm{I}}, \sigma_{\mathrm{II}}) \wedge (A(\sigma_{\mathrm{I}} \otimes^{\mathrm{I}}_{W} \sigma_{\mathrm{II}}) \leftrightarrow B(\sigma_{\mathrm{I}} \otimes^{\mathrm{II}}_{W} \sigma_{\mathrm{II}}))\big] \end{aligned}$$

where σ_{I} and σ_{II} range over strategies for player I and strategies for player II, respectively.

We are now in a position to give the precise axiomatizations of the theories for Wadge determinacy we will be interested in.

Definition 3.8 (Wadge Determinacy Theories) *Let Γ_1 and Γ_2 be classes of formulas with distinguished function variables $f, g \in X^{\mathbb{N}}$, respectively.*

1. *The scheme of (Γ_1, Γ_2)-Wadge determinacy in $X^{\mathbb{N}}$, (Γ_1, Γ_2)-$\mathbf{Det}^X_W$, is given by the axiom scheme*

   $$Det^X_W(A,B)$$

   where $A(f)$ is in Γ_1 and $B(g)$ is in Γ_2.

   For simplicity, if $\Gamma_1 = \Gamma_2 = \Gamma$, we will write Γ-$\mathbf{Det}^X_W$ instead of (Γ, Γ)-$\mathbf{Det}^X_W$.

2. *The scheme of (Δ^0_n, Δ^0_m)-Wadge determinacy in $X^{\mathbb{N}}$, (Δ^0_n, Δ^0_m)-$\mathbf{Det}^X_W$, is given by the axiom scheme*

$$\forall f \in X^{\mathbb{N}}(A(f) \leftrightarrow C(f)) \wedge \forall g \in X^{\mathbb{N}}(B(g) \leftrightarrow D(g)) \;\rightarrow\; Det^X_W(A, B)$$

where $A(f)$ is in Σ^0_n, $C(f)$ is in Π^0_n, $B(g)$ is in Σ^0_m and $D(g)$ is in Π^0_m.

For simplicity, if $n = m$, we will write Δ^0_n-$\mathbf{Det}^X_W$ instead of (Δ^0_n, Δ^0_n)-$\mathbf{Det}^X_W$.

3. *The scheme of (Γ, Δ^0_n)-Wadge determinacy in $X^{\mathbb{N}}$, (Γ, Δ^0_n)-$\mathbf{Det}^X_W$, is given by the axiom scheme*

$$\forall g \in X^{\mathbb{N}}(B(g) \leftrightarrow D(g)) \;\rightarrow\; Det^X_W(A, B)$$

where $A(f)$ is in Γ, $B(g)$ is in Σ^0_n and $D(g)$ is in Π^0_n.

4. *The scheme of (Δ^0_n, Γ)-Wadge determinacy in $X^{\mathbb{N}}$, (Δ^0_n, Γ)-$\mathbf{Det}^X_W$, is given by the axiom scheme*

$$\forall f \in X^{\mathbb{N}}(A(f) \leftrightarrow C(f)) \;\rightarrow\; Det^X_W(A, B)$$

where $A(f)$ is in Σ^0_n, $C(f)$ is in Π^0_n and $B(g)$ is in Γ.

5. *If $X = \mathbb{N}$ (determinacy in the Baire space), we will omit the superscript X and write (Γ_1, Γ_2)-$\mathbf{Det}_W$, (Δ^0_n, Δ^0_m)-$\mathbf{Det}_W$, and so on. If $X = \{0, 1\}$ (determinacy in the Cantor space), we will write (Γ_1, Γ_2)-$\mathbf{Det}^*_W$, (Δ^0_n, Δ^0_m)-$\mathbf{Det}^*_W$, and so on.*

The next step is to give some natural axiomatizations of the semilinear ordering principle for the Wadge reducibility relation. Recall that a set $A \subseteq X^{\mathbb{N}}$ is Wadge reducible to a set $B \subseteq X^{\mathbb{N}}$, written $A \leq_W B$, if and only if player II has a winning strategy in the game $G^X_W(A, B)$. Hence, given two formulas $A(f)$ and $B(g)$, the fact that $A \leq_W B$ can be formalized by the following axiom

$$Red^X_W(A, B) \equiv \exists \sigma_{\mathrm{II}}\, \forall \sigma_{\mathrm{I}}\, [Inf(\sigma_{\mathrm{I}}, \sigma_{\mathrm{II}}) \wedge (A(\sigma_{\mathrm{I}} \otimes^{\mathrm{I}}_W \sigma_{\mathrm{II}}) \leftrightarrow B(\sigma_{\mathrm{I}} \otimes^{\mathrm{II}}_W \sigma_{\mathrm{II}}))],$$

where σ_{I} and σ_{II} range over strategies for player I and strategies for player II in the game $G^X_W(A, B)$, respectively.

The semilinear ordering principle SLO^X_W says that given $A, B \subseteq X^{\mathbb{N}}$, either $A \leq_W B$ or $\neg B \leq_W A$. Thus, the following axiomatizations appear naturally.

Definition 3.9 *Let Γ_1 and Γ_2 be classes of formulas with distinguished function variables $f, g \in X^{\mathbb{N}}$, respectively.*

1. *The scheme of (Γ_1, Γ_2)-Wadge semilinear ordering principle in $X^{\mathbb{N}}$, (Γ_1, Γ_2)-$\mathbf{SLO}^X_W$, is given by the axiom scheme*

$$Red^X_W(A, B) \vee Red^X_W(\neg B, A)$$

where $A(f)$ is in Γ_1 and $B(g)$ is in Γ_2.

For simplicity, if $\Gamma_1 = \Gamma_2 = \Gamma$, we will write Γ-$\mathbf{SLO}^X_W$ instead of (Γ, Γ)-$\mathbf{SLO}^X_W$.

2. *The scheme of* (Δ^0_n, Δ^0_m)*-Wadge semilinear ordering principle in* $X^{\mathbb{N}}$*,* (Δ^0_n, Δ^0_m)-$\mathbf{SLO}^X_W$*, is given by the axiom scheme*

   $$\forall f \in X^{\mathbb{N}}(A(f) \leftrightarrow C(f)) \wedge \forall g \in X^{\mathbb{N}}(B(g) \leftrightarrow D(g)) \rightarrow \mathit{Red}^X_W(A, B) \vee \mathit{Red}^X_W(\neg B, A)$$

   where $A(f)$ *is in* Σ^0_n*,* $C(f)$ *is in* Π^0_n*,* $B(g)$ *is in* Σ^0_m *and* $D(g)$ *is in* Π^0_m*.*

   For simplicity, if $n = m$*, we will write* Δ^0_n-$\mathbf{SLO}^X_W$ *instead of* (Δ^0_n, Δ^0_n)-$\mathbf{SLO}^X_W$*.*

3. *The scheme of* (Γ, Δ^0_n)*-Wadge semilinear ordering principle in* $X^{\mathbb{N}}$*,* (Γ, Δ^0_n)-$\mathbf{SLO}^X_W$*, is given by the axiom scheme*

   $$\forall g \in X^{\mathbb{N}}(B(g) \leftrightarrow D(g)) \rightarrow \mathit{Red}^X_W(A, B) \vee \mathit{Red}^X_W(\neg B, A)$$

   where $A(f)$ *is in* Γ*,* $B(g)$ *is in* Σ^0_n *and* $D(g)$ *is in* Π^0_n*.*

4. *The scheme of* (Δ^0_n, Γ)*-Wadge semilinear ordering principle in* $X^{\mathbb{N}}$*,* (Δ^0_n, Γ)-$\mathbf{SLO}^X_W$*, is given by the axiom scheme*

   $$\forall f \in X^{\mathbb{N}}(A(f) \leftrightarrow C(f)) \rightarrow \mathit{Red}^X_W(A, B) \vee \mathit{Red}^X_W(\neg B, A)$$

   where $A(f)$ *is in* Σ^0_n*,* $C(f)$ *is in* Π^0_n *and* $B(g)$ *is in* Γ*.*

5. *If* $X = \mathbb{N}$ *(determinacy in the Baire space), we will omit the superscript* X *and write* (Γ_1, Γ_2)-$\mathbf{SLO}_W$*,* (Δ^0_n, Δ^0_m)-$\mathbf{SLO}_W$*, and so on. If* $X = \{0, 1\}$ *(determinacy in the Cantor space), we will write* (Γ_1, Γ_2)-$\mathbf{SLO}^*_W$*,* (Δ^0_n, Δ^0_m)-$\mathbf{SLO}^*_W$*, and so on.*

3.4 Basic properties of Lipschitz and Wadge determinacy

In what follows, we state some basic properties of Lipschitz and Wadge determinacy that will be used in the next chapters, sometime without explicit reference.

The fact that Γ-determinacy and $\neg\Gamma$-determinacy are equivalent principles is immediate for Lipschitz and Wadge games, as the games $G^X_{L/W}(A, B)$ and $G^X_{L/W}(\neg A, \neg B)$ turn out to be the same game.

Lemma 3.10 *It is provable over* $\mathbf{RCA}_0$ *that*

1. (Γ_1, Γ_2)-$\mathbf{Det}^X_{L/W}$ *and* $(\neg\Gamma_1, \neg\Gamma_2)$-$\mathbf{Det}^X_{L/W}$ *are equivalent.*

2. (Γ_1, Γ_2)-$\mathbf{SLO}^X_{L/W}$ *and* $(\neg\Gamma_1, \neg\Gamma_2)$-$\mathbf{SLO}^X_{L/W}$ *are equivalent.*

3. *The same holds for the complementary pair of classes* (Γ, Δ^0_m) *and* $(\neg\Gamma, \Delta^0_m)$*, and for* (Δ^0_n, Γ) *and* $(\Delta^0_n, \neg\Gamma)$*.*

Proof. Given $A, B \subseteq X^{\mathbb{N}}$, note that $G^X_{L/W}(A, B)$ and $G^X_{L/W}(\neg A, \neg B)$ coincide, for $A(f) \leftrightarrow B(g)$ and $\neg A(f) \leftrightarrow \neg B(g)$ are logically equivalent. ■

The argument that games in the Cantor space can be reduced to games in the Baire space without increasing the payoff complexity given in the proof of Lemma 3.4 applies equally well in the present context. Thus, we have

Lemma 3.11 *It is provable over* $\mathbf{RCA}_0$ *that*

1. (Γ_1,Γ_2)-$\mathbf{Det}_{L/W}$ *implies* (Γ_1,Γ_2)-$\mathbf{Det}^*_{L/W}$.

2. (Γ_1,Γ_2)-$\mathbf{SLO}_{L/W}$ *implies* (Γ_1,Γ_2)-$\mathbf{SLO}^*_{L/W}$.

3. *The same holds for classes* (Δ^0_n,Δ^0_m), (Γ,Δ^0_m), (Δ^0_n,Γ).

The well known fact that the semilinear ordering principle can be inferred from determinacy can be formalized inside $\mathbf{RCA}_0$. Indeed, we have

Lemma 3.12 *It is provable over* $\mathbf{RCA}_0$ *that*

1. (Γ_1,Γ_2)-$\mathbf{Det}^X_{L/W}$ *implies* (Γ_1,Γ_2)-$\mathbf{SLO}^X_{L/W}$.

2. *The same holds for classes* (Δ^0_n,Δ^0_m), (Γ,Δ^0_m), (Δ^0_n,Γ).

Proof. We only write the proof for the Lipschitz determinacy case. The Wadge determinacy case is analogous. Assume (Γ_1,Γ_2)-$\mathbf{Det}^X_L$. Let $A(f)\in\Gamma_1$ and $B(g)\in\Gamma_2$. We must show that either $Red^X_L(A,B)$ or $Red^X_L(\neg B,A)$ holds. By hypothesis, the game $G^X_L(A,B)$ is determined.

Case 1: $\exists\sigma_{\mathrm{II}}\,\forall\sigma_{\mathrm{I}}\,(A(\sigma_{\mathrm{I}}\otimes^{I}_{L}\sigma_{\mathrm{II}})\leftrightarrow B(\sigma_{\mathrm{I}}\otimes^{\mathrm{II}}_{L}\sigma_{\mathrm{II}}))$.
Then, $Red^X_L(A,B)$ holds by definition.

Case 2: $\exists\sigma_{\mathrm{I}}\,\forall\sigma_{\mathrm{II}}\,\neg(A(\sigma_{\mathrm{I}}\otimes^{\mathrm{I}}_{L}\sigma_{\mathrm{II}})\leftrightarrow B(\sigma_{\mathrm{I}}\otimes^{\mathrm{II}}_{L}\sigma_{\mathrm{II}}))$.
Define τ_{II} to be the strategy for player II given by:

$$\tau_{\mathrm{II}}(\langle x_0\rangle)=\sigma_{\mathrm{I}}(\langle\rangle)$$
$$\tau_{\mathrm{II}}(\langle x_0,y_0,\ldots,x_k,y_k,x_{k+1}\rangle)=\sigma_{\mathrm{I}}(\langle y_0,x_0,\ldots,y_k,x_k\rangle)$$

We claim that τ_{II} is a winning strategy for player II in the game $G^X_L(\neg B,A)$. To see this, pick a strategy for player I, τ_{I} say, and consider σ_{II} to be the strategy for player II defined from τ_{I} as above:

$$\sigma_{\mathrm{II}}(\langle x_0\rangle)=\tau_{\mathrm{I}}(\langle\rangle)$$
$$\sigma_{\mathrm{II}}(\langle x_0,y_0,\ldots,x_k,y_k,x_{k+1})=\tau_{\mathrm{I}}(y_0,x_0,\ldots,y_k,x_k\rangle)$$

By Δ^0_0-induction, it is easy to check that

$$\tau_{\mathrm{I}}\otimes^{\mathrm{I}}_{L}\tau_{\mathrm{II}}=\sigma_{\mathrm{I}}\otimes^{\mathrm{II}}_{L}\sigma_{\mathrm{II}}$$
$$\tau_{\mathrm{I}}\otimes^{\mathrm{II}}_{L}\tau_{\mathrm{II}}=\sigma_{\mathrm{I}}\otimes^{\mathrm{I}}_{L}\sigma_{\mathrm{II}}$$

So, $\neg(A(\tau_{\mathrm{I}}\otimes^{\mathrm{II}}_{L}\tau_{\mathrm{II}})\leftrightarrow B(\tau_{\mathrm{I}}\otimes^{\mathrm{I}}_{L}\sigma_{\mathrm{II}}))$. Hence, $\neg B(\tau_{\mathrm{I}}\otimes^{\mathrm{I}}_{L}\sigma_{\mathrm{II}})\leftrightarrow A(\tau_{\mathrm{I}}\otimes^{\mathrm{II}}_{L}\tau_{\mathrm{II}})$, as required. We have thus shown that there exists a winning strategy for player II in the game $G^X_L(\neg B,A)$ and so $Red^X_L(\neg B,A)$ holds. ■

Remark 3.13 *Notice that the previous argument actually shows that over* $\mathbf{RCA}_0$, (Γ_1, Γ_2)-$\mathbf{Det}_W^X$ *implies* $Red_W^X(A, B) \vee Red_L^X(\neg B, A)$.

Concerning the relation between Lipschitz and Wadge determinacy, observe that *i*) a winning strategy for player I in a Wadge game automatically gives rise to a winning strategy for player I in the corresponding Lipschitz game, and *ii*) a winning strategy for player II in a Lipschitz game automatically gives rise to a winning strategy for player I in the corresponding Wadge game. Thus, we have

Lemma 3.14 *Given formulas* $A(f), B(g)$, *it is provable in* $\mathbf{RCA}_0$ *that*

1. $\exists \sigma_{\mathrm{I}} \forall \sigma_{\mathrm{II}} \neg(A(\sigma_{\mathrm{I}} \otimes_W^{\mathrm{I}} \sigma_{\mathrm{II}}) \leftrightarrow B(\sigma_{\mathrm{I}} \otimes_W^{\mathrm{II}} \sigma_{\mathrm{II}}))$ *implies* $\exists \sigma_{\mathrm{I}} \forall \sigma_{\mathrm{II}} \neg(A(\sigma_{\mathrm{I}} \otimes_L^{\mathrm{I}} \sigma_{\mathrm{II}}) \leftrightarrow B(\sigma_{\mathrm{I}} \otimes_L^{\mathrm{II}} \sigma_{\mathrm{II}}))$.
2. $\exists \sigma_{\mathrm{II}} \forall \sigma_{\mathrm{I}} (A(\sigma_{\mathrm{I}} \otimes_L^{\mathrm{I}} \sigma_{\mathrm{II}}) \leftrightarrow B(\sigma_{\mathrm{I}} \otimes_L^{\mathrm{II}} \sigma_{\mathrm{II}}))$ *implies* $\exists \sigma_{\mathrm{II}} \forall \sigma_{\mathrm{I}} (A(\sigma_{\mathrm{I}} \otimes_W^{\mathrm{I}} \sigma_{\mathrm{II}}) \leftrightarrow B(\sigma_{\mathrm{I}} \otimes_W^{\mathrm{II}} \sigma_{\mathrm{II}}))$.

Proof. Immediate. ∎

As a consequence, we obtain that

Lemma 3.15 *It is provable over* $\mathbf{RCA}_0$ *that*

1. (Γ_1, Γ_2)-$\mathbf{SLO}_L^X$ *implies* (Γ_1, Γ_2)-$\mathbf{SLO}_W^X$.
2. *The same holds for classes* (Δ_n^0, Δ_m^0), (Γ, Δ_m^0), (Δ_n^0, Γ).

Proof. It follows from Lemma 3.14 that $Red_L^X(A, B)$ implies $Red_W^X(A, B)$. ∎

3.5 Reducing Lipschitz and Wadge games to Gale-Stewart games

Lipschitz and Wadge games can be naturally reduced to Gale-Stewart games. This reduction is effective and checkable in, say, $\mathbf{RCA}_0$. However, there is a price to pay: a possible increase of the payoff set complexity.

Fix $X \subseteq \mathbb{N}$. Let us start by analyzing the Lipschitz case, which is simpler because player II is not allowed to pass. Consider the game $G_L^X(A, B)$, where $A(f), B(g)$ are formulas and f, g range over $X^{\mathbb{N}}$. Recall that the resulting play in a Lipschitz game can be effectively recovered from the resulting play in a Gale-Stewart game. Indeed, given a function $h \in X^{\mathbb{N}}$, it suffices to consider the functions h^1, h^2 defined by composition from h as follows

$$h^1(n) = h(2n)$$
$$h^2(n) = h(2n+1).$$

Now put

$$Trans_L(A, B)(h) = \neg(A(h^1) \leftrightarrow B(h^2)),$$

where h ranges over functions in $X^{\mathbb{N}}$. It is clear that a winning strategy for player I (respectively, for player II) in the game $G^X(Trans_L(A, B))$ is a winning strategy for player I (respectively, for player II) in the game $G^X_L(A, B)$. Thus, we have

$$Det^X(Trans_L(A, B)) \to Det^X_L(A, B).$$

Since $\neg(A \leftrightarrow B)$ is logically equivalent to $(A \vee B) \wedge (\neg A \vee \neg B)$, the previous reduction allows one to infer Lipschitz determinacy for Γ sets from Gale-Stewart determinacy for $\Gamma \wedge (\neg\Gamma)$ sets (We are assuming that Γ is closed under conjunction and disjunctions.) This provides us with an upper bound on the strength of Lipschitz determinacy for Γ sets. This upper bound needn't be, however, optimal. Even worse, a Lipschitz game whose determinacy can be proved in second order arithmetic may well unravel to a Gale-Stewart game whose determinacy cannot be established within second order arithmetic. Consider, for example, payoff complexity $\Gamma = \Sigma^0_4$. By a result of Louveau and Saint-Raymond [LSR87] $\mathbf{Z_2}$ proves the principle of Σ^0_4-$\mathbf{Det}_L$, whereas by a result of Montalbán and Shore [MS12] $\mathbf{Z_2}$ does not prove $\Sigma^0_4 \wedge \Pi^0_4$-$\mathbf{Det}$.

As to the Wadge case, consider the game $G^X_W(A, B)$, where $A(f), B(g)$ are formulas and f, g range over functions in $X^{\mathbb{N}}$. Given a function $h \in X^{\mathbb{N}}$, let $Inf(h)$ denote the Π^0_2 formula given by

$$\forall n\, \exists k\, (k > n \wedge h(k) \neq 0),$$

let h^1, h^2 denote the function given by

$$h^1(n) = h(2n)$$
$$h^2(n) = h(2n + 1)$$

and let $h^{2,W}$ denote the function given by

$$move(h)(0) = \mu i\, [h(2i + 1) \neq 0]$$
$$move(h)(n + 1) = \mu i\, [i > move(h)(n) \wedge h(2i + 1) \neq 0]$$
$$h^{2,W}(n) = h(2move(h)(n) + 1)) - 1.$$

Now put

$$Trans_W(A, B)(h) = Inf(h^2) \to \neg(A(h^1) \leftrightarrow B(h^{2,W})),$$

where h ranges over functions in $X^{\mathbb{N}}$. It is easy to see that a winning strategy for player I (respectively, for player II) in the game $G^X(Trans_W(A, B))$ gives rise to a winning strategy for player I (respectively, for player II) in the game $G^X_W(A, B)$. Thus, we have

$$Det^X(Trans_W(A,B)) \rightarrow Det^X_W(A,B).$$

As in the Lipschitz case, the previous reduction allows one to infer Wadge determinacy for Γ sets from Gale-Stewart determinacy for an appropriate playoff complexity. But observe that now the possible increase of the payoff complexity is even worse due to the presence of the Π^0_2 formula $Inf(h^2)$ in the translation of the game. For example, the translation of a clopen Wadge game would already give rise to a Σ^0_2 Gale-Stewart game.

3.6 Wadge's lemma in $\mathbf{RCA}_0$

In this final section we show that Wadge's lemma can be formalized and proved in $\mathbf{RCA}_0$. We only present a proof of Wadge's lemma for Lipschitz reductions, the proof for Wadge reductions is similar. In order to do that, the notion of a Lipschitz function $F : X^{\mathbb{N}} \rightarrow X^{\mathbb{N}}$, must be defined within $\mathbf{RCA}_0$. This will be done by adapting the definition of continuous function given in Simpson in [Smp99], Definition II.6.1. Previously we have to view $X^{\mathbb{N}}$ within $\mathbf{RCA}_0$ as a complete separable metric space. For this end we need to formalize within $\mathbf{RCA}_0$ several concepts that we take from [Smp99], pp. 74-81.)

The set of real numbers, $\mathbb{R}$, does not formally exist within $\mathbf{RCA}_0$, since $\mathbf{RCA}_0$ is limited to the language of second order arithmetic. So the real numbers are defined in $\mathbf{RCA}_0$ as Cauchy sequences of rational numbers. The latter numbers are obtained from $\mathbb{N}$ via the usual Dedekind construction of the number systems. In this way, each integer number is coded by a natural number coding a pair of natural numbers; and similarly, a rational number is coded by a natural number coding a pair of integers.

Within $\mathbf{RCA}_0$, a real number is a sequence of rational numbers $\langle q_k : k \in \mathbb{N}\rangle$ such that

$$\forall k \forall i \ |q_k - q_{k+i}| \leq 2^{-k}.$$

We say that two real numbers $\langle q_k : k \in \mathbb{N}\rangle$ and $\langle q'_k : k \in \mathbb{N}\rangle$ are equal if

$$\forall k \ \left|q_k - q'_k\right| \leq 2^{-k+1}.$$

We embed $\mathbb{Q}$ in $\mathbb{R}$ by identifying each rational number q with the real $x_q = \langle q_k : k \in \mathbb{N}\rangle$, with $q_k = q$ for all $k \in \mathbb{N}$.

Now we can formalize sequences of real numbers within $\mathbf{RCA}_0$ as functions $f : \mathbb{N}\times\mathbb{N} \rightarrow \mathbb{Q}$ such that for each $n \in \mathbb{N}$ the function $f_n : \mathbb{N} \rightarrow \mathbb{Q}$ defined by

$$f_n(k) = f(k,n)$$

is a real number in the sense of the former definition of real number. The notation $\langle x_k : k \in \mathbb{N}\rangle$ will be used to represent the sequence of real numbers f with $f_n = x_n$.

Using the previous definition we can define convergence within $\mathbf{RCA}_0$. A sequence $\langle x_k : k \in \mathbb{N}\rangle$ converges to the real number x, written $\lim x_n = x$, if

$$\forall k \, \exists n \, \forall i \ |x - x_{n+i}| < 2^{-k}.$$

Like $\mathbb{R}$, the metric space $X^{\mathbb{N}}$ also does not formally exist within $\mathbf{RCA}_0$. However we can formalize it within $\mathbf{RCA}_0$ by using a coding machinery for complete separable metric spaces. For this purpose let us observe that each complete separable metric space M is determined by a countable dense set $X \subseteq M$ and the function $d : X \times X \to \mathbb{R}$ obtained by restricting to X the distance function of M.

Formally, a code for a complete separable metric space is given by a set X and sequence of real numbers $d : X \times X \to \mathbb{R}$ such that

$$d(a,a) = 0,\ d(a,b) = d(b,a) \geq 0, \text{ and } d(a,b) + d(b,c) \geq d(a,c)$$

for all $a, b, c \in X$. A code (X, d) determines a metric space that we denote by $\widehat{X}$ (although formally there is no concrete object corresponding to it). An element of $\widehat{X}$ is a sequence $\langle a_k : k \in \mathbb{N}\rangle$ of elements of X, such that

$$\forall i\, \forall j\ \left(i < j \to d(a_i, a_j) \leq 2^{-i}\right).$$

For points $x = \langle a_k : k \in \mathbb{N}\rangle$ and $y = \langle b_k : k \in \mathbb{N}\rangle$ of $\widehat{X}$ we define $d(x, y) = \lim_k d(a_k, b_k)$. The equality $x = y$ is defined to mean that $d(x, y) = 0$.

We embed X in $\widehat{X}$ by identifying each $a \in X$ with the point $x_a = \langle a_k : k \in \mathbb{N}\rangle$ of $\widehat{X}$, with $x_k = a$ for all $k \in \mathbb{N}$. Under this identification, X is a (countable) dense subset of $\widehat{X}$.

Baire space (and similarly Cantor space) can be described in this framework as follows.

First of all, recall that the set X_0 of all functions $f : \omega \to \omega$ such that for some $k_f \in \omega$, we have $f(n) = 0$ for all $n \geq k_f$, is a dense subset of Baire space. Each function $f \in X_0$ can be put in correspondence with the finite sequence $f[k_f]$ and so X_0 is countable and can be identified with the set $\omega^{<\omega}$ of all finite sequences. (A similar argument is also valid for Cantor space.)

Therefore, working in $\mathbf{RCA}_0$, if $X = \{0, 1\}$ or $X = \mathbb{N}$ a code for the complete separable metric space $X^{\mathbb{N}}$ can be obtained by taking $X^{<\mathbb{N}}$ as a code for a countable dense subset and a distance function given by $d : X^{<\mathbb{N}} \times X^{<\mathbb{N}} \to \mathbb{R}$ with

$$d(s,t) = \begin{cases} \frac{1}{2^{k+1}} & \text{if } \exists j \leq \max(|s|, |t|)\,(s_j \neq t_j) \text{ and } k = \min\{j : s_j \neq t_j\} \\ 0 & \text{if } \forall j \leq \max(|s|, |t|)\,(s_j = t_j) \end{cases}$$

where

$$s_i = \begin{cases} s(i) & \text{if } i < |s| \\ 0 & \text{if } i \geq |s| \end{cases} \quad \text{and} \quad t_i = \begin{cases} t(i) & \text{if } i < |t| \\ 0 & \text{if } i \geq |t| \end{cases}$$

Lemma 3.16 *Within* $\mathbf{RCA}_0$ *is provable that:*

1. *$(X^{<\mathbb{N}}, d)$ is a code for a complete separable metric space, $\widehat{X^{<\mathbb{N}}}$;*

2. *The points of $\widehat{X^{<\mathbb{N}}}$ can be identified with functions from $\mathbb{N}$ to X. Namely,*

- *If* $x = \langle a_k : k \in \mathbb{N} \rangle$ *is a point of* $\widehat{X^{<\mathbb{N}}}$, *we identify* x *with the function* $f : \mathbb{N} \to X$, *defined by* $f(i) = a_{i+2}(i)$, *for all* i*; and*
- *Given* $f : \mathbb{N} \to X$, *we identify* f *with the point* $x = \langle a_k : k \in \mathbb{N} \rangle$ *of* $\widehat{X^{<\mathbb{N}}}$ *defined by* $a_i = f[i]$, *for all* i.

So we shall denote $\widehat{X^{<\mathbb{N}}}$ by using the more natural notation $X^{\mathbb{N}}$. Under this identification, the metric of the metric space $\widehat{X^{\mathbb{N}}}$ coincides with the usual distance for Baire and Cantor space defined in Chapter 1.

Within $\mathbf{RCA}_0$, each pair $(s, r) \in X^{<\mathbb{N}} \times \mathbb{Q}^+$ is regarded as a code for an open ball $B(s, r)$ consisting of all $f \in X^{<\mathbb{N}}$ such that $d(s, f) < r$. The notation $(s, r_1) < (t, r_2)$ is used to mean $d(s, t) + r_1 < r_2$, i.e. the closure of the open ball $B(s, r_1)$ is included in the open ball $B(t, r_2)$.

Next we formalize the concept of a Lipschitz function (see [Smp99], Definition II.6.1). Within $\mathbf{RCA}_0$, a code for a Lipschitz function $F : X^{\mathbb{N}} \to X^{\mathbb{N}}$ is a subset $\Phi \subseteq \mathbb{N} \times \left(X^{<\mathbb{N}} \times \mathbb{Q}^+\right)^2$ which can be viewed as an union of pairs of open balls. The set Φ is required to satisfy several properties as we state below. We write $(s, r_1)\Phi(t, r_2)$ as an abbreviation for $\exists n \, (n, s, r_1, t, r_2) \in \Phi$.

Definition 3.17 *The set* $\Phi \subseteq \mathbb{N} \times \left(X^{<\mathbb{N}} \times \mathbb{Q}^+\right)^2$ *is a code for a continuous function* $F_\Phi : X^{\mathbb{N}} \to X^{\mathbb{N}}$ *if* Φ *satisfies the following properties:*

1. *If* $(s, r_1)\Phi(t, r_2)$ *and* $(s, r_1)\Phi(t', r_3)$, *then* $d(t, t') \leq r_2 + r_3$.
2. *If* $(s, r_1)\Phi(t, r_2)$ *and* $(s', r_3) < (s, r_1)$, *then* $(s', r_3)\Phi(t, r_2)$.
3. *If* $(s, r_1)\Phi(t, r_2)$ *and* $(t, r_2) < (t', r_3)$, *then* $(s, r_1)\Phi(t', r_3)$.

The idea of the above definition is the following: the conditions make of the set Φ a code for a partially defined continuous function from $X^{\mathbb{N}}$ to $X^{\mathbb{N}}$. If f is a point of $X^{\mathbb{N}}$ (that is, a function from $\mathbb{N}$ to X), we write $f \in \operatorname{dom} F_\Phi$ to mean that for every k there exists in Φ a pair of open balls, $(t, r_1)\Phi(t', r_2)$ such that $d(f, t) < r_1$ and $r_2 < \frac{1}{2^{k+1}}$. If $f \in \operatorname{dom} F_\Phi$ it can be proved within $\mathbf{RCA}_0$ that the three conditions in the definition ensure that there exists a unique $g \in X^{\mathbb{N}}$ such that $d(g, t') \leq r_2$ for all $(t, r_1)\Phi(t', r_2)$ with $d(f, t) < r_1$. Namely, $g : \mathbb{N} \to X$ is defined by

$$\forall i \, (g(i) = b^i(i))$$

where $b^i \in X^{<\mathbb{N}}$ is such that for some n, a, r_1 and r_2, $u = (n, a, r_1, b, r_2) \in \Phi$ is the least element of Φ such that

$$r_2 < \frac{1}{2^{i+1}} \wedge d(f, a) < r_1$$

We refer this element by writing $F_\Phi(f) = g$. As usual, to say that $F_\Phi(f)$ is defined means that $f \in \operatorname{dom} F_\Phi$, and to say that F_Φ is totally defined on $X^{\mathbb{N}}$ means that $F_\Phi(f)$

is defined for every $f \in X^{\mathbb{N}}$. We write $F_\Phi : X^{\mathbb{N}} \to X^{\mathbb{N}}$ to mean that Φ a code for a continuous function and $F : X^{\mathbb{N}} \to X^{\mathbb{N}}$ is a continuous, totally defined function from $X^{\mathbb{N}}$ to $X^{\mathbb{N}}$.

We say that Φ is a code for *Lipschitz function* if Φ is code for a continuous function $F_\Phi : X^{\mathbb{N}} \to X^{\mathbb{N}}$ and there exists $c \leq 1$ such that the following fourth condition is satisfied

4. If $f \neq f'$, $F_\Phi(f) = g$, and $F_\Phi(f') = g'$, then $d(g, g') \leq c \cdot d(f, f')$.

If $c < 1$ we say that F_Φ is a *contraction*.

Now let us consider the Lipschitz game $G_L^X(A, B)$ in the space $X^{\mathbb{N}}$ determined by two formulas A and B.

In order to establish a connection between Lipschitz reductions and winning strategies we shall slightly modify our notion of strategy (although it can be easily proved within $\mathbf{RCA}_0$ that both notions are equivalent).

A strategy for player I in the game $G_L^X(A, B)$ is a function $\mathcal{E} : X^{<\mathbb{N}} \to X^{<\mathbb{N}}$ such that

$$s \subseteq s' \to \mathcal{E}(s) \subseteq \mathcal{E}(s') \quad \text{and} \quad |\mathcal{E}(s)| = |s| + 1$$

for all $s, s' \in X^{<\mathbb{N}}$. The intuition here is that for any finite sequence s played by the opponent of player I, the strategy $\mathcal{E}$ instructs player I to play $(\mathcal{E}(s))_{|s|}$, i.e. the last element of the sequence $\mathcal{E}(s)$. If $s = \langle\rangle$, then $(\mathcal{E}(\langle\rangle))_0$ is the move of player I with which the game starts.

A strategy for player II in a Lipschitz game is a function $\mathcal{E} : X^{<\mathbb{N}} - \{\langle\rangle\} \to X^{<\mathbb{N}}$ such that

$$s \subseteq t \to \mathcal{E}(s) \subseteq \mathcal{E}(t) \quad \text{and} \quad |\mathcal{E}(s)| = |s|$$

for all $s, t \in X^{<\mathbb{N}}$. As former, the intuition here is that for any finite sequence s played by the opponent of player II, the strategy $\mathcal{E}$ instructs player II to play $(\mathcal{E}(s))_{|s|-1}$, i.e. the last element of the sequence $\mathcal{E}(s)$.

Now we define the concept of winning strategy. Let $\mathcal{E} : X^{<\mathbb{N}} - \{\langle\rangle\} \to X^{<\mathbb{N}}$ be a strategy for player II in the game $G_L^X(A, B)$. Given $f : \mathbb{N} \to X$, there exists a unique function $g : \mathbb{N} \to X$ such that for all $k \in \mathbb{N}$,

$$g[k] = \mathcal{E}(f([k])).$$

We denote this function g by $F_{\mathcal{E}}(f)$.

The function $\mathcal{E} : X^{<\mathbb{N}} - \{\langle\rangle\} \to X^{<\mathbb{N}}$ is said to be a winning strategy for player II in the Lipschitz game $G_L^X(A, B)$ if for all $f \in X^{\mathbb{N}}$

$$A(f) \leftrightarrow B(F_{\mathcal{E}}(f)).$$

Analogously a strategy $\mathcal{E} : X^{<\mathbb{N}} \to X^{<\mathbb{N}}$ for player I is said to be a winning strategy for player I in the Lipschitz game $G_L^X(A, B)$ if for all $g \in X^{\mathbb{N}}$

$$A(F_{\mathcal{E}}(g)) \leftrightarrow \neg B(g).$$

where now $F_{\mathcal{E}}(g)$ is defined to be the unique $f : \mathbb{N} \to X$ such that for all $k \in \mathbb{N}$,

$$f[k+1] = \mathcal{E}(g[k]).$$

Now we establish the link to the Lipschitz functions.

Lemma 3.18 *The following is provable in* $\mathbf{RCA}_0$.

1. *Let* $\mathcal{E} : X^{<\mathbb{N}} - \{\langle\rangle\} \to X^{<\mathbb{N}}$ *be a strategy for player* II *in a Lipschitz game. Then there is a code* Φ *for Lipschitz function such that for every* $f \in X^{\mathbb{N}}$, $F_{\mathcal{E}}(f) = F_{\Phi}(f)$.
2. *Let* $\mathcal{E} : X^{<\mathbb{N}} \to X^{<\mathbb{N}}$ *be a strategy for* I *in a Lipschitz game. Then there is a code* Φ *for a contraction (i.e. a Lipschitz function with constant* $c < 1$*) such that for every* $f \in X^{\mathbb{N}}$, $F_{\mathcal{E}}(f) = F_{\Phi}(f)$.

Proof. The proofs of both parts are similar.
Working in $\mathbf{RCA}_0$, by Δ^0_1-comprehension there exists a set Φ such that

$$u \in \Phi \leftrightarrow \exists k, s, t, r_1, r_2 \leq u \begin{cases} u = (k, s, r_1, t, r_2) \wedge k \geq 1 \wedge \\ r_1, r_2 \in \mathbb{Q}^+ \wedge r_1, r_2 < \frac{1}{2^k} \wedge \\ \mathcal{E}\left(s\left[k\right]\right) \subseteq t \end{cases}$$

Now it is straightforward to check that Φ is a code for a continuous function such that for all $f \in X^{\mathbb{N}}$, $F_{\mathcal{E}}(f) = F_{\Phi}(f)$, and, as a consequence Φ is a code for a Lipschitz function. If $\mathcal{E}$ is a strategy for player I then F_{Φ} is a contraction. ∎

To prove the Wadge lemma we need one more result.

Lemma 3.19 *Let* Φ *be a code for a Lipschitz function. Then there is a strategy* $\mathcal{E} : X^{<\mathbb{N}} - \{\langle\rangle\} \to X^{<\mathbb{N}}$ *for player* II *in a Lipschitz game such that* $F_{\Phi} = F_{\mathcal{E}}$.

Proof. Let us define $\mathcal{E} : X^{<\mathbb{N}} - \{\langle\rangle\} \to X^{<\mathbb{N}}$ as follows. For each $s \in X^{<\mathbb{N}} - \{\langle\rangle\}$ let $f_s : \mathbb{N} \to X$ be defined by

$$f_s(i) = \begin{cases} s(i) & \text{if } i < |s| \\ 0 & \text{if } i \geq |s| \end{cases}$$

Since F_{Φ} is a totally defined function, $f_s \in \operatorname{dom} F_{\Phi}$ and, thus, there exists $g_s = F_{\Phi}(f_s)$.
Let $k = |s|$. Then, since $f_s \in \operatorname{dom} F_{\Phi}$, by Σ^0_1-minimization (available in $\mathbf{RCA}_0$) there exists the least $u = (n, a, r_1, b, r_2) \in \Phi$ such that $d(f_s, a) = d(s, a) < r_1$ and $r_2 < \frac{1}{2^{k+1}}$. We define $\mathcal{E}\left(s\right) = b\left[k\right]$. By definition of g_s, $\mathcal{E}(s) = F_{\Phi}(f_s)\left[|s|\right]$ and $\mathcal{E}(t) = F_{\Phi}(f_t)\left[|t|\right]$. Since $s \subseteq t$, $d(f_s, f_t) < \frac{1}{2^{|s|+1}}$ and since F_{Φ} is Lipschitz we conclude that

$$d(F_{\Phi}(f_s), F_{\Phi}(f_t)) \leq d(f_s, f_t) \leq \frac{1}{2^{|s|+1}}.$$

Thus we have shown that $F_\Phi(f_s)[|s|] = F_\Phi(f_t)[|s|]$ and so

$$\mathcal{E}(s) = F_\Phi(f_s)[|s|] \subseteq \mathcal{E}(t) = F_\Phi(f_t)[|t|].$$

It is obvious that $|\mathcal{E}(s)| = |F_\Phi(f_s)[|s|]| = |s|$, for every $s \in X^{<\mathbb{N}} - \{\langle\rangle\}$, and so $\mathcal{E}$ is a strategy for player II, and, by definition, for all $f \in X^{\mathbb{N}}$

$$F_\Phi(f) = F_{\mathcal{E}}(f).$$

■

Note that if we would like to obtain a strategy for player I in a Lipschitz game from a Lipschitz function F, as we did for player II in the former lemma, the Lipschitz function F would be required to be a contraction with $c < 1$. This is why the equivalence between the existence of a winning strategy for player I in a Lipschitz game and the reducibility relation $\leq_L$ is problematic.

Lemma 3.20 (Wadge) *Within* $\mathbf{RCA}_0$*, if* $A(f)$ *and* $B(g)$ *are formulas then:*

1. *Player* II *has a winning strategy in the Lipschitz game* $G_L^X(A,B)$ *iff* $A \leq_L B$.
2. *If player* I *has a winning strategy in the Lipschitz game* $G_L^X(A,B)$*, then* $\neg B \leq_L A$.

Proof. Part 1 follows from part 1 of Lemma 3.18 and from Lemma 3.19. To prove part 2 let $\mathcal{E}$ be a strategy for player I and let $\mathcal{E}' : X^{<\mathbb{N}} - \{\langle\rangle\} \to X^{<\mathbb{N}}$ be defined by $\mathcal{E}'(s) = \mathcal{E}(s)$. Then $\mathcal{E}'$ is a strategy for player II in the Lipschitz game $G_L^X(\neg B, A)$ and part 2 follows using part 1 of Lemma 3.18. ■

Since Wadge´s lemma is provable within $\mathbf{RCA}_0$, we have chosen to assume it as a basic result and to formalize the relations $A \leq_W B$ and $A \leq_L B$, respectively, as "player II has a wining strategy in Wadge game $G_W(A,B)$" and "player II has a wining strategy in Lipschitz game $G_L(A,B)$". This way the formalization becomes simpler since we deal directly with games avoiding functions.

Chapter 4

Lipschitz and Wadge games in Cantor space

In this chapter we begin the task of calibrating the strength of determinacy principles in terms of subsystems of second order arithmetic. The characterization of closed sets in terms of trees and the fact that these trees can be pruned within subsystems of second order arithmetic play a major role in the proofs of determinacy. Thus we start by studying the axiomatic strength needed to prove that a tree can be pruned. In Section 1 we show that $\mathbf{WKL}_0$ suffices to prove that trees which correspond to clopen sets can be pruned and that within $\mathbf{RCA}_0$ the subsystem $\mathbf{ACA}_0$ can even be fully characterized by an assertion that states that binary trees can be pruned.

In Section 2 we show that the subsystem of second arithmetic $\mathbf{WKL}_0$ is strong enough to prove the determinacy of Lipschitz and Wadge games in the Cantor space. As a consequence we obtain that $\mathbf{WKL}_0$ also proves Lipschitz and Wadge semilinear order principle.

In Section 3 we analyze the strength needed to prove Lipschitz and Wadge determinacy for open sets. We obtain several partial determinacy results from $\mathbf{WKL}_0$, but full determinacy for open sets seems to require the strength of the second order subsystem $\mathbf{ACA}_0$. We formulate a dichotomy principle (**DP**) which together with $\mathbf{WKL}_0$ implies Lipschitz and Wadge determinacy in Cantor space. (**DP**) follows from several assertions, all of them equivalent to $\mathbf{ACA}_0$. We do not know if $\mathbf{WKL}_0$ is strong enough for proving (**DP**). Would that be the case, $\mathbf{WKL}_0$ would suffice to deduce Lipschitz and Wadge determinacy and semilinear order principle for open sets in Cantor space.

In the last two sections we prove the main result of this chapter, namely that Lipschitz determinacy and Lipschitz semilinear order principle for $(\Sigma^0_1)_2$ sets in Cantor space are equivalent to $\mathbf{ACA}_0$ within base system $\mathbf{RCA}_0$. Both directions are interesting. The direction from $\mathbf{ACA}_0$ to determinacy for $(\Sigma^0_1)_2$ sets is interesting since it cannot be derived from known results on Gale-Stewart determinacy. The other direction is interesting because it yields a reversal for $\mathbf{ACA}_0$.

In this chapter we continue to prefer f's and s's to denote variables which range over $2^{\mathbb{N}}$ and $2^{<\mathbb{N}}$, respectively, for the plays of player I, and g's and t's for plays of player II.

4.1 Trees and closed sets

A key fact for the topological analysis of Lipschitz and Wadge games developed in Chapter 2 is that closed sets in the Cantor and Baire spaces correspond to the sets of paths of trees. This fact can be proved in $\mathbf{RCA}_0$ and it is, indeed, an immediate consequence of the normal form theorem for Σ^0_1 formulas (Theorem II.2.7 of [Smp99]).

Proposition 4.1 *The following is provable in* $\mathbf{RCA}_0$*. Suppose* $X \subseteq \mathbb{N}$*. Assume* $\varphi(f) \in \Pi^0_1$*, with* $f \in X^{\mathbb{N}}$*. Then, there is a tree* $T \subseteq X^{<\mathbb{N}}$ *satisfying that* $[T] = \{f \in X^{\mathbb{N}} : \varphi(f)\}$*.*

Proof. See Lemma VI.1.5 of [Smp99]. ■

Thus, we identify points in the Cantor space with functions $f \in 2^{\mathbb{N}}$, and we identify closed sets in the Cantor space with Π^0_1 formulas containing a second order free variable f which ranges over $2^{\mathbb{N}}$. Similarly, open sets will correspond to Σ^0_1 formulas and so on. Mutatis mutandis, we consider the same conventions for the Baire space.

We will also identify a closed set with the set of paths of a tree, $[T]$, and, by abuse of language, use set theoretic notations to mean the arithmetic formula expressing the corresponding set (For instance, a term of the form $[T] - [S]$ is to be understood as the $\Pi^0_1 \wedge \Sigma^0_1$ formula expressing that f is a path of T and is not a path of S.)

Definition 4.2 *The following definition is made in* $\mathbf{RCA}_0$*. Given* $X \subseteq \mathbb{N}$*, we say that a tree* $T \subseteq X^{<\mathbb{N}}$ *defines a clopen set if there exists another tree* $S \subseteq X^{<\mathbb{N}}$ *such that*

$$\forall f \in X^{\mathbb{N}} \, (f \notin [T] \leftrightarrow f \in [S]).$$

Definition 4.3 *The following definition is made in* $\mathbf{RCA}_0$*. A tree* T *is said to be pruned if every sequence of* T *lies on a path of* T*. More formally, given* $X \subseteq \mathbb{N}$*, we say that a tree* $T \subseteq X^{<\mathbb{N}}$ *is pruned if*

$$\forall s \in X^{<\mathbb{N}} \, (s \in T \rightarrow \exists f \in X^{\mathbb{N}} \, (s \subseteq f \wedge f \in [T])).$$

It is well known that the assertion that every tree $T \subseteq \mathbb{N}^{<\mathbb{N}}$ can be pruned (that is to say, for every tree there exists some pruned subtree with the same set of paths) is equivalent over $\mathbf{RCA}_0$ to Π^1_1-$\mathbf{CA}_0$ (see, e.g., Lemma VI.4.4 of [Smp99]). However, here we show that if we restrict ourselves to the Cantor space or to trees defining clopen sets, we can prune a tree at a lower price. We first need the following lemma asserting that Π^0_1 formulas are closed in $\mathbf{WKL}_0$ under existential quantifiers of the form $\exists f \in 2^{\mathbb{N}}$.

Lemma 4.4 *Let* ψ *be a* Π^0_1 *formula. Within* $\mathbf{WKL}_0$*,* $\exists f \in 2^{\mathbb{N}} \, \psi(f)$ *is equivalent to a* Π^0_1 *formula.*

Proof. See Lemma VIII.2.4 in [Smp99] or Lemma 3.2 in [NMT07]. ■

In Section 4.4 we will also need the following version of the previous lemma.

Lemma 4.5 *Let ψ be a $\Sigma^0_1 \wedge \Pi^0_1$ formula. Within* $\mathbf{WKL}_0$*, $\exists f \in 2^{\mathbb{N}}\, \psi(f)$ is equivalent to an arithmetical (in fact, Σ^0_2) formula.*

Proof. Write $\psi = \varphi(f) \wedge \exists x\, \theta(x, f)$, with $\varphi \in \Pi^0_1$ and $\theta \in \Delta^0_0$. Then:

$$\begin{aligned}\exists f \in 2^{\mathbb{N}}\, \psi(f) &\leftrightarrow \exists f \in 2^{\mathbb{N}}\, (\varphi(f) \wedge \exists x\, \theta(x, f))\\ &\leftrightarrow \exists x\, \exists f \in 2^{\mathbb{N}}\, (\varphi(f) \wedge \theta(x, f))\end{aligned}$$

Now the result follows from Lemma 4.4. ∎

Proposition 4.6

1. *The following is provable in* $\mathbf{ACA}_0$*. Let $\varphi(f) \in \Pi^0_1$, with $f \in 2^{\mathbb{N}}$. Then, there exists a pruned binary tree T satisfying that $[T] = \{f \in 2^{\mathbb{N}} : \varphi(f)\}$.*

2. *The following is provable in* $\mathbf{WKL}_0$*. Let $\varphi(f) \in \Sigma^0_1$ and $\psi(f) \in \Pi^0_1$ such that $\forall f \in 2^{\mathbb{N}}(\varphi(f) \leftrightarrow \psi(f))$, with $f \in 2^{\mathbb{N}}$. Then, there exists a pruned binary tree T satisfying that $[T] = \{f \in 2^{\mathbb{N}} : \varphi(f)\}$.*

Proof. (1): We work in $\mathbf{ACA}_0$. By Proposition 4.1, let S be a binary tree such that $[S] = \{f \in 2^{\mathbb{N}} : \varphi(f)\}$. It suffices to consider the obvious definition

$$T = \{s \in 2^{<\mathbb{N}} : s \in S \wedge \exists f \in 2^{\mathbb{N}}\, (f \in [S] \wedge s \subset f)\}.$$

It is clear that T is a pruned subtree of S with $[S] = [T]$. In addition, the existence of T follows by Π^0_1-comprehension (available in $\mathbf{ACA}_0$), for the formula defining T is equivalent to a Π^0_1 formula by Lemma 4.4.

(2): We work in $\mathbf{WKL}_0$. By Proposition 4.1, let S and S' be binary trees such that

(-) $[S] = \{f \in 2^{\mathbb{N}} : \psi(f)\}$, and

(-) $[S'] = \{f \in 2^{\mathbb{N}} : \neg\varphi(f)\}$.

Define

$$A(s) = \{s \in 2^{<\mathbb{N}} : s \in S \wedge \exists f \in 2^{\mathbb{N}}\, (f \in [S] \wedge s \subset f)\}$$

$$B(s) = \{s \in 2^{<\mathbb{N}} : s \in S \wedge \exists t\, (s \subseteq t \wedge t \notin S')\}$$

It is clear that $B(s) \in \Sigma^0_1$ and it follows from Lemma 4.4 that $A(s)$ is equivalent to a Π^0_1 formula. In addition, we claim that

- $\forall s \in 2^{<\mathbb{N}} (A(s) \leftrightarrow B(s))$.

  Pick $s_1 \in 2^{<\mathbb{N}}$ satisfying $A(s_1)$. There exists $g \in [S]$ such that $s_1 \subset g$. Then, $g \notin [S']$ since $[S] \cap [S'] = \emptyset$. Consequently, there must exist some $k > |s_1|$ such that $g[k] \notin S'$. We have $s_1 \subseteq g[k]$ and $g[k] \notin S'$ and so $B(s_1)$ holds.

  Now pick $s_2 \in 2^{<\mathbb{N}}$ satisfying $B(s_2)$. There exists t_1 such that $s_2 \subseteq t_1$ and $t_1 \notin S'$. Define $h : \mathbb{N} \to \{0, 1\}$ by putting $h(i) = t_1(i)$ if $i < |t_1|$ and $h(i) = 0$ otherwise. Clearly, $h \notin [S']$ and so h must be in $[S]$. But then we have $s_2 \subset h$ and $h \in [S]$ and hence $A(s_2)$ holds.

It follows from the claim that $\{s : A(s)\}$ exists by Δ^0_1-comprehension. It is immediate to check that $A(s)$ defines a pruned binary tree, T, with $[T] = [S]$, as required. ■

Proposition 4.7 *The following are equivalent over* $\mathbf{RCA}_0$*.*

1. $\mathbf{ACA}_0$*.*
2. *Every binary tree can be pruned, i.e., for every binary tree* T *there exists a pruned binary tree* T' *such that* $[T] = [T']$*.*

Proof. In view of Proposition 4.6, we only have to prove that part (2) implies part (1). We reason in $\mathbf{RCA}_0$. Let $\varphi(x) \in \Pi^0_1$ (we disregard possible parameters). We must show that $\{x : \varphi(x)\}$ exists. Put $\varphi(x) = \forall y\, \varphi_0(x, y)$, with $\varphi_0(x, y) \in \Delta^0_0$. Define a binary tree T by putting $s \in T$ if and only if

$$\forall i < |s|\, (s(i) = 0)$$
$$\vee$$
$$\exists i < |s|\, [s(i) = 1 \wedge \forall j < |s| (j \neq i \to s(j) = 0) \wedge \forall j < |s| - (i + 1)\, \varphi_0(i, j)]$$

That is to say, a binary finite sequence s is in T if and only if either

$$s = 0^{(k)},$$

for some $k \in \omega$, or

$$s = 0^{(i)} * \langle 1 \rangle * 0^{(l)}$$

with $\forall y < l\, \varphi_0(i, y)$. Clearly, T exists by Δ^0_1-comprehension. Since we are assuming that every binary tree can be pruned, let T' be a binary pruned tree with $[T] = [T']$. Then, we have

$$\varphi(k) \leftrightarrow 0^{(k)} * \langle 1 \rangle \in T'$$

and so $\{x : \varphi(x)\}$ exists by Δ^0_1-comprehension. ■

Let us observe that concerning pruning trees in the Baire space at a lower price, in Chapter 5 we will show that $\mathbf{ATR_0}$ suffices for pruning a tree T defining a clopen set (an analogous to part 2 of Proposition 4.6 above).

These facts together give us a nice picture of the strength needed to prove that a tree can be pruned.

	Cantor space	Baire space
Clopen case	$\mathbf{WKL}_0$	$\mathbf{ATR}_0$
General case	$\mathbf{ACA}_0$	Π^1_1-$\mathbf{CA}_0$

Moreover, the subsystems in the second row are known to be precisely equivalent to the corresponding "pruning tree" assertion. We finish by asking whether the same holds for the systems in the first row.

Problem 4.8

1. *Is* $\mathbf{WKL}_0$ *equivalent over* $\mathbf{RCA}_0$ *to the assertion that every binary tree defining a clopen set can be pruned?*

2. *Is* $\mathbf{ATR}_0$ *equivalent over* $\mathbf{RCA}_0$ *to the assertion that every tree defining a clopen set can be pruned?*

4.2 Determinacy for clopen sets

In this section we show that $\mathbf{WKL}_0$ suffices for establishing the structure of clopen Lipschitz and Wadge degrees in the Cantor space. To that end, we will follow the topological analysis of Lipschitz games developed in Chapter 2 and show that the reasoning involved can be formalized within $\mathbf{WKL}_0$.

We start with an easy observation. We say that a set is *trivial* if either it is empty or it is the total set. Determinacy for games with some trivial payoff set turns out to be trivial (and provable in, say, $\mathbf{RCA}_0$).

Lemma 4.9 *Let* $Empty(\varphi)$ *denote the formula* $\neg\exists f \in 2^{\mathbb{N}}\varphi(f)$ *and let* $Total(\varphi)$ *denote the formula* $\forall f \in 2^{\mathbb{N}}\varphi(f)$*. The following facts are provable in* $\mathbf{RCA}_0$*.*

1. $Empty(\varphi_1) \wedge \neg Total(\varphi_2) \rightarrow Red^{\star}_{L/W}(\varphi_1, \varphi_2)$.

2. $Empty(\varphi_1) \wedge Total(\varphi_2) \rightarrow \exists\sigma_{\mathrm{I}}\,\forall\sigma_{\mathrm{II}}\,\neg(\varphi_1(\sigma_{\mathrm{I}} \otimes^{\mathrm{I}}_{L/W} \sigma_{\mathrm{II}}) \leftrightarrow \varphi_2(\sigma_{\mathrm{I}} \otimes^{\mathrm{II}}_{L/W} \sigma_{\mathrm{II}}))$.

3. $Total(\varphi_1) \wedge \neg Empty(\varphi_2) \rightarrow Red^{\star}_{L/W}(\varphi_1, \varphi_2)$.

4. $Total(\varphi_1) \wedge Empty(\varphi_2) \rightarrow \exists\sigma_{\mathrm{I}}\,\forall\sigma_{\mathrm{II}}\,\neg(\varphi_1(\sigma_{\mathrm{I}} \otimes^{\mathrm{I}}_{L/W} \sigma_{\mathrm{II}}) \leftrightarrow \varphi_2(\sigma_{\mathrm{I}} \otimes^{\mathrm{II}}_{L/W} \sigma_{\mathrm{II}}))$.

Proof. Immediate. ∎

As a consequence, it is provable in $\mathbf{RCA}_0$ that $\{\emptyset\}$ and $\{X\}$ form a pair of incomparable degrees which are reducible to any other degree.

In what follows we show that some basic topological properties of clopen sets in the Cantor space can be proved in $\mathbf{WKL}_0$. The use of $\mathbf{WKL}_0$ is not surprising, as this subsystem is well known to be closely related to *compactness arguments*.

Our first result shows that it is provable in $\mathbf{WKL}_0$ that in Cantor space every clopen set is a finite union of basic opens.

Proposition 4.10 *The following is provable in* $\mathbf{WKL_0}$. *Let* $\varphi(f) \in \Sigma^0_1$ *and* $\psi(f) \in \Pi^0_1$ *such that* $\forall f \in 2^{\mathbb{N}}(\varphi(f) \leftrightarrow \psi(f))$. *Then, there exists a finite set* $A \subseteq 2^{<\mathbb{N}}$ *such that*

$$\forall f \in 2^{\mathbb{N}}(\varphi(f) \leftrightarrow \exists s\,(s \in A \wedge s \subset f)).$$

Proof. First of all, observe that we may assume that $\exists f \in 2^{\mathbb{N}}\,\varphi(f)$ (otherwise take $A = \emptyset$) and that $\exists f \in 2^{\mathbb{N}}\,\neg\varphi(f)$ (otherwise take $A = \{\langle\rangle\}$). By Proposition 4.1, there exist some nonempty binary trees S and T satisfying that

- $[S] = \{f \in 2^{\mathbb{N}} : \varphi(f)\}$, and
- $[T] = \{f \in 2^{\mathbb{N}} : \neg\varphi(f)\}$.

Then, $S \cap T$ is a nonempty binary tree with no path, for otherwise there would be $f \in 2^{\mathbb{N}}$ such that both $\varphi(f)$ and $\neg\varphi(f)$ hold. Also, note that $S \cap T$ exists by Δ^0_0-comprehension. Hence, by applying $\mathbf{WKL_0}$, we obtain that $S \cap T$ must be finite. Put

$$A = \{s \in 2^{<\mathbb{N}} \,:\, s[|s|-1] \in S \cap T \wedge s \in S \wedge s \notin T\}$$

(Note that the set A exists by Δ^0_0-comprehension.) It is clear that A is a finite set. In addition, we claim that

- $\forall f \in 2^{\mathbb{N}}(\varphi(f) \leftrightarrow \exists s\,(s \in A \wedge s \subset f))$.

  Pick $f \in 2^{\mathbb{N}}$ satisfying $\varphi(f)$. Then, $f \notin [T]$ and so there is some $k \in \mathbb{N}$ such that $f[k] \notin T$. By Δ^0_0-minimization, let m be the least k satisfying $f[k] \notin T$. Take s_0 to be $f[m]$. Clearly, we have $s_0[m-1] \in S \cap T$, $s_0 \in S$ and $s_0 \notin T$. By definition, s_0 is in A. Thus, s_0 is the desired finite sequence satisfying that $s_0 \in A$ and $s_0 \subset f$.

  For the opposite direction, pick $f \in 2^{\mathbb{N}}$ and $s_1 \in 2^{<\mathbb{N}}$ such that $s_1 \in A$ and $s_1 \subset f$. It follows from $s_1 \notin T$ and $s_1 \subset f$ that $f \notin [T]$ and hence $\varphi(f)$ holds.

This completes the proof of the Proposition. ■

As an application, we obtain:

Corollary 4.11 *The following is provable in* $\mathbf{WKL_0}$. *Let* $\varphi(f) \in \Sigma^0_1$ *and* $\psi(f) \in \Pi^0_1$ *such that* $\forall f \in 2^{\mathbb{N}}(\varphi(f) \leftrightarrow \psi(f))$. *Then, there exist* $X \subseteq 2^{<\mathbb{N}}$ *finite and* $k \in \mathbb{N}$ *satisfying*

$$\forall f \in 2^{\mathbb{N}}\,(\varphi(f) \leftrightarrow f[k] \in X).$$

Proof. By Proposition 4.10, there exists $A \subseteq 2^{<\mathbb{N}}$ finite satisfying that

$$\forall f \in 2^{\mathbb{N}}(\varphi(f) \leftrightarrow \exists s\,(s \in A \wedge s \subset f)). \tag{†}$$

Since A is finite, there exists $k \in \mathbb{N}$ such that $\forall s\,(s \in A \to |t| \leq k)$. Put

$$X = \{t \in 2^{<\mathbb{N}} : |t| = k \wedge \exists s \subseteq t\,(s \in A)\}$$

(Note that X exists by Δ^0_0-comprehension.) Clearly, X is a finite set. In addition, we claim that

- $\forall f \in 2^{\mathbb{N}}\,(\varphi(f) \leftrightarrow f[k] \in X)$.

  Pick $f \in 2^{\mathbb{N}}$ satisfying $\varphi(f)$. By (†), there is some $s \in A$ such that $s \subseteq f$. In particular, $|s| \leq k$ and so $s \subseteq f[k]$. Thus, $f[k] \in X$.

  For the opposite direction, pick $f \in 2^{\mathbb{N}}$ satisfying that $f[k] \in X$. Then, there exists $s \subseteq f[k]$ with $s \in A$. Thus, $s \in A$ and $s \subset f$. So, $\varphi(f)$ holds by (†).

This proves the Corollary. ■

We are ready for the main result of the section.

Theorem 4.12

1. $\mathbf{WKL_0}$ *proves* Δ^0_1-$\mathbf{Det}^*_L$.
2. $\mathbf{RCA_0}$ *proves* Δ^0_1-$\mathbf{Det}^*_W$.

Proof. (1): We work in an arbitrary model of $\mathbf{WKL_0}$.
Let $A(f), B(g) \in \Sigma^0_1$ and $A'(f), B'(g) \in \Pi^0_1$ satisfying that

(-) $\forall f \in 2^{\mathbb{N}}(A(f) \leftrightarrow A'(f))$, and

(-) $\forall g \in 2^{\mathbb{N}}(B(g) \leftrightarrow B'(g))$.

In view of Lemma 4.9, we can safely assume that all of A, A', B, and B are different from the empty set and the total set. That is to say, for each A, A', B, and B there exists some $f \in 2^{\mathbb{N}}$ satisfying the corresponding formula. By Proposition 4.6, there are nonempty pruned binary trees $S, S', T, T' \subseteq 2^{<\mathbb{N}}$ such that

$$[S] = \{f \in 2^{\mathbb{N}} \,:\, A(f)\} \text{ and } \left[S'\right] = \{f \in 2^{\mathbb{N}} \,:\, \neg A(f)\}$$

$$[T] = \{g \in 2^{\mathbb{N}} \,:\, B(g)\} \text{ and } \left[T'\right] = \{g \in 2^{\mathbb{N}} \,:\, \neg B(g)\}$$

We must show that the Lipschitz game $G_L([S],[T])$ is determined.

Firstly, consider the set $S \cap S'$ (such a set exists by Δ^0_0-comprehension.) Note that $S \cap S'$ is a nonempty binary tree with no path, as S and S' do not have any common paths. Hence, $S \cap S'$ must be finite by $\mathbf{WKL_0}$. By applying Δ^0_0-induction, we obtain that there are $k \in \mathbb{N}$ and $s_k \in 2^{<\mathbb{N}}$ such that

1. $k = \max\{|s| \,:\, s \in S \cap S'\}$, and
2. $s_k \in S \cap S'$ and $|s| \ = k$.

We claim that

- $\forall f \in 2^{\mathbb{N}}\,(A(f) \leftrightarrow f[k+1] \in S)$.
- One of the following holds:

(-) $\forall f \in 2^{\mathbb{N}} \, [(s_k * \langle 0 \rangle \subset f \to A(f)) \wedge (s_k * \langle 1 \rangle \subset f \to \neg A(f))]$, or

(-) $\forall f \in 2^{\mathbb{N}} \, [(s_k * \langle 0 \rangle \subset f \to \neg A(f)) \wedge (s_k * \langle 1 \rangle \subset f \to A(f))]$.

Firstly, pick $f \in 2^{\mathbb{N}}$. It is clear that if $A(f)$ holds then $f \in [S]$ and so $f[k+1] \in S$. Now suppose that $f[k+1] \in S$ but $A(f)$ does not hold. Then, $f \in [S']$ and so $f[k+1] \in S'$. As a consequence, $f[k+1]$ would be a finite sequence of length $k+1$ belonging to $S \cap S'$, which contradicts the definition of k. This proves the first part of the claim.

Let us now prove the second part of the claim. We consider two cases.

<u>Case 1</u>: $s_k * \langle 0 \rangle \in S$.
Since $|s_k * \langle 0 \rangle| = k+1$ and $s_k * \langle 0 \rangle \in S$, it follows from the first part of the claim that for each $f \in 2^{\mathbb{N}}$, $s_k * \langle 0 \rangle \subset f \to A(f)$. Now pick $f \in 2^{\mathbb{N}}$ satisfying $s_k * \langle 1 \rangle \subset f$. We must show that $\neg A(f)$ holds. Notice that $s_k * \langle 0 \rangle \notin S'$, for otherwise $s_k * \langle 0 \rangle$ would be in $S \cap S'$, contradicting the maximality of k. Hence, since $s_k \in S'$ and S' is a pruned tree, $s_k * \langle 1 \rangle$ must be in S'. But then $s_k * \langle 1 \rangle$ is not in S, for otherwise $s_k * \langle 1 \rangle$ would be in $S \cap S'$, again contradicting the maximality of k. As a consequence, $f \notin [S]$ and so $\neg A(f)$ holds.

<u>Case 2</u>: $s_k * \langle 0 \rangle \notin S$.
Since $s_k \in S$ and S is a pruned tree, $s_k * \langle 1 \rangle$ must be in S. By reasoning as in the previous case, we obtain that for each $f \in 2^{\mathbb{N}}$, $s_k * \langle 1 \rangle \subset f \to A(f)$ and $s_k * \langle 0 \rangle \subset f \to \neg A(f)$.

This proves the second part of the claim.

By repeating the previous reasoning for the tree $T \cap T'$, we obtain that there exist $m \in \mathbb{N}$ and $s_m \in 2^{<\mathbb{N}}$ such that

- $\forall g \in 2^{\mathbb{N}} \, (B(g) \leftrightarrow g[m+1] \in T)$.
- One of the following holds:

  (-) $\forall g \in 2^{\mathbb{N}} \, [(s_m * \langle 0 \rangle \subset g \to B(g)) \wedge (s_m * \langle 1 \rangle \subset g \to \neg B(g))]$, or

  (-) $\forall g \in 2^{\mathbb{N}} \, [(s_m * \langle 0 \rangle \subset g \to \neg B(g)) \wedge (s_m * \langle 1 \rangle \subset g \to B(g))]$.

We are now in a position to show that the game $G_L([S],[T])$ is determined.

<u>Case A</u>: $k > m$.

Then, player I has a winning strategy in the game $G_L([S],[T])$. Namely, we define a strategy σ_{I} as follows. Assume that

$$\forall f \in 2^{\mathbb{N}} \, [(s_k * \langle 0 \rangle \subset f \to A(f)) \wedge (s_k * \langle 1 \rangle \subset f \to \neg A(f))] \qquad (\dagger)$$

(If the other possibility holds, the definition of σ_{I} is to be modified accordingly.) Put

$$\sigma_{\mathrm{I}}(\langle x_0, y_0, \ldots, x_{n-1}, y_{n-1}\rangle) = \begin{cases} s_k(n) & \text{if } n < k \\ 1 & \text{if } n = k \text{ and } \langle y_0, \ldots, y_{n-1}\rangle \in T \\ 0 & \text{if } n = k \text{ and } \langle y_0, \ldots, y_{n-1}\rangle \notin T \\ 0 & \text{if } n > k \end{cases}$$

It is clear that σ_{I} exists by Δ^0_0-comprehension and it follows from the properties of the sequences s_k and s_m and the assumption (†) that σ_{I} is winning for player I.

Case B: $k \leq m$.

Then, player II has a winning strategy in the game $G_L([S], [T])$. Namely, we define a strategy σ_{II} as follows. Assume that

$$\forall g \in 2^{\mathbb{N}} \left[(s_m * \langle 0\rangle \subset g \to B(g)) \wedge (s_m * \langle 1\rangle \subset g \to \neg B(g))\right] \tag{††}$$

(If the other possibility holds, the definition of σ_{II} is to be modified accordingly.) Put

$$\sigma_{\mathrm{II}}(\langle x_0, y_0, \ldots, x_{n-1}, y_{n-1}, x_n\rangle) = \begin{cases} s_m(n) & \text{if } n < m \\ 0 & \text{if } n = m \text{ and } \langle x_0, \ldots, x_n\rangle \in S \\ 1 & \text{if } n = m \text{ and } \langle x_0, \ldots, x_n\rangle \notin S \\ 0 & \text{if } n > m \end{cases}$$

Clearly, σ_{II} exists by Δ^0_0-comprehension and it follows from the properties of the sequences s_k and s_m and the assumption (††) that σ_{II} is winning for player II.

(2): We work in an arbitrary model of $\mathbf{RCA}_0$. Let $A(f), B(f) \in \Sigma^0_1$ and $A'(f), B'(g) \in \Pi^0_1$ satisfying that

(-) $\forall f \in 2^{\mathbb{N}}(A(f) \leftrightarrow A'(f))$, and

(-) $\forall g \in 2^{\mathbb{N}}(B(g) \leftrightarrow B'(g))$.

By Lemma 4.9 we may assume that there exists some $g_{in} \in 2^{\mathbb{N}}$ satisfying $B(g_{in})$ and there exists some $g_{out} \in 2^{\mathbb{N}}$ satisfying $\neg B(g_{out})$. By Proposition 4.1, there are binary trees $S, S' \subseteq 2^{<\mathbb{N}}$ such that

$$[S] = \{f \in 2^{\mathbb{N}} : A(f)\} \text{ and } [S'] = \{f \in 2^{\mathbb{N}} : \neg A(f)\} .$$

We must show that the Wadge game $G_W([S], B)$ is determined.

Observe that the binary tree $S \cap S'$ cannot have any path. As a consequence, we can define a winning strategy for player II. The idea is simple: while player I plays inside $S \cap S'$ player II passes; and when player I leaves $S \cap S'$ (this has to happen sooner or later as $S \cap S'$ has no path) player II plays accordingly by using either g_{in} or g_{out}. In order to give a precise definition of the strategy, recall that by our conventions in Chapter 3, the following correspondence holds.

Formalized strategy	Player I's real move	Player II's real move
0	0	p
1	0	0
2	1	1

Having this in mind, given any sequence of odd length $s = \langle x_0, y_0, \ldots, x_{n-1}, y_{n-1}, x_n \rangle$, we put

$$\sigma_{\mathrm{II}}(s) = \begin{cases} 0 & \text{if } \langle x_0 \dot{-} 1, \ldots, x_n \dot{-} 1 \rangle \in S \cap S' \\ g_{out}(n-k)+1 & \text{if } \langle x_0 \dot{-} 1, \ldots, x_n \dot{-} 1 \rangle \notin S \text{ and } k = \mu j \ \big(\langle x_0 \dot{-} 1, \ldots, x_j \dot{-} 1 \rangle \notin S\big) \\ g_{in}(n-k)+1 & \text{if } \langle x_0 \dot{-} 1, \ldots, x_n \dot{-} 1 \rangle \notin S' \text{ and } k = \mu j \ \big(\langle x_0 \dot{-} 1, \ldots, x_j \dot{-} 1 \rangle \notin S'\big) \end{cases}$$

Clearly, σ_{II} exists by Δ^0_0-comprehension and it is easy to see that σ_{II} is winning for player II. This completes the proof of the theorem. ∎

Corollary 4.13

1. $\mathbf{WKL}_0$ *proves* Δ^0_1-$\mathbf{SLO}^*_L$.
2. $\mathbf{RCA}_0$ *proves* Δ^0_1-$\mathbf{SLO}^*_W$.

Proof. See Lemma 3.12. ∎

Remark 4.14

1. *The scheme of Σ^0_1-induction has not been used so far. Thus, it follows from the proof of Theorem 4.12 that $\mathbf{WKL}^*_0$ already proves Δ^0_1-$\mathbf{Det}^*_L$. The system $\mathbf{WKL}^*_0$ consists of basic recursive axioms for addition, multiplication, and exponentiation; augmented with the schemes of Δ^0_1-Comprehension and Δ^0_0-Induction and with Weak König Lemma. That is to say, $\mathbf{WKL}^*_0$ is essentially $\mathbf{WKL}_0$ with Σ^0_1 induction weakened to Δ^0_0 induction.*

2. *By a result of Nemoto (see Proposition 3.1 of [N09a]), $\mathbf{WKL}^*_0$ proves Δ^0_1-$\mathbf{Det}^*$. By an observation in Section 6 of Chapter 3, a clopen Lipschitz game can be effectively reduced to a clopen Gale-Stewart game. Putting these two facts together, we obtain another proof of part 1 of Theorem 4.12.*

3. *In the proof of part 2 of Theorem 4.12, we do not make use of the fact that φ_2 defines a Δ^0_1-set. Actually, the proof shows that it is provable in $\mathbf{RCA}_0$ that Δ^0_1-sets are Wadge reducible to any nontrivial set.*

In particular, it follows from the proof of part 2 of Theorem 4.12 that it is provable in $\mathbf{RCA}_0$ that the nontrivial Δ^0_1-sets form a Wadge degree. That is to say, write $A \equiv_W B$ to denote the formula $Red^\star_W(A, B) \wedge Red^\star_W(B, A)$. Then, we have

Proposition 4.15 *It is provable in* $\mathbf{RCA}_0$ *that if* S *and* T *are binary trees defining nontrivial clopen sets then* $[S] \equiv_W [T]$.

Concerning the structure of the clopen Lipschitz degrees, the following result holds. Write $A <_L B$ to denote the formula $Red^\star_L(A,B) \wedge \neg Red^\star_L(B,A)$ and write $A \equiv_L B$ to denote the formula $Red^\star_L(A,B) \wedge Red^\star_L(B,A)$. Then, we have

Proposition 4.16 *It is provable in* $\mathbf{WKL}_0$ *that there exists a sequence of binary trees,* $\{T_k : k \in \mathbb{N}\}$, *satisfying that*

1. *for each* k, $[T_k]$ *defines a nontrivial clopen set;*
2. *for each* k, $[T_k] <_L [T_{k+1}]$; *and*
3. *for each binary tree* S *defining a nontrivial clopen set, there exists* $k \in \mathbb{N}$ *such that* $[S] \equiv_L [T_k]$.

Proof. We work in an arbitrary model of $\mathbf{WKL}_0$. For each $k \in \mathbb{N}$ it suffices to consider the sequence of binary trees given by

$$t \in T_k \;\leftrightarrow\; t \subseteq 0^{(k+1)} \vee 0^{(k+1)} \subseteq t.$$

It is clear that such a sequence exists by Δ^0_1-comprehension and it is easy to see that each T_k defines a nontrivial clopen set. Finally, by inspection of the proof part 1 of Theorem 4.12, it follows that properties (2) and (3) above hold too. ∎

We have been unable to obtain a reversal for $\mathbf{WKL}_0$ in terms of Lipschitz or Wadge determinacy or semilinear ordering principle. We then pose the following questions.

Problem 4.17

1. *Is* $\mathbf{WKL}_0$ *equivalent over* $\mathbf{RCA}_0$ *to* Δ^0_1-$\mathbf{Det}^*_L$*?*
2. *Is* $\mathbf{WKL}_0$ *equivalent over* $\mathbf{RCA}_0$ *to* Δ^0_1-$\mathbf{SLO}^*_L$*?*
3. *Is* Δ^0_1-$\mathbf{Det}^*_L$ *equivalent over* $\mathbf{RCA}_0$ *to* Δ^0_1-$\mathbf{SLO}^*_L$*?*

4.3 Determinacy for open sets

In this section we show that the system $\mathbf{ACA}_0$ suffices for proving determinacy for closed sets (and so also for open sets) in the Cantor space. However, we start by showing that $\mathbf{WKL}_0$ is still sufficient in most of the cases when one player plays in an open set and the opponent plays in a clopen set. Namely, we have

Proposition 4.18

1. $\mathbf{WKL}_0$ *proves* (Δ^0_1, Σ^0_1)-$\mathbf{Det}^*_L$.

2. $\mathbf{WKL}_0$ *proves* (Δ^0_1, Σ^0_1)-$\mathbf{Det}^*_W$.

3. $\mathbf{WKL}_0$ *proves* (Σ^0_1, Δ^0_1)-$\mathbf{Det}^*_L$.

Proof. We work in an arbitrary model of $\mathbf{WKL}_0$.
(1): We will prove (Δ^0_1, Π^0_1)-$\mathbf{Det}^*_L$, which is equivalent to (Δ^0_1, Σ^0_1)-$\mathbf{Det}^*_L$. Let $A(f) \in \Sigma^0_1$ and $A'(f) \in \Pi^0_1$ satisfying that

(-) $\forall f \in 2^{\mathbb{N}} \, (A(f) \leftrightarrow A'(f))$.

Let $B(g) \in \Pi^0_1$ and let T be a binary tree such that

(-) $[T] = \{g \in 2^{\mathbb{N}} \, : \, B(g)\}$.

Since $A(f)$ is Δ^0_1, it follows from Proposition 4.6 that there are pruned binary trees $S, S'^{<\mathbb{N}}$ such that

$$[S] = \{f \in 2^{\mathbb{N}} : A(f)\} \text{ and } [S'] = \{f \in 2^{\mathbb{N}} : \neg A(f)\}$$

Hence, reasoning as in the proof of Theorem 4.12 we obtain that there are $k \in \omega$ and $s_k \in S \cap S'$ satisfying that

(a) $\forall f \in 2^{\mathbb{N}} \, (A(f) \leftrightarrow f[k+1] \in S)$.

(b) One of the following holds:

  (-) $\forall f \in 2^{\mathbb{N}} \, [(s_k * \langle 0 \rangle \subseteq f \rightarrow A(f)) \wedge (s_k * \langle 1 \rangle \subseteq f \rightarrow \neg A(f))]$, or
  (-) $\forall f \in 2^{\mathbb{N}} \, [(s_k * \langle 0 \rangle \subseteq f \rightarrow \neg A(f)) \wedge (s_k * \langle 1 \rangle \subseteq f \rightarrow A(f))]$.

Now consider the following sets

$$X_{in} = \{t \in 2^{<\mathbb{N}} : |t| = k \wedge t \in T \wedge \exists g \in 2^{\mathbb{N}} \, (t \subseteq g \wedge g \in [T])\}$$

$$X_{out} = \{t \in 2^{<\mathbb{N}} : |t| = k \wedge t \in T \wedge \exists t \, (t \subseteq t' \wedge t' \notin T)\}$$

$$X = X_{in} \cap X_{out}$$

The existence of such sets follows by *bounded* Σ^0_1 or *bounded* Π^0_1 comprehension (which are well known to be provable from $\mathbf{RCA}_0$) and by the fact that Π^0_1 formulas are closed in $\mathbf{WKL}_0$ under quantifiers of the form $\exists g \in 2^{\mathbb{N}}$ (see Lemma 4.4). Intuitively, X comprises those positions of length k for which player II still has the possibility of playing inside or outside the closed set $[T]$.

Case 1: X is nonempty.
Then player II has a winning strategy. Namely, we define σ_{II} as follows. Pick $t_k \in X$ and g_{in} , $g_{out} \in 2^{\mathbb{N}}$ such that $t_k \subseteq g_{in}$, $t_k \subseteq g_{out}$, $g_{in} \in [T]$ and $g_{out} \notin [T]$. Given any sequence of odd length, $s = \langle x_0, y_0, \ldots, x_{n-1}, y_{n-1}, x_n \rangle$, we define

$$\sigma_{\mathrm{II}}(s) = \begin{cases} t_k(n) & \text{if } n < k \\ g_{in}(n) & \text{if } n \geq k \text{ and } \langle x_0, \ldots, x_k \rangle \in S \\ g_{out}(n) & \text{if } n \geq k \text{ and } \langle x_0, \ldots, x_k \rangle \notin S \end{cases}$$

It is clear that $\sigma_{\rm II}$ exists by Δ^0_1-comprehension and in view of (a), it is immediate to see that $\sigma_{\rm II}$ is winning for player II.

Case 2: X is empty.

Then player I has a winning strategy. On the one hand, since $X = \emptyset$, we have $\neg\exists t\,(t \in X_{in} \wedge t \in X_{out})$ and thus

$$\forall t\,(t \in X_{in} \rightarrow \forall g \in 2^{\mathbb{N}}\,(t \subseteq g \rightarrow g \in [T]))$$

$$\forall t\,(t \in X_{out} \rightarrow \forall g \in 2^{\mathbb{N}}\,(t \subseteq g \rightarrow g \notin [T])).$$

On the other hand, it follows by (b) that there are f_{in} , $f_{out} \in 2^{\mathbb{N}}$ such that $s_k \subseteq f_{in}$, $s_k \subseteq f_{out}$, $f_{in} \in [S]$ and $f_{out} \notin [S]$. Having these facts in mind, given any sequence of even length, $s = \langle x_0, y_0, \ldots, x_{n-1}, y_{n-1}\rangle$, we define

$$\sigma_{\rm I}(s) = \begin{cases} s_k(n) & \text{if } n < k \\ f_{in}(n) & \text{if } n \geq k \text{ and } \langle y_0, \ldots, y_{n-1}\rangle \notin T \\ f_{out}(n) & \text{if } n \geq k \text{ and } \langle y_0, \ldots, y_{n-1}\rangle \in X_{in} \\ f_{in}(n) & \text{if } n \geq k \text{ and } \langle y_0, \ldots, y_{n-1}\rangle \in X_{out} \end{cases}$$

Again, $\sigma_{\rm I}$ exists by Δ^0_1-comprehension and it easy to see that $\sigma_{\rm I}$ is winning for player I.

(2): In view of Remark 4.14, it suffices to repeat the proof of part 2 of Theorem 4.12.

(3): The proof is similar to that of part 1 and we omit it. ■

In view of Lemma 3.12 we obtain:

Corollary 4.19 *Each of the following is provable in* $\mathbf{WKL}_0$*:*

$$(\Delta^0_1, \Sigma^0_1) - \mathbf{SLO}^*_L,\ \ (\Delta^0_1, \Sigma^0_1) - \mathbf{SLO}^*_W,\ \textit{and}\ (\Sigma^0_1, \Delta^0_1) - \mathbf{SLO}^*_L.$$

Remark 4.20

1. *The scheme of Σ^0_1-induction has been used in the proof of parts 1 and 3 in Proposition 4.18, but not for the proof of part 2.*
2. *We do not know whether (Σ^0_1, Δ^0_1)-$\mathbf{Det}^*_W$ is provable from $\mathbf{WKL}_0$. We shall show, however, that it is provable from $\mathbf{ACA}_0$.*

Definition 4.21 *We say that a binary tree T defines a true closed set if*

$$\exists f \in 2^{\mathbb{N}}\,[f \in [T] \wedge \forall k\,\exists s\,(f[k] \subset s \wedge s \notin T)].$$

We will write $TrueClosed(T)$ to denote the above formula.

Proposition 4.22 *Assume that T is a binary tree. It is provable in $\mathbf{ACA}_0$ that*

$$TrueClosed(T) \vee \exists k \forall f \in 2^{\mathbb{N}} (f \in [T] \leftrightarrow f[k] \in T).$$

Proof. We work in $\mathbf{ACA}_0$. Suppose that $\exists k \forall f \in 2^{\mathbb{N}} (f \in [T] \leftrightarrow f[k] \in T)$ does not hold. Then, we have

$$\forall k \exists f \in 2^{\mathbb{N}} (f[k] \in T \wedge f \notin [T])$$

and so

$$\forall k \exists s, t \in 2^{<\mathbb{N}} (|s| = k \wedge s \subset t \wedge s \in T \wedge t \notin T). \tag{†}$$

Define T' to be

$$\{s \in 2^{<\mathbb{N}} : s \in T \wedge \exists t (s \subset t \wedge t \notin T)\}.$$

Note that T' exists by Σ^0_1-comprehension (which is available thanks to $\mathbf{ACA}_0$). Clearly, T' is a binary tree and it follows by (†) that T' is infinite. By applying Weak König Lemma we obtain that T' has a path, say $g \in 2^{\mathbb{N}}$. Since $T' \subseteq T$, $g \in [T]$. In addition, by the definition of T' we have

$$\forall k \exists s (g[k] \subset s \wedge s \notin T).$$

Thus, we have shown that $TrueClosed(T)$ holds, as required. ∎

Corollary 4.23 *Assume that T is a binary tree. It is provable in $\mathbf{ACA}_0$ that*

$$TrueClosed(T) \vee T \ defines \ a \ clopen \ set.$$

We are in a position to prove the main result of the section.

Theorem 4.24

1. $\mathbf{ACA}_0$ *proves* Σ^0_1-$\mathbf{Det}^*_L$.
2. $\mathbf{ACA}_0$ *proves* Σ^0_1-$\mathbf{Det}^*_W$.

Proof. We work in an arbitrary model of $\mathbf{ACA}_0$.

(1): We will prove Π^0_1-$\mathbf{Det}^*_L$, which is equivalent to Σ^0_1-$\mathbf{Det}^*_L$. Let $A(f), B(g) \in \Pi^0_1$. By Proposition 4.1 there are binary trees S and T satisfying that

$$[S] = \{f \in 2^{\mathbb{N}} : A(f)\} \text{ and } [T] = \{g \in 2^{\mathbb{N}} : B(g)\}$$

We must show that the game $G_L([S], [T])$ is determined.

Case A: $TrueClosed(T)$ holds.

Then, player II has a winning strategy in the game $G_L([S], [T])$. Actually, pick $g_0 \in 2^{\mathbb{N}}$ satisfying that

$$g_0 \in [T] \wedge \forall k \exists t (g_0[k] \subset t \wedge t \notin T)].$$

By Δ^0_1-comprehension, there exists $h : \mathbb{N} \to 2^{<\mathbb{N}}$ satisfying that

$$\forall k\,(g_0[k] \subset h(k) \wedge h(k) \notin T).$$

Define H to be the set given by

$$(k,n,i) \in H \;\leftrightarrow\; (n < |h(k)| \;\wedge i = h(n)) \vee (n \geq |h(k)| \wedge i = 0).$$

Clearly, H exists by Δ^0_1-comprehension. We will write $H_k(n) = i$ for $(k,n,i) \in H$. Thus, each function H_k extends the finite sequence $h(k)$ by putting zeros on the end. We are now in a position to define a strategy for player II, σ_{II}, as follows. Given any sequence of odd length $s = \langle x_0, y_0, \ldots, x_{n-1}, y_{n-1}, x_n\rangle$, we define

$$\sigma_{\mathrm{II}}(s) = \begin{cases} g_0(n) & \text{if } \langle x_0,\ldots,x_n\rangle \in S \\ H_k(n) & \text{if } \langle x_0,\ldots,x_n\rangle \notin S \text{ and } k = \mu j\ (\langle x_0,\ldots,x_j\rangle \notin S) \end{cases}$$

Again, σ_{II} exists by Δ^0_1-comprehension and it is straightforward to see that σ_{II} is winning for player II.

Case B: $TrueClosed(T)$ does not hold.
Then, it follows from Proposition 4.22 that there exists some $k_0 \in \mathbb{N}$ satisfying that

$$\forall g \in 2^{\mathbb{N}}\,(g \in [T] \leftrightarrow g[k_0] \in T).$$

Consequently, $B(g)$ defines a Δ^0_1-set.

Case B.1: $TrueClosed(S)$ holds.
Then, player I has a winning strategy in the game $G_L([S],[T])$. To see this, pick $f_0 \in 2^{\mathbb{N}}$ such that

$$f_0 \in [S] \wedge \forall k\, \exists s\,(f_0[k] \subseteq s \wedge s \notin S)].$$

In particular, there is some finite sequence s' such that $f_0[k_0] \subseteq s'$ and $s' \notin S$. Define f_{in} to be f_0 and define f_{out} to be the function obtained from the finite sequence s' by putting zeros on the end. Then, we have $s' \subseteq f_{in}$, $s' \subseteq f_{out}$, $f_{in} \in [S]$ and $f_{out} \notin [S]$. Given any finite sequence of even length, $s = \langle x_0, y_0, \ldots, x_{n-1}, y_{n-1}\rangle$, we define

$$\sigma_{\mathrm{I}}(s) = \begin{cases} s'(n) & \text{if } n < |s'| \\ f_{out}(n) & \text{if } n \geq |s'| \text{ and } \langle y_0,\ldots,y_{k_0-1}\rangle \in T \\ f_{in}(n) & \text{if } n \geq |s'| \text{ and } \langle y_0,\ldots,y_{k_0-1}\rangle \notin T \end{cases}$$

Then, σ_{I} exists by Δ^0_1-comprehension and σ_{I} is winning for player I.

Case B.2: $TrueClosed(S)$ does not hold.
It then follows from Proposition 4.22 that

$$\exists k\, \forall f \in 2^{\mathbb{N}}\,(f \in [S] \leftrightarrow f[k] \in S).$$

Hence, both $A(f)$ and $B(g)$ define a Δ^0_1-set and the determinacy of the game $G_L([S],[T])$ follows by Theorem 4.12.

(2): We will prove Π^0_1-$\mathbf{Det}^\star_W$, which is equivalent to Σ^0_1-$\mathbf{Det}^\star_W$. Let $A(f)$, $B(g) \in \Pi^0_1$. By Proposition 4.1 there are binary trees S and T satisfying that

$$[S] = \{f \in 2^{\mathbb{N}} : A(f)\} \text{ and } [T] = \{g \in 2^{\mathbb{N}} : B(g)\}$$

We must show that the game $G_W([S],[T])$ is determined. The proof is similar to that of the Lipschitz case. Since a winning strategy for player II in $G_L([S],[T])$ immediately gives rise to a winning strategy for player II in $G_W([S],[T])$, the only situation that deserves some explanations is the case where player I wins, i.e., case B.1 above. Thus, assume $TrueClosed(S)$ holds and $TrueClosed(T)$ does not. On the one hand, there exists $f_0 \in 2^{\mathbb{N}}$ such that

$$f_0 \in [S] \wedge \forall k\, \exists s\, (f_0[k] \subseteq s \wedge s \notin S)].$$

On the other hand, it follows from Proposition 4.22 that there exists some $k_0 \in \mathbb{N}$ satisfying that

$$\forall g \in 2^{\mathbb{N}}\, (g \in [T] \leftrightarrow g[k_0] \in T).$$

By Δ^0_1-comprehension, there exists $h : \mathbb{N} \to 2^{<\mathbb{N}}$ satisfying that

$$\forall k\, (f_0[k] \subseteq h(k) \wedge h(k) \notin S).$$

As in the proof of Case A of part (1), there exists a sequence of functions, $\{H_k : k \in \mathbb{N}\}$, such that each H_k extends the finite sequence $h(k)$ by putting zeros on the end. Since now player II is allowed to pass, we also need a function $ext : 2^{<\mathbb{N}} \to 2^{<\mathbb{N}}$ such that $ext(s)$ is the finite sequence obtained by dropping the zeros of the finite sequence s and decreasing the values by 1 (Recall that we identify passing with picking the number 0 and we identify picking i with picking $i+1$.) We are now in a position to define a winning strategy for player I. Given any sequence of even length, $s = \langle x_0, y_0, \ldots, x_{n-1}, y_{n-1}\rangle$, we put

$$\sigma_{\mathrm{I}}(s) = \begin{cases} f_0(n)+1 & \text{if } |ext(\langle y_0,\ldots,y_{n-1}\rangle)| < k_0 \\ H_k(n)+1 & \text{if } |ext(\langle y_0,\ldots,y_{n-1}\rangle)| < k_0 \text{ and } ext(\langle y_0,\ldots,y_{n-1}\rangle)[k_0] \in T \\ f_0(n)+1 & \text{if } |ext(\langle y_0,\ldots,y_{n-1}\rangle)| < k_0 \text{ and } ext(\langle y_0,\ldots,y_{n-1}\rangle)[k_0] \notin T \end{cases}$$

Then, σ_{I} exists by Δ^0_1-comprehension and it is easy to verify that σ_{I} is winning for player I. This completes the proof of the theorem. ∎

Corollary 4.25

1. $\mathbf{ACA}_0$ *proves* Σ^0_1-$\mathbf{SLO}^*_L$.
2. $\mathbf{ACA}_0$ *proves* Σ^0_1-$\mathbf{SLO}^*_W$.

Proof. See Lemma 3.12. ∎

Remark 4.26 *Part 1 of Theorem 4.24 can also be derived from known results on Gale-Stewart determinacy. On the one the hand, in [NMT07] Nemoto showed that* $\mathbf{ACA}_0$ *proves Gale-Stewart determinacy for the second level of the difference hierarchy* $(\Sigma^0_1)_2$*. On the other hand, we showed in Chapter 3 that an open Lipschitz game can be reduced to a Gale-Stewart game of payoff complexity* $(\Sigma^0_1)_2$*. Part 2 of Theorem 4.24 is, to the best of our knowledge, new.*

Although Part 1 of Theorem 4.24 is a consequence of already known results, we think that the alternative proof we give here is interesting because it suggests a natural strategy to improve the result. In fact, let us observe that in our proof of Σ^0_1-$\mathbf{Det}^\star_{L/W}$, we only need $\mathbf{ACA}_0$ to justify the use of Proposition 4.22. The rest of the proof only requires $\mathbf{WKL}_0$. Proposition 4.22 says that in the Cantor space, every closed set is either true closed or clopen. This dichotomy principle can be formalized as follows.

Definition 4.27 *Let* ($\mathbf{DP}$) *denote the formula*

$$BinaryTree(T) \;\rightarrow\; TrueClosed(T) \,\vee\, \exists k\, \forall f \in 2^{\mathbb{N}}\, (f \in [T] \leftrightarrow f[k] \in T),$$

where BinaryTree(T) *is a formula declaring that* T *is a binary tree.*

Then, the proof of Theorem 4.24 gives us

Corollary 4.28 $\mathbf{WKL}_0 + (\mathbf{DP})$ *proves* Σ^0_1-$\mathbf{Det}^*_{L/W}$.

We do not know whether Σ^0_1-$\mathbf{Det}^*_{L/W}$ is provable from plain $\mathbf{WKL}_0$. In view of Corollary 6.2.1, a natural idea for searching for a proof of Σ^0_1-$\mathbf{Det}^*_{L/W}$ in $\mathbf{WKL}_0$ emerges: Can we find a principle implying ($\mathbf{DP}$) and provable in $\mathbf{WKL}_0$? In what follows, we present a number of natural principles implying ($\mathbf{DP}$). However, all of them have turned out to be equivalent to $\mathbf{ACA}_0$!

Proposition 4.29 *Over* $\mathbf{RCA}_0$*, each of the following assertions implies* ($\mathbf{DP}$).

1. *(Weak König Lemma for* Σ^0_1 *trees) Let* $\varphi(s)$ *be a* Σ^0_1 *formula. If* $\varphi(s)$ *defines a binary tree then*
$$\forall k\, \exists s\, (|s| = k \wedge \varphi(s)) \;\rightarrow\; \exists f \in 2^{\mathbb{N}}\, \forall k\, \varphi(f[k]).$$

2. *(Weak Radó selection lemma) Given a sequence of finite functions* $\langle f_k : k \in \mathbb{N}\rangle$, $f_k : \{0, 1, \ldots, k\} \rightarrow \{0, 1\}$*, there exists* $f : \mathbb{N} \rightarrow \{0, 1\}$ *such that*
$$\forall m\, \exists k\, (k \geq m \wedge f[m] = f_k[m]).$$

3. *(The scheme of* $\Sigma^0_1 \wedge \Pi^0_1$ *choice) Let* $\varphi(k, Y)$ *be a* $\Sigma^0_1 \wedge \Pi^0_1$ *formula in which* Z *does not occur. Then*
$$\forall k\, \exists Y\, \varphi(k, Y) \rightarrow \exists Z\, \forall k\, \varphi(k, (Z)_k),$$
where we are using the notation $(Z)_k = \{i : (i, k) \in Z\}$.

Proof. We work in $\mathbf{RCA}_0$. Let T be a binary tree. Suppose that

$$\exists k\, \forall f \in 2^{\mathbb{N}}\, (f \in [T] \leftrightarrow f[k] \in T)$$

does not hold. Then, we have $\forall k\, \exists f \in 2^{\mathbb{N}}\, (f[k] \in [T] \wedge f \notin [T])$ and so

$$\forall k\, \exists s, t \in 2^{<\mathbb{N}}\, (|s| = k \wedge s \subseteq t \wedge s \in T \wedge t \notin T). \qquad (\dagger)$$

(1): Put $\varphi(s) \equiv s \in T \wedge \exists t\, (s \subseteq t \wedge t \notin T)$. Clearly, $\varphi(s)$ is in Σ^0_1 and $\varphi(s)$ defines a binary tree. It follows by ($\dagger$) that $\forall k\, \exists s\, (|s| = k \wedge \varphi(s))$. Hence, by applying Weak König Lemma for Σ^0_1 trees, we obtain that $\exists g \in 2^{\mathbb{N}}\, \forall k\, \varphi(g[k])$. Such a function g is a witness that $TrueClosed(T)$ holds.

(2): Since the s in the $\exists s$ quantifier in ($\dagger$) can be bounded by a function of k, using $\mathbf{RCA}_0$, we get a function $h : \mathbb{N} \to \mathbb{N}$ such that

$$\forall k\, \exists s\, \exists t \leq h(k)\, (|s| = k \wedge s \subseteq t \wedge s \in T \wedge t \notin T).$$

Again by $\mathbf{RCA}_0$, there exists a sequence of finite functions $\langle f_k : k \in \mathbb{N} \rangle$, $f_k : \{0, 1, \ldots, k\} \to \{0, 1\}$, satisfying that

$$\forall k\, \exists t\, (f_k \subseteq t \wedge f_k \in T \wedge t \notin T)$$

(Notice that we identify finite sequences with finite functions.) By using Weak Radó selection lemma, we pick $g : \mathbb{N} \to \{0, 1\}$ such that $\forall m\, \exists k\, (k \geq m \wedge g[m] = g_k[m])$. Clearly, g is a witness that $TrueClosed(T)$ holds.

(3): Let $bd : \mathbb{N} \to \mathbb{N}$ be a function such that (the code of) each finite sequence of length k is bounded by $bd(k)$. We claim that

- $\forall k\, \exists Y \begin{cases} \exists u\, (u \in Y) \wedge \\ \forall u\, (u \in Y \to \forall s \leq bd(k)\, (\exists t\, (s \subseteq t \wedge t \notin T) \to \exists t \leq u\, (s \subseteq t \wedge t \notin T))) \end{cases}$

To see this, fix $k \in \mathbb{N}$. Since $\mathbf{RCA}_0$ contains the scheme of Σ^0_1 induction, $\mathbf{RCA}_0$ proves the scheme of strong Σ^0_1 collection (see, e.g., Exercise II.3.14 in [Smp99]). Hence, there exists $k' \in \mathbb{N}$ such that $\forall s \leq bd(k)\, (\exists t\, (s \subseteq t \wedge t \notin T) \to \exists t \leq k'\, (s \subseteq t \wedge t \notin T))$. It suffices to consider $Y = \{k'\}$. This proves the claim.

By applying $\Sigma^0_1 \wedge \Pi^0_1$ choice, we obtain that there is a set Z satisfying that

$$\forall k \begin{cases} \exists u\, (u \in (Z)_k) \wedge \\ \forall u\, (u \in (Z)_k \to \forall s \leq bd(k)\, (\exists t\, (s \subseteq t \wedge t \notin T) \to \exists t \leq u\, (s \subseteq t \wedge t \notin T))) \end{cases}$$

Define a function $h : \mathbb{N} \to \mathbb{N}$ by putting $h(i) =$ least k such that $(i, k) \in Z$. Then, we have

$$\forall s\, (\exists t\, (s \subseteq t \wedge t \notin T) \to \exists t \leq h(|s|)\, (s \subseteq t \wedge t \notin T)).$$

Define $A(s) = \{s : s \in T \wedge \exists t \leq h(\notin)\, (s \subseteq t \wedge t \notin T)\}$ (such a set exists by Δ^0_1-comprehension). Then, A is an infinite binary tree. By Lemma VIII.2.5 in [Smp99] the scheme of Π^0_1 choice already implies the weak König Lemma. Thus, the binary tree A has a path. But any path of A is a witness that $TrueClosed(T)$ holds. ■

Proposition 4.30 *The following assertions are pairwise equivalent over* $\mathbf{RCA}_0$.

1. $\mathbf{ACA}_0$.
2. *Weak König Lemma for* Σ^0_1 *trees.*
3. *Weak Radó selection lemma.*
4. *The scheme of* $\Sigma^0_1 \wedge \Pi^0_1$ *choice.*

Proof. We reason in $\mathbf{RCA}_0$.

(1) $\Rightarrow$ (2): Immediate.

(2) $\Rightarrow$ (3): Consider a sequence of finite functions $\langle f_k : k \in \mathbb{N}\rangle$, $f_k : \{0, 1, \ldots, k\} \to \{0, 1\}$. Define $\varphi(s)$ to be $\{s \in 2^{<\mathbb{N}} : \exists k\, (s \subset f_k)\}$. Then, $\varphi(s) \in \Sigma^0_1$ defines a binary tree and $\forall k\, \exists s\, (|s| = k \wedge \varphi(s))$. By Weak König lemma for Σ^0_1 trees, $\varphi(s)$ has a path, say $f \in 2^{\mathbb{N}}$. Clearly, $\forall m\, \exists k\, (k \geq m \wedge f[m] = f_k[m])$.

(3) $\Rightarrow$ (1): It is well known that $\mathbf{ACA}_0$ is equivalent over $\mathbf{RCA}_0$ to the assertion that for all injective functions $g : \mathbb{N} \to \mathbb{N}$, the range of g exists (see, e.g., Lemma III.1.3 of [Smp99]). Consider $g : \mathbb{N} \to \mathbb{N}$ injective. We define a sequence $\langle f_k : k \in \mathbb{N}\rangle$, $f_k : \{0, 1, \ldots, k\} \to \{0, 1\}$, as follows

$$f_k(i) = 1 \;\leftrightarrow\; \exists y \leq k\, g(y) = i$$

By Weak Radó lemma there is $f \in 2^{\mathbb{N}}$ such that $\forall m\, \exists k\, (k \geq m \wedge f[m] = f_k[m])$. It is easy to see that $X = \{i : f(i) = 1\}$ defines the range of g.

(1) $\Rightarrow$ (4): Immediate.

(4) $\Rightarrow$ (1): By Lemma III.1.3 of [Smp99], it suffices to show Σ^0_1-comprehension. Let $\varphi(x)$ be a Σ^0_1 formula. By bounded Σ^0_1-comprehension (available thanks to $\mathbf{RCA}_0$), for every $k \in \mathbb{N}$ the set $\{x : x \leq k \wedge \varphi(x)\}$ exists. Thus, have

$$\forall k\, \exists Y\, [\forall u\, (u \leq k \wedge \varphi(x) \to u \in Y) \wedge \forall u \leq k\, (u \in Y \to \varphi(u))]$$

Note that by Σ^0_1 collection (again available thanks to $\mathbf{RCA}_0$), Σ^0_1 formulas are closed under bounded quantification. Hence, applying $\Sigma^0_1 \wedge \Pi^0_1$ choice, we get that there exists Z satisfying that

$$\forall k\, [\forall u\, (u \leq k \wedge \varphi(x) \to u \in (Z)_k) \wedge \forall u \leq k\, (u \in (Z)_k \to \varphi(u))]$$

Then, $\{x : \varphi(x)\} = \{x : x \in (Z)_x\}$ and this set exists by Δ^0_1-comprehension. ∎

The previous Proposition seems to suggest that *natural* formalizations of (**DP**) require arithmetical comprehension. Of course, this does not rule out the possibility of finding other, perhaps more elaborated proofs of (**DP**) (and hence proofs of Σ^0_1-$\mathbf{Det}^\star_L$ too) which can be formalized within $\mathbf{WKL}_0$. Thus, we pose the following questions:

Problem 4.31

1. *Is* (**DP**) *provable in* $\mathbf{WKL}_0$*?*

2. *Is* (**DP**) *equivalent over* $\mathbf{RCA}_0$ *to* $\mathbf{ACA}_0$*?*

Problem 4.32

1. *Is* Σ^0_1-$\mathbf{Det}^\star_{L/W}$ *provable in* $\mathbf{WKL}_0$*?*

2. *Is* Σ^0_1-$\mathbf{SLO}^\star_{L/W}$ *provable in* $\mathbf{WKL}_0$*?*

3. *Is* Σ^0_1-$\mathbf{Det}^\star_{L/W}$ *equivalent over* $\mathbf{RCA}_0$ *to* $\mathbf{ACA}_0$*?*

4. *Is* Σ^0_1-$\mathbf{Det}^\star_{L/W}$ *equivalent over* $\mathbf{RCA}_0$ *to* Σ^0_1-$\mathbf{SLO}^*_L$*?*

4.4 Determinacy for $(\Sigma^0_1)_2$ sets

In this section we show that $\mathbf{ACA}_0$ proves determinacy for the second level in the Difference Hierarchy, $(\Sigma^0_1)_2$. This result is particularly interesting since, to the best of our knowledge, it cannot be derived from known results on Gale-Stewart determinacy.

Our starting point is the next Proposition.

Proposition 4.33

1. $\mathbf{ACA}_0$ *proves* (Σ^0_1, Π^0_1)-$\mathbf{Det}^\star_L$.

2. $\mathbf{ACA}_0$ *proves* (Π^0_1, Σ^0_1)-$\mathbf{Det}^\star_L$.

3. $\mathbf{ACA}_0$ *proves* (Σ^0_1, Π^0_1)-$\mathbf{Det}^\star_W$.

4. $\mathbf{ACA}_0$ *proves* (Π^0_1, Σ^0_1)-$\mathbf{Det}^\star_W$.

Proof. We work in an arbitrary model of $\mathbf{ACA}_0$.

(1): Let $A(f) \in \Sigma^0_1$ and $B(g) \in \Pi^0_1$. We must show that the game $G_L(A, B)$ is determined. By Proposition 4.1, there are binary trees S, T satisfying that

$$[S] = \{f \in 2^{\mathbb{N}} \,:\, \neg A(f)\} \text{ and } [T] = \{g \in 2^{\mathbb{N}} \,:\, B(g)\}$$

<u>Case A</u>: $TrueClosed(S)$ holds.
Then, player I has a winning strategy. To see this, pick $f_0 \in 2^{\mathbb{N}}$ such that

$$f_0 \in [S] \wedge \forall k \, \exists s \, (f_0[k] \subseteq s \wedge s \notin S).$$

By Δ^0_1-comprehension, there exists $h : \mathbb{N} \to 2^{<\mathbb{N}}$ such that

$$\forall k\, (f_0[k] \subseteq h(k) \wedge h(k) \notin S)$$

and so

$$\neg A(f_0) \wedge \forall k\, \forall f \in 2^{\mathbb{N}}\, (h(k) \subset f \to A(f)). \tag{$\dagger$}$$

As in the proof of Theorem 4.24, consider a sequence of functions, $\{H_k : k \in \mathbb{N}\}$, such that each function H_k extends the finite sequence $h(k)$ by putting zeros on the end. We define a strategy for player I as follows. Given any sequence of even length $s = \langle x_0, y_0, \dots, x_{n-1}, y_{n-1}\rangle$, we define

$$\sigma_{\mathrm{I}}(s) = \begin{cases} f_0(n) & \text{if } \langle y_0, \dots, y_{n-1}\rangle \in T \\ H_k(n) & \text{if } \langle y_0, \dots, y_{n-1}\rangle \notin T \text{ and } k = \mu j\ (\langle y_0, \dots, y_{j-1}\rangle \notin T) \end{cases}$$

Note that σ_{I} exists by Δ^0_1-comprehension and it follows by ($\dagger$) that σ_{I} is winning for player I.

<u>Case B</u>: $TrueClosed(S)$ does not hold.
Then, by Proposition 4.22, $A(f)$ defines a Δ^0_1-set. Thus, the fact that game $G_L(A, B)$ is determined follows from (Δ^0_1, Π^0_1)-$\mathbf{Det}^\star_L$ (which is available in $\mathbf{WKL}_0$ by Proposition 4.18).

(2): It follows by part (1), for (Π^0_1, Σ^0_1)-$\mathbf{Det}^\star_L$ is equivalent to (Σ^0_1, Π^0_1)-$\mathbf{Det}^\star_L$.

(3): Let $A(f) \in \Sigma^0_1$ and $B(g) \in \Pi^0_1$. We must show that the game $G_W(A, B)$ is determined. By Proposition 4.1, there are binary trees S, T satisfying that

$$[S] = \{f \in 2^{\mathbb{N}} : \neg A(f)\} \text{ and } [T] = \{g \in 2^{\mathbb{N}} : B(g)\}.$$

<u>Case A</u>: $TrueClosed(S)$ holds.
Then, player I has a winning strategy. The proof is similar to that of part (1), but now we have to take into account that player II is allowed to pass. Consider $f_0 \in 2^{\mathbb{N}}$ such that

$$f_0 \in [S] \wedge \forall k\, \exists s\, (f_0[k] \subseteq s \wedge s \notin S)$$

and $h : \mathbb{N} \to 2^{<\mathbb{N}}$ such that

$$\forall k\, (f_0[k] \subseteq h(k) \wedge h(k) \notin S).$$

Consider a sequence of functions, $\{H_k : k \in \mathbb{N}\}$, such that each function H_k extends the finite sequence $h(k)$ by putting zeros on the end. Since now player II is allowed to pass, we also need a function $ext : 2^{<\mathbb{N}} \to 2^{<\mathbb{N}}$ such that $ext(s)$ is the finite sequence obtained by dropping the zeros of the finite sequence s and decreasing the values by 1 (Recall that we identify passing with picking the number 0 and we identify picking i with picking $i + 1$.) We are now in a position to define a winning strategy for player I. Given any sequence of even length, $s = \langle x_0, y_0, \dots, x_{n-1}, y_{n-1}\rangle$, we put

$$\sigma_{\mathrm{I}}(s) = \begin{cases} f_0(n) + 1 & \text{if } ext(\langle y_0, \dots, y_{n-1}\rangle) \in T \\ H_k(n) + 1 & \text{if } ext(\langle y_0, \dots, y_{n-1}\rangle) \notin T \text{ and } k = \mu j\ (ext\,(\langle y_0, \dots, y_{j-1}\rangle) \notin T) \end{cases}$$

It is easy to see that σ_{I} is a winning strategy for player I.

Case B: $TrueClosed(S)$ does not hold.
Then, by Proposition 4.22, $A(f)$ defines a Δ^0_1-set. Thus, the fact that game $G_W(A,B)$ is determined follows from (Δ^0_1,Π^0_1)-$\mathbf{Det}^{\star}_W$ (which is available in $\mathbf{WKL}_0$ by Proposition 4.18).

(4): It follows by part (3), for (Π^0_1,Σ^0_1)-$\mathbf{Det}^{\star}_W$ is equivalent to (Σ^0_1,Π^0_1)-$\mathbf{Det}^{\star}_W$. ■

Corollary 4.34

1. $\mathbf{ACA}_0$ *proves* (Σ^0_1,Π^0_1)-$\mathbf{SLO}^{\star}_L$.
2. $\mathbf{ACA}_0$ *proves* (Π^0_1,Σ^0_1)-$\mathbf{SLO}^{\star}_L$.
3. $\mathbf{ACA}_0$ *proves* (Σ^0_1,Π^0_1)-$\mathbf{SLO}^{\star}_W$.
4. $\mathbf{ACA}_0$ *proves* (Π^0_1,Σ^0_1)-$\mathbf{SLO}^{\star}_W$.

Definition 4.35 *The following definition is made in* $\mathbf{ACA}_0$. *Let* $T\subseteq 2^{<\mathbb{N}}$ *be a binary tree. The boundary of* T *is the following set*

$$\delta\,(T)=\{t\in T:\ \exists t'(t\subseteq t'\wedge t'\notin T)\}.$$

Given $S\subseteq 2^{<\mathbb{N}}$, *we define*

$$\delta_S T=\{t\in T:\ \exists t'(t'\in S-T\wedge t\subseteq t')\}.$$

If $T_1,T_0\subseteq 2^{<\mathbb{N}}$ *are trees such that* $T_1\subseteq T_0$ *then we define*

$$\delta(T_0,T_1)=\{s\in T_1:\ \exists h,g\,(h\in[\delta_{T_0}T_1]\wedge g\in[\delta\,(T_0)]\wedge g\notin[T_1]\wedge s\subset h\wedge s\subset g)\}.$$

Let us notice that if T and S are trees then δT and $\delta_S T$ are also trees. The existence of $\delta(T_0,T_1)$ in $\mathbf{ACA}_0$ follows from Lemma 4.5. It is easily checked that $\delta(T_0,T_1)$ is a tree.

Lemma 4.36 *The following is provable in* $\mathbf{ACA}_0$ *Let* $S_0,S_1,T_0,T_1\subseteq 2^{<\mathbb{N}}$ *be pruned trees such that*

(-) $S_1\subseteq S_0$ *and* $T_1\subseteq T_0$, *and*

(-) $\delta(T_0,T_1)$ *and* $\delta(S_0,S_1)$ *are finite.*

Then:

1. $G_L([S_0]-[S_1],[T_0]-[T_1])$ *is determined.*
2. $G_W([S_0]-[S_1],[T_0]-[T_1])$ *is determined.*

Proof. (1): By hypothesis $\delta(S_0, S_1)$ and $\delta(T_0, T_1)$ are finite trees and, therefore, by applying Δ^0_0-induction, we obtain that there are $k_1, k_2 \in \mathbb{N}$ and $s_{k_1}, s_{k_2} \in 2^{<\mathbb{N}}$ such that

1. $k_1 = \max\{|s| : s \in \delta(S_0, S_1)\}$,
2. $s_{k_1} \in \delta(S_0, S_1)\}$ with $|s_{k_1}| = k_1$.
3. $k_2 = \max\{|s| : s \in \delta(T_0, T_1)\}$, and
4. $s_{k_2} \in \delta(T_0, T_1)\}$ with $|s_{k_2}| = k_2$.

Then,

- one of the following holds:

  (-) $\exists f_0, f_1\, (s_{k_1} * \langle 0\rangle \subset f_0 \wedge f_0 \in [\delta_{S_0} S_1] \wedge s_{k_1} * \langle 1\rangle \subset f_1 \wedge f_1 \in [\delta\,(S_0)] - [S_1])$,
  (-) $\exists f_0, f_1\, (s_{k_1} * \langle 0\rangle \subset f_0 \wedge f_0 \in [\delta\,(S_0)] - [S_1] \wedge s_{k_1} * \langle 1\rangle \subset f_1 \wedge f_1 \in [\delta_{S_0} S_1])$.

- and one of the following holds:

  (-) $\exists g_0, g_1\, (s_{k_2} * \langle 0\rangle \subset g_0 \wedge g_0 \in [\delta_{T_0} T_1] \wedge s_{k_2} * \langle 1\rangle \subset g_1 \wedge g_1 \in [\delta\,(T_0)] - [T_1])$,
  (-) $\exists g_0, g_1\, (s_{k_2} * \langle 0\rangle \subset g_0 \wedge g_0 \in [\delta\,(T_0)] - [T_1] \wedge s_{k_2} * \langle 1\rangle \subset g_1 \wedge g_1 \in [\delta_{T_0} T_1])$.

Now, we distinguish several cases:

Case 1: $[\delta_{T_0} T_1] \neq \emptyset$, $[\delta\,(T_0)] - [T_1] \neq \emptyset$, $[\delta_{S_0} S_1] \neq \emptyset$, and $[\delta\,(S_0)] - [S_1] \neq \emptyset$.

We distinguish two subcases:

1. $k_1 \leq k_2$.

   Then player II has a winning strategy in the game $G_L([S_0]-[S_1], [T_0]-[T_1])$. Namely, we define a strategy σ_{II} as follows. Assume that

   $$\exists g_0, g_1\, (s_{k_2} * \langle 0\rangle \subset g_0 \wedge g_0 \in [\delta_{T_0} T_1] \wedge s_{k_2} * \langle 1\rangle \subset g_1 \wedge g_1 \in [\delta\,(T_0)] - [T_1])$$

   Thus we can fix a $g_0 \in 2^{\mathbb{N}}$ such that $s_{k_2} * \langle 0\rangle \subset g_0 \wedge g_0 \in [\delta_{T_0} T_1]$ and a $g_1 \in 2^{\mathbb{N}}$ such that $s_{k_2} * \langle 1\rangle \subset g_1 \wedge g_1 \in [\delta\,(T_0)] - [T_1]$.

   We will need a formula $(*)$ which says that there is no $f \in [\delta_{S_0} S_1]$ such that $s[k_2+1] \subset f$. Namely

   $$\forall s'((s\,[k_2+1] \subset s' \wedge s' \in \delta_{S_0} S_1) \rightarrow \exists k \forall s''(s'\,[k] \subseteq s'' \rightarrow s'' \in S_1))\ . \qquad (*)$$

   Now let $s, t \in 2^{<\mathbb{N}}$ be such that $|s| = j+1$ and $|t| = j$.

$$\sigma_{\mathrm{II}}(s\otimes t) = \begin{cases} (s_{k_2})_j & \text{if } j < k_2 \\ g_0(j) & \text{if } j \geq b \wedge s[k_2+1] \notin S_0 \\ g_1(j) & \text{if } j \geq b \wedge s[k_2+1] \in S_0 - S_1 \wedge s \in S_0 \\ g_1(j) & \text{if } j \geq b \wedge s[k_2+1] \in S_0 - S_1 \wedge s \notin S_0 \wedge t * \langle 0 \rangle \in T_0 \wedge t * \langle 1 \rangle \in T_0 \\ k & \text{if } j \geq b \wedge s[k_2+1] \in S_0 - S_1 \wedge s \notin S_0 \wedge (t * \langle 0 \rangle \notin T_0 \vee t * \langle 1 \rangle \notin T_0) \\ & \text{and } k = \min\{i : t * \langle i \rangle \notin T_0\} \\ g_0(j) & \text{if } j \geq b \wedge s[k_2+1] \in S_1 \wedge \exists f\,(f \in [\delta_{S_0} S_1] \wedge s[k_2+1] \subset f) \wedge \\ & (s \in S_1 \vee s \notin S_0) \\ g_0(j) & \text{if } j \geq b \wedge s[k_2+1] \in S_1 \wedge \exists f\,(f \in [\delta_{S_0} S_1] \wedge s[k_2+1] \subset f) \wedge \\ & s \in S_0 - S_1 \wedge \exists s'(s \subseteq s' \wedge s' \notin S_0) \\ g_0(j) & \text{if } j \geq b \wedge s[k_2+1] \in S_1 \wedge \exists f\,(f \in [\delta_{S_0} S_1] \wedge s[k_2+1] \subset f) \wedge \\ & s \in S_0 - S_1 \wedge\ \forall s'(s \subset s' \rightarrow s' \in S_0) \wedge \\ & (t * \langle 0 \rangle \notin T_0 - T_1 \wedge t * \langle 1 \rangle \notin T_0 - T_1) \\ k & \text{if } j \geq b \wedge s[k_2+1] \in S_1 \wedge \exists f\,(f \in [\delta_{S_0} S_1] \wedge s[k_2+1] \subset f) \wedge \\ & s \in S_0 - S_1 \wedge\ \forall s'(s \subset s' \rightarrow s' \in S_0) \wedge \\ & (t * \langle 0 \rangle \in T_0 - T_1 \vee t * \langle 1 \rangle \in T_0 - T_1) \\ & \text{and } k = \min\{i : t * \langle i \rangle \in T_0 - T_1\} \\ g_1(j) & \text{if } j \geq b \wedge s[k_2+1] \in S_1 \wedge (*) \wedge (s \in \delta_{S_0} S_1 \vee s \in S_0 - S_1) \\ g_1(j) & \text{if } j \geq b \wedge s[k_2+1] \in S_1 \wedge (*) \wedge s \notin \delta_{S_0} S_1 \wedge (s \in S_1 \vee s \notin S_0) \wedge \\ & t * \langle 0 \rangle \in T_0 \wedge t * \langle 1 \rangle \in T_0 \\ k & \text{if } j \geq b \wedge s[k_2+1] \in S_1 \wedge (*) \wedge s \notin \delta_{S_0} S_1 \wedge (s \in S_1 \vee s \notin S_0) \wedge \\ & (t * \langle 0 \rangle \notin T_0 \vee t * \langle 1 \rangle \notin T_0) \\ & \text{and } k = \min\{i : t * \langle i \rangle \notin T_0\} \end{cases}$$

Observe that the strategy σ_{II} exists by $\mathbf{ACA}_0$ (since $\exists f\,(f \in [\delta_{S_0} S_1] \wedge s[k_2+1] \subset f)$ is equivalent to a Π^0_1 formula thanks to Lemma 4.4) and that σ_{II} is a winning strategy for player II in $G_L([S_0]-[S_1],[T_0]-[T_1])$. The proof of this assertion is analogous to the one of part 1 of Case 1 in Lemma 2.10.

2. $k_2 < k_1$.

Then player I has a winning strategy in the game $G_L([S_0]-[S_1],[T_0]-[T_1])$. Namely, we define a strategy σ_{I} as follows. Assume that

$$\exists f_0, f_1\,(s_{k_1} * \langle 0\rangle \subset f_0 \wedge f_0 \in [\delta_{S_0} S_1] \wedge s_{k_1} * \langle 1\rangle \subset f_1 \wedge f_1 \in [\delta\,(S_0)] - [S_1])$$

Thus we can fix a $f_0 \in 2^{\mathbb{N}}$ such that $s_{k_1} * \langle 0\rangle \subset f_0 \wedge f_0 \in [\delta_{T_0} T_1]$ and a $f_1 \in 2^{\mathbb{N}}$ such that $s_{k_1} * \langle 1\rangle \subset f_1 \wedge f_1 \in [\delta\,(T_0)] - [T_1]$.

Again we need a formula $(**)$ which says that there is no $g \in [\delta_{T_0} T_1]$ such that $t[a] \subset g$. Namely

$$\forall t'(t\,[k_1] \subset t' \wedge t' \in \delta_{T_0} T_1 \rightarrow \exists k \forall t''(t'\,[k] \subseteq t'' \rightarrow t'' \in T_1))\ . \qquad (**)$$

Let us now define a strategy σ_{I} for player I. Firstly we put

$$\sigma_{\mathrm{I}}\,(\langle\rangle) = s_{k_1}(0).$$

Now for all $s, t \in 2^{<\mathbb{N}}$ with $|s| = |t| = j \geq 1$.

$$
\sigma_{\mathrm{I}}(s \otimes t) = \begin{cases}
(s_{k_1})_j & \text{if } j < k_1 \\
f_1(j) & \text{if } j \geq k_1 \wedge t[k_1] \notin T_0 \\
f_0(j) & \text{if } j \geq k_1 \wedge t[k_1] \in T_0 - T_1 \wedge t \in T_0 \\
f_0(j) & \text{if } j \geq k_1 \wedge t[k_1] \in T_0 - T_1 \wedge t \notin T_0 \wedge s * \langle 0 \rangle \notin S_0 - S_1 \wedge \\
 & s * \langle 1 \rangle \notin S_0 - S_1 \\
k & \text{if } j \geq k_1 \wedge t[k_1] \in T_0 - T_1 \wedge t \notin T_0 \wedge \\
 & (s * \langle 0 \rangle \in S_0 - S_1 \vee s * \langle 1 \rangle \in S_0 - S_1) \\
 & \text{and } k = \min\{i : s * \langle i \rangle \in S_0 - S_1\} \\
f_1(j) & \text{if } j \geq k_1 \wedge t[k_1] \in T_1 \wedge \exists g \ (g \in [\delta_{T_0} T_1] \wedge t[k_1] \subset g) \wedge (t \in T_1 \vee t \notin T_0) \\
f_1(j) & \text{if } j \geq k_1 \wedge t[k_1] \in T_1 \wedge \exists g \ (g \in [\delta_{T_0} T_1] \wedge t[k_1] \subset g) \wedge \\
 & t \in T_0 - T_1 \wedge \exists t' (t \subseteq t' \wedge t' \notin T_0) \\
f_1(j) & \text{if } j \geq k_1 \wedge t[k_1] \in T_1 \wedge \exists g \ (g \in [\delta_{T_0} T_1] \wedge t[k_1] \subset g) \wedge \\
 & t \in T_0 - T_1 \wedge \ \forall t' (t \subset t' \rightarrow t' \in T_0) \wedge \\
 & (s * \langle 0 \rangle \in S_0 - S_1 \wedge s * \langle 1 \rangle \in S_0 - S_1) \\
k & \text{if } j \geq k_1 \wedge t[k_1] \in T_1 \wedge \exists g \ (g \in [\delta_{T_0} T_1] \wedge t[k_1] \subset g) \wedge \\
 & t \in T_0 - T_1 \wedge \ \forall t' (t \subset t' \rightarrow t' \in T_0) \wedge \\
 & (s * \langle 0 \rangle \notin S_0 - S_1 \vee s * \langle 1 \rangle \notin S_0 - S_1) \\
 & \text{and } k = \min\{i : s * \langle i \rangle \in S_0 - S_1\} \\
f_0(j) & \text{if } j \geq k_1 \wedge t[k_1] \in T_1 \wedge (**) \wedge (t \in \delta_{T_0} T_1 \vee t \in T_0 - T_1) \\
f_0(j) & \text{if } j \geq k_1 \wedge t[k_1] \in T_1 \wedge (**) \wedge t \notin \delta_{T_0} T_1 \wedge (t \in T_1 \vee t \notin T_0) \wedge \\
 & s * \langle 0 \rangle \notin S_0 - S_1 \wedge s * \langle 1 \rangle \notin S_0 - S_1 \\
k & \text{if } j \geq k_1 \wedge t[k_1] \in T_1 \wedge (**) \wedge t \notin \delta_{T_0} T_1 \wedge (t \in T_1 \vee t \notin T_0) \wedge \\
 & (s * \langle 0 \rangle \in S_0 - S_1 \vee s * \langle 1 \rangle \in S_0 - S_1) \wedge \\
 & \text{and } k = \min\{i : t * \langle i \rangle \in S_0 - S_1\}
\end{cases}
$$

The strategy σ_{I} does exist (by $\mathbf{ACA}_0$) and it is straightforward to check that it is a winning strategy for player I in in $G_L([S_0]-[S_1],[T_0]-[T_1])$. The proof of this assertion is analogous to the one of part 2 of Case 1 in Lemma 2.10.

Case 2: (T_0,T_1) or (S_0,S_1) is in a degenerated position, but not both of them.
As we did in the previous chapter we say that (T_0,T_1) (and similarly (S_0,S_1)) is in a *degenerated position* if $[\delta_{T_0}T_1]=\emptyset$ or $[\delta\,(T_0)]-[T_1]=\emptyset$.
Recall that if (T_0,T_1) is in a degenerated position then the formula $g\in[T_0]-[T_1]$ is equivalent to a Σ^0_1 formula or a Π^0_1 formula:

- If $[\delta_{T_0}T_1]=\emptyset$ then $\delta_{T_0}T_1$ is finite and, as a consequence, we have for each $g\in 2^{\mathbb{N}}$,

$$g\in[T_0]-[T_1]\leftrightarrow g\in[T_0]\wedge\forall k\,(g[k]\notin\delta_{T_0}T_1\rightarrow g[k]\notin T_1)$$

- If $[\delta\,(T_0)]-[T_1]=\emptyset$ then let $T_0'=\{s\in T_0:\forall s'\,(s\subseteq s'\rightarrow s'\in T_0\}$. For each $g\in 2^{\mathbb{N}}$,

$$g\in[T_0]-[T_1]\leftrightarrow\exists k\,(g[k]\in T_0'\wedge g[k]\notin T_1)$$

If player I plays in a degenerated position, then player II has a winning strategy (essentially, player II plays simulating the strategy described in Lemma 4.24 (case A)):

- If $f\in[S_0]-[S_1]$ is equivalent to a Π^0_1 formula then, since (T_0,T_1) is not in a degenerate position there exists $g\in[\delta\,(T_0)]-[T_1]$ and player II can win the game using g.
- If $f\in[S_0]-[S_1]$ is equivalent to a Σ^0_1 formula then, since (T_0,T_1) is not in a degenerate position there exists $g\in[\delta_{T_0}T_1]$ and player II can win the game using g.

In a similar way it can be proved that if player II plays in a degenerated position then player I has a winning strategy:

- If $g\in[T_0]-[T_1]$ is equivalent to a Σ^0_1 formula then, since (S_0,S_1) is not in a degenerate position there exists $f\in[\delta\,(S_0)]-[S_1]$ and player I can win the game using f.
- If $g\in[T_0]-[T_1]$ is equivalent to a Π^0_1 formula then, since (S_0,S_1) is not in a degenerate position there exists $f\in[\delta_{S_0}S_1]$ and player I can win the game using f.

Case 3: (T_0,T_1) and (S_0,S_1) are in a degenerated position.
Recall that in these degenerated cases $f\in[T_0]-[T_1]$ and $f\in[S_0]-[S_1]$ are equivalent to some formulas in $\Sigma^0_1\cup\Pi^0_1$, so, the corresponding game is determined by Proposition 4.33.

(2): Now we prove that under the same hypothesis the Wadge game $G_W([S_0]-[S_1],[T_0]-[T_1])$ is determined. As before, let $k_1,k_2\in\mathbb{N}$ and $s_{k_1},s_{k_2}\in 2^{<\mathbb{N}}$ such that

1. $k_1=\max\{|s|\,:\,s\in\delta(S_0,S_1)\}$,

2. $s_{k_1} \in \delta(S_0, S_1)\}$ with $|s_{k_1}| = k_1$.

3. $k_2 = \max\{|s| : s \in \delta(T_0, T_1)\}$, and

4. $s_{k_2} \in \delta(T_0, T_1)\}$ with $|s_{k_2}| = k_2$.

We have to consider the same cases:

Case 1: $[\delta_{T_0}T_1] \neq \emptyset$, $[\delta(T_0)] - [T_1] \neq \emptyset$, $[\delta_{S_0}S_1] \neq \emptyset$, and $[\delta(S_0)] - [S_1] \neq \emptyset$.

We distinguish two subcases:

1. $k_1 \leq k_2$.

   Since a winning strategy for player II in $G_L([S_0] - [S_1], [T_0] - [T_1])$ immediately yields a winning strategy for player II in $G_W([S_0]-[S_1], [T_0]-[T_1])$, the proof of this subcase is similar to that of the Lipschitz subcase.

2. $k_2 < k_1$.

   In contrast to the corresponding Lipschitz subcase, in this subcase player II has a winning strategy in the Wadge game $G_W([S_0] - [S_1], [T_0] - [T_1])$, because player II can pass while player I is playing inside $\delta(S_0, S_1)$. As soon as player I starts playing outside $\delta(S_0, S_1)$, which must eventually happen, since $\delta(S_0, S_1)$ is well-founded, player II uses the strategy described in the former subcase and wins the game.

Cases 2 and 3: In these cases the fact that player II can pass in a Wadge game does not change anything essential in the proofs in comparison to the corresponding Lipschitz cases. The winning strategies for player I and player II in the Wadge game $G_W([S_0]-[S_1], [T_0]-[T_1])$ remain the same. ■

Theorem 4.37

1. $\mathbf{ACA}_0$ *proves* $\left(\Sigma^0_1\right)_2$-$\mathbf{Det}^*_L$.

2. $\mathbf{ACA}_0$ *proves* $\left(\Sigma^0_1\right)_2$-$\mathbf{Det}^*_W$.

Proof. We work in an arbitrary model of $\mathbf{ACA}_0$.
(1): Let $A(f), B(f) \in \left(\Sigma^0_1\right)_2$. We must show that the game $G_L(A, B)$ is determined. Since $A(f)$ and $B(f)$ are differences of closed sets, there exist, by Proposition 4.7, binary pruned trees S_0, S_1, T_0, and T_1 such that

1. $S_1 \subseteq S_0$ and $T_1 \subseteq T_0$.

2. $A(f) \leftrightarrow f \in [S_0] - [S_1]$, and $B(g) \leftrightarrow g \in [T_0] - [T_1]$.

We distinguish several cases:
<u>Case 1</u>: $[\delta(T_0, T_1)] \neq \emptyset$.
Let $g_0 \in [\delta(T_0, T_1)]$. Then, $g_0 \in [\delta_{T_0} T_1]$ and

$$\forall k\, \exists g\, (g_0[k] \subset g \wedge g \in [\delta\,(T_0)] - [T_1]).$$

By Δ^0_1-comprehension, there exists $h_0 : \mathbb{N} \to 2^{<\mathbb{N}}$ such that

$$\forall k\, (g[k] \subseteq h_0(k) \wedge h_0(k) \in \delta\,(T_0) - T_1).$$

Thus, we have

$$\forall k\, \exists h\, (h \in 2^{\mathbb{N}} \wedge h_0\,[k] \subset h \wedge h \in [\delta\,(T_0)] \wedge h \notin [T_1]).$$

Therefore, by $\Sigma^0_1 \wedge \Pi^0_1$ choice (available in $\mathbf{ACA}_0$ in by Proposition 4.30), there exists a set H such that

$$\forall k\, ((H)_k \in 2^{\mathbb{N}} \wedge h_0(k) \subset (H)_k \wedge (H)_k \in [\delta\,(T_0)] \wedge (H)_k \notin [T_1]).$$

We will write $H_k(n) = i$ for $(k, n, i) \in H$. Thus, each function H_k extends the finite sequence $h_0(k)$ to a path in $[T_0]$ and in the boundary of $[T_0]$, leaving player II the possibility of playing still inside or outside $[T_0]$. We are now in the position to define a strategy for player II, σ_{II}, as follows. Given $s, t \in 2^{<\mathbb{N}}$, with $|s| = j + 1$ and $|t| = j$, we define

$$\sigma_{\mathrm{II}}(s \otimes t) = \begin{cases} g_0(j) & \text{if } s \in S_1 \vee (s \notin S_0 \wedge \forall l\,(l < \min\{i : s\,[i] \notin S_0\} \to s\,[l] \in S_1)) \\[2ex] H_b(j) & \text{if } s \in S_0 - S_1 \wedge b = \min\{i : s\,[i+1] \notin S_1\}) \\[2ex] H_b(j) & \text{if } s \notin S_0 \wedge \exists l\,(s\,[l] \in S_0 - S_1) \wedge \forall i\,(t * \langle i \rangle \in T_0) \\ & \text{and } b = \min\{i : s\,[i+1] \notin S_0\} \\[2ex] k & \text{if } s \notin S_0 \wedge \exists l\,(s\,[l] \in S_0 - S_1) \wedge \exists i\,(t * \langle i \rangle \notin T_0) \\ & \text{and } b = \min\{i : t * \langle i \rangle \notin T_0\} \end{cases}$$

Observe that σ_{II} formalizes the strategy described in case 1 of the proof of Lemma 2.13. It is straightforward to check that it is winning strategy for player II.

<u>Case 2</u>: $[\delta(T_0, T_1)] = \emptyset$ but $[\delta(S_0, S_1)] \neq \emptyset$.
Thus, Weak König Lemma implies that $\delta(T_0, T_1)$ is a nonempty finite binary tree. On the other hand, let $f_0 \in [\delta(S_0, S_1)]$. Then,

$$\forall k\, \exists f\, (f_0[k] \subset f \wedge f \in [\delta\,(S_0)] - [S_1])$$

By Δ^0_1-comprehension, there exists $h_0 : \mathbb{N} \to 2^{<\mathbb{N}}$ such that

$$\forall k\, (f[k] \subseteq h_0(k) \wedge h_0(k) \in \delta\,(S_0) - S_1)$$

As in case A, using $\Sigma^0_1 \wedge \Pi^0_1$ choice, we define H to be the set satisfying the following condition

$$\forall k\, ((H)_k \in 2^{\mathbb{N}} \wedge h_0(k) \subset (H)_k \wedge (H)_k \in [\delta\,(S_0)] \wedge (H)_k \notin [S_1]).$$

We write $H_k(n) = i$ for $(k, n, i) \in H$. Then player I wins the game with the following strategy, σ_{I}:
First $\sigma_{\mathrm{I}}(\langle\rangle) = f_0(0)$ and given $s, t \in 2^{<\mathbb{N}}$, with $|s| = |t| = j \geq 1$, let us take $b = \min\{|t'| : t' \subseteq t \wedge t' \notin \delta(T_0, T_1)\}$ and define

$$\sigma_{\mathrm{I}}(s \otimes t) = \begin{cases} f_0(j) & \text{if } t \in \delta(T_0, T_1) \\ H_b(j) & \text{if } t \notin \delta(T_0, T_1) \wedge \exists g \in [\delta_{T_0} T_1]\,(t\,[b] \subset g) \wedge t \notin T_0 - T_1 \\ H_b(j) & \text{if } t \notin \delta(T_0, T_1) \wedge \exists g \in [\delta_{T_0} T_1]\,(t\,[b] \subset g) \wedge t \in T_0 - T_1 \wedge \\ & \exists t'\,(t \subseteq t' \wedge t' \notin T_0) \\ H_b(j) & \text{if } t \notin \delta(T_0, T_1) \wedge \exists g \in [\delta_{T_0} T_1]\,(t\,[b] \subset g) \wedge t \in T_0 - T_1 \wedge \\ & \forall t'\,(t \subseteq t' \to t' \in T_0) \wedge \forall i\,(s * \langle i \rangle \in S_0) \\ k & \text{if } t \notin \delta(T_0, T_1) \wedge \exists g \in [\delta_{T_0} T_1]\,(t\,[b] \subset g) \wedge t \in T_0 - T_1 \wedge \\ & \forall t'\,(t \subseteq t' \to t' \in T_0) \wedge \exists i\,(s * \langle i \rangle \notin S_0) \\ & \text{and } k = \min\{i :\ s * \langle i \rangle \notin S_0\} \\ f_0(j) & \text{if } t \notin \delta(T_0, T_1) \wedge \neg\exists g \in [\delta_{T_0} T_1]\,(t\,[b] \subset g) \wedge t \notin T_0 - T_1 \wedge \\ & \exists t' \in T_0\,(t \subseteq t' \wedge t' \notin T_1) \\ H_c(j) & \text{if } t \notin \delta(T_0, T_1) \wedge \neg\exists g \in [\delta_{T_0} T_1]\,(t\,[b] \subset g) \wedge t \notin T_0 - T_1 \wedge \\ & \forall t' \in T_0\,(t \subseteq t' \to t' \in T_1) \\ & \text{and } c = \min\{|t'| : t' \subseteq t \wedge \forall t'' \in T_0\,(t' \subseteq t'' \to t'' \in T_1)\} \\ f_0(j) & \text{if } t \notin \delta(T_0, T_1) \wedge \neg\exists g \in [\delta_{T_0} T_1]\,(t\,[b] \subset g) \wedge t \in T_0 - T_1 \wedge \\ & \exists t'\,(t \subseteq t' \wedge t' \notin T_0) \\ H_d(j) & \text{if } t \notin \delta(T_0, T_1) \wedge \neg\exists g \in [\delta_{T_0} T_1]\,(t\,[b] \subset g) \wedge t \in T_0 - T_1 \wedge \\ & \forall t' \in T_0\,(t \subseteq t' \to t' \in T_0 - T_1) \\ & \text{and } d = \min\{|t'| : t' \subseteq t \wedge \forall t'' \in T_0\,(t' \subseteq t'' \to t'' \in T_0 - T_1)\} \end{cases}$$

Case 3: $[\delta(T_0, T_1)] = \emptyset$ and $[\delta(S_0, S_1)] = \emptyset$.
Then $G_L(A, B)$ is determined by Lemma 4.36.

(2): Let $A(f), B(f) \in \left(\Sigma^0_1\right)_2$. We must show that the game $G_W(A, B)$ is determined. Since $A(f)$ and $B(f)$ are differences of closed sets, there exist, by Proposition 4.7, binary pruned trees S_0, S_1, T_0, and T_1 such that

1. $S_1 \subseteq S_0$ and $T_1 \subseteq T_0$.

2. $A(f) \leftrightarrow f \in [S_0] - [S_1]$, and $B(g) \leftrightarrow g \in [T_0] - [T_1]$.

We distinguish the same cases as in the proof of Lipschitz determinacy:

Case 1: $[\delta(T_0, T_1)] \neq \emptyset$.

The proof of this case is the same as the corresponding Lipschitz case since for all $A(f), B(f) \in \left(\Sigma^0_1\right)_2$ a winning strategy for player II in $G_L(A, B)$ yields a winning strategy for player II in $G_W(A, B)$.

Case 2: $[\delta(T_0, T_1)] = \emptyset$ but $[\delta(S_0, S_1)] \neq \emptyset$.

This case deserves some explanation because player II can pass. By Weak König Lemma $\delta(T_0, T_1)$ is a nonempty finite binary tree which implies that player II soon or later will play outside the tree $\delta(T_0, T_1)$, but she can delay this a finite number of times. On the other hand, let $f_0 \in [\delta(S_0, S_1)]$. Then,

$$\forall k \, \exists f \, (f_0[k] \subset f \wedge f \in [\delta\,(S_0)] - [S_1])$$

By Δ^0_1-comprehension, there exists $h_0 : \mathbb{N} \to 2^{<\mathbb{N}}$ such that

$$\forall k \, (f[k] \subseteq h_0(k) \wedge h_0(k) \in \delta\,(S_0) - S_1)$$

As in case A we define the corresponding set H and write $H_k(n) = i$ for $(k, n, i) \in H$. Thus there exists a sequence of functions, $\{H_k : k \in \mathbb{N}\}$, such that each H_k extends the finite sequence $h(k)$ to a path in $[S_0]$ and in the boundary of $[S_0]$, leaving player II the possibility of playing still inside or outside $[S_0]$. Let $ext : 2^{<\mathbb{N}} \to 2^{<\mathbb{N}}$ such that $ext(s)$ is the finite sequence obtained by dropping the zeros of the finite sequence s and decreasing the values by 1. This sequence is needed since in a Wadge game player II is allowed to pass and we identify passing with picking zero and we identify picking i with picking $i+1$. Then I wins the game with the following strategy, σ_{I}:

First $\sigma_{\mathrm{I}}(\langle\rangle) = f_0(0) + 1$ and given $s, t \in 2^{<\mathbb{N}}$, with $|s| = |t| = j \geq 1$, let us take

$b = \min\{|t'| : t' \subseteq t \wedge t' \in \delta(T_0, T_1)\}$ and define

$$\sigma_{\mathrm{I}}(s \otimes t) = \begin{cases} f_0(j)+1 & \text{if } ext(t) \in \delta(T_0,T_1) \\ H_b(j)+1 & \text{if } ext(t) \notin \delta(T_0,T_1) \wedge \exists g \in [\delta_{T_0}T_1]\,(ext(t)\,[b] \subset g) \wedge ext(t) \notin T_0 - T_1 \\ H_b(j)+1 & \text{if } ext(t) \notin \delta(T_0,T_1) \wedge \exists g \in [\delta_{T_0}T_1]\,(ext(t)\,[b] \subset g) \wedge ext(t) \in T_0 - T_1 \wedge \\ & \exists t'\,(ext(t) \subseteq t' \wedge t' \notin T_0) \\ H_b(j)+1 & \text{if } ext(t) \notin \delta(T_0,T_1) \wedge \exists g \in [\delta_{T_0}T_1]\,(ext(t)\,[b] \subset g) \wedge ext(t) \in T_0 - T_1 \wedge \\ & \forall t'\,(ext(t) \subseteq t' \rightarrow t' \in T_0) \wedge \forall i\,(s * \langle i \rangle \in S_0) \\ k+1 & \text{if } ext(t) \notin \delta(T_0,T_1) \wedge \exists g \in [\delta_{T_0}T_1]\,(ext(t)\,[b] \subset g) \wedge ext(t) \in T_0 - T_1 \wedge \\ & \forall t'\,(ext(t) \subseteq t' \rightarrow t' \in T_0) \wedge \forall i\,(s * \langle i \rangle \notin S_0) \\ & \text{and } k = \{i :\ s * \langle i \rangle \notin S_0\} \\ f_0(j)+1 & \text{if } ext(t) \notin \delta(T_0,T_1) \wedge \neg\exists g \in [\delta_{T_0}T_1]\,(ext(t)\,[b] \subset g) \wedge ext(t) \notin T_0 - T_1 \wedge \\ & \exists t' \in T_0\,(ext(t) \subseteq t' \wedge t' \notin T_1) \\ H_c(j)+1 & \text{if } ext(t) \notin \delta(T_0,T_1) \wedge \neg\exists g \in [\delta_{T_0}T_1]\,(ext(t)\,[b] \subset g) \wedge ext(t) \notin T_0 - T_1 \wedge \\ & \forall t' \in T_0\,(t \subseteq t' \rightarrow t' \in T_1) \\ & \text{and } c = \min\{|t'| : t' \subseteq ext(t) \wedge \forall t'' \in T_0\,(t' \subseteq t'' \rightarrow t'' \in T_1)\} \\ f_0(j)+1 & \text{if } ext(t) \notin \delta(T_0,T_1) \wedge \neg\exists g \in [\delta_{T_0}T_1]\,(ext(t)\,[b] \subset g) \wedge ext(t) \in T_0 - T_1 \wedge \\ & \exists t'\,(ext(t) \subseteq t' \wedge t' \notin T_0) \\ H_d(j) & \text{if } ext(t) \notin \delta(T_0,T_1) \wedge \neg\exists g \in [\delta_{T_0}T_1]\,(ext(t)\,[b] \subset g) \wedge ext(t) \in T_0 - T_1 \wedge \\ & \forall t' \in T_0\,(ext(t) \subseteq t' \rightarrow t' \in T_0 - T_1) \\ & \text{and } d = \min\{|t'| : t' \subseteq ext(t) \wedge \forall t'' \in T_0\,(t' \subseteq t'' \rightarrow t'' \in T_0 - T_1)\} \end{cases}$$

<u>Case 3</u>: $[\delta(T_0, T_1)] = \emptyset$ and $[\delta(S_0, S_1)] = \emptyset$.
Then $G_W(A, B)$ is determined by Lemma 4.36. ■

Corollary 4.38

1. $\mathbf{ACA}_0$ *proves* $\left(\Sigma^0_1\right)_2$-$\mathbf{SLO}^*_L$.
2. $\mathbf{ACA}_0$ *proves* $\left(\Sigma^0_1\right)_2$-$\mathbf{SLO}^*_W$.

Again, we observe that it follows from the proof of part 2 of Theorem 4.37 that it is provable in $\mathbf{ACA}_0$ that the nontrivial $\left(\Sigma^0_1\right)_2$-sets form a Wadge degree. That is to say, write $A \equiv_W B$ to denote the formula $Red^{\star}_W(A, B) \wedge Red^{\star}_W(B, A)$. Then, we have

Proposition 4.39 *It is provable in* $\mathbf{ACA}_0$ *that if* S, S', T, *and* T' *are binary trees such that* $[S]-[S']$ *and* $[T]-[T']$ *are neither open nor closed, then* $[S]-[S'] \equiv_W [T]-[T']$.

As above write $A <_L B$ to denote the formula $Red^\star_L(A,B) \wedge \neg Red^\star_L(B,A)$ and write $A \equiv_L B$ to denote the formula $Red^\star_L(A,B) \wedge Red^\star_L(B,A)$. Then, we have

Proposition 4.40 *It is provable in* $\mathbf{ACA}_0$ *that there exists a sequence of pairs of binary trees* $\{(T_k, T'_k) : k \in \mathbb{N}\}$, *such that*

1. *for each* $k \in \mathbb{N}$, $[T_k]-[T'_k]$ *is neither open nor closed,*
2. *for each* $k \in \mathbb{N}$, $[T_k]-[T'_k] <_L [T_{k+1}] - \left[T'_{k+1}\right]$, *and*
3. *for each pair of binary trees* (S,S') *such that* $[S]-[S']$ *is neither open nor closed, there exists* $k \in \mathbb{N}$ *such that* $[S]-[S'] \equiv_L [T_k]-[T'_k]$.

Proof. We work in an arbitrary model of $\mathbf{ACA}_0$. Define binary trees T_0 and T'_0 by

$$t \in T_0 \leftrightarrow \begin{cases} \exists m \left(\langle 0\rangle * 0^{(m)} * \langle 1\rangle * \langle 0\rangle \subseteq t\right) \vee \exists m \left(\langle 1\rangle * 0^{(m)} * \langle 1\rangle * \langle 0\rangle \subseteq t\right) \\ \vee \exists m \left(t \subseteq 0^{(m)}\right) \vee \exists m \left(t \subseteq \langle 1\rangle * 0^{(m)}\right) \end{cases}$$

and

$$t \in T'_0 \leftrightarrow \exists m \left(t \subseteq 0^{(m)}\right).$$

Now assuming that the pair (T_k, T'_k) has already been defined we define T_{k+1} and T'_{k+1} by

$$t \in T_{k+1} \leftrightarrow \exists t' \left(t \subseteq \langle 0\rangle * t' \wedge t' \in T_k\right)$$

and

$$t \in T'_{k+1} \leftrightarrow \exists m \left(t \subseteq 0^{(m)}\right).$$

It is clear that such a sequence exists by Σ^0_1-comprehension and it is easy to see that each $[T_k]-[T'_k]$ defines a set that satisfies property 1. Finally, by inspection of the proof of part 1 of Theorem 4.37, it follows that properties 2 and 3 above hold too. ■

4.5 A reversal for $\mathbf{ACA}_0$

We close this chapter with one of the main results of this thesis. Namely, a reversal for $\mathbf{ACA}_0$ which calibrates the exact strength in terms of Reverse Mathematics of Lipschitz determinacy and semilinear ordering principle for $(\Sigma^0_1)_2$ sets.

Theorem 4.41 *It is provable in* $\mathbf{RCA}_0$ *that* $(\Sigma^0_1)_2$-$\mathbf{SLO}^*_L$ *implies* $\mathbf{ACA}_0$.

Proof. Reasoning in $\mathbf{RCA}_0$, assume $(\Sigma^0_1)_2$-$\mathbf{SLO}^*_L$. Take $\varphi(x) \in \Sigma^0_1$ (we disregard parameters). We must show that the set $\{x : \varphi(x)\}$ exists. To this end, define $A(f)$ and $B(g)$ to be

$$\exists k\,(f(k) = 1 \wedge \forall k' < k\, f(k') = 0)$$

and

$$\exists k\,(g(k) = 1 \wedge \forall k' < k\, g(k') = 0 \wedge \forall i \leq k\,(g(k+i+1) = 1 \rightarrow \varphi(i)) \wedge$$
$$\forall k\,(g(k) = 1 \wedge \forall k' < k\, g(k') = 0 \rightarrow \forall i \leq k\,(\varphi(i) \rightarrow g(k+i+1) = 1)),$$

respectively. That is to say, a play for player I is in A if it is of the form

$$\mathrm{I}:\ 0^{(k)} * \langle 1 \rangle * f'$$

for some $k \in \mathbb{N}$ and $f' \in 2^{\mathbb{N}}$. On the other hand, a play for player II is in B if it is of the form

$$\mathrm{II}:\ 0^{(l)} * \langle 1 \rangle * \langle t_0,\, t_1\, \ldots\, t_l \rangle * g'$$

for some $g' \in 2^{\mathbb{N}}$ and for each $i \leq l$, $t_i = 1$ iff $\varphi(i)$ holds.

It is clear that A and B are in $(\Sigma^0_1)_2$.

We claim that

- Player II cannot have a winning strategy in the game $G_L(\neg B, A)$.

  We must show that $\forall \sigma_{\mathrm{II}}\, \exists \sigma_{\mathrm{I}}\, \neg(\neg B(\sigma_{\mathrm{I}} \otimes^{\mathrm{I}} \sigma_{\mathrm{II}}) \leftrightarrow A(\sigma_{\mathrm{I}} \otimes^{\mathrm{II}} \sigma_{\mathrm{II}}))$. Consider any strategy for player II, σ_{II}. We distinguish two cases.

  Case 1: For all $k \in \mathbb{N}$, $\sigma_{\mathrm{II}}(0^{(2k+1)}) = 0$.

  It suffices to consider the strategy for player I given by $\sigma_{\mathrm{I}}(s) = 0$ for all $s \in$ $\mathrm{Seq}_{\mathrm{even}}$. Then, $(\sigma_{\mathrm{I}} \otimes^{\mathrm{I}} \sigma_{\mathrm{II}})(i) = (\sigma_{\mathrm{I}} \otimes^{\mathrm{II}} \sigma_{\mathrm{II}})(i) = 0$ for all $i \in \omega$. Hence, both $\neg B(\sigma_{\mathrm{I}} \otimes^{\mathrm{I}} \sigma_{\mathrm{II}})$ and $\neg A(\sigma_{\mathrm{I}} \otimes^{\mathrm{II}} \sigma_{\mathrm{II}})$ hold. Thus, $\neg(\neg B(\sigma_{\mathrm{I}} \otimes^{\mathrm{I}} \sigma_{\mathrm{II}}) \leftrightarrow A(\sigma_{\mathrm{I}} \otimes^{\mathrm{II}} \sigma_{\mathrm{II}}))$.

  Case 2: There is some $k \in \mathbb{N}$ such that $\sigma_{\mathrm{II}}(0^{(2k+1)}) = 1$. Let k_0 denote a minimal such element.

  By bounded Σ^0_1-comprehension (available in $\mathbf{RCA}_0$), there exists $C = \{x : x \leq k_0 \wedge \varphi(x)\}$. Let σ_{I} be any strategy for player I satisfying that

  $\sigma_{\mathrm{I}}(0^{(2m)}) = 0$, for each $m \leq k$

  $\sigma_{\mathrm{I}}(0^{(2k+1)} * \langle 1 \rangle) = 1$

  $\sigma_{\mathrm{I}}(s) = \begin{cases} 1 & \text{if } i \in C \\ 0 & \text{if } i \notin C \end{cases}$ if $|s| = 2k_0 + 4 + 2i$ with $i \leq k_0$.

  Using Δ^0_0-induction, we obtain that

  $$\forall x \leq k\,((\sigma_{\mathrm{I}} \otimes^{\mathrm{I}} \sigma_{\mathrm{II}})(k+i+1) = 1 \leftrightarrow x \in C).$$

  Then, both $B(\sigma_{\mathrm{I}} \otimes^{\mathrm{I}} \sigma_{\mathrm{II}})$ and $A(\sigma_{\mathrm{I}} \otimes^{\mathrm{II}} \sigma_{\mathrm{II}})$ hold. Thus, $\neg(\neg B(\sigma_{\mathrm{I}} \otimes^{\mathrm{I}} \sigma_{\mathrm{II}}) \leftrightarrow A(\sigma_{\mathrm{I}} \otimes^{\mathrm{II}} \sigma_{\mathrm{II}}))$, as required.

  This proves the claim.

Hence by $(\Sigma^0_1)_2$-$\mathbf{SLO}^*_L$, player II must have a winning strategy in $G_L(A,B)$. Let σ_{II} denote a such winning strategy. Pick $k \in \mathbb{N}$. We will use the winning strategy σ_{II} to decide whether or not $\varphi(k)$ holds. To that end, for each $k \in \mathbb{N}$, we consider a strategy for player I, σ^k_{I}, satisfying that

$$\sigma^k_{\mathrm{I}}(\langle x_0, y_0, ..., x_{n-1}, y_{n-1}\rangle) = \begin{cases} 0 & \text{if } n < k \\ 1 & \text{if } n = k \\ 0 & \text{if } n > k \end{cases}$$

That is to say, according to σ^k_{I} player I plays as follows:

$$\mathrm{I}: \; 0^{(k)} * \langle 1 \rangle * \vec{0}$$

It is clear that $A(\sigma^k_{\mathrm{I}} \otimes^{\mathrm{I}} \sigma_{\mathrm{II}})$ holds. So $B(\sigma^k_{\mathrm{I}} \otimes^{\mathrm{II}} \sigma_{\mathrm{II}})$ holds as well, for σ_{II} is a winning strategy for player II in $G_L(A,B)$. Consequently, there exists $l \in \mathbb{N}$ such that

$$g(l) = 1 \wedge \forall l' < l \, g(l') = 0 \wedge \forall i \leq l \, (\varphi(i) \leftrightarrow g(i+l+1) = 1),$$

where $g = \sigma^k_{\mathrm{I}} \otimes^{\mathrm{II}} \sigma_{\mathrm{II}}$.

Now we claim that

- We have $k \leq l$.

  Assume not. Then, $\tau(0^{(2l+1)}) = 1$. By bounded Σ^0_1-comprehension there exists $D = \{x : x \leq l \wedge \varphi(x)\}$. Consider a new strategy for player I, σ'_{I}, given by

$$\sigma'_{\mathrm{I}}(\langle x_0, y_0, ..., x_{n-1}, y_{n-1}\rangle) = \begin{cases} 0 & \text{if } n < l+2 \\ 0 & \text{if } n = l+2 \text{ and } \forall i \leq l \, (i \in D \leftrightarrow y_{l+i+1} \neq 0) \\ 1 & \text{if } n = l+2 \text{ and } \neg\forall i \leq l \, (i \in D \leftrightarrow y_{l+i+1} \neq 0) \\ 0 & \text{if } n > l+2 \end{cases}$$

  In other words, at any stage different from $2l+2$ player I picks 0. At stage $2l+2$ player I picks 0 if player II has played inside B; or 1 otherwise.

  It is clear that σ'_{I} exists by Δ^0_1-comprehension. In addition, it is immediate to see that $A(\sigma'_{\mathrm{I}} \otimes^{\mathrm{I}} \sigma_{\mathrm{II}})$ holds if and only if $B(\sigma'_{\mathrm{I}} \otimes^{\mathrm{I}} \sigma_{\mathrm{II}})$ does not hold, contradicting the fact that σ_{II} is a winning strategy for player II in $G_L(A,B)$.

  This proves our second claim.

Observe that by Δ^0_1-comprehension there exists $S \subseteq \mathrm{Seq}_{\mathrm{even}} \times \mathbb{N} \times \mathbb{N}$ such that $(S)_k = \sigma^k_{\mathrm{I}}$ for each k. Hence, we have

$$\varphi(k) \leftrightarrow \exists k' \begin{cases} (k \leq l \wedge ((S)_k \otimes^{\mathrm{II}} \sigma_{\mathrm{II}})(l) = 1 \wedge \forall l' < l \, ((S)_k \otimes^{\mathrm{II}} \sigma_{\mathrm{II}})(l') = 0) \wedge \\ ((S)_k \otimes^{\mathrm{II}} \sigma_{\mathrm{II}})(k+l+1) = 1 \end{cases}$$

and

$$\varphi(k) \leftrightarrow \exists k' \begin{cases} (k \leq l \wedge ((S)_k \otimes^{\mathrm{II}} \sigma_{\mathrm{II}})(l) = 1 \wedge \forall l' < l \, ((S)_k \otimes^{\mathrm{II}} \sigma_{\mathrm{II}})(l') = 0) \rightarrow \\ ((S)_k \otimes^{\mathrm{II}} \tau)(k + l + 1) = 1 \end{cases}$$

Thus the set $\{x : \varphi(x)\}$ exists by Δ^0_1-comprehension using the winning strategy σ_{II} as a parameter. ■

Corollary 4.42 *The following assertions are pairwise equivalent over* $\mathbf{RCA}_0$*:*

1. $\mathbf{ACA}_0$.
2. $(\Sigma^0_1)_2$-$\mathbf{Det}^*_L$.
3. $(\Sigma^0_1)_2$-$\mathbf{SLO}^*_L$.

Remark 4.43

1. *Nemoto proved* $\mathbf{ACA}_0$ *to be equivalent to* $(\Sigma^0_1)_2$-$\mathbf{Det}^*$ *over* $\mathbf{RCA}_0$*. As a consequence, it follows from Corollary 4.42 that over* $\mathbf{RCA}_0$*, Gale-Stewart determinacy and Lipschitz determinacy for* $(\Sigma^0_1)_2$ *sets in the Cantor space are equivalent principles.*
2. *Andretta proved in Set Theory (actually, in* **ZF** *plus the axiom of dependent choices over the reals and the statement that "every set has the property of Baire") that Borel determinacy and the Borel semilinear ordering principle for Lipschitz games are equivalent. Corollary 4.42 says that when restricted to* $(\Sigma^0_1)_2$ *sets, this equivalence can be proved already in* $\mathbf{RCA}_0$*.*

It seems natural to pose the following questions:

Problem 4.44

1. *Is* $\mathbf{ACA}_0$ *equivalent over* $\mathbf{RCA}_0$ *to* $(\Sigma^0_1)_2$-$\mathbf{Det}^*_W$*?*
2. *Are* $(\Sigma^0_1)_2$-$\mathbf{Det}^*_L$ *and* $(\Sigma^0_1)_2$-$\mathbf{Det}^*_W$ *equivalent over* $\mathbf{RCA}_0$*?*
3. *Are* $(\Sigma^0_1)_2$-$\mathbf{SLO}^*_L$ *and* $(\Sigma^0_1)_2$-$\mathbf{SLO}^*_W$ *equivalent over* $\mathbf{RCA}_0$*?*

Chapter 5

Lipschitz and Wadge games in Baire space

The analysis of determinacy of Lipschitz and Wadge games in Baire space we have presented in Chapter 2 rests on basic properties of well-founded trees and ordinal rank functions associated with them. Therefore, in order to obtain a formalization of these results in a subsystem of second order arithmetic it is natural to aim at subsystems that are strong enough to deal with the basic properties of countable ordinals. One of such subsystems is $\mathbf{ATR}_0$. This system is axiomatized over $\mathbf{ACA}_0$ by a principle stating the existence of sets defined by iterating arithmetical comprehension along countable well orderings. By a result of Friedman (Theorem V.6.8 of [Smp99]), over $\mathbf{RCA}_0$, $\mathbf{ATR}_0$ is equivalent to the principle of comparability of countable well orderings and, as a matter of fact, it encapsulates most of the basic theory of countable ordinals. Thus, we shall begin this chapter by providing a survey of some basic facts on ordinals and well-founded trees that are provable in $\mathbf{ATR}_0$ and that will be needed for the formalization of the proofs of determinacy given from Section 2 onwards. This introductory part (Section 1) ends with the proof that $\mathbf{ATR}_0$ implies that a tree defining a clopen set in Baire space can be pruned. In Chapter 4 we proved a similar result concerning trees that define clopen sets in Cantor space.

In Section 2 we prove one of the main results of this chapter, namely that Lipschitz semilinear order principle for clopen sets in Baire space is equivalent to $\mathbf{ATR}_0$. This reversal is obtained within the subsystem of second order arithmetic $\mathbf{ACA}_0$. As consequence we also have that Δ^0_1-$\mathbf{SLO}_L$ and Δ^0_1-$\mathbf{Det}_L$ are equivalent principles over $\mathbf{ACA}_0$.

The analysis of Lipschitz determinacy of closed sets in Baire space will allow us to improve the reversal obtained in Section 2 by weakening the base theory. In fact, in the next section (Section 3) we prove the main result of the chapter, namely that Lipschitz determinacy for closed sets in Baire space is equivalent to $\mathbf{ATR}_0$ over $\mathbf{RCA}_0$.

In the last section of the present chapter, we formalize the concepts developed in the topological analysis of Chapter 2 concerning Lipschitz determinacy for sets which are differences of two closed sets in Baire space. Finally we obtain the result that subsystem Π^1_1-$\mathbf{CA}_0$ proves Lipschitz determinacy for all sets which are differences of two closed sets.

5.1 Well-founded trees and ranks in second order arithmetic

As usual in second order arithmetic, we identify $\mathbb{N} \times \mathbb{N}$ with a subset of $\mathbb{N}$ using a pairing function $(i, j) = (i + j)^2 + i$. Thus, a binary relation R on $\mathbb{N}$ is identified with a subset of $\mathbb{N} \times \mathbb{N}$.

Let us observe that, given a binary relation R on $\mathbb{N}$, we cannot assume, working over $\mathbf{RCA}_0$, the existence (as sets) of the domain or the range of R. To deal with this difficulty in $\mathbf{RCA}_0$ an ordering is defined to be a *reflexive* relation (of course, satisfying other additional properties). Working in $\mathbf{RCA}_0$ we make the following definitions:

Let $R \subseteq \mathbb{N} \times \mathbb{N}$. We say that R is *reflexive* if

$$\forall i \, \forall j \, [(i, j) \in R \to ((i, i) \in R \land (j, j) \in R)].$$

If R is reflexive then, by Δ^0_0–comprehension, there exists a set

$$\text{field}(R) = \{i : \ (i, i) \in R\}.$$

We also write

$$i \leq_R j \leftrightarrow (i, j) \in R$$
$$i <_R j \leftrightarrow (i, j) \in R \land (j, i) \notin R.$$

Definition 5.1 *The following definitions are made in* $\mathbf{RCA}_0$*. Let R be a reflexive binary relation.*

1. *We say that R is well-founded if there is no $f : \mathbb{N} \to \text{field}(R)$ such that*

$$\forall n \, (f(n + 1) <_R f(n)) \, .$$

2. *We say that R is a countable linear ordering if it is a reflexive ordering over its field, i.e. R is a reflexive, symmetric, transitive, and total relation.*

3. *We say that R is a countable well ordering if it is a countable linear ordering and it is well-founded.*

Observe that there is an arithmetical formula $\text{LO}(X)$ expressing that X is a countable linear ordering. It can be easily checked that there exist Π^1_1 formulas $\text{WF}(X)$ and $\text{WO}(X)$ (with a single free variable X) expressing, respectively, that X is a well-founded (reflexive) relation and X is a countable well ordering.

A useful principle closely related to the notion of countable well ordering is the principle of transfinite induction. This principle is available in $\mathbf{ACA}_0$ for arithmetical formulas:

Lemma 5.2 (Arithmetical transfinite induction) *For each arithmetical formula $\varphi(u)$,* $\mathbf{ACA}_0$ *proves*

$$\text{WO}(X) \land \forall v \, (\forall u (u <_X v \to \varphi(u)) \to \varphi(v)) \to \forall v \varphi(v)$$

The system $\mathbf{ATR}_0$ is axiomatized over $\mathbf{ACA}_0$ by the following principle of Arithmetical Transfinite Recursion. We need some notation for a complete description of this principle.

Let $\theta(u, Y)$ be any formula. We define $H_\theta(X, Y)$ to be the formula

$$\mathrm{LO}(X) \wedge \forall z\, (z \in Y \leftrightarrow \exists u,\, j \leq z\, (z = (u, j) \wedge j \in \mathrm{field}(X) \wedge \theta(u, Y^j)))$$

where $Y^j = \{(m, i) : i <_X j \wedge (m, i) \in Y\}$.

Using the normal form theorem it can be shown that if $\theta(u, Y)$ is arithmetical then $H_\theta(X, Y)$ is also an arithmetical formula.

Definition 5.3 *The system* $\mathbf{ATR}_0$ *is axiomatized over* $\mathbf{ACA}_0$ *by the scheme:*

$$\forall X\ (\mathrm{WO}(X) \to \exists Y\, H_\theta(X, Y))$$

where $\theta(u, Y)$ *is arithmetical.*

We shall use Greek letters α, β, $\gamma, \ldots$ to denote countable well orderings. If α is a well ordering then $\alpha + 1$ denotes a well ordering obtained from α by adding an upper bound as follows:

$$\alpha + 1 = \{(2m, 2n) : (m, n) \in \alpha\} \cup \{(1, 1)\} \cup \{(2m, 1) : m \in \mathrm{field}(\alpha)\}.$$

Let us now introduce two natural comparability notions between ordinals that turn out to be equivalent to Arithmetical Transfinite Recursion.

Definition 5.4 *The following definition is made in* $\mathbf{RCA}_0$*. Let* α *and* β *be countable well orderings. We say that* α *is weakly less than or equal to* β*,* $\alpha \leq_w \beta$*, if there is an injection* $f : \mathrm{field}(\alpha) \to \mathrm{field}(\beta)$ *such that*

$$\forall i, j \in \mathrm{field}(\alpha)\ (i \leq_\alpha j \leftrightarrow f(i) \leq_\beta f(j))\,.$$

We write $\alpha <_w \beta$ *if* $\alpha + 1 \leq_w \beta$*.*

Definition 5.5 *The following definition is made in* $\mathbf{RCA}_0$*. Let* α *and* β *be countable well orderings. We say that* α *is strongly less than or equal to* β*,* $\alpha \leq_s \beta$*, if there is* $f : \mathrm{field}(\alpha) \to \mathrm{field}(\beta)$ *such that:*

1. $\forall i, j \in \mathrm{field}(\alpha)\ (i \leq_\alpha j \leftrightarrow f(i) \leq_\beta f(j))$*, and*
2. f *is bijective or, there exists* $k \in \mathrm{field}(\beta)$ *such that* f *is a bijection from* $\mathrm{field}(\alpha)$ *onto the initial segment determined by* k *in* β*, i.e the set* $\{i \in \mathrm{field}(\beta) : i <_\beta k\}$*.*

We write $\alpha <_s \beta$ *if* $\alpha + 1 \leq_s \beta$*.*

Theorem 5.6 ([Hir01]) *Over* $\mathbf{RCA}_0$ *the following principles are equivalent:*

1. $\mathbf{ATR}_0$.
2. $\forall \alpha, \beta\ (\alpha \leq_s \beta \vee \beta \leq_s \alpha)$.
3. $\forall \alpha, \beta\ (\alpha \leq_w \beta \vee \beta \leq_w \alpha)$.

In what follows we shall see that $\mathbf{ATR}_0$ is strong enough to prove the basic results on well-founded trees and ordinal rank functions. Our exposition here follows [Hir00] and Section 2 of [GM08]. The following definitions are made in $\mathbf{RCA}_0$.

Definition 5.7 *A tree* $T \subseteq \mathbb{N}^{\mathbb{N}}$ *is well-founded if it has no path.*

Let us observe that for each tree T, the reverse inclusion $\supseteq$ defines a reflexive binary relation on T and T is a well-founded tree if and only if $\supseteq$ is a well-founded relation.

Definition 5.8 *Let* $S, T \subseteq \mathbb{N}^{\mathbb{N}}$ *be trees. We shall write* $S \preceq T$ *if there is a function* $f : S \to T$ *such that*

$$\forall s_1, s_2 \in S\ (s_1 \subset s_2 \to f(s_1) \subset f(s_2)).$$

Definition 5.9 *Let* T *be a tree. A rank function for* T *is a pair* (rk, α) *where* α *is a countable well ordering and* $\mathrm{rk} : T \to \mathrm{field}(\alpha)$, *is a function such that* $\alpha = \mathrm{rk}(\langle\rangle) + 1$ *and for every* $t \in T$,

$$\mathrm{rk}(t) = \sup\{\mathrm{rk}(s) + 1 : t \subset s \wedge |s| = |t| + 1\}.$$

We say that T *is a ranked tree if there exists some rank function for* T.

The following basic properties of rank functions can be proved in $\mathbf{RCA}_0$. In particular, from part 2 in the next proposition we see that $\mathbf{RCA}_0$ essentially proves uniqueness of rank functions.

Proposition 5.10 *Let* T *be a tree. The following is provable in* $\mathbf{RCA}_0$.

1. *If* (rk, α) *is a rank function for* T, *then*

$$\forall t_1, t_2 \in T\, (t_1 \subset t_2 \to \mathrm{rk}(t_2) <_\alpha \mathrm{rk}(t_1)).$$

2. *If* (rk_1, α) *and* (rk_2, β) *are rank functions for* T, *then there is an order preserving bijection* $h : \mathrm{field}(\alpha) \to \mathrm{field}(\beta)$ *such that for all* $t \in T$, $\mathrm{rk}_1(t) = h(\mathrm{rk}_2(t))$.

Proof. We reason in $\mathbf{RCA}_0$.
(1): Let $\psi(x) \in \Pi^0_1$ be the formula

$$\forall t_2 \in T\,(|t_2| = x \to \forall t_1 \in T\,(t_1 \subset t_2 \to \mathrm{rk}(t_2) <_\alpha \mathrm{rk}(t_1))).$$

Then by Π^0_1–induction we obtain that $\forall x\, \psi(x)$:
Obviously $\psi(0)$ holds. Let us assume that $\psi(x)$ holds and let $t_1, t_2 \in T$ be such that $t_1 \subset t_2$ and $|t_2| = x+1$. If $|t_1| = x$ then, since rk is a rank function, we have

$$\mathrm{rk}(t_2) <_\alpha \mathrm{rk}(t_2) + 1 \leq_\alpha \mathrm{rk}(t_1).$$

In other case, there exists $t_0 \in T$ such that $t_1 \subset t_0 \subset t_2$ and $|t_0| = x$. By induction hypothesis, $\mathrm{rk}(t_0) <_\alpha \mathrm{rk}(t_1)$ and, since rk is a rank function, we have

$$\mathrm{rk}(t_2) <_\alpha \mathrm{rk}(t_2) + 1 \leq_\alpha \mathrm{rk}(t_0) <_\alpha \mathrm{rk}(t_1).$$

Thus, $\psi(x+1)$ holds, as required.

(2): Let us define $h : \mathrm{field}(\alpha) \to \mathrm{field}(\beta)$ by

$$h(u) = \mathrm{rk}_2(\min\{t \in T :\ \mathrm{rk}_1(t) = u\})$$

for $u \in \mathrm{field}(\alpha)$. Then the function h exists by Δ^0_1–comprehension and it is an order isomorphism from α to β, see theorem 11 of [Hir00] for details. ∎

As a consequence, the following definition makes sense.

Definition 5.11 *Let T be a ranked tree and (rk, α) a rank function for T. We define the rank of T as $\mathrm{rk}(T) = \mathrm{rk}(\langle\rangle)$.*

Remark 5.12 *The following family of trees provides us with natural examples of ranked trees with a given rank. It will appear again in the proof of a reversal for Δ^0_1-$\mathbf{SLO}_L$ and $\mathbf{ATR}_0$ (see Theorem 5.21).*
We work in $\mathbf{RCA}_0$. For each linear ordering X, we define a tree

$$T(X) = \{s \in \mathrm{Seq}^X :\ \forall i, j < |s|\ (i < j \to (s)_j <_X (s)_i)\}.$$

Then $\mathbf{RCA}_0$ can prove that, for each X such that $LO(X)$, we have:

1. *$WO(X) \leftrightarrow T(X)$ is well-founded.*
2. *For every countable well ordering α, $T(\alpha)$ is ranked and has rank α.*
   Indeed a rank function for $T(\alpha)$ is $\mathrm{rk} : T(\alpha) \to \alpha + 1$, defined by

$$\mathrm{rk}(s) = \begin{cases} \alpha & \text{if } s = \langle\rangle \\ (s)_l & \text{if } |s| = l+1 \end{cases}$$

It can be easily checked that (in $\mathbf{RCA}_0$) every ranked tree is well-founded. The converse can be derived in $\mathbf{ATR}_0$. As a matter of fact a stronger result holds (this is, essentially, Theorem 7 of [Hir00]):

Theorem 5.13 *The following assertions are equivalent over* $\mathbf{RCA}_0$*:*

1. $\mathbf{ATR}_0$.
2. *Every well-founded tree is ranked.*

Rank functions provide a powerful tool in the study of immersions between well-founded trees and they will be crucial in our research on determinacy of Lipschitz games between clopen sets of the Baire space.

Lemma 5.14 *Let* $S, T \subseteq \mathbb{N}^{<\mathbb{N}}$ *be ranked trees. Then*

1. *The following is provable in* $\mathbf{RCA}_0$. *If* $\mathrm{rk}(S) \leq_w \mathrm{rk}(T)$ *then* $S \preceq T$.
2. *The following is provable in* $\mathbf{ACA}_0$. *If* $S \preceq T$ *then* $\mathrm{rk}(S) \leq_w \mathrm{rk}(T)$.

Proof. See Lemma 3.6 and Lemma 3.7 in [GM08]. ■

As an application we shall use the machinery we have just developed in a proof of a result of existence of pruned trees. In Chapter 4 we have shown that in $\mathbf{ACA}_0$, every closed set in the Cantor space is the set of paths of some pruned binary tree. A similar result can be proved in $\mathbf{WKL}_0$ for clopen sets. In the Baire space, it is known that Π^1_1-$\mathbf{CA}_0$ is equivalent to the fact that every closed set coincides with the set of paths of some pruned tree. Now we shall see that the corresponding result for clopen sets can be proved in $\mathbf{ATR}_0$.

Proposition 5.15 *The following is provable in* $\mathbf{ATR}_0$. *Let* $\varphi(f) \in \Sigma^0_1$ *and* $\psi(f) \in \Pi^0_1$ *such that* $\forall f \in \mathbb{N}^{\mathbb{N}} (\varphi(f) \leftrightarrow \psi(f))$. *Then there exists a pruned tree* T *satisfying that* $[T] = \{f \in \mathbb{N}^{\mathbb{N}} : \varphi(f)\}$.

Proof. Let S_1 and S_2 be trees such that

$$[S_1] = \{f \in \mathbb{N}^{\mathbb{N}} : \neg\varphi(f)\} \quad \text{and} \quad [S_2] = \{f \in \mathbb{N}^{\mathbb{N}} : \psi(f)\}.$$

Then $S_1 \cap S_2$ is a well-founded tree and, by Theorem 5.13, it is ranked. Let (rk, α) be a rank function for $S_1 \cap S_2$ and let us consider the following formula $\theta(s, Y)$

$$s \in S_2 \wedge \forall i \in \mathrm{field}(\alpha) \, (\exists z \, ((z, i) \in Y) \rightarrow \exists k \, (s * \langle k \rangle, i) \in Y)).$$

Since α is a countable well ordering, by $\mathbf{ATR}_0$, there exists a set Y such that $H_\theta(\alpha, Y)$ and, as a consequence,

$$\forall u \, \big(u \in Y \leftrightarrow \exists z, i \, (u = (s, j) \wedge j \in \mathrm{field}(\alpha) \wedge \theta(s, Y^j)\big)$$

where $Y^j = \{(s, i) : i \in \mathrm{field}(\alpha) \wedge i <_\alpha j \wedge (s, i) \in Y\}$. For each $j \in \mathrm{field}(\alpha)$, let us define $Y_j = \{s : \theta(s, Y^j)\}$. Then

1. $\forall j \in \text{field}(\alpha)\,(Y_j \subseteq S_2 \wedge Y_j$ is a tree).

   Indeed, $s \in Y_j \leftrightarrow \theta(s, Y^j)$, and therefore $s \in Y_j \rightarrow s \in S_2$. The second assertion follows form the following one, that we prove by transfinite induction,

$$\forall i \in \text{field}(\alpha)\,\forall t, s\,\left(\theta(s, Y^j) \wedge t \subseteq s \rightarrow \theta(t, Y^j)\right).$$

   Let $j \in \text{field}(\alpha)$ and let us assume that

$$\forall i <_\alpha j\,\left(\forall t, s\,\left(\theta(s, Y^i) \wedge t \subseteq s \rightarrow \theta(t, Y^i)\right)\right).$$

   If $\theta(t, Y^j) \wedge t \subseteq s$ then $s \in S_2$ and, since S_2 is a tree, $t \in S_2$. Let $i \in \text{field}(\alpha)$ such that $\exists z\,((z, i) \in Y^j)$. Then $i <_\alpha j$ and $\exists k\,((s * \langle k \rangle, i) \in Y)$. But, by the very definition of the set Y, $(s * \langle k \rangle, i) \in Y \rightarrow \theta(s * \langle k \rangle, Y^i)$, and, therefore, by induction hypothesis, $\theta(t * \langle k \rangle, Y^i)$ (recall that $t * \langle k \rangle \subseteq s * \langle k \rangle$). Thus $\theta(t, Y^j)$, as required.

2. For all $j \in \text{field}(\alpha)$, $Y_j \neq \emptyset$ and

$$\forall s\,\left(s \in Y_j \leftrightarrow ((s \in S_1 \cap S_2 \wedge \text{rk}(s) \geq j) \vee \exists f \in [S_2]\,(s \subseteq f))\right). \qquad (\dagger)$$

   We prove this by transfinite induction. Let us assume that for all $i <_\alpha j$, $Y_i \neq \emptyset$ and Y_i satisfies the corresponding equivalence

$$\forall s\,\left(s \in Y_i \leftrightarrow ((s \in S_1 \cap S_2 \wedge \text{rk}(s) \geq i) \vee \exists f \in [S_2]\,(s \subseteq f))\right).$$

   Let $s \in S_2$. Then, since we are assuming that $\forall i <_\alpha j\,(Y_i \neq \emptyset)$, we get

$$s \in Y_j \leftrightarrow \theta(s, Y^j) \leftrightarrow \forall i <_\alpha j\,\exists k\,(s * \langle k \rangle \in Y).$$

   $(\leftarrow)$: Given $s \in S_2$, we distinguish two cases:

   - If there exists $f \in [S_2]$ such that $s \subseteq f$, then, taking $k = f(|s|)$, we get that $s * \langle k \rangle \subseteq f$ and, by induction hypothesis, $\forall i <_\alpha j\,\exists k\,(s * \langle k \rangle \in Y_i)$. By definition of Y_i, this implies $\forall i <_\alpha j\,\theta(s, Y^i)$ and, so $\forall i <_\alpha j\,((s * \langle k \rangle, i) \in Y)$, as required.
   - If $s \in S_1 \cap S_2$ and $\text{rk}(s) \geq j$, then, using the recursion equations that satisfy rk, we obtain that $\forall i <_\alpha j\,\exists k\,(\text{rk}(s * \langle k \rangle) \geq i)$. By induction hypothesis, it follows that $\forall i <_\alpha j\,\exists k\,(s * \langle k \rangle \in Y_i)$ and we conclude $s \in Y_j$ as in previous case.

   $(\rightarrow)$: Let $s \in Y_j$, then $\forall i <_\alpha j\,\exists k\,((s * \langle k \rangle, i) \in Y)$. By definition of Y_i we get that $\forall i <_\alpha j\,\exists k\,(s * \langle k \rangle \in Y_i)$. By induction hypothesis, it follows that

$$\forall i <_\alpha j\,\exists k\,\left((s * \langle k \rangle \in S_1 \wedge \text{rk}(s * \langle k \rangle) \geq i) \vee \exists f \in [S_2]\,(s * \langle k \rangle \subseteq f)\right)$$

   and as a consequence, if $\exists f \in [S_2]\,(s \subseteq f)$ does not hold then

$$\forall i <_\alpha j\,\exists k\,(s * \langle k \rangle \in S_1 \wedge \text{rk}(s * \langle k \rangle) \geq i),$$

   and it follows that $s \in S_1$ and $\text{rk}(s) \geq j$.

   Finally let us observe that we can easily prove by transfinite induction that for all $j \in \text{field}(\alpha)$, $\{s \in S_1 \cap S_2 \wedge \text{rk}(s) \geq j\} \neq \emptyset$, and thus $Y_j \neq \emptyset$.

Now let $T = \{s \in S_2 : \forall j \in \text{field}(\alpha)\,((s,j) \in Y)\}$. Then, using (†), we see that

$$T = \{s \in S_2 : \forall j \in \text{field}(\alpha)\,(s \in T_j)\} = \{\langle\rangle\} \cup \{s \in S_2 : \exists f \in [S_2]\ (s \subseteq f)\}.$$

So, T is a pruned tree and $[T] = [S_2]$, as required. ■

This result partially completes the first line of the table presented in the previous chapter concerning how much arithmetic is needed to prune a tree:

	Cantor space	Baire space
Clopen case	$\mathbf{WKL_0}$	$\mathbf{ATR_0}$
General case	$\mathbf{ACA_0}$	Π^1_1–$\mathbf{CA_0}$

We now know that $\mathbf{WKL_0}$ and $\mathbf{ATR_0}$ suffice for pruning a tree T defining a clopen set in Cantor space (part 2 of Proposition 4.6) and in Baire space (Proposition 5.15), respectively. However we do not know if these systems are precisely equivalent to the corresponding "pruning tree" assertions.

Concerning the second row we proved in the previous chapter that $\mathbf{ACA_0}$ is equivalent to the assertion stating that every binary tree can be pruned and it is well known that a similar assertion for trees in Baire space is equivalent to Π^1_1-$\mathbf{CA_0}$ (see Lemma VI.4.4 in [Smp99]). Thus, as we have already pointed out in the previous chapter, it remains us to determine whether the same holds for the systems in the first row.

Problem 5.16

1. *Is* $\mathbf{WKL_0}$ *equivalent over* $\mathbf{RCA_0}$ *to the assertion that every binary tree defining a clopen set can be pruned?*
2. *Is* $\mathbf{ATR_0}$ *equivalent over* $\mathbf{RCA_0}$ *to the assertion that every tree defining a clopen set can be pruned?*

5.2 Determinacy for clopen games

By a result of Steel we know that Δ^0_1-$\mathbf{Det}$ can be proved in $\mathbf{ATR_0}$ (and as matter of fact both principles are equivalent over $\mathbf{RCA_0}$, see Theorems V.8.2 and V.8.7 in [Smp99]). As a consequence $\mathbf{ATR_0}$ is strong enough to prove determinacy for clopen Lipschitz games, for a clopen Lipschitz game can be effectively reduced to a clopen Gale-Stewart game. In this section we shall present an alternative proof of determinacy for clopen Lipschitz games based on the topological arguments developed in Chapter 2. We shall obtain also a reversal result (over $\mathbf{ACA_0}$) for $\mathbf{ATR_0}$ and Δ^0_1-$\mathbf{SLO}_L$.

As in the previous chapter let us start with an easy observation. We recall that a set is said to be *trivial* if either it is empty or it is the total set. Determinacy for games with some trivial payoff set is as we already know trivial.

Lemma 5.17 *Let* $Empty(\varphi)$ *denote the formula* $\neg\exists f \in \mathbb{N}^{\mathbb{N}}\varphi(f)$ *and let* $Total(\varphi)$ *denote the formula* $\forall f \in \mathbb{N}^{\mathbb{N}}\varphi(f)$*. The following facts are provable in* $\mathbf{RCA_0}$*.*

1. $Empty(\varphi_1) \wedge \neg Total(\varphi_2) \to Red^{\star}_{L/W}(\varphi_1, \varphi_2)$.

2. $Empty(\varphi_1) \wedge Total(\varphi_2) \to \exists \sigma_{\mathrm{I}} \forall \sigma_{\mathrm{II}} \neg(\varphi_1(\sigma_{\mathrm{I}} \otimes^{\mathrm{I}}_{L/W} \sigma_{\mathrm{II}}) \leftrightarrow \varphi_2(\sigma_{\mathrm{I}} \otimes^{\mathrm{II}}_{L/W} \sigma_{\mathrm{II}}))$.

3. $Total(\varphi_1) \wedge \neg Empty(\varphi_2) \to Red^{\star}_{L/W}(\varphi_1, \varphi_2)$.

4. $Total(\varphi_1) \wedge Empty(\varphi_2) \to \exists \sigma_{\mathrm{I}} \forall \sigma_{\mathrm{II}} \neg(\varphi_1(\sigma_{\mathrm{I}} \otimes^{\mathrm{I}}_{L/W} \sigma_{\mathrm{II}}) \leftrightarrow \varphi_2(\sigma_{\mathrm{I}} \otimes^{\mathrm{II}}_{L/W} \sigma_{\mathrm{II}}))$.

Proof. Immediate. ■

Theorem 5.18

1. $\mathbf{ATR}_0$ *proves* Δ^0_1-$\mathbf{Det}_L$.

2. $\mathbf{ACA}_0$ *proves* Δ^0_1-$\mathbf{Det}_W$.

Proof. (1): We work in an arbitrary model of $\mathbf{ATR}_0$.
Let $A(f)$, $B(f) \in \Sigma^0_1$ and $A'(g)$, $B'(g) \in \Pi^0_1$ satisfying that

(-) $\forall f \in \mathbb{N}^{\mathbb{N}}(A(f) \leftrightarrow A'(f))$, and

(-) $\forall g \in \mathbb{N}^{\mathbb{N}}(B(g) \leftrightarrow B'(g))$.

In view of Lemma 5.17, we can safely assume that all of A, A', B, and B are different from the empty set and the total set. Then, by Proposition 5.15, there are nonempty pruned trees $S, S', T, T' \subseteq \mathbb{N}^{<\mathbb{N}}$ such that

$$[S] = \{f \in \mathbb{N}^{\mathbb{N}} : A(f)\} \text{ and } [S'] = \{f \in \mathbb{N}^{\mathbb{N}} : \neg A(f)\}.$$

$$[T] = \{g \in \mathbb{N}^{\mathbb{N}} : B(g)\} \text{ and } [T'] = \{g \in \mathbb{N}^{\mathbb{N}} : \neg B(g)\}.$$

We must show that the Lipschitz game $G_L([S],[T])$ is determined.
Let us observe that $S_0 = S \cap S'$ and $T_0 = T \cap T'$ are well-founded trees. Thus, by $\mathbf{ATR}_0$, S_0 and T_0 are ranked trees. Let $(\mathrm{rk}_{S_0}, \alpha)$ and $(\mathrm{rk}_{T_0}, \beta)$ be rank functions for S_0 and T_0 respectively. Again by $\mathbf{ATR}_0$, we have $\alpha \leq_s \beta \vee \beta \leq_s \alpha$. We distinguish two cases:

<u>Case 1</u>: $\alpha \leq_s \beta$.
Then player II has a winning strategy in the game $G_L([S],[T])$. Let us observe that, if $t_0 \notin T_0 = T \cap T'$ then

$$\forall t\,(t_0 \subset t \to t \in T) \vee \forall t\,(t_0 \subset t \to t \in T').$$

Similarly if $s_0 \notin S_0 = S \cap S'$ then

$$\forall s\,(s_0 \subset s \to s \in S) \vee \forall s\,(s_0 \subset s \to s \in S').$$

As a consequence a winning strategy for player II, σ_{II}, can be defined as follows: For all $s,t \in \mathbb{N}^{<\mathbb{N}}$ with $|s| = l+1$ and $|t| = l$,

$$\sigma_{\mathrm{II}}(s \otimes t) = \begin{cases} \min\{k :\ t * \langle k\rangle \in T_0 \wedge \mathrm{rk}_{S_0}(s) \leq_\beta \mathrm{rk}_{T_0}(t * \langle k\rangle)\} & \text{if } s \in S_0 \\ \min\{k :\ \exists t_1 \in T\, (t * \langle k\rangle \subseteq t_1)\} & \text{if } s \in S - S_0 \wedge \\ & t \in T_0 \wedge \mathrm{rk}_{T_0}(t) \neq 0 \\ \min\{k :\ \exists t_2 \in T'\, (t * \langle k\rangle \subseteq t_2)\} & \text{if } s \in S' - S_0 \wedge \\ & t \in T_0 \wedge \mathrm{rk}_{T_0}(t) \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

Since $\alpha \leq_s \beta$ we can identify field(α) with an initial segment of field(β) (and $<_\alpha$ is the restriction of $<_\beta$ to field(α)). The strategy σ_{II} exists by arithmetical comprehension.

Case 2: $\beta <_s \alpha$.
Then player I has a winning strategy in the game $G_L([S],[T])$.
Since $\beta <_s \alpha$ there exists $j_\beta \in$ field(α) such that β is order isomorphic to the initial segment defined by j_β in α.
The first move of player I is $\langle i\rangle$, with

$$i = \min\{k :\ \langle k\rangle \in S_0 \wedge j_\beta \leq_\alpha \mathrm{rk}_{S_0}(\langle k\rangle)\}.$$

For all $s,t \in \mathbb{N}^{<\mathbb{N}}$ with $|s| = |t| = j \geq 1$,

$$\sigma_{\mathrm{I}}(s \otimes t) = \begin{cases} \min\{k :\ s * \langle k\rangle \in S_0 \wedge \mathrm{rk}_{T_0}(t) \leq_\alpha \mathrm{rk}_{S_0}(s * \langle k\rangle)\} & \text{if } t \in T_0 \\ \min\{k :\ \exists s_1 \in S'\, (s * \langle k\rangle \subseteq s_1)\} & \text{if } t \in T - T_0 \wedge \\ & s \in S_0 \wedge \mathrm{rk}_{S_0}(s) \neq 0 \\ \min\{k :\ \exists s_2 \in S\, (s * \langle k\rangle \subseteq s_2)\} & \text{if } t \in T' - T_0 \wedge \\ & s \in S_0 \wedge \mathrm{rk}_{S_0}(s) \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

(2): We work in an arbitrary model of $\mathbf{ACA}_0$. Let $A(f)$, $B(f) \in \Sigma^0_1$ and $A'(g)$, $B'(g) \in \Pi^0_1$ such that

(-) $\forall f \in \mathbb{N}^{\mathbb{N}}(A(f) \leftrightarrow A'(f))$, and

(-) $\forall g \in \mathbb{N}^{\mathbb{N}}(B(g) \leftrightarrow B'(g))$.

In view of Lemma 5.17, we may assume that there exists some $g_{in} \in \mathbb{N}^{\mathbb{N}}$ satisfying $B(g_{in})$ and there exists some $g_{out} \in \mathbb{N}^{\mathbb{N}}$ satisfying $\neg B(g_{out})$. By Proposition 4.1, there are non-empty pruned trees S, and S' such that

$$[S] = \{f \in \mathbb{N}^{\mathbb{N}} :\ A(f)\} \text{ and } [S'] = \{f \in \mathbb{N}^{\mathbb{N}} : \neg A(f)\}\ .$$

We must show that the Wadge game $G_W([S], B)$ is determined.

Observe that $S \cap S'$ cannot have any path. Thus the following strategy σ_{II} will be winning for player II. While player I plays inside $S \cap S'$ player II passes; and when player I leaves $S \cap S'$ (this has to happen sooner or later as $S \cap S'$ has no path) player II plays accordingly by using either g_{in} or g_{out}. In order to give a precise definition of the strategy, recall that by our conventions in Chapter 3, the following correspondence holds.

Formalized strategy	Player I's real move	Player II's real move
k	$k \dot{-} 1$	$\mathtt{p}$, if $k = 0$ $k - 1$, otherwise

Having this in mind, given any sequence of odd length $s = \langle x_0, y_0, \dots, x_{n-1}, y_{n-1}, x_n \rangle$, we put

$$\sigma_{\mathrm{II}}(s) = \begin{cases} 0 & \text{if } \langle x_0 \dot{-} 1, \dots, x_n \dot{-} 1 \rangle \in S \cap S' \\ g_{out}(n-k)+1 & \text{if } \langle x_0 \dot{-} 1, \dots, x_n \dot{-} 1 \rangle \notin S \\ & \text{and } k = \mu j \ \big(\langle x_0 \dot{-} 1, \dots, x_j \dot{-} 1 \rangle \notin S\big) \\ g_{in}(n-k)+1 & \text{if } \langle x_0 \dot{-} 1, \dots, x_n \dot{-} 1 \rangle \notin S' \\ & \text{and } k = \mu j \ \big(\langle x_0 \dot{-} 1, \dots, x_j \dot{-} 1 \rangle \notin S'\big) \end{cases}$$

Clearly, σ_{II} exists by Δ^0_1–comprehension and it is easy to see that σ_{II} is winning for player II. This completes the proof of the theorem. ■

Corollary 5.19

1. **ATR**$_0$ *proves* Δ^0_1-**SLO**$_L$.
2. **ACA**$_0$ *proves* Δ^0_1-**SLO**$_W$.

In particular, it follows from the proof of part 2 of Theorem 5.18 that it is provable in **ACA**$_0$ that the nontrivial Δ^0_1–sets form a Wadge degree. That is to say, write $A \equiv_W B$ to denote the formula $Red_W(A, B) \wedge Red_W(B, A)$. Then, we have

Proposition 5.20 *It is provable in* **ACA**$_0$ *that if S and T are trees defining nontrivial clopen sets then $[S] \equiv_W [T]$.*

Now we derive a reversal for **ATR**$_0$ over **ACA**$_0$ in terms of Lipschitz determinacy and semilinear order principle for clopen sets. Let us observe that in the next section we shall obtain another reversal for **ATR**$_0$ over the weaker base theory **RCA**$_0$.

Theorem 5.21 *The following are equivalent over* **ACA**$_0$*:*

1. **ATR**$_0$.

2. Δ^0_1-$\mathbf{Det}_L$.

3. Δ^0_1-$\mathbf{SLO}_L$.

Proof. By Theorem 5.18 we know that (1) implies (2) and, as we already saw in chapter 3, Δ^0_1-$\mathbf{Det}_L$ implies Δ^0_1-$\mathbf{SLO}_L$ (that is, (2) implies (3)). Therefore, we only must show how to derive $\mathbf{ATR}_0$ from Δ^0_1-$\mathbf{SLO}_L$ (working over $\mathbf{ACA}_0$). Let α and β be countable well orderings. We shall prove that $\alpha \leq_s \beta \vee \beta \leq_s \alpha$. By Friedman–Hirst theorem (see Theorem 4 of [Hir00]) this suffices to derive $\mathbf{ATR}_0$.
Let $S(\alpha)$ be the tree of decreasing sequences (w.r.t. $<_\alpha$) of elements of field(α) (the tree $T(\beta)$ is defined using β accordingly). Then, as noticed in Remark 5.12, $\mathbf{RCA}_0$ suffices to show that $S(\alpha)$ and $T(\beta)$ are ranked trees and that there are rank functions (rk, α) and (rk, β) for $S(\alpha)$ and $T(\beta)$ respectively. Let us define the following trees

$$S = \{s : s \in S(\alpha) \vee \exists t \in S(\alpha) \exists t' \in \mathbb{N}^{<\mathbb{N}} \exists j\, (t * \langle 2j \rangle \notin S(\alpha) \wedge s = t * \langle 2j \rangle * t')\},$$

$$S' = \{s : s \in S(\alpha) \vee \exists t \in S(\alpha) \exists t' \in \mathbb{N}^{<\mathbb{N}} \exists j\, (t * \langle 2j+1 \rangle \notin S(\alpha) \wedge s = t * \langle 2j+1 \rangle * t')\}.$$

Then S and S' are pruned trees, $[S]$ is a clopen set (since $[S']$ is its complement) and, in addition, $S \cap S' = S(\alpha)$.
In a similar way we define

$$T = \{s : s \in T(\beta) \vee \exists t \in T(\beta) \exists t' \in \mathbb{N}^{<\mathbb{N}} \exists j\, (t * \langle 2j \rangle \notin T(\beta) \wedge s = t * \langle 2j \rangle * t')\},$$

$$T' = \{s : s \in T(\beta) \vee \exists t \in T(\beta) \exists t' \in \mathbb{N}^{<\mathbb{N}} \exists j\, (t * \langle 2j+1 \rangle \notin T(\beta) \wedge s = t * \langle 2j+1 \rangle * t')\}.$$

Once again T and T' are pruned trees, $[T]$ and $[T']$ are clopen sets and $T(\beta) = T \cap T'$.
By Δ^0_1-$\mathbf{SLO}_L$ we have that $Red_L([S],[T])$ or $Red_L([T'],[S])$ (recall that $[T']$ coincides with the complement of $[T]$).
If $Red_L([S],[T])$ holds then player II has a winning strategy σ_{II} in the Lipschitz game $G_L([S],[T])$. In such a case, we define by primitive recursion a function $F : \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$ as follows

$$F(\langle\rangle) = \langle\rangle$$

$$F(s * \langle k \rangle) = F(s) * \langle \sigma_{\mathrm{II}}((s \otimes F(s)) * \langle k \rangle) \rangle$$

(Here, recall that if s and t are sequences with $|s| = |t|$, $s \otimes t$ denotes the sequence of length $2|s|$ and elements

$$(s \otimes t)_{2j} = (s)_j \quad \text{and} \quad (s \otimes t)_{2i-1} = (t)_{i-1} \quad (\text{if } 1 < i < |s|).$$

Obviously, if $s_1 \subset s_2$ then $F(s_1) \subset F(s_2)$ and it can be easily checked that

$$\forall s\, (s \in S(\alpha) \to F(s) \in T(\beta)).$$

Indeed, if $s \in S(\alpha)$ but $F(s) \notin T(\beta)$ then, $F(s) \in T - T'$ or $F(s) \in T' - T$ say. If $F(s) \in T - T'$ then there exists $s' \in S(\alpha)$ such that $s \subseteq s'$ and $s' * \langle 1 \rangle \in S'$ and we can define a strategy σ_{I} for player I as follows:

$$\sigma_{\mathrm{I}}(s \otimes t) = \begin{cases} (s')_i & \text{if } |s| = i < |s'| \\ 1 & \text{otherwise} \end{cases}$$

Then $\sigma_{\mathrm{I}} \otimes^{\mathrm{I}}_{L} \sigma_{\mathrm{II}} \in [S']$ but $\sigma_{\mathrm{I}} \otimes^{\mathrm{II}}_{L} \sigma_{\mathrm{II}} \in [T]$. This is a contradiction since σ_{II} is a winning strategy for player II in the game $G_L([S],[T])$.
Thus, using F we show that $S(\alpha) \preceq T(\beta)$ and, as a consequence, $\mathrm{rk}(S(\alpha)) = \alpha \leq_s \mathrm{rk}(T(\beta)) = \beta$.
If $Red_L([T'],[S])$ holds then there exists a winning strategy for player II in the Lipschitz game $G_L([T'],[S])$, and we can prove reasoning as in the previous case that $T(\beta) \preceq S(\alpha)$ and, as a consequence, $\beta = \mathrm{rk}(T(\beta)) \leq_s \mathrm{rk}(S(\alpha)) = \alpha$.

This completes the proof of the lemma. ■

Remark 5.22 *Andretta proved in* $\mathbf{ZF} + \mathbf{BP} + \mathbf{DC}$ *that Lipschitz determinacy and the Lipschitz semilinear ordering principle are equivalent. Theorem 5.21 says that when restricted to* Δ^0_1 *sets, this equivalence can be proved already in* $\mathbf{ACA}_0$.

Since the above reversal was not obtained over the ideal base theory $\mathbf{RCA}_0$, it is natural to ask whether the result can be improved. Thus we pose the following questions.

Problem 5.23

1. *Is* $\mathbf{ATR}_0$ *equivalent over* $\mathbf{RCA}_0$ *to* Δ^0_1-$\mathbf{SLO}_L$*?*

2. *Does* Δ^0_1-$\mathbf{SLO}_L$ *or* Δ^0_1-$\mathbf{Det}_L$ *imply* $\mathbf{ACA}_0$ *over* $\mathbf{RCA}_0$*?*

3. *Are* Δ^0_1-$\mathbf{SLO}_L$ *and* Δ^0_1-$\mathbf{Det}_L$ *equivalent over* $\mathbf{RCA}_0$*?*

5.3 Determinacy for closed and open sets

In this section we shall study determinacy of Lipschitz and Wadge games for open or closed sets in the Baire space. We shall obtain a reversal (over $\mathbf{RCA}_0$) for $\mathbf{ATR}_0$ using Π^0_1-$\mathbf{Det}_L$. This improves the previous reversal for $\mathbf{ATR}_0$ that was derived over $\mathbf{ACA}_0$.

In the analysis of determinacy for closed sets that we developed in Chapter 4 the notion $TrueClosed(T)$ and the associated dichotomy property **(DP)** played an important role. We proved there that $\mathbf{ACA}_0$ proves the dichotomy property for closed sets in the Cantor space. A similar property for closed sets in the Baire space can be also derived in $\mathbf{ACA}_0$.

Definition 5.24 *We say that a tree* T *defines a true closed set if*

$$\exists f \in \mathbb{N}^{\mathbb{N}}\, [f \in [T] \wedge \forall k\, \exists s\, (f[k] \subseteq s \wedge s \notin T)].$$

We will write $TrueClosed(T)$ *to denote the above formula.*

Lemma 5.25 *Assume that* T *is a tree. It is provable in* $\mathbf{ACA}_0$ *that*

$$TrueClosed(T) \vee T \text{ defines a clopen set.}$$

Proof. We work in $\mathbf{ACA}_0$. Suppose that $\neg TrueClosed(T)$ holds. Let us define

$$S = \{s \in \mathbb{N}^{<\mathbb{N}} : \exists t\,(s \subseteq t \wedge t \notin T)\}.$$

The set S exists by Σ^0_1–comprehension, and it is a tree. Finally it is easily checked that

$$\forall f \in \mathbb{N}^{\mathbb{N}}\,(f \notin [T] \leftrightarrow f \in [S]).$$

So, T defines a clopen set. ■

Theorem 5.26

1. $\mathbf{ATR}_0$ *proves* $(\Sigma^0_1 \cup \Pi^0_1)$-$\mathbf{Det}_L$.
2. $\mathbf{ACA}_0$ *proves* $(\Sigma^0_1 \cup \Pi^0_1)$-$\mathbf{Det}_W$.

Proof. (1): Let $A(f), B(g) \in \Sigma^0_1 \cup \Pi^0_1$ and let us consider the game $G_L(A, B)$. First we deal with the case $A(f)$, $B(g) \in \Pi^0_1$. It is enough to show that, working in an arbitrary model of $\mathbf{ATR}_0$ if T, $S \subseteq \mathbb{N}^{<\mathbb{N}}$ are pruned trees then the Lipschitz game $G_L([S], [T])$ is determined. We distinguish two cases:

Case 1: $TrueClosed(T)$ holds, i.e., there exists $g \in [T]$ such that $\forall k\,\exists s\,(g[k] \subseteq s \wedge s \notin T)$. Let us see that in this case there exists a winning strategy for player II, σ_{II}, defined as follows:
For all $s, t \in \mathbb{N}^{<\mathbb{N}}$ with $|s| = j + 1$ and $|t| = j$,

$$\sigma_{\mathrm{II}}(s \otimes t) = \begin{cases} g(j) & \text{if } s \in S \\ \min\{k : t * \langle k\rangle \notin T\} & \text{if } s \notin S \wedge \exists k\,(t * \langle k\rangle \notin T) \\ g(j) & \text{if } s \notin S \wedge \forall k\,(t * \langle k\rangle \in T) \end{cases}$$

The existence of such a σ_{II} is granted by $\mathbf{ACA}_0$ and it is straightforward to check that σ_{II} is a winning strategy for player II.

Case 2: Case 1 does not hold but $TrueClosed(S)$ holds, i.e., there exists $f \in [S]$ such that $\forall k\,\exists s\,(f[k] \subseteq s \wedge s \notin S)$.
Let us define $T' = \{t \in T : \exists t'\,(t \subseteq t' \wedge t' \notin T)\}$. Then, by arithmetical comprehension, T' is a well-founded tree (since Case 1 fails) and a winning strategy for player I, σ_{I}, can be defined as follows:
Let $\sigma_{\mathrm{I}}(\langle\rangle) = f(0)$ and for all $s, t \in \mathbb{N}^{<\mathbb{N}}$ with $|s| = |t| = j \geq 1$,

$$\sigma_{\mathrm{I}}(s \otimes t) = \begin{cases} f(j) & \text{if } t \notin T \vee t \in T' \\ \min\{k : s * \langle k\rangle \notin S\} & \text{if } t \in T - T' \wedge \exists k\,(s * \langle k\rangle \notin S) \\ f(j) & \text{if } t \in T - T' \wedge \forall k\,(s * \langle k\rangle \in S) \end{cases}$$

Since T' is well–founded player II must eventually play outside T'. Let us denote by t the sequence of movements of player II. Then $t \notin T'$ and

$$\forall t\,(t \subset t' \rightarrow t' \notin T) \vee \forall t\,(t \subset t' \rightarrow t' \in T).$$

Thus σ_{I} is a winning strategy for player I.

Case 3: Both $TrueClosed(T)$ and $TrueClosed(S)$ fail.
Since Case 1 fails, by Lemma 5.25 $[T]$ is a clopen set and, by a similar argument (recall that Case 2 also fails) we have that $[S]$ is also a clopen set. Therefore $G_L([S],[T])$ is determined by Theorem 5.18.

We have showed in this way that $\mathbf{ATR}_0$ proves Π^0_1-$\mathbf{Det}_L$. Now, bearing in mind that the strategies for a game $G_L(A(f),B(g))$ are also strategies for the corresponding dual game $G_L(\neg A(f),\neg B(g))$ we obtain from Π^0_1-$\mathbf{Det}_L$ that $G_L(A(f),B(g))$ is also determined when $A(f),B(g)\in\Sigma^0_1$. To conclude let us prove that $G_L(A(f),B(g))$ is determined for $A(f)\in\Sigma^0_1$ and $B(g)\in\Pi^0_1$ (the remaining case follows from this one by duality).
It is enough to show that if T, $S\subseteq\mathbb{N}^{<\mathbb{N}}$ are pruned trees then the Lipschitz game $G_L([S]^c,[T])$ is determined. We distinguish two cases:

Case 1: There exists $f\in[S]$ such that $\forall k\,\exists s\,(f[k]\subseteq s\wedge s\notin S)$, i.e., $TrueClosed(S)$ holds.
Then, a winning strategy for player I, σ_{I}, can be defined as follows:
Let $\sigma_{\mathrm{I}}(\langle\rangle)=f(0)$ and for all $s,t\in\mathbb{N}^{<\mathbb{N}}$ with $|s|=|t|=j$,

$$\sigma_{\mathrm{I}}(s\otimes t)=\begin{cases} f(j) & \text{if } t\in T\\ \min\{k: s*\langle k\rangle\notin S\} & \text{if } t\notin T\wedge\exists k\,(s*\langle k\rangle\notin S)\\ f(j) & \text{if } t\notin T\wedge\forall k\,(s*\langle k\rangle\in S)\end{cases}$$

Once more, the existence of σ_{I} is granted by $\mathbf{ACA}_0$ and it is straightforward to check that σ_{I} is a winning strategy for player I in $G_L([S]^c,[T])$.

Case 2: Case 1 fails.
Then $[S]^c$ is closed and $G_L([S]^c,[T])$ is determined as proved in the first part of this proof.

(2): Firstly, observe that the winning strategies for player I given in the part (1) of the present proof are also winning for player I even if player II is allowed to pass a finite number of times. Secondly, observe that in the previous argument $\mathbf{ATR}_0$ was only used to deal with determinacy of Lipschitz games for clopen sets. Since determinacy of Wadge games for clopen sets can be proved in $\mathbf{ACA}_0$ the result follows.
This completes the proof of the theorem. ■

Corollary 5.27

1. $\mathbf{ATR}_0$ *proves* $(\Sigma^0_1\cup\Pi^0_1)$-$\mathbf{SLO}_L$.
2. $\mathbf{ACA}_0$ *proves* $(\Sigma^0_1\cup\Pi^0_1)$-$\mathbf{SLO}_W$.

Our next result is the promised reversal for $\mathbf{ATR}_0$ over the base theory $\mathbf{RCA}_0$.

Theorem 5.28 *The following are equivalent over* $\mathbf{RCA}_0$*:*

1. $\mathbf{ATR}_0$.

2. $(\Sigma^0_1 \cup \Pi^0_1)$-$\mathbf{Det}_L$.

3. Π^0_1-$\mathbf{Det}_L$.

4. (Δ^0_1, Π^0_1)-$\mathbf{Det}_L$.

Proof. By Theorem 5.26 it is enough to show that (4) implies (1). Since (Δ^0_1, Π^0_1)-$\mathbf{Det}_L$ extends Δ^0_1-$\mathbf{Det}_L$ and, by Theorem 5.21, Δ^0_1-$\mathbf{Det}_L$ implies $\mathbf{ATR}_0$ over $\mathbf{ACA}_0$ it will suffice to show how to derive, over $\mathbf{RCA}_0$, $\mathbf{ACA}_0$ from (Δ^0_1, Π^0_1)-$\mathbf{Det}_L$.
Reasoning in $\mathbf{RCA}_0$, assume (Δ^0_1, Π^0_1)-$\mathbf{Det}_L$. We take $\varphi(x) \in \Sigma^0_1$ (we disregard parameters) and show that the set $\{x : \varphi(x)\}$ exists. To this end, we assume that $\varphi(x)$ is $\exists y\, \varphi_0(x, y)$, with $\varphi_0(x, y) \in \Delta^0_0$, and define $A(f)$ and $B(g)$ to be

$$\exists k\, \forall j \leq k\, (f(j) = k - j)$$

and

$$\begin{aligned}&\forall l\, \forall m\, (g(0) = l \wedge g(l+1) = m \rightarrow \forall j \leq l\, (g(j) = l - j) \wedge \forall i \leq l\, (g(l+i+2) \leq 1) \wedge \\ &\quad \forall i \leq l\, (\varphi(i) \rightarrow g(l+i+2) = 1) \wedge \\ &\quad \forall i \leq l\, (g(l+i+2) = 1 \rightarrow \exists y \leq m\, \varphi_0(i, y))),\end{aligned}$$

respectively. That is to say, a play for player I is in A if it is of the form

$$\mathrm{I}: \quad \overbrace{\langle k, (k-1), (k-2), \ldots, 0\rangle}^{\text{length } k+1} * f'$$

for some $f' \in \mathbb{N}^{\mathbb{N}}$.
A play for player II is in B if it is of the form

$$\mathrm{II}: \quad \overbrace{\langle l, (l-1),\ (l-2),\ \ldots,\ 0\rangle}^{\text{length } l+1} * \langle m\rangle * \langle t_0,\ t_1,\ \ldots,\ t_l\rangle * g'$$

for some $g' \in \mathbb{N}^{\mathbb{N}}$ and, for each $i \leq l$, if $t_i = 1$ then $\exists y \leq m\ \varphi_0(i, y)$ holds, and if $\varphi(i)$ holds then $t_i = 1$.
It is clear that B is in Π^0_1 and that A is equivalent to

$$\forall k\, (f(0) = k \rightarrow \forall j \leq k\, (f(j) = k - j)).$$

Therefore A is in Δ^0_1

We claim that

- Player I cannot have a winning strategy in the game $G_L(A, B)$.

  We must show that $\forall \sigma_{\mathrm{I}}\, \exists \sigma_{\mathrm{II}}\, (B(\sigma_{\mathrm{I}} \otimes^{\mathrm{I}} \sigma_{\mathrm{II}}) \leftrightarrow A(\sigma_{\mathrm{I}} \otimes^{\mathrm{II}} \sigma_{\mathrm{II}}))$. Consider any strategy for player I, σ_{I}, and $k = \sigma_{\mathrm{I}}(\langle\rangle)$. Let σ_{II} be a strategy for player II that mimics the moves of player I until round k (included), that is, given $s, t \in \mathbb{N}^{<\mathbb{N}}$ with $|t| = i < k$ and $|s| = i + 1$,

$$\sigma_{\mathrm{II}}(s \otimes t) = (s)_i\ .$$

We distinguish two cases.

Case 1: There is $i \leq k$ such that $\sigma_{\mathrm{I}} \otimes^{\mathrm{I}} \sigma_{\mathrm{II}}(i) = \sigma_{\mathrm{I}} \otimes^{\mathrm{II}} \sigma_{\mathrm{II}}(i) \neq k - i$.

Then, both $\neg B(\sigma_{\mathrm{I}} \otimes^{\mathrm{I}} \sigma_{\mathrm{II}})$ and $\neg A(\sigma_{\mathrm{I}} \otimes^{\mathrm{II}} \sigma_{\mathrm{II}})$ hold. Therefore, $(A(\sigma_{\mathrm{I}} \otimes^{\mathrm{I}} \sigma_{\mathrm{II}}) \leftrightarrow B(\sigma_{\mathrm{I}} \otimes^{\mathrm{II}} \sigma_{\mathrm{II}}))$ holds and, thus, σ_{I} is not a winning strategy for player I, as required.

Case 2: $\forall i \leq k \ \big[(\sigma_{\mathrm{I}} \otimes^{\mathrm{I}} \sigma_{\mathrm{II}})(i) = (\sigma_{\mathrm{I}} \otimes^{\mathrm{II}} \sigma_{\mathrm{II}})(i) = k - i\big]$.

Then, using Π^0_1-induction (which is available in $\mathbf{RCA}_0$), we can prove that there is some $m \in \mathbb{N}$ such that

$$\forall i \leq k \, (\exists y \, \varphi_0(i, y) \rightarrow \exists y \leq m \, \varphi_0(i, y)).$$

By bounded Σ^0_1-comprehension (available thanks to $\mathbf{RCA}_0$), there exists

$$C = \{x \, : \, x \leq k \wedge \exists y \leq m \, \varphi_0(x, y)\}.$$

Let σ_{II} be a strategy for player II that mimics σ_{I} until round k (included) and continues by playing as follows:

Given s, $t \in \mathbb{N}^{<\mathbb{N}}$ with $|t| = i < k$ and $|s| = i + 1$,

$$\sigma_{\mathrm{II}}(s \otimes t) = \begin{cases} m, & \text{if } i = k + 1 \\ 1, & \text{if } k + 1 < i \leq 2k + 2 \wedge i - (k + 2) \in C \\ 0, & \text{if } k + 1 < i \leq 2k + 2 \wedge i - (k + 2) \notin C \end{cases}$$

Then, we have $B(\sigma_{\mathrm{I}} \otimes^{\mathrm{I}} \sigma_{\mathrm{II}})$ and $A(\sigma_{\mathrm{I}} \otimes^{\mathrm{II}} \sigma_{\mathrm{II}})$. Therefore, we also have $A(\sigma_{\mathrm{I}} \otimes^{\mathrm{I}} \sigma_{\mathrm{II}}) \leftrightarrow B(\sigma_{\mathrm{I}} \otimes^{\mathrm{II}} \sigma_{\mathrm{II}}))$ and, thus σ_{I} is not a winning strategy for player II, as required.

This proves the claim.

Hence, by (Δ^0_1, Π^0_1)-$\mathbf{Det}_L$, player II must have a winning strategy in the game $G_L(A, B)$. Let σ_{II} denote such a winning strategy. Pick $k \in \mathbb{N}$. We will use the winning strategy σ_{II} to decide whether or not $\varphi(k)$ holds. To that end, for each $k \in \mathbb{N}$, we consider a strategy for player I, σ^k_{I}, satisfying that for each s, $t \in \mathbb{N}^{<\mathbb{N}}$ with $|s|\,| = i = |t|$,

$$\sigma^k_{\mathrm{I}}(s \otimes t) = \begin{cases} k - i & \text{if } i \leq k \\ 0 & \text{if } i > k \end{cases}$$

That is to say, according to σ^k_{I} player I plays as follows:

$$\mathrm{I}: \quad \overbrace{\langle k, \, (k-1), \, (k-2), \, \ldots, \, 0\rangle}^{k+1 \text{ moves}} * \vec{0}$$

It is clear that $A(\sigma^k_{\mathrm{I}} \otimes^{\mathrm{I}} \sigma_{\mathrm{II}})$ holds. So $B(\sigma^k_{\mathrm{I}} \otimes^{\mathrm{II}} \sigma_{\mathrm{II}})$ holds as well, for σ_{II} is a winning strategy for player II in $G_L(A, B)$. Consequently, there exist $l, m \in \mathbb{N}$ such that

$$\begin{aligned} &g(0) = l \wedge g(l + 1) = m \wedge \forall i \leq l \, (g(i) = l - i) \wedge \\ &\quad \forall i \leq l \, ((\varphi(i) \rightarrow g(i + l + 2) = 1) \wedge g(l + i + 2) = 1 \rightarrow \exists y \leq m \, \varphi_0(i, y)), \end{aligned}$$

where $g = \sigma_{\mathrm{I}}^k \otimes^{\mathrm{II}} \sigma_{\mathrm{II}}$.

Now we claim that

- We have $k \leq 2(l+2)$.

  Assume not. Then, $2(l+2) < k$ and, by bounded Σ_1^0-comprehension, there exists $D = \{x : x \leq l \wedge \varphi(x)\}$. Consider a new strategy for player I, σ_I', given by

$$\sigma_{\mathrm{I}}'(\langle x_0, y_0, \ldots, x_{n-1}, y_{n-1}\rangle) = \begin{cases} k-n & \text{if } n < 2l+3 \\ k-n & \text{if } 2l+3 \leq n \leq k \wedge \\ & \quad \forall i \leq l\ ((i \in D \to y_{l+i+2} = 1) \wedge \\ & \quad (i \notin D \to y_{l+i+2} = 0)) \\ k+1 & \text{otherwise} \end{cases}$$

  It is clear that σ_{I}' exists by Δ_1^0-comprehension. In addition, it is immediate to see that $A(\sigma_{\mathrm{I}}' \otimes^{\mathrm{I}} \sigma_{\mathrm{II}})$ holds if and only if $B(\sigma_{\mathrm{I}}' \otimes^{\mathrm{II}} \sigma_{\mathrm{II}})$ does not hold, thus contradicting the fact that σ_{II} is a winning strategy for player II in $G_L(A, B)$.

  This proves our second claim.

Observe that by Δ_1^0–comprehension, there exists $S \subseteq \mathrm{Seq}_{\mathrm{even}} \times \mathbb{N} \times \mathbb{N}$ such that $(S)_k = \sigma_{\mathrm{I}}^k$ for each k, where $(S)_k = \{(s, n) \in \mathrm{Seq}_{\mathrm{even}} \times \mathbb{N} : (s, n, k) \in S\}$. Then, we have

$$\varphi(k) \leftrightarrow \exists l\, (k \leq l \wedge ((S)_{2(k+2)} \otimes^{\mathrm{II}} \sigma_{\mathrm{II}})(l+2+k) = 1)$$

and

$$\varphi(k) \leftrightarrow \forall l\, (k \leq l \wedge ((S)_{2(k+2)} \otimes^{\mathrm{II}} \sigma_{\mathrm{II}})(l+2+k) = 1).$$

Thus, the set $\{x : \varphi(x)\}$ exists by Δ_1^0-comprehension using the winning strategy σ_{II} as a parameter.
This completes the proof of the theorem. ■

Remark 5.29 *According to Simpson [Smp99]* $\mathbf{ATR}_0$ *is equivalent to* Δ_1^0-**Det** *and* Σ_1^0-**Det** *over* $\mathbf{RCA}_0$. *As a consequence, it follows from the above theorem that over* $\mathbf{RCA}_0$, *Gale-Stewart determinacy for open sets and Lipschitz determinacy for open sets are equivalent principles in Baire space. Hence, the huge difference of strength between Gale-Stewart and Lipschitz determinacy must appear in higher levels of Borel determinacy.*

The following questions emerge naturally:

Problem 5.30

1. *Is* $(\Sigma_1^0 \cup \Pi_1^0)$-$\mathbf{Det}_W$ *equivalent over* $\mathbf{RCA}_0$ *to* $\mathbf{ACA}_0$*?*

2. *Is* $(\Sigma^0_1 \cup \Pi^0_1)$-$\mathbf{SLO}_W$ *equivalent over* $\mathbf{RCA}_0$ *to* $\mathbf{ACA}_0$*?*

3. *Is* $(\Sigma^0_1 \cup \Pi^0_1)$-$\mathbf{SLO}_L$ *equivalent over* $\mathbf{RCA}_0$ *to* $\mathbf{ATR}_0$*?*

4. *Is* Π^0_1-$\mathbf{SLO}_L$ *equivalent over* $\mathbf{RCA}_0$ *to* $\mathbf{ATR}_0$*?*

5. *Is* (Δ^0_1, Π^0_1)-$\mathbf{SLO}_L$ *equivalent over* $\mathbf{RCA}_0$ *to* $\mathbf{ATR}_0$*?*

5.4 Determinacy for $(\Sigma^0_1)_2$ sets

In this section we show that the topological argument presented in Chapter 2 can be formalized in the system Π^1_1-$\mathbf{CA}_0$, obtaining in this way a proof of $((\Sigma^0_1)_2 \cup \neg(\Sigma^0_1)_2)$-$\mathbf{Det}_{L/W}$ in this system.

First of all, we show that some auxiliary trees used in the topological argument can be proved to exist in models of Π^1_1-$\mathbf{CA}_0$.

Definition 5.31 *The following definition is made in* Π^1_1-$\mathbf{CA}_0$*. Let* $T \subseteq \mathbb{N}^{<\mathbb{N}}$ *be a tree. The boundary of* T *is the following set*

$$\delta(T) = \{t \in T : \exists t'(t \subseteq t' \wedge t' \notin T)\}$$

Given $S \subseteq \mathbb{N}^{<\mathbb{N}}$*, we define*

$$\delta_S T = \{t \in T : \exists t'(t' \in S - T \wedge t \subseteq t')\}.$$

If $T_1, T_0 \subseteq \mathbb{N}^{<\mathbb{N}}$ *are trees such that* $T_1 \subseteq T_0$ *then we define*

$$\delta(T_0, T_1) = \{s \in T_1 : \exists h, g\,(h \in [\delta_{T_0} T_1] \wedge g \in [\delta(T_0)] \wedge g \notin [T_1] \wedge s \subset h \wedge s \subset g)\}.$$

Let us notice that $\delta(T)$ and $\delta_S T$ exist (as sets) by $\mathbf{ACA}_0$ and that they are trees if so are T and S. The existence of $\delta(T_0, T_1)$ requires Σ^1_1-comprehension, which is available in Π^1_1-$\mathbf{CA}_0$. It is easily checked that $\delta(T_0, T_1)$ is a tree.

Lemma 5.32 *The following is provable in* Π^1_1-$\mathbf{CA}_0$*. Let* S_0*,* S_1*,* T_0*,* $T_1 \subseteq \mathbb{N}^{<\mathbb{N}}$ *be pruned trees such that*

(-) $S_1 \subseteq S_0$ *and* $T_1 \subseteq T_0$*, and*

(-) $\delta(T_0, T_1)$ *and* $\delta(S_0, S_1)$ *are well-founded.*

Then:

1. $G_L([S_0] - [S_1], [T_0] - [T_1])$ *is determined.*

2. $G_W([S_0] - [S_1], [T_0] - [T_1])$ *is determined.*

Proof. (1): By hypothesis $\delta(S_0, S_1)$ and $\delta(T_0, T_1)$ are well–founded trees and, therefore, by $\mathbf{ATR}_0$ they are ranked trees. Thus, there are rank functions (rk_1, α) and (rk_2, β) for $\delta(S_0, S_1)$ and $\delta(T_0, T_1)$ respectively. Now, we distinguish several cases:

Case 1: $[\delta_{T_0} T_1] \neq \emptyset$, $[\delta\,(T_0)] - [T_1] \neq \emptyset$, $[\delta_{S_0} S_1] \neq \emptyset$, and $[\delta\,(S_0)] - [S_1] \neq \emptyset$.

We distinguish two subcases:

1. $\alpha \leq_s \beta$.

   It follows from the definition of $\delta(T_0, T_1)$ that for each $t \in \delta(T_0, T_1)$ there exist $g_t \in [\delta_{T_0} T_1]$ with $t \subset g_t$ and $g'_t \in [\delta\,(T_0)] - [T_1]$ such that $t \subset g'_t$. Using Σ^1_1-$\mathbf{AC}$, available by $\mathbf{ATR}_0$ (see theorem V.8.3 in [Smp99]) the functions g_t and g'_t can be selected uniformly. Indeed, let $\theta(t, g)$ be the formula

   $$t \in \delta(T_0, T_1) \to t \subset g \wedge g \in [\delta_{T_0} T_1].$$

   Then $\forall t\, \exists g\, \theta(t, g)$ and by Σ^1_1-$\mathbf{AC}$ there exists g such that $\forall t\, \theta(t, (g)_t)$.

   In a similar way we can prove that there exists g' such that

   $$t \in \delta(T_0, T_1) \to t \subset (g')_t \wedge (g')_t \in [\delta\,(T_0)] - [T_1].$$

   Then player II has a winning strategy, σ_{II}, defined as follows:

   For all $s, t \in \mathbb{N}^{<\mathbb{N}}$ with $|s| = j + 1$ and $|t| = j$, if player I plays $s \in \delta(S_0, S_1)$ then

   $$\sigma_{\mathrm{II}}(s \otimes t) = \min\{k : \, t * \langle k \rangle \in \delta(T_0, T_1) \wedge \mathrm{rk}_2(t * \langle k \rangle) \geq \mathrm{rk}_1(s)\}.$$

   If at some point player I plays outside $\delta(S_0, S_1)$ (this must be eventually the case, since $\delta(S_0, S_1)$ is well–founded) and player II is still playing inside $\delta(T_0, T_1)$, with $\mathrm{rk}_2(t) \neq 0$, then

   $$\sigma_{\mathrm{II}}(s \otimes t) = \min\{k : \, t * \langle k \rangle \in \delta(T_0, T_1)\}.$$

   If $\mathrm{rk}_2(t) = 0$, or $t \notin \delta(T_0, T_1)$, then player II decides her move depending on the position, s, of player I, as follows.

   For each $t' \in \delta(T_0, T_1)$ such that $\rho_{\delta(T_0,T_1)}(t') = 0$ let us fix $(g_1)_{t'} \in [\delta_{T_0} T_1]$ with $t' \subseteq (g_1)_{t'}$ and $(g_2)_{t'} \in [\delta(T_0)] - [T_1]$ with $t' \subseteq (g_2)_{t'}$. Let $b_{t'} = \max\{k : t'\,[k] \in \delta(T_0, T_1)\}$ and let $(*)$ be the formula

   $$\forall s'((s\,[b_{t'} + 1] \subset s' \wedge s' \in \delta_{S_0} S_1) \to \exists k \forall s''(s'\,[k] \subseteq s'' \to s'' \in S_1)). \qquad (*)$$

   Then the strategy of player II continues as follows:

$$\sigma_{\mathrm{II}}(s\otimes t) = \begin{cases} (g_1)_{t'}(j) & \text{if } j \geq b_{t'} \wedge s[b_{t'}+1] \notin S_0 \\ (g_2)_{t'}(j) & \text{if } j \geq b_{t'} \wedge s[b_{t'}+1] \in S_0 - S_1 \wedge s \in S_0 \\ (g_2)_{t'}(j) & \text{if } j \geq b_{t'} \wedge s[b_{t'}+1] \in S_0 - S_1 \wedge s \notin S_0 \wedge \forall k\,(t * \langle k\rangle \in T_0) \\ k & \text{if } j \geq b_{t'} \wedge s[b_{t'}+1] \in S_0 - S_1 \wedge s \notin S_0 \wedge \exists k\,(t * \langle k\rangle \notin T_0) \\ & \text{and } k = \min\{i : t * \langle i\rangle \notin T_0\} \\ (g_1)_{t'}(j) & \text{if } j \geq b_{t'} \wedge s[b_{t'}+1] \in S_1 \wedge \exists f\ (f \in [\delta_{S_0}S_1] \wedge s[b_{t'}+1] \subset f) \wedge \\ & (s \in S_1 \vee s \notin S_0) \\ (g_1)_{t'}(j) & \text{if } j \geq b_{t'} \wedge s[b_{t'}+1] \in S_1 \wedge \exists f\ (f \in [\delta_{S_0}S_1] \wedge s[b_{t'}+1] \subset f) \wedge \\ & s \in S_0 - S_1 \wedge \exists s'(s \subseteq s' \wedge s' \notin S_0) \\ (g_1)_{t'}(j) & \text{if } j \geq b_{t'} \wedge s[b_{t'}+1] \in S_1 \wedge \exists f\ (f \in [\delta_{S_0}S_1] \wedge s[b_{t'}+1] \subset f) \wedge \\ & s \in S_0 - S_1 \wedge\ \forall s'(s \subset s' \to s' \in S_0) \wedge \forall k\,(t * \langle k\rangle \notin T_0 - T_1) \\ k & \text{if } j \geq b_{t'} \wedge s[b_{t'}+1] \in S_1 \wedge \exists f\ (f \in [\delta_{S_0}S_1] \wedge s[b_{t'}+1] \subset f) \wedge \\ & s \in S_0 - S_1 \wedge\ \forall s'(s \subset s' \to s' \in S_0) \wedge \exists k\,(t * \langle k\rangle \in T_0 - T_1) \\ & \text{and } k = \min\{i : t * \langle i\rangle \in T_0 - T_1\} \\ (g_2)_{t'}(j) & \text{if } j \geq b_{t'} \wedge s[b_{t'}+1] \in S_1 \wedge \neg\exists f\,(f \in [\delta_{S_0}S_1] \wedge s[b_{t'}+1] \subset f) \wedge \\ & (s \in \delta_{S_0}S_1 \vee s \in S_0 - S_1) \\ (g_2)_{t'}(j) & \text{if } j \geq b_{t'} \wedge s[b_{t'}+1] \in S_1 \wedge \neg\exists f\,(f \in [\delta_{S_0}S_1] \wedge s[b_{t'}+1] \subset f) \wedge \\ & s \notin \delta_{S_0}S_1 \wedge (s \in S_1 \vee s \notin S_0) \wedge \forall k\,(t * \langle k\rangle \in T_0) \\ k & \text{if } j \geq b_{t'} \wedge s[b_{t'}+1] \in S_1 \wedge \neg\exists f\,(f \in [\delta_{S_0}S_1] \wedge s[b_{t'}+1] \subset f) \wedge \\ & s \notin \delta_{S_0}S_1 \wedge (s \in S_1 \vee s \notin S_0) \wedge \exists k\,(t * \langle k\rangle \notin T_0) \\ & \text{and } k = \min\{i : t * \langle i\rangle \notin T_0\} \end{cases}$$

Let us observe that the strategy σ_{II} exists by Σ^1_1–$\mathbf{CA}_0$. Let us see that σ_{II} is a winning strategy for player II. We must check that

$$\forall \sigma_{\mathrm{I}}\,(\sigma_{\mathrm{I}} \otimes^{\mathrm{I}}_{L} \sigma_{\mathrm{II}} \in [S_0] - [S_1] \leftrightarrow \sigma_{\mathrm{I}} \otimes^{\mathrm{II}}_{L} \sigma_{\mathrm{II}} \in [T_0] - [T_1]).$$

Let σ_{I} be such that $\sigma_{\mathrm{I}} \otimes^{\mathrm{I}}_{L} \sigma_{\mathrm{II}} \in [S_0] - [S_1]$. This means that either player I has been playing all the time inside $S_0 - S_1$ or he started playing inside S_1 and at some later moment he switched to $S_0 - S_1$. Player II always starts playing inside T_1, but if player I commits himself to $S_0 - S_1$ then player II will finish the game in $T_0 - T_1$ as we can see checking the second, the fourth, the eighth, and the ninth conditions in the definition of the strategy σ_{II}. Suppose now that $\sigma_{\mathrm{I}} \otimes_L^{\mathrm{I}} \sigma_{\mathrm{II}} \in S_1 \cup S_0^c$. Then, as we can check observing the first, the fifth, the sixth, and the eleventh conditions, player II will either play in T_1 or leave T_0. The remaining conditions represent intermediate steps. Hence $\sigma_{\mathrm{I}} \otimes_L^{\mathrm{II}} \sigma_{\mathrm{II}} \notin [T_0] - [T_1]$.

2. $\beta <_s \alpha$.

Then player I has a winning strategy.

Using again Σ_1^1-**AC** we can prove that there exist f and g such that

$$\forall s\,(s \in \delta(S_0, S_1) \rightarrow s \subset (f)_s \wedge (f)_s \in [\delta_{S_0} S_1]), \text{ and}$$

$$\forall s\,(s \in \delta(S_0, S_1) \rightarrow s \subset (f')_s \wedge (f')_s \in [\delta\,(S_0)] - [S_1]).$$

Let us define the following strategy σ_{I} for player I. First, we put

$$\sigma_{\mathrm{I}}(\langle\rangle) = \min\{k : \beta \leq_s \mathrm{rk}_1(\langle k \rangle)\}.$$

Let $s, t \in \mathbb{N}^{<\mathbb{N}}$ with $|s| = |t| = j \geq 1$. If player II plays $t \in \delta(T_0, T_1)$ then

$$\sigma_{\mathrm{I}}(s \otimes t) = \min\{k :\ s * \langle k \rangle \in \delta(S_0, S_1) \wedge \mathrm{rk}_2(t) \leq_s \mathrm{rk}_1(s * \langle k \rangle)\}.$$

If at some point player II plays outside $\delta(T_0, T_1)$ (this must be eventually the case, since $\delta(T_0, T_1)$ is well–founded) and player I is still playing inside $\delta(S_0, S_1)$, with $\mathrm{rk}_1(s) \neq 0$, then

$$\sigma_{\mathrm{I}}(s \otimes t) = \min\{k :\ s * \langle k \rangle \in \delta(S_0, S_1)\}.$$

If $\mathrm{rk}_1(s) = 0$, or $s \notin \delta(S_0, S_1)$, then player I decides his move depending on the position, t, of player II, as follows.

For each $s' \in \delta(S_0, S_1)$ such that $\rho_\delta(S_0, S_1)(s') = 0$ let us fix $(f_1)_{s'} \in [\delta_{S_0} S_1]$ with $s' \subseteq (f_1)_{s'}$ and $(f_2)_{s'} \in [\delta(S_0)] - [S_1]$ such that $s' \subseteq (f_2)_{s'}$. Let $a_{s'} = \max\{k : s'\,[k] \in \delta(S_0, S_1)\}$ and let $(**)$ be the formula

$$\forall t'(t\,[a_{s'}] \subset t' \wedge t' \in \delta_{T_0} T_1 \rightarrow \exists k \forall t''(t'\,[k] \subseteq t'' \rightarrow t'' \in T_1)). \qquad (**)$$

Then the strategy of player II continues as follows:

$$\sigma_{\mathrm{I}}(s \otimes t) = \begin{cases} (f_2)_{s'}(j) & \text{if } j \geq a_{s'} \wedge t[a_{s'}] \notin T_0 \\ (f_1)_{s'}(j) & \text{if } j \geq a_{s'} \wedge t[a_{s'}] \in T_0 - T_1 \wedge t \in T_0 \\ (f_1)_{s'}(j) & \text{if } j \geq a_{s'} \wedge t[a_{s'}] \in T_0 - T_1 \wedge t \notin T_0 \wedge \forall k\, (s * \langle k \rangle \notin S_0 - S_1) \\ k & \text{if } j \geq a_{s'} \wedge t[a_{s'}] \in T_0 - T_1 \wedge t \notin T_0 \wedge \exists k\, (s * \langle k \rangle \in S_0 - S_1) \\ & \text{and } k = \min\{i : s * \langle i \rangle \in S_0 - S_1\} \\ (f_2)_{s'}(j) & \text{if } j \geq a_{s'} \wedge t[a_{s'}] \in T_1 \wedge \exists g\ (g \in [\delta_{T_0} T_1] \wedge t[a_{s'}] \subset g) \wedge \\ & (t \in T_1 \vee t \notin T_0) \\ (f_2)_{s'}(j) & \text{if } j \geq a_{s'} \wedge t[a_{s'}] \in T_1 \wedge \exists g\ (g \in [\delta_{T_0} T_1] \wedge t[a_{s'}] \subset g) \wedge \\ & t \in T_0 - T_1 \wedge \exists t'(t \subseteq t' \wedge t' \notin T_0) \\ (f_2)_{s'}(j) & \text{if } j \geq a_{s'} \wedge t[a_{s'}] \in T_1 \wedge \exists g\ (g \in [\delta_{T_0} T_1] \wedge t[a_{s'}] \subset g) \wedge \\ & t \in T_0 - T_1 \wedge\ \forall t'(t \subset t' \to t' \in T_0) \wedge \forall k\, (s * \langle k \rangle \in S_0 - S_1) \\ k & \text{if } j \geq a_{s'} \wedge t[a_{s'}] \in T_1 \wedge \exists g\ (g \in [\delta_{T_0} T_1] \wedge t[a_{s'}] \subset g) \wedge \\ & t \in T_0 - T_1 \wedge\ \forall t'(t \subset t' \to t' \in T_0) \wedge \exists k\, (s * \langle k \rangle \notin S_0 - S_1) \\ & \text{and } k = \min\{i : s * \langle i \rangle \in S_0 - S_1\} \\ (f_1)_{s'}(j) & \text{if } j \geq a_{s'} \wedge t[a_{s'}] \in T_1 \wedge \neg\exists g\, (g \in [\delta_{T_0} T_1] \wedge t[a_{s'}] \subset g) \wedge \\ & (t \in \delta_{T_0} T_1 \vee t \in T_0 - T_1) \\ (f_1)_{s'}(j) & \text{if } j \geq a_{s'} \wedge t[a_{s'}] \in T_1 \wedge \neg\exists g\, (g \in [\delta_{T_0} T_1] \wedge t[a_{s'}] \subset g) \wedge \\ & t \notin \delta_{T_0} T_1 \wedge (t \in T_1 \vee t \notin T_0) \wedge \forall k\, (s * \langle k \rangle \notin S_0 - S_1) \\ k & \text{if } j \geq a_{s'} \wedge t[a_{s'}] \in T_1 \wedge \neg\exists g\, (g \in [\delta_{T_0} T_1] \wedge t[a_{s'}] \subset g) \wedge \\ & t \notin \delta_{T_0} T_1 \wedge (t \in T_1 \vee t \notin T_0) \wedge \exists k\, (s * \langle k \rangle \in S_0 - S_1) \\ & \text{and } k = \min\{i : t * \langle i \rangle \in S_0 - S_1\} \end{cases}$$

The strategy σ_{I} does exist by Σ^1_1–$\mathbf{CA}_0$. To see that it is a winning strategy for player I, we must check that

$$\forall \sigma_{\mathrm{II}}\, (\sigma_{\mathrm{I}} \otimes^{\mathrm{I}}_L \sigma_{\mathrm{II}} \in [S_0] - [S_1] \leftrightarrow \sigma_{\mathrm{I}} \otimes^{\mathrm{II}}_L \sigma_{\mathrm{II}} \notin [T_0] - [T_1]).$$

Let σ_{II} be such that $\sigma_{\mathrm{I}} \otimes^{\mathrm{II}}_L \sigma_{\mathrm{II}} \in [T_0] - [T_1]$. This means that either player II has been playing all the time inside $T_0 - T_1$ or he started playing inside T_1 and at some later moment he switched to $T_0 - T_1$. Player I starts playing inside S_1 and if player

II commits himself to $T_0 - T_1$ then player II will finish the game in S_1 as we can see checking the second, the eighth, and the ninth conditions in the definition of the strategy σ_{I}. Thus player I finishes the game outside $[S_0] - [S_1]$. Suppose now that $\sigma_{\mathrm{I}} \otimes_L^{\mathrm{II}} \sigma_{\mathrm{II}} \notin [T_0] - [T_1]$, that is $\sigma_{\mathrm{I}} \otimes_L^{\mathrm{II}} \sigma_{\mathrm{II}} \in [T_1] \cup [T_0^c]$. Then, as we can check observing the first, the fourth, the fifth, the sixth, and the eleventh conditions, player I will play in $[S_0] - [S_1]$. The remaining conditions represent intermediate steps. Hence $\sigma_{\mathrm{I}} \otimes_L^{\mathrm{I}} \sigma_{\mathrm{II}} \in [S_0] - [S_1]$.

Case 2: (T_0, T_1) or (S_0, S_1) is in a degenerated position, but not both of them.
Here we say that (T_0, T_1) (and similarly (S_0, S_1)) is in a *degenerated position* if $[\delta_{T_0} T_1] = \emptyset$ or $[\delta(T_0)] - [T_1] = \emptyset$ ($[\delta_{S_0} S_1] = \emptyset$ or $[\delta(S_0)] - [S_1] = \emptyset$).
Let us observe that if (T_0, T_1) is in a degenerated position then the formula $g \in [T_0] - [T_1]$ is equivalent to a Σ_1^0 formula or a Π_1^0 formula:

- If $[\delta_{T_0} T_1] = \emptyset$ then $\delta_{T_0} T_1$ is well-founded and, as a consequence, we have for each $g \in \mathbb{N}^{\mathbb{N}}$,
$$g \in [T_0] - [T_1] \leftrightarrow g \in [T_0] \wedge \forall k\, (g[k] \notin \delta_{T_0} T_1 \rightarrow g[k] \notin T_1).$$
- If $[\delta(T_0)] - [T_1] = \emptyset$ then let $T_0' = \{s \in T_0 : \forall s'\, (s \subseteq s' \rightarrow s' \in T_0)\}$. For each $g \in \mathbb{N}^{\mathbb{N}}$,
$$g \in [T_0] - [T_1] \leftrightarrow \exists k\, (g[k] \in T_0' \wedge g[k] \notin T_1).$$

If player I plays in a degenerated position, then player II has a winning strategy (essentially, player II plays simulating the strategy described in Lemma **??** (case 1)):

- If $f \in [S_0] - [S_1]$ is equivalent to a Π_1^0 formula then, since (T_0, T_1) is not in a degenerate position there exists $g \in [\delta(T_0)] - [T_1]$ and player II can win the game using g.
- If $f \in [S_0] - [S_1]$ is equivalent to a Σ_1^0 formula then, since (T_0, T_1) is not in a degenerate position there exists $g \in [\delta_{T_0} T_1]$ and player II can win the game using g.

In a similar way it can be proved that if player II plays in a degenerated position then player I has a winning strategy.

Case 3: (T_0, T_1) and (S_0, S_1) are in a degenerated position.
Recall that in these degenerated cases $f \in [T_0] - [T_1]$ and $f \in [S_0] - [S_1]$ are equivalent to some formulas in $\Sigma_1^0 \cup \Pi_1^0$, so, the corresponding game is determined by Theorem 5.28.

(2): Now we prove that under the same hypothesis $G_W([S_0] - [S_1], [T_0] - [T_1])$ is determined. As before, let (rk_1, α) and (rk_2, β) be rank functions for $\delta(S_0, S_1)$ and $\delta(T_0, T_1)$ respectively. We have to consider the same cases:

Case 1: $[\delta_{T_0} T_1] \neq \emptyset$, $[\delta(T_0)] - [T_1] \neq \emptyset$, $[\delta_{S_0} S_1] \neq \emptyset$, and $[\delta(S_0)] - [S_1] \neq \emptyset$.

We distinguish two subcases:

1. $\alpha \leq_s \beta$.

   Since a winning strategy for player II in $G_L([S_0]-[S_1],[T_0]-[T_1])$ immediately yields a winning strategy for player II in $G_W([S_0]-[S_1],[T_0]-[T_1])$, the proof of this subcase is similar to that of the Lipschitz subcase.

2. $\beta <_s \alpha$.

   In contrast to the corresponding Lipschitz subcase, in this subcase player II has a winning strategy in the Wadge game $G_W([S_0]-[S_1],[T_0]-[T_1])$, because player II can pass while player I is playing inside $\delta(S_0,S_1)$. As soon as player I starts playing outside $\delta(S_0,S_1)$, which must eventually happen, since $\delta(S_0,S_1)$ is well-founded, player II uses the strategy described in the former subcase and wins the game.

Cases 2 and 3: In these cases the fact that player II can pass in a Wadge game does not change anything essential in the proofs in comparison to the corresponding Lipschitz cases. The winning strategies for player I and player II in $G_W([S_0]-[S_1],[T_0]-[T_1])$ remain basically the same. ■

Proposition 5.33

1. Π^1_1-$\mathbf{CA}_0$ *proves* $((\Sigma^0_1)_2 \cup \neg(\Sigma^0_1)_2)$-$\mathbf{Det}_L$.
2. Π^1_1-$\mathbf{CA}_0$ *proves* $((\Sigma^0_1)_2 \cup \neg(\Sigma^0_1)_2)$-$\mathbf{Det}_W$.

Proof. (1): Let $A(f), B(g) \in (\Sigma^0_1)_2 \cup \neg(\Sigma^0_1)_2$. We shall prove that $G_L(A,B)$ is determined. By duality it is enough to show determinacy in the following cases:

(A) $A(f), B(g) \in (\Sigma^0_1)_2$, and

(B) $A(f) \in \neg(\Sigma^0_1)_2$ and $B(g) \in (\Sigma^0_1)_2$.

Case (A): Let us assume that $A(f), B(g) \in (\Sigma^0_1)_2$.
Without loss of generality, by Π^1_1–$\mathbf{CA}_0$, we can assume that there exist *pruned* trees S_0, S_1, T_0, T_1 such that

1. $S_1 \subseteq S_0$ and $T_1 \subseteq T_0$.
2. $A(f) \leftrightarrow f \in [S_0]-[S_1]$, and $B(g) \leftrightarrow g \in [T_0]-[T_1]$.

We distinguish several subcases:

- Case A.1: $[\delta(T_0,T_1)] \neq \emptyset$.

  Let $g \in [\delta(T_0,T_1)]$. Then,

$$\forall k\, \exists g' \,(g[k] \subseteq g' \wedge g' \in [\delta\,(T_0)] - [T_1]).$$

By Σ^1_1–**AC**, there exists G such that

$$\forall k\,(g[k] \subseteq (G)_k \wedge (G)_k \in [\delta\,(T_0)] - [T_1]).$$

Using g and G a winning strategy, σ_{II}, for player II can be informally described as follows. Given $s, t \in \mathbb{N}^{<\mathbb{N}}$, with $|s| = j+1$ and $|t| = j$, we define

$$\sigma_{\mathrm{II}}(s \otimes t) = \begin{cases} g(j) & \text{if } s \in S_1 \vee (s \notin S_0 \wedge \forall l\,(l < \min\{i : s[i] \notin S_0\} \rightarrow s[l] \in S_1)) \\ (G)_k(j) & \text{if } k = \min\{i : s\,[i+1] \notin S_1\}) \wedge (s \in S_0 - S_1 \vee \\ & s \notin S_0 \wedge \exists l\, s\,[l] \in S_0 - S_1 \wedge \forall i\,(t * \langle i \rangle \in T_0) \\ k & \text{if } s \notin S_0 \wedge \exists l\, s\,[l] \in S_0 - S_1 \wedge \\ & k = \min\{i : t * \langle i \rangle \notin T_0\} \end{cases}$$

<u>Case A.2</u>: $[\delta(T_0, T_1)] = \emptyset$ but $[\delta(S_0, S_1)] \neq \emptyset$.

Then $\delta(T_0, T_1)$ is a well–founded tree. Pick $f \in [\delta(S_0, S_1)]$. Then, $f \in [\delta_{S_0} S_1)]$ and

$$\forall k\, \exists f'\,(f[k] \subseteq f' \wedge f' \in [\delta\,(S_0)] - [S_1]).$$

By Σ^1_1–**AC**, there exists F such that

$$\forall k\,(f[k] \subseteq (F)_k \wedge (F)_k \in [\delta\,(S_0)] - [S_1]).$$

Then I wins the game with the following strategy, σ_{I}.

First we put $\sigma_{\mathrm{I}}(\langle\rangle) = f(0)$, and for all $s, t \in \mathbb{N}^{<\mathbb{N}}$ with $|s| = |t| = j \geq 1$ and $t \in \delta(T_0, T_1)$ we define

$$\sigma_{\mathrm{I}}(s \otimes t) = f(j).$$

Now at some point player II plays outside $\delta(T_0, T_1)$, since $\delta(T_0, T_1)$ is well–founded. So in this case player I decides his move depending on the position $t \notin \delta(T_0, T_1)$ of player II using a strategy similar to that of the proof of Theorem 4.37, part (1), case 2.

<u>Case A.3</u>: $[\delta(T_0, T_1)] = \emptyset$ and $[\delta(S_0, S_1)] = \emptyset$.

Then $G_L(A, B)$ is determined by Lemma 5.32.

<u>Case B</u>: $A(f) \in \neg(\Sigma^0_1)_2$ and $B(g) \in (\Sigma^0_1)_2$.
Then there exist *pruned* trees S_0, S_1, T_0, T_1 such that

1. $S_1 \subseteq S_0$ and $T_1 \subseteq T_0$.
2. $\neg A(f) \leftrightarrow f \in [S_0] - [S_1]$, and
3. $B(g) \leftrightarrow g \in [T_0] - [T_1]$.

We distinguish again three cases:

- Case B.1: $[\delta(S_0, S_1)] \neq \emptyset$.
  Let $f \in [\delta(S_0, S_1)]$. By Σ^1_1–**AC**, there exists F such that

$$\forall k\,(f[k] \subseteq (F)_k \wedge (F)_k \in [\delta\,(S_0)] - [S_1]).$$

Then I wins the game with the following strategy:
Fist $\sigma_{\mathrm{I}}(\langle\rangle) = f(0)$. Given $s, t \in \mathbb{N}^{<\mathbb{N}}$, with $|s| = |t| = j \geq 1$, we define

$$\sigma_{\mathrm{I}}(s \otimes t) = \begin{cases} f(j) & \text{if } t \in T_1 \vee (t \notin T_0 \wedge \forall l\,(l < \min\{i : t[i] \notin T_0\} \rightarrow t[l] \in T_1)) \\ (F)_k(j) & \text{if } k = \min\{i : t\,[i+1] \notin T_1\}) \wedge (t \in T_0 - T_1 \vee \\ & t \notin T_0 \wedge \exists l\, t\,[l] \in T_0 - T_1 \wedge \forall i\,(s * \langle i \rangle \in S_0) \\ k & \text{if } t \notin T_0 \wedge \exists l\, t\,[l] \in T_0 - T_1 \wedge \\ & k = \min\{i : s * \langle i \rangle \notin S_0\} \end{cases}$$

Case B.2: $[\delta(S_0, S_1)] = \emptyset$ and $[\delta(T_0, T_1)] \neq \emptyset$.
Let $g \in [\delta(T_0, T_1)]$ and, as in previous cases, let us fix G such that

$$\forall k\,(g[k] \subseteq (G)_k \wedge (G)_k \in [\delta\,(T_0)] - [T_1]).$$

Then, player II wins using the following strategy:
Given $s, t \in \mathbb{N}^{<\mathbb{N}}$, with $|s| = j+1$, $|t| = j$, and $s \in \delta(S_0, S_1)$ we define

$$\sigma_{\mathrm{II}}(s \otimes t) = g(j).$$

We know that at some point player I plays outside $\delta(S_0, S_1)$, since $\delta(S_0, S_1)$ is well–founded. So from that point player II decides her move depending on the position $s \notin \delta(S_0, S_1)$ of player I following a strategy similar to that of the proof of Theorem 4.37, part (1), case 2.

Case B.3: $[\delta(S_0, S_1)] = \emptyset$ and $[\delta(T_0, T_1)] = \emptyset$.
Then $G_L(A, B)$ is determined by Lemma 5.32.

(2): The cases to be considered are exactly the same as in the latter proof of Lipschitz determinacy. Since a winning strategy for player II in a Lipschitz game automatically gives rise to a strategy for player II in a Wadge game, the proofs in the cases where player II has a winning strategy are very similar. In the other cases, where player I has a winning strategy, it is not hard to see that the fact that player II can pass in a Wadge game can not be used to alter the final result. The strategies for player I are defined as in Theorem 4.37 using a function $ext : \mathbb{N}^{<\mathbb{N}} \rightarrow \mathbb{N}^{<\mathbb{N}}$ such that $ext(s)$ is the finite sequence obtained by dropping the zeros of the finite sequence s and decreasing the values by 1.
This completes the proof of the proposition. ∎

Corollary 5.34

1. Π^1_1-$\mathbf{CA}_0$ *proves* $((\Sigma^0_1)_2 \cup \neg(\Sigma^0_1)_2)$-$\mathbf{SLO}_L$.
2. Π^1_1-$\mathbf{CA}_0$ *proves* $((\Sigma^0_1)_2 \cup \neg(\Sigma^0_1)_2)$-$\mathbf{SLO}_W$.

Althought it seems plausible that Π^1_1-$\mathbf{CA}_0$ is already equivalent to $(\Sigma^0_1)_2$-$\mathbf{Det}_L$, we have not been able to obtain a reversal for Π^1_1-$\mathbf{CA}_0$ in terms of Lipschitz determinacy in Baire space. We then pose the following questions.

Problem 5.35

1. *Is* $((\Sigma^0_1)_2 \cup \neg(\Sigma^0_1)_2)$-$\mathbf{Det}_L$ *equivalent over* $\mathbf{RCA}_0$ *to* Π^1_1-$\mathbf{CA}_0$*?*
2. *Is* $((\Sigma^0_1)_2 \cup \neg(\Sigma^0_1)_2)$-$\mathbf{SLO}_L$ *equivalent over* $\mathbf{RCA}_0$ *to* Π^1_1-$\mathbf{CA}_0$*?*
3. *Is* $((\Sigma^0_1)_2 \cup \neg(\Sigma^0_1)_2)$-$\mathbf{Det}_W$ *equivalent over* $\mathbf{RCA}_0$ *to* Π^1_1-$\mathbf{CA}_0$*?*
4. *Is* $((\Sigma^0_1)_2 \cup \neg(\Sigma^0_1)_2)$-$\mathbf{SLO}_W$ *equivalent over* $\mathbf{RCA}_0$ *to* Π^1_1-$\mathbf{CA}_0$*?*

Chapter 6

Concluding remarks

In this chapter we recapitulate the main questions that have been raised throughout the thesis and that have been left unanswered, as well as the ways in which this future research can be developed. First of all, we recall the main goals of the thesis referred in the introduction:

- *to give direct proofs of the determinacy of Lipschitz and Wadge games for the first levels of the Wadge hierarchy;*
- *to formalize these proofs in the setting of second order arithmetic in order to calibrate the strength of Lipschitz and Wadge determinacy in terms of the (set existence) axioms needed to prove them, as a new contribution to the research program of Reverse Mathematics; and*
- t*o examine the relation between the semilinear order principle and the determinacy of Lipschitz and Wadge games in the formal context of second order arithmetic and to search for the axioms needed to prove the equivalence between these principles.*

The first goal was completely fulfilled for the first five levels of the Wadge hierarchy both in Cantor and in Baire space (Chapter 2). This result was grounded on a topological analysis of the structure of the residues of those sets which, together with their complements, are differences of closed sets. In order to extend this approach to further levels of the Wadge hierarchy, the underlying topological analysis would have to be extended too. We give some more details in Section 1 below.

The second and third goals were fulfilled only partially. We gave explicit formalizations of determinacy for Lipschitz and Wadge games, and the semilinear ordering principle, in second order arithmetic (Chapter 3). We obtained a good number of results showing natural subsystems of $\mathbf{Z}_2$ in which these principles are provable (Chapter 4 and 5). But we only were able to calibrate the *exact* strength of these principles in the following cases:

- Lipschitz determinacy and semilinear ordering principle for differences of closed sets in Cantor space (Corollary 4.42),

- Lipschitz determinacy for closed sets in Baire space (Theorem 5.28), and
- over the stronger base theory $\mathbf{ACA}_0$, Lipschitz determinacy and semilinear ordering principle for clopen sets in Baire space (Theorem 5.21).

A number of other calibrating questions remained open. We give some more details in Section 2 below.

6.1 Topological analysis of Lipschitz and Wadge games

The topological analysis of Chapter 2 was only carried out to the extent we needed for obtaining determinacy proofs for sets located in the first five Wadge degrees. Of course, it would be interesting to know if this analysis can be further extended to all finite levels, or even to the infinite ordinal levels, of the difference hierarchy, thus covering all complexity levels below $\boldsymbol{\Delta}^0_2$.

Let us formulate in some detail the finite case. As before $A \in \mathbf{Df}_n(X^\omega)$ means that $A \subseteq X^\omega$ is a finite difference of a sequence of closed sets F_0, ..., F_n of X^ω, i.e.

$$A = (F_0 - (F_1 - \cdots (F_{n-1} - F_n) \cdots))$$

Suppose $F, E \subseteq X^\omega$. We define a sequence $\{C_i : n \geq 1\}$ by putting $C_1 = \partial F - E$, and for any $n > 1$,

$$C_n = \begin{cases} \overline{C_{n-1}} \cap E & \text{if } n \text{ is even} \\ \overline{C_{n-1}} \cap E^c & \text{if } n \text{ is odd} \end{cases}$$

Thus, we have

$$\begin{aligned} C_1 &= \partial F - E \\ C_2 &= \overline{\partial F - E} \cap E \\ C_3 &= \overline{\overline{\partial F - E} \cap E} \cap E^c \\ &\vdots \end{aligned}$$

The key property we used in Chapter 2 and that we conjecture to be true for all finite levels is the following.

Proposition 6.1 *Let $A \in \mathbf{Df}_n(X^\omega)$ such that $A = F - E$ for some closed set F and $E \in \mathbf{Df}_{n-1}(X^\omega)$. Then:*

1. *If n is even, $A^c \in \mathbf{Df}_n(X^\omega)$ iff*

$$\mathrm{Rs}_n(A^c) = C_n \textit{ and } \mathrm{Rs}_n(A^c) = \emptyset.$$

2. *If n is odd, $A^c \in \mathbf{Df}_n(X^\omega)$ iff*

$$\mathrm{Rs}_n(A) = C_n \textit{ and } \mathrm{Rs}_n(A) = \emptyset.$$

Case $n = 1$ is immediate, and we have already proved case $n = 2$ in Chapter 2. We have also checked cases $n = 3$ and $n = 4$, but an inductive proof for all finite levels has been left pending.

Equipped with the above results, a natural line of future research could be

(L1) *to extend the topological analysis of games developed in Chapter 2 to all levels of the difference hierarchy in order to obtain a direct proof of Lipschitz and Wadge determinacy for* $\boldsymbol{\Delta}^0_2$ *sets.*

Hopefully, such a direct proof could help us obtain further reverse mathematics results for Lipschitz determinacy restricted to sets in the difference hierarchy.

6.2 Lipschitz and Wadge games in second order arithmetic

Several interesting problems concerning the reverse mathematics of Lipschitz and Wadge games are left to solve. Among others, the problems particularly interesting for the author are the following ones.

6.2.1 Games in Cantor space: subsystems $\mathbf{WKL}_0$ and $\mathbf{ACA}_0$

In Chapter 4 we have shown that $\mathbf{WKL}_0$ suffices for establishing the structure of clopen Lipschitz and Wadge degrees in the Cantor space. As consequence, we have also shown that the semilinear ordering principle for clopen Lipschitz and Wadge games is provable in $\mathbf{WKL}_0$. However, we were not able to obtain a reversal for $\mathbf{WKL}_0$ in terms of Lipschitz or Wadge determinacy or semilinear ordering principle. A natural candidate, we think, is Δ^0_1-$\mathbf{SLO}^*_L$. If Δ^0_1-$\mathbf{SLO}^*_L$ would imply $\mathbf{WKL}_0$ over $\mathbf{RCA}_0$, then $\mathbf{WKL}_0$, Δ^0_1-$\mathbf{Det}^*_L$, and Δ^0_1-$\mathbf{SLO}^*_L$ would be all equivalent over $\mathbf{RCA}_0$. Thus, the following questions are in order:

- Is $\mathbf{WKL}_0$ equivalent over $\mathbf{RCA}_0$ to Δ^0_1-$\mathbf{Det}^*_L$?
- Is $\mathbf{WKL}_0$ equivalent over $\mathbf{RCA}_0$ to Δ^0_1-$\mathbf{SLO}^*_L$?
- Is Δ^0_1-$\mathbf{Det}^*_L$ equivalent over $\mathbf{RCA}_0$ to Δ^0_1-$\mathbf{SLO}^*_L$?

We have also shown that subsystem $\mathbf{WKL}_0$ would be sufficient for obtaining Lipschitz and Wadge for open sets in Cantor space if we could prove the dichotomy principle ($\mathbf{DP}$) within $\mathbf{WKL}_0$ (see Corollary of Theorem 4.24).

Recall that ($\mathbf{DP}$) formalizes the basic topological property: "every closed set in Cantor space either has a nonempty boundary or is clopen."

We have only been able to prove ($\mathbf{DP}$) within $\mathbf{ACA}_0$. So, determining the exact strength of this principle seems to be of interest. However, it is also possible that, after all, ($\mathbf{DP}$) turns out to be too strong to be provable from $\mathbf{WKL}_0$; but, on the other hand, one could prove open Lipschitz determinacy within $\mathbf{WKL}_0$ by using a different argument. Thus several questions regarding determinacy for open sets are in order:

- Is (**DP**) provable in $\mathbf{WKL_0}$?
- Is Σ^0_1-$\mathbf{Det}^\star_{L/W}$ provable in $\mathbf{WKL_0}$?
- Is Σ^0_1-$\mathbf{SLO}^\star_{L/W}$ provable in $\mathbf{WKL_0}$?
- Is (**DP**) equivalent over $\mathbf{RCA_0}$ to $\mathbf{ACA_0}$?
- Is Σ^0_1-$\mathbf{Det}^\star_{L/W}$ equivalent over $\mathbf{RCA_0}$ to $\mathbf{ACA_0}$?
- Is Σ^0_1-$\mathbf{Det}^\star_{L/W}$ equivalent over $\mathbf{RCA_0}$ to Σ^0_1-$\mathbf{SLO}^*_L$?

Subsuming, we think that clopen/open Lipschitz determinacy in Cantor space deserves further investigations in a near future.

As to subsystem $\mathbf{ACA_0}$, the analysis of Lipschitz determinacy and semilinear ordering principle for differences of closed sets in Cantor space has been remarkably successful. We have shown that both principles are equivalent over $\mathbf{RCA_0}$ to $\mathbf{ACA_0}$ (see Corollary 4.42). Nonetheless several natural problems concerning Wadge determinacy for differences of closed sets in Cantor space remain still open. Namely:

- Is $\mathbf{ACA_0}$ equivalent over $\mathbf{RCA_0}$ to $(\Sigma^0_1)_2$-$\mathbf{Det}^*_W$?
- Are $(\Sigma^0_1)_2$-$\mathbf{Det}^*_L$ and $(\Sigma^0_1)_2$-$\mathbf{Det}^*_W$ equivalent over $\mathbf{RCA_0}$?
- Are $(\Sigma^0_1)_2$-$\mathbf{SLO}^*_L$ and $(\Sigma^0_1)_2$-$\mathbf{SLO}^*_W$ equivalent over $\mathbf{RCA_0}$?

Finally, we state a second line of future research, in connection with line **L1** stated above. We think that it is plausible to extend our methods in order to show that $\mathbf{ACA_0}$ can prove Lipschitz and Wadge determinacy not only for $(\Sigma^0_1)_2$ sets, but also for *all the finite levels* of the difference hierarchy of closed sets.

(L2) *To show that* $\mathbf{ACA_0}$ *is strong enough to formalize out topological analysis of games up to finite levels of the difference hierarchy, and thus obtain Lipschitz and Wadge determinacy for all finite levels of the difference hierarchy within* $\mathbf{ACA_0}$.

This result would be particularly interesting, because it can be seen as a variant of an old result announced by Steel. In fact, in his PhD [Stee77], Steel stated without a proof that $\mathbf{ACA_0}$ plus full induction can prove *Gale-Stewart* determinacy for all finite levels of the difference hierarchy. However, to the best of our knowledge, no precise proof of that result has been given in the literature. An argument pointing out that Steel's conjecture might be, after all, false cannot be found either.

6.2.2 Games in Baire space: subsystems $\mathbf{ATR}_0$ and Π^1_1-$\mathbf{CA}_0$

We have obtained two reversals in terms of Lipschitz determinacy for subsystem $\mathbf{ATR}_0$. The first one is proved over the ideal base theory $\mathbf{RCA}_0$ (see Theorem 5.28). The second one is proved over the stronger base theory $\mathbf{ACA}_0$, but it has the advantage of including an equivalence in terms of the semilinear ordering principle (see Theorem 5.21). Natural improvements of these results could be:

- Is $\mathbf{ATR}_0$ equivalent over $\mathbf{RCA}_0$ to Δ^0_1-$\mathbf{SLO}_L$?
- Is $\mathbf{ATR}_0$ equivalent over $\mathbf{RCA}_0$ to Σ^0_1-$\mathbf{SLO}_L$?
- Are Δ^0_1-$\mathbf{Det}_L$ and Σ^0_1-$\mathbf{Det}_L$ equivalent over $\mathbf{RCA}_0$?
- Are Δ^0_1-$\mathbf{SLO}_L$ and Σ^0_1-$\mathbf{SLO}_L$ equivalent over $\mathbf{RCA}_0$?
- Does Δ^0_1-$\mathbf{SLO}_L$ or Δ^0_1-$\mathbf{Det}_L$ imply $\mathbf{ACA}_0$ over $\mathbf{RCA}_0$?

Regarding Π^1_1–$\mathbf{CA}_0$, we have not been able to obtain a reversal for this subsystem in terms of Lipschitz determinacy or semilinear ordering principle in Baire space. So, this is a clear line of research to explore in a near future.

By analogy with our result for $\mathbf{ACA}_0$ in Cantor space (see Corollary 4.42), natural candidates for obtaining such a reversal are the principle of Lipschitz determinacy and the principle of semilinear order restricted to the second sevel of the difference hierarchy $(\Sigma^0_1)_2$. Indded, we also think that it is plausible to extend our topological analysis of games in Baire space in order to show that Π^1_1-$\mathbf{CA}_0$ can prove Lipschitz determinacy for all finite levels of the difference hierarchy. We pose both conjectures as pending questions.

- Is Π^1_1-$\mathbf{CA}_0$ equivalent over $\mathbf{RCA}_0$ to $(\Sigma^0_1)_2$-$\mathbf{Det}_L$?
- Is Π^1_1-$\mathbf{CA}_0$ equivalent over $\mathbf{RCA}_0$ to $(\Sigma^0_1)_2$-$\mathbf{SLO}_L$?
- Does Π^1_1-$\mathbf{CA}_0$ imply over $\mathbf{RCA}_0$ the principle of Lipschitz determinacy in Baire space for all finite levels of the difference hierarchy?

Finally, we would like to discuss two general lines of future research in the area which are natural extensions of the present work.

Firstly, note that we have obtained a number of reversals in terms of Lipschitz determinacy, but we have not been able to prove any reversal in terms of Wadge determinacy or Wadge semilinear ordering principle. Hence, a better understanding of the exact strength of Wadge determinacy is pending.

Let us observe that in the setting of set theory, Andretta [AA03] showed that the determinacy of all Lipschitz games, the determinacy of all Wadge games, and the semilinear ordering principle for Lipschitz maps are all equivalent. In addition, Andretta's proof is 'local' in the sense that it remains true when the sets range over some reasonably closed pointclass (e.g., $\mathbf{\Sigma}^0_n$ or $\mathbf{\Sigma}^1_n$). Thus, we propose:

(L3) *to formalize Andretta's proof within* $\mathbf{Z}_2$ *in order to obtain equivalences between subsystems of second order arithmetic and Wadge determinacy principles.*

Secondly, one of the main questions in the area is still open. Louveau and Saint-Raymond [LSR87] showed that $\mathbf{Z}_2$ can prove Lipschitz determinacy for all Borel sets. But determining the exact strength of Borel Lipschitz determinacy is still pending. Thus, an interesting line of research for the not distant future is:

(L4) *to study in detail Louveau and Saint-Raymond's proof in order to isolate a natural subsystem of second order arithmetic which can already prove full Borel Lipschitz determinacy, and to investigate whether that subsystem would turn out to be actually equivalent to Borel Lipschitz determinacy over a weak base theory.*

At first glance, this task seems to be quite difficult, since Louveau and Saint-Raymond's proof is very technical and elaborated. But a satisfactory answer would be, we think, a very nice result in the reverse mathematics of infinite games.

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