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Steiner distance and convexity in graphs

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Abstract

We use the Steiner distance to define a convexity in the vertex set of a graph, which has a nice behavior in the well-known class of HHD-free graphs. For this graph class, we prove that any Steiner tree of a vertex set is included into the geodesical convex hull of the set, which extends the well-known fact that the Euclidean convex hull contains at least one Steiner tree for any planar point set. We also characterize the graph class where Steiner convexity becomes a convex geometry, and provide a vertex set that allows us to rebuild any convex set, using convex hull operation, in any graph.

1. Introduction

The Steiner tree problem in networks, and particularly in graphs, was formulated quite recently – in 1971 – by Hakimi (see [10]) and Levi (see [13]). In the case of an unweighted, undirected graph, this problem consists of finding, for a subset of vertices A, a minimal-size connected subgraph that contains the vertices in A. Such a subgraph is a tree called a *Steiner tree* of A. The computational side of this problem has been widely studied, and it is known that it is an NP-hard problem for general graphs (see [12]).

Abstract convexity started to develop in the early fifties, with the searching of an axiom system to define a set to be convex, in order to generalize, in some way, the classical concept of Euclidean convex set. These concepts can be found in [15]. Among the wide variety of structures that has been studied under this point of view, such as metric spaces, ordered sets or lattices, we are particularly interested in graphs, where several convexities associated to the vertex set are well-known.

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Distance optimization properties of Steiner trees have given a way to define the Steiner distance as a generalization of the usual distance in graphs (see [3]). Following that, in Section 2 we define an abstract convexity in the context of graphs by means of the Steiner distance, and we compare it with other well-known convexities in graphs. Section 3 is devoted to studying the relation between Steiner trees and convex hulls in the class of HHD-free graphs, and we prove that any Steiner tree of a vertex set is included in the geodesical convex hull of the set. This result extends the well-known fact that the Euclidean convex hull contains at least one Steiner tree for any planar point set, and relates a polynomially solvable problem, such as the computation of the geodesical convex hull, with a NP-hard problem, the Steiner tree problem in graphs. Finally, in Section 4 we characterize the class of graphs where Steiner convexity becomes a convex geometry; that means that every convex set is the convex hull of its extreme points. Just some chordal graphs have this property, so we provide a kind of generalization of it, in the sense of finding a vertex set, bigger than extreme points, that plays the same role in any graph. All graphs considered here are finite, simple, unweighted and undirected.

2. The convexity associated to the Steiner distance

There are several well-known definitions of convex vertex sets in graphs, and these convexities are usually defined by means of certain paths. In this fashion, a subset S of vertices of a graph G is *monophonically* (*geodesically*) *convex* (see [7]) if S contains every vertex of any chordless (shortest) path between vertices in S. These sets are called m-convex sets (g-convex sets).

The definitions above follow the general scheme of abstract convexities. A family $\mathcal{C}$ of subsets of a set X is called a *convexity* (see [15]) on X if contains the empty set and universal set X, is closed under intersections, and is closed under nested unions; that is, if $\mathcal{D} \subseteq \mathcal{C}$ is non-empty and totally ordered by inclusion, then $\bigcup \mathcal{D}$ is in $\mathcal{C}$. Note that last property is trivial if X is a finite set. The elements of $\mathcal{C}$ are called *convex sets*. It is clear that any subset A of a convex structure is included in a smallest convex set, $\mathrm{CH}_{\mathcal{C}}(A) = \bigcap\{C \in \mathcal{C}: A \subseteq C\}$, called the *convex hull* of A. A point p in a convex set S is said to be an *extreme point* if $S \setminus \{p\}$ is convex. The preceding definitions correspond to m-convexity and g-convexity in graphs and their convex hulls are denoted by CH_m and CH_g respectively.

If G is a connected graph and A is a subset of vertices of G, then the *Steiner distance* of A denoted by $d_G(A)$ (see [3]) is the size $|T|$ (that is, the number of edges) of a smallest connected subgraph T of G that contains A. Such a subgraph is obviously a tree and is called a *Steiner tree* of A. Endvertices of any Steiner tree T of A belong to A, because an endvertex of T not in A can be removed from the tree in order to obtain a smaller tree.

Notice that the Steiner distance is, in some sense, a generalization of the usual distance between vertices, simply by considering that Steiner trees of 2-vertex subsets are shortest paths. In fact, there are a number of usual distance properties, such as being a distance stable graph (see [6,4]) or a distance-hereditary graph (see [11]), and distance invariants, such as eccentricity, radius, diameter, center or median, that have been extended to the Steiner distance (see [4,14]). This suggests us that a convexity can be defined using the Steiner distance in a similar way as usual distance is used to defined g-convexity.

Definition 1. Let G be a connected graph. A subset $S \subseteq V(G)$ is said to be *St-convex* if, for any $A \subseteq S$, all vertices in every Steiner tree of A belong to S. The family of all St-convex sets of $V(G)$ defines a convexity called St-convexity.

The usual convexities in the vertex set of a graph are path-convexities, defined by a system $\mathcal{P}$ of paths. Thus a subset S of vertices is said to be $\mathcal{P}$-convex if S contains all vertices on every $\mathcal{P}$-path between all pairs $u, v \in S$. This scheme is valid for m-convexity and g-convexity, among some others ones. However Steiner-convexity is defined in a different way, characterized by the use of subsets A of S with any cardinality, not just pairs of elements of S.

In spite of the different nature of the definitions, we can find a close relation between St-convexity and geodesic and monophonic convexities. Clearly, any m-convex set is also g-convex but not vice versa in general. So convex hulls satisfy the inclusion relation $\mathrm{CH}_g(S) \subseteq \mathrm{CH}_m(S)$, for any graph G and any subset $S \subseteq V(G)$. We will show that the St-convex hull can be properly placed between the two terms of the inclusion above. This relation becomes an equality, between St-convexity and g-convexity, or even between the three convexities, in some special cases.

A first approximation to the relation between g-convexity and St-convexity comes from the way the Steiner distance is a generalization of the geodesic distance in a graph, as we will see. On the other hand, the following lemma is the key to relating St-convexity and m-convexity. Recall that a *chord* on a path P in a graph G is an edge of G joining nonconsecutive vertices of P.

Lemma 2. *Let G be a connected graph and let $A \subseteq V(G)$. Then every internal point of a Steiner tree of A, is in A or lies on a chordless path between two vertices in A.*

Proof. Let u be an internal point of a Steiner tree T of A, $u \notin A$, and suppose that it does not lie on any chordless path between two vertices in A. Order the neighbors of u in T arbitrarily $\{u_1, u_2, \ldots, u_k\}$, and for each u_i, consider an endvertex a_i, which is necessarily in A, such that the path between a_i and u contains u_i. Then the path in T between any vertex pair a_i, a_{i+1}, $1 \leq i \leq k-1$, contains u, and by our hypothesis, it has a chord joining two vertices which leave u in the middle. Finally build a new tree removing from T the vertex u and the k edges which are incident with u, and adding the $k-1$ chords in paths between a_i and a_{i+1}, $1 \leq i \leq k-1$. So this new tree has smaller size than T and contains all vertices in A, which contradicts the choice of T as a Steiner tree of A. □

Now we can formulate the relation between three convex hulls in any graph.

Proposition 3. *Let G be a connected graph and let $S \subseteq V(G)$. Then the following chain of inclusions holds:* $\mathrm{CH}_g(S) \subseteq \mathrm{CH}_{\mathrm{St}}(S) \subseteq \mathrm{CH}_m(S)$.

Proof. It is clear that $\mathrm{CH}_{\mathrm{St}}(S)$ is a g-convex set, since Steiner trees of 2-element subsets of S are shortest paths. This gives the first inclusion.

For the second one, we will show that $\mathrm{CH}_m(S)$ is a St-convex set. Let $A \subseteq \mathrm{CH}_m(S)$ and T be a Steiner tree of A; then, using Lemma 2, any vertex in T is in A or in a chordless path between two vertices in A. So the vertex set of T is contained in $\mathrm{CH}_m(S)$. □

A whole class of graphs, where the inclusions in the proposition above are equalities, are distance-hereditary graphs. A graph G is said to be *distance-hereditary* (see [11]) if each connected induced subgraph F of G has the property that $d_F(u, v) = d_G(u, v)$ for each $u, v \in V(F)$. Therefore, in these graphs, the chordless paths between vertices are shortest paths, the geodesic and monophonic convexities coincide, and so also does Steiner convexity. Also, the next example shows that the inclusions are not equalities in general.

Example 4. In the graph in Fig. 1, the geodesic convex hull of the subset $\{a, b, c\}$ comprises the black vertices, the Steiner convex hull comprises the black and white vertices and the monophonic convex hull is the whole set of vertices.

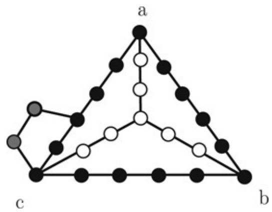

Fig. 1. A graph with non-equal geodesic, Steiner and monophonic convex hulls.

3. St-convexity in HHD-free graphs

A graph is called *house-hole-domino free (HHD-free)* (see [1]) if it contains no induced house, domino, or induced cycle C_k, $k \geq 5$. Several types of graphs are contained in the class of HHD-free graphs (see [1]), such as chordal and distance hereditary graphs among others. The relation between geodesic and Steiner convexities becomes an equality in the class of HHD-free graphs. The key to show this relation is the following result that relates Steiner trees of HHD-free graphs with geodesical convex hulls, in a similar way as they are related in the case of the Euclidean plane.

For any set S of vertices of a graph G, let T be an Steiner tree for S. We call vertices of S terminals and vertices of $T - S$ of degree ≥ 3 in T *Steiner points*. We will say that a Steiner point s is *adjacent* in T to a terminal t if the path connecting s and t in T does not contain other Steiner points. We call *peripheral* those Steiner points adjacent to at least two terminals.

Theorem 5. *For any set of vertices S of a connected HHD-free graph G, any Steiner tree T of S is contained in the geodesical convex hull of S.*

Proof. We will prove the assertion by induction on the cardinality of S, starting from the obvious case $|S| = 2$.

Let T be a Steiner tree of S, with $|S| = k \geq 3$. Firstly, we may assume that all terminals are placed on leaves of T. If some terminal $t \in S$ is an inner vertex of T, then T can be viewed as the union of subtrees $T_1, \ldots, T_m$ having t as a leaf. They are Steiner trees for sets of terminals $S_1, \ldots S_m$ located in these subtrees (notice that $T_i \cap T_j = S_i \cap S_j = \{t\}$, for all $1 \leq i < j \leq m$ and $S_1 \cup \cdots \cup S_m = S$). Our induction hypothesis gives $T_i \subseteq \mathrm{CH}_g(S_i)$, $i = 1, \ldots, m$. Since $\mathrm{CH}_g(S_i) \subseteq \mathrm{CH}_g(S)$, we obtain the desired inclusion $V(T) \subseteq \mathrm{CH}_g(S)$.

Let T be a Steiner tree of S where all terminals are leaves, so $|S| \geq 3$ assures us that T has at least one peripheral Steiner point s, adjacent to terminals t, t'.

Firstly suppose that a peripheral point s satisfies $s \in \mathrm{CH}_g(S)$, and let P be the path on T between t and t'. The minimality of T gives that P is the union of a shortest path P_t between t and s, and a shortest path $P_{t'}$ between t' and s, so in this case $V(P) \subseteq \mathrm{CH}_g(S)$.

Let T' be the tree obtained from T removing the path P, but keeping vertex s. Then T' is an Steiner tree for the set $S' = (S \setminus \{t, t'\}) \cup \{s\}$, because if T'_0 is a smaller tree, then adjoining to T'_0 the path P we obtain a connected subgraph spanning S, but having less vertices than T, contradicting the optimality of T. Hence T' is minimal, and the inductive hypothesis gives $V(T') \subseteq \mathrm{CH}_g(S')$. So, if $s \in \mathrm{CH}_g(S)$ we are done, because in this case $V(T') \subseteq \mathrm{CH}_g(S') \subseteq \mathrm{CH}_g(S)$ and $P \subseteq \mathrm{CH}_g(S)$, thus $V(T) = V(T') \cup V(P) \subseteq \mathrm{CH}_g(S)$.

Now we suppose that all peripheral Steiner points of T are located outside $\mathrm{CH}_g(S)$, and let s be such a point, adjacent to two terminals t and t'. Then the path P on T between t and t' is not a shortest path on G, because it contains s, but subpaths P_t and $P_{t'}$ between s and t and t'

respectively are shortest paths by minimality of T. Let u_1 be the neighbor of s in P_t, let v_1 be the neighbor of s in $P_{t'}$ and let Q be a shortest path in G between t and t'. Note that in case $u_1 = t$, the tree obtained from T removing subpath $P_{t'}$ but keeping vertex s and adding path Q, or has smaller size than T (if $|Q| < |P| - 1 = |P_{t'}|$) which is not possible, or it has the same size as T (if $|Q| = |P| - 1 = |P_{t'}|$) and it is a Steiner tree of S with a terminal $t = u_1$ which is not a leaf, so $s \in \mathrm{CH}_g(S)$, a contradiction with our hypothesis. Thus we may assume that $u_1 \neq t$, and using a similar argument, $v_1 \neq t'$.

Using that G is a HHD-free graph, we obtain that peripheral vertex s is a neighbor of a vertex q in $Q \setminus \{t, t'\}$ or u_1 and v_1 are adjacent in G (see Appendix).

Let us see that the peripheral vertex s, adjacent to two terminals t and t', cannot be a neighbor of any vertex q ($q \neq t, t'$) lying in a shortest path Q between both terminals. Suppose to the contrary that it can and call $S^* = (S \setminus \{t, t'\}) \cup \{q\}$, so $|S^*| < |S|$ and the inductive hypothesis gives that any Steiner tree T^* of S^* satisfies $V(T^*) \subseteq \mathrm{CH}_g(S^*) \subseteq \mathrm{CH}_g(S)$. We build a new tree T° removing from T path P, but keeping vertex s and adding edge sq. It is clear that T° contains vertex s, so $V(T^\circ) \nsubseteq \mathrm{CH}_g(S)$, but on the other hand it spans S^*, so T° is not a Steiner tree of S^*. Thus $|T^*| < |T^\circ| = |T'| + 1$ (remember that T' is the tree obtained from T removing path P but keeping vertex s). So $|T^*| \leq |T'|$ and $|T^*| + |Q| < |T'| + |P| = |T|$ (note that $|Q| < |P|$). Finally, the size of the tree spanned by $V(T^*) \cup V(Q)$, that covers S, is smaller than the size of T, which is a contradiction with the optimality of T.

Let us see now that vertices u_1 and v_1 cannot be neighbors in G, which is a contradiction with the hypothesis "all peripheral Steiner points of T are located outside $\mathrm{CH}_g(S)$", which concludes induction. Suppose that u_1 and v_1 are neighbors, and call T_1 the Steiner tree of S obtained from T by removing edge sv_1 and adding edge u_1v_1. Note that vertex sets of T and T_1 agree and u_1 becomes a peripheral Steiner point of T_1 adjacent to t and t'. If $u_1 \in \mathrm{CH}_g(S)$, using a preceding argument, $V(T) = V(T_1) \subseteq \mathrm{CH}_g(S)$, which is a contradiction with the hypothesis $s \notin \mathrm{CH}_g(S)$. So $u_1 \notin \mathrm{CH}_g(S)$ and T_1 is a Steiner tree on the same conditions than T. So if u_2 is the neighbor of u_1 in the path to t, again u_2 and v_1 are neighbors in G (because if u_1 is a neighbor of a vertex q in a shortest path between t and t', using a preceding argument, we obtain a tree smaller than T_1, so than T, spanning S, which is not possible). But if u_2 and v_1 are neighbors in G, the tree obtained from T removing vertex u_1 and edges su_1, u_1u_2 and adding edge v_1u_2, spans S and it has smaller size than T, which is not possible. □

Theorem 6. *Let G be a connected HHD-free graph and let $S \subseteq V(G)$. Then S is a g-convex set if and only if it is a St-convex set.*

Proof. The inclusion $\mathrm{CH}_g(S) \subseteq \mathrm{CH}_{\mathrm{St}}(S)$ in Proposition 3 gives the sufficiency. Conversely, suppose that S is a g-convex set and let T a Steiner tree in G of $A \subseteq S$; then $V(T) \subseteq \mathrm{CH}_g(A) \subseteq \mathrm{CH}_g(S) = S$, so S is St-convex. □

4. St-convexity as a convex geometry

A convexity is said to be a *convex geometry* (see [15]) if every convex set is the convex hull of its extreme points. This property gives good behavior to a convexity in graphs, because in this case we can keep all information about a convex vertex sets just in its extreme points. To find conditions under which St-convexity shares this property, we firstly need to characterize extreme points of a St-convex set. Recall that a vertex is called *simplicial* if its neighborhood is a complete subgraph. The next lemma shows that St-extreme points are simplicial vertices, the same condition as in case of g-convex sets.

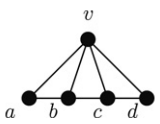

Fig. 2. 3-fan.

Lemma 7. *Let S be a St-convex vertex set of a connected graph G, then St-extreme points of S are the simplicial vertices in the subgraph induced by S.*

Proof. If $p \in S$ is a St-extreme point, then $S \setminus \{p\}$ is St-convex and so it is g-convex. Thus if p is a g-extreme point of S, it is simplicial (see [7]).

Reciprocally, let $p \in S$ be a simplicial vertex, and $A \subseteq S \setminus \{p\}$. Suppose that p is in a Steiner tree T of A. Using that p is simplicial in S, build a new tree T', removing from T edges which are incident to p and adding edges from one of the neighbors of p to all the other ones, and them to p. It is clear that T' has the same vertex set as T, and p is an endvertex of T'. Finally the tree that results by removing p from T' contains A and it is smaller than T, which is not possible because T is a Steiner tree of A. So p is not in any Steiner tree of $A \subseteq S \setminus \{p\}$, which means that $S \setminus \{p\}$ is a St-convex set and p is an St-extreme point of S. □

Note that, in any case, the extreme points of the convex hull of a set of vertices A belong to A, because if $p \in \mathrm{CH}(A) \setminus A$ is an extreme point; then $\mathrm{CH}(A) \setminus \{p\}$ is a convex set containing A, which contradicts the minimality of convex hull. Now we can characterize the class of graphs in which St-convexity becomes a convex geometry.

Theorem 8. *The St-convexity in a connected graph G is a convex geometry if and only if G is chordal and contains no induced 3-fan.*

Proof. For the sufficiency, note that a chordal graph with no induced 3-fan is distance-hereditary, so St-convexity coincides with g-convexity, which is a convex geometry (see [7]).

For the necessity, let G be a graph such that any St-convex vertex-set is the St-convex hull of its extreme points. Let us first see that G has no induced 3-fan. Suppose on the contrary, that $F = \{a, b, c, d, v\}$ (see Fig. 2) is the vertex set of an induced 3-fan in G.

We consider two cases.

1. If F is St-convex, then extreme points of F are a and d, whose St-convex hull does not contain b nor c, in contradiction with the hypothesis.
2. If F is not St-convex, then the extreme points of $\mathrm{CH}_{\mathrm{St}}(F)$ belong to $\{a, d\}$. If any of a or d are not extreme points, the contradiction with the hypothesis is clear. So suppose that both of them are the extreme points of $\mathrm{CH}_{\mathrm{St}}(F)$. Then $\mathrm{CH}_{\mathrm{St}}(\{a, d\})$ consists of a, d and their common neighbors, and hence contains neither b nor c.

Finally we show that G is chordal using the well-known following characterization of chordal graphs: every induced subgraph has a simplicial vertex. So let G' be an induced subgraph of G and $A = V(G')$. There are two cases.

1. If A is a St-convex set, by hypothesis, A is the St-convex hull of its extreme points, so A has some extreme points, which are simplicial vertices of G'.
2. If A is not St-convex, then let $S = \mathrm{CH}_{\mathrm{St}}(A)$ be. Again S is the St-convex hull of its extreme points, which are simplicial vertices of the subgraph induced by S. Let p be one of these extreme points; then $p \in A$ and so p is a simplicial vertex of G'. □

In order to generalize this property, we look for a vertex set, other than St-extreme points, that allows us to rebuild a St-convex set by means of St-convex hull operation further than chordal graphs without an induced 3-fan. In view of Theorem 8, this new vertex set must be bigger than the St-extreme vertices, and in order to obtain it, we will follow the ideas in [2], where contour vertices of a graph G are defined. Remember that, if v is a vertex of a connected graph G, the eccentricity $e(v)$ of v is defined by $e(v) = \max\{d(u, v) : u \in V(G)\}$. A vertex v is called a *contour vertex* if $e(v) \geq e(u),\ \forall u \in N[v]$ (see [2]).

We now use the n-eccentricity of a vertex, as defined in [3], to translate this concept to Steiner distance. Until the end of the paper, G will be a connected graph of order $p \geq 2$ and n will be an integer with $2 \leq n \leq p$. Let $S \subseteq V(G)$ and $v \in S$, the *n-eccentricity* $e_{n,S}(v)$ de v en S is defined by $e_{n,S}(v) = \max\{d_S(K) : K \subseteq S, |K| = n, v \in K\}$. In case $S = V(G)$, we denote $e_{n,S}$ simply by e_n. This concept is defined in [3], and it is used to obtain graph invariants similar to radius and diameter, and plays the role of eccentricity because it shares many of its properties. One of them, as can be seen in the next result, is that n-eccentricities of neighbors differ, at most, by one unit.

Proposition 9. *Let G be a connected graph and let uv be an edge of G, then $e_n(u) - 1 \leq e_n(v) \leq e_n(u) + 1$.*

Proof. Suppose on the contrary that $e_n(u) + 2 \leq e_n(v)$ and let $R \subseteq V(G)$ such that $|R| = n,\ v \in R, e_n(v) = d(R)$. We define $R' = R$ if $u \in R$, or $R' = (R \setminus \{v\}) \cup \{u\}$ if $u \notin R$. Thus $|R'| = n$ and $u \in R'$. Let T' be a Steiner tree of R'; then $|T'| = d(R') \leq e_n(u)$. Let us call T the tree obtained from T' adding edge uv, then T spans R and satisfies the inequality $|T| \leq |T'| + 1 \leq e_n(u) + 1 < e_n(u) + 2 \leq e_n(v) = d(R)$, which is not possible by the definition of $d(R)$. So $e_n(v) \leq e_n(u) + 1$, and analogously $e_n(u) \leq e_n(v) + 1$, as desired. □

Our main interest about n-eccentricity is that it will allow us to translate the definition of contour vertices to the environment of Steiner distance.

Definition 10. Let G be a connected graph and $S \subseteq V(G)$. A vertex $v \in S$ is called n-contour of S if it satisfies $e_{n,S}(v) \geq e_{n,S}(u),\ \forall u \in N_S[v]$. The set $\mathrm{Ct}_{n,S}(G)$ of n-contour vertices of S is called the n-contour set of S.

The first property we can observe in the n-contour set is that it is an enlargement of St-extreme point set.

Proposition 11. *Let G be a connected graph and $S \subseteq V(G)$. Then $\mathrm{Ct}_{n,S}(G)$ contains all St-extreme points of S.*

Proof. Let v be an St-extreme point of S; this means $N_S[v]$ induces a complete subgraph; let $u \in N_S[v]$. We will see that $e_{n,S}(v) \geq e_{n,S}(u)$. Let $R \subseteq S$ such that $|R| = n, u \in R,\ e_{n,S}(u) = d_S(R)$. We consider two different cases.

Firstly, if $v \in R$, it is clear that $e_{n,S}(v) \geq d_S(R) = e_{n,S}(u)$, as desired. So suppose now that $v \notin R$. Then we build $R' = (R \setminus \{u\}) \cup \{v\}$, and let T' be a Steiner tree of R' in S. If $u \in V(T')$, then T' spans R, and so $e_{n,S}(u) = d_S(R) \leq |T'| = d_S(R') \leq e_{n,S}(v)$, as desired. On the contrary, if $u \notin V(T')$, let w be a neighbor of v in T' and we build a tree T, from T', as follow: remove edges $xv,\ \forall x$ neighbor of v in T', remove vertex v and add vertex u. Using the fact that v is an St-extreme point of S, neighbors of v in S are also neighbors of w, so add edges $xw,\ \forall x$ neighbor of v in T', and edge uw.

It is clear that resulting subgraph T is a tree in S, with the same size than T' and spans R, so $e_{n,S}(u) = d_S(R) \leq |T| = |T'| = d_S(R') \leq e_{n,S}(v)$, which concludes proof. □

Now we prove the main result for n-contour vertices: these vertices can rebuild any St-convex set by means of a Steiner convex hull operation. This result provides, in some sense, a generalization of Theorem 8, using a vertex set bigger than St-extreme points, that works in any connected graph.

Theorem 12. *Let G be a connected graph and $S \subseteq V(G)$ a St-convex set. Then $\mathrm{CH_{St}}(\mathrm{Ct}_{n,S}(S)) = S$.*

Proof. Note that $\mathrm{Ct}_{n,S}(S) \subseteq S$, S St-convex, implies $\mathrm{CH_{St}}(\mathrm{Ct}_{n,S}(S)) \subseteq S$.

To complete the proof, suppose on the contrary that $S \setminus \mathrm{CH_{St}}(\mathrm{Ct}_{n,S}(S)) \neq \emptyset$ and let $u \in S \setminus \mathrm{CH_{St}}(\mathrm{Ct}_{n,S}(S))$ such that $e_{n,S}(u) \geq e_{n,S}(v)$, $\forall v \in S \setminus \mathrm{CH_{St}}(\mathrm{Ct}_{n,S}(S))$. Then $u \notin \mathrm{Ct}_{n,S}(S)$ and there exists $v \in N_S[u]$ such that $e_{n,S}(v) > e_{n,S}(u)$, so $v \in \mathrm{CH_{St}}(\mathrm{Ct}_{n,S}(S))$, by definition of u.

Let $R \subseteq S$ such that $|R| = n$, $v \in R$, $e_{n,S}(v) = d_S(R)$. We will see that $R \subseteq \mathrm{CH_{St}}(\mathrm{Ct}_{n,S}(S))$. Let $x \in R$, by definition of n-eccentricity, it is true that $e_{n,S}(x) \geq d(R) = e_{n,S}(v) > e_{n,S}(u)$, and so $x \in \mathrm{CH_{St}}(\mathrm{Ct}_{n,S}(S))$, as desired. In particular $u \notin R$, and we define $R' = (R \setminus \{v\}) \cup \{u\}$. It is clear that $R' \subseteq S$, $|R'| = n$, $u \in R'$, and thus $d_S(R') \leq e_{n,S}(u) < e_{n,S}(v) = d_S(R)$. Then $d_S(R') + 1 \leq d_S(R)$.

Let T' be an Steiner tree of R' in S, and we build a new tree T adding T' edge uv. Then T is a tree in S that spans R and it satisfies the inequality $|T| \leq |T'| + 1 = d_S(R') + 1 \leq d_S(R)$. So T is a Steiner tree of R in S that contains u.

Now, using that $R \subseteq \mathrm{CH_{St}}(\mathrm{Ct}_{n,S}(S))$, vertices in any Steiner tree of R belong to $\mathrm{CH_{St}}(\mathrm{Ct}_{n,S}(S))$, and so $u \in \mathrm{CH_{St}}(\mathrm{Ct}_{n,S}(S))$, which is a contradiction with the election of u. Finally $S \setminus \mathrm{CH_{St}}(\mathrm{Ct}_{n,S}(S)) = \emptyset$, and we are done. □

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Appendix

Using that G is a HHD-free graph, let us see that peripheral vertex s is a neighbor of a vertex q in $Q \setminus \{t, t'\}$ or u_1 and v_1 are adjacent in G. Note that, using the minimality of tree T, vertex v_1 cannot be a neighbor of any vertex in P_t other than s and (perhaps) u_1. In the same way, u_1 cannot be a neighbor of any vertex in $P_{t'}$ other than s and (perhaps) v_1. We denote by z the vertex in $V(P_t) \cap V(Q)$ further from t (maybe $z = t$), and we denote by z' the vertex in $V(P_{t'}) \cap V(Q)$ further from t' (maybe $z' = t'$). Consider the cycle C consisting of the subpath of P_t from s to z, the subpath of Q from z to z' and the subpath of $P_{t'}$ from z' to s.

Case 1: C is a 3-cycle, so it is $(z = u_1)s(v_1 = z')$, and u_1, v_1 are adjacent in G.

Note that if C is a n-cycle, with $n \geq 4$, we cannot have both $u_1 = z$ and $v_1 = z'$, because Q is a shortest path with $|Q| < |P|$, so $d_Q(z, z') < d_P(z, z')$. Thus, in the case where $u_1 = z$ and $v_1 = z'$, we obtain $d_P(z, z') = 2$ and $d_Q(z, z') = 1$, and C is a 3-cycle. In the following cases we may assume, without loss of generality, that $z' \neq v_1$.

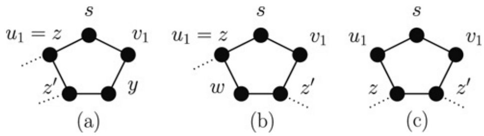

Fig. 3. C is a 5-cycle. Path Q in dotted line.

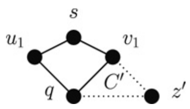

Fig. 4. The cycle C' in case C is a k-cycle, $k \geq 6$.

Case 2: C is a 4-cycle, so it is $(z = u_1)sv_1z'$. Then u_1 is a neighbor of vertex z', lying in $P_{t'}$, different from s and v_1, which is not possible. So C cannot be a 4-cycle.
Case 3: C is a 5-cycle. Then C is $(z = u_1)sv_1yz'$, with y a vertex in $P_{t'}$, neighbor v_1 and z', or $(z = u_1)sv_1z'w$, with w a vertex in Q neighbor of z and z', or zu_1sv_1z'. In the first case (see Fig. 3(a)) u_1 is a neighbor of z', lying in $P_{t'}$, different from s and v_1, which is not possible. In second case (see Fig. 3(b)), using that G is hole-free, the 5-cycle must have a chord which cannot be sz' by the minimality of $P_{t'}$, nor u_1z' by the minimality of Q, so it must be that u_1v_1, and u_1, v_1 are neighbors in G, or sw and s is a neighbor of a vertex in Q other than t, t', or v_1w, and we obtain a house, which cannot be induced, so u_1v_1 or sw must be edges (no other one is suitable). In the last case (see Fig. 3(c)), the 5-cycle must have a chord, which cannot be v_1z, u_1z', sz or sz' (by similar arguments that above), so it must be that u_1v_1 and u_1, v_1 are adjacent in G.
Case 4: C is a k-cycle with $k \geq 6$. Suppose, on the contrary, that neither s is a neighbor of any vertex in $Q \setminus \{t, t'\}$, and nor are u_1 and v_1 adjacent in G. Using that G is hole-free, edge u_1s must be contained in a triangle or in a 4-cycle in C (see Lemma 2.2 in [5]). Under our hypothesis, a triangle is not possible, because s has no neighbors in C other than u_1, v_1 and they are not neighbors, and there is just one suitable 4-cycle: u_1 and v_1 have a common neighbor q in the subpath of Q between z and z', so the 4-cycle is qu_1sv_1 (see Fig. 4). We pick vertex the q as close as possible to z'. Note that $q \neq z'$, because q is a neighbor of u_1 and u_1 cannot be a neighbor of z', which lies on $P_{t'}$ and it is not s or v_1. So edge qv_1, the subpath of $P_{t'}$ from v_1 to z' and the subpath of Q from z' to q, make a cycle C' (with length at least 3) (see Fig. 4). Using again that G is hole-free, edge v_1q must be contained in a triangle or in a 4-cycle with vertices in C'.

Firstly suppose that edge v_1q is contained in a triangle with vertices in C'. If the third vertex in the triangle is on Q, it must be q' the neighbor of q in C' other than v_1, by minimality of Q (see Fig. 5(a)). On the other hand, if the third vertex is on $P_{t'}$ it must be v_1' the neighbor of v_1 in C' other than q, by minimality of $P_{t'}$ (see Fig. 5(b)). In both cases following conditions gives an induced house, which is not possible in G: u_1 is no neighbor of v_1 by hypothesis, it is no neighbor of q', which is a neighbor of v_1 closer to z' than q, and it is no neighbor of v_1' by minimality of T, on the other hand vertex s is no neighbor of q nor q' by hypothesis and it is no neighbor of v_1' by minimality of $P_{t'}$. So edge v_1q is not contained in any triangle in C'.

Therefore edge v_1q must be contained in a 4-cycle in C'. The other two vertices in the four cycle can be both in $P_{t'}$, both in Q or one in $P_{t'}$ and the other one in Q. If both are in $P_{t'}$, the 4-cycle is $qv_1v_1'v_1''$ with v_1'' as the neighbor of v_1' in C' other than v_1 (see Fig. 6(a)). If they are both in Q, the 4-cycle is $v_1qq'q''$ with q'' as the neighbor of q' in C' other than q (see Fig. 6(b)).

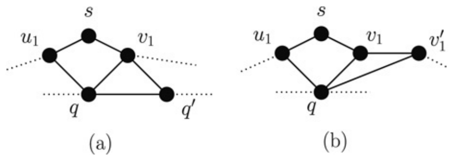

Fig. 5. Induced houses in G.

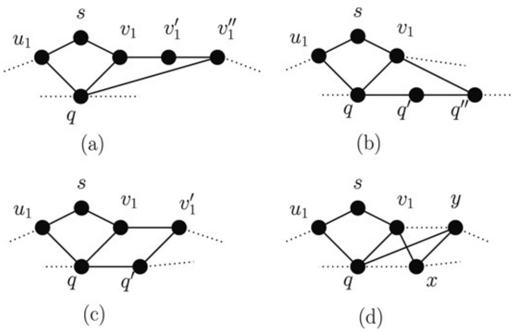

Fig. 6. Induced dominos in G.

And finally if one of them is on $P_{t'}$ and the other one is on Q, there are two suitable 4-cycles: $v_1 v_1' q' q$ (see Fig. 6(c)) and $v_1 x y q$ with x some vertex in Q between q and z' (other than q'), and y some vertex in $P_{t'}$ between v_1 and z' (other than v_1')(see Fig. 6(d)). In all cases, the following conditions give an induced domino, which is not possible in G: u_1 is not a neighbor of v_1 by hypothesis, it is not a neighbor of v_1', v_1'' nor y by the minimality of T, it is not a neighbor of q'' nor x which are neighbors of v_1 closer to z' than q and it is not a neighbor of q' because in this case $u_1 q' q'' v_1 s$ would be an induced 5-cycle. Vertex s is not a neighbor of q, q', q'' nor x by hypothesis, and it is not a neighbor of v_1', v_1'' or y by the minimality of $P_{t'}$. Vertex v_1 is not a neighbor of v_1'' or y by the minimality of $P_{t'}$, and it is not a neighbor of q' because edge $q v_1$ is not in any triangle of C'. And finally vertex q is not a neighbor of q'' or x by the minimality of Q, and it is not a neighbor of v_1' because edge $q v_1$ is not in any triangle of C'. This means that $v_1 q$ is not contained in a 4-cycle in C'.

But this is a contradiction with G being a hole-free graph. So in the case that C is a k-cycle with $k \geq 6$, our supposition is false and thus s is a neighbor of some vertex in $Q \setminus \{t, t'\}$, or u_1 and v_1 are adjacent in G, as desired. □

It is clear that algorithmic computation of the geodesic convex hull in a HHD-free graph is a polynomially solvable problem; however it is closely related to an NP-hard problem such as the Steiner tree problem (see [8]) (note that the class of HHD-free graphs includes chordal graphs). This result allows us to find, for $A \subseteq V(G)$, a particular set of vertices where any Steiner tree of A lies, but does not gives any Steiner tree in particular.

A similar result holds in the case of the Euclidean Steiner problem (see [9]) and the Rectilinear Steiner problem (see [12]), but note that in both these cases just one Steiner tree can be placed for sure into the convex hull, while for HHD-free graphs, all Steiner trees hold this property. Even in [15], it is shown that for every set of terminals in a median space, there exists at least one Steiner tree contained in the median hull of the terminals. Now we can easily deduce the equality between geodesic and Steiner convexities for HHD-free graphs.

References

[1] A. Brandstädt, V.B. Le, J.P. Spinrad, Graph Classes: A Survey, in: SIAM Monographs on Discrete Mathematics and Applications, Philadelphia, 1999.
[2] J. Cáceres, A. Márquez, O. Oellermann, M.L. Puertas, Rebuilding convex sets in graphs, Discrete Math. 297 (2005) 26–37.
[3] G. Chartrand, O.R. Oellermann, S. Tian, H.-B. Zou, Steiner distance in graphs, Math. Bohem. 144 (4) (1989) 399–410.
[4] W. Goddard, O.R. Oellrmann, H.C. Swart, Steiner distance stable graphs, Discrete Math. 132 (1994) 65–73.
[5] F.F. Dragan, F. Nicolai, A.A. Brandstädt, Convexity and HHD-free graphs, SIAM J. Discrete Math. 12 (1) (1999) 119–135.
[6] R.C. Entriger, D.E. Jackson, P. Slater, Geodetic connectivity of graphs, IEEE Trans. Circuits Syst. 27 (8) (1977) 460–463.
[7] M. Farber, R.E. Jamison, Convexity in graphs and hypergraphs, SIAM J. Alg. Disc. Math. 7 (3) (1986) 433–444.
[8] M. Farber, K. White, W. Pulleyblank, Steiner tree, connected domination and strongly chordal graphs, Networks 6 (1985) 109–124.
[9] E.N. Gilbert, H.O. Pollak, Steiner minimal trees, SIAM J. Appl. Math. 16 (1968) 1–29.
[10] S.L. Hakimi, Steiner's problem in graph and its implications, Networks 1 (1971) 113–133.
[11] E. Howorka, A characterization of distance-hereditary graphs, Q. J. Math. Oxford 28 (1977) 417–420.
[12] F.K. Hwang, D.S. Richards, P. Winter, The Steiner Tree Problem, North-Holland, Amsterdam, 1992.
[13] A.Y. Levi, Algorithm for shortest connection of a group of graph vertices, Sov. Math. Dokl. 12 (1971) 1477–1481.
[14] O.R. Oellermann, Steiner distance in graphs with emphasis on eccentricity measures: A survey, Cong. Numer. 99 (1994) 179–189.
[15] M.J.L. Van de Vel, Theory of Convex Structures, North-Holland, Amsterdam, 1993.