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Matrix valued orthogonal polynomials satisfying differential equations

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Resumen

The theory of matrix valued orthogonal polynomials goes back to the fundamental works of M. G. Krein [9, 10]. If one is considering possible applications of these polynomials, it is natural to concentrate on those cases where some extra property holds. In [5] the problem of characterizing those positive definite matrix valued weights whose matrix valued orthogonal polynomials satisfy second order differential equations is raised. The scalar situation brings the very well known families of Hermite, Laguerre and Jacobi polynomials (see, for instance [2]). Nevertheless, the matrix case is entirely different. The noncommutative product and the existence of singular matrices make us think that we are very far away from a classification theorem.

In this communication, we will show recent advances in this subject, focusing on new phenomena that are not possible in the scalar case. For instance, we can obtain several linearly independent second order differential having a fixed family of orthogonal polynomials as eigenfunctions or we have found families of orthogonal polynomials satisfying odd order differential operators.

1. Introduction

Given a self-adjoint positive definite matrix valued smooth weight function $W = W(t)$ with finite moments we can consider the skew symmetric bilinear form defined for any pair of matrix valued polynomial functions $P(t)$ and $Q(t)$ by the numerical matrix

$$(P, Q) = \int_{\mathbb{R}} P(t)W(t)Q^*(t)dt,$$

where $Q^*(t)$ denotes the conjugate transpose of $Q(t)$.

This leads, using the Gram–Schmidt process, to the existence of a sequence of matrix valued orthogonal polynomials $(P_n)_{n\geq 0}$ with non–singular leading coefficients such that

$$\int_{\mathbb{R}} P_n(t)W(t)P_m^*(t)dt = \Pi_n\delta_{n,m}.$$

Like in the scalar case, given an orthogonal sequence one gets a three-term recursion relation

$$tP_n(t) = A_nP_{n+1}(t) + B_nP_n(t) + C_nP_{n-1}(t), \quad n \geq 0, \quad P_{-1} = 0, \tag{1}$$

where A_n is non-singular.

In the quest of possible applications it is natural to concentrate on those cases where some additional property holds, such as the subject of this communication. Although initiated in [5], several years have been necessary to discover the first nontrivial examples of matrix valued orthogonal polynomials $(P_n)_n$ satisfying right–hand side second order differential equations of the form

$$P_n''(t)F_2(t) + P_n'(t)F_1(t) + P_n(t)F_0 = \Gamma_nP_n(t). \tag{2}$$

The coefficients F_2, F_1 and F_0 are matrix polynomials (which do not depend on n) of degrees less than or equal to 2, 1 and 0, respectively. This is equivalent to the symmetry of the second order differential operator

$$D = \partial^2F_2(t) + \partial^1F_1(t) + \partial^0F_0, \quad \partial = \frac{d}{dt}, \tag{3}$$

with respect to the weight matrix W. The symmetry of D with respect to W is defined by $\int D(P)dWQ^* = \int PdW(D(Q))^*$, for any matrix polynomials P and Q.

There are different ways to produce examples:

1. Solving an appropriate set of differential equations (see [6] or next section).
2. Coming directly from the study of matrix valued spherical functions (see [8] and [11]).
3. Solving an appropriate set of moment equations (equivalent to Point 1).
4. Solving the so called *ad-conditions*, relating coefficients of the three-term recurrence relation (1) and the eigenvalues Γ_n in (2) (see [4] for the statement and solution of the problem in the continuous–continuous scalar valued case).

We will concentrate on Point 1, as we will describe in the next section. These families of matrix valued orthogonal polynomials are among those that are likely to play in the case of matrix orthogonality the role of the classical families of Hermite, Laguerre and Jacobi in the case of scalar orthogonality.

2. Symmetry equations

Let $W(t)$ be a weight matrix. The symmetry of a second order differential operator D like (3) is equivalent to the following set of three matrix differential equations

$$F_2(t)W(t) = W(t)F_2^*(t), \tag{4}$$

$$2(F_2(t)W(t))' = F_1(t)W(t) + W(t)F_1^*(t), \tag{5}$$

$$(F_2(t)W(t))'' - (F_1(t)W(t))' + F_0W(t) = W(t)F_0^*. \tag{6}$$

with the corresponding boundary conditions that

$$F_2(t)W(t), \qquad (F_2(t)W(t))' - F_1(t)W(t) \tag{7}$$

should have vanishing limits at each of the endpoints of the support of $W(t)$. We remark that if we consider that everything commutes, the first equation (4) is trivial, the third equation (6) is the derivative of the second equation, which transform in the so-called *Pearson equation*. Solving the above differential equations is not a trivial task. A general method for solving these equations in the case of a leading scalar polynomial $F_2(t)$ can be found in [6], as well as a collection of illustrative new examples. All of those examples have the following structure

$$W(t) = \rho(t)T(t)T^*(t)$$

where ρ is one of the scalar weights of Hermite, Laguerre or Jacobi and T certain matrix polynomial (see [6] for details).

3. New phenomena

As more families of matrix valued orthogonal polynomials satisfying second order differential equations become available many new and certainly interesting phenomena are being discovered. These phenomena are absent in the well known scalar theory. We focus in one example for the two new phenomena in the list below:

a) For a fixed family of orthogonal polynomials one can find several linearly independent second order differential operators having them as common eigenfunctions. A collection of instructive examples can be found in [3]. We give here the more recent example in [7]. Let W be the following $N \times N$ weight matrix

$$W(t) = t^\alpha e^{-t} e^{At} t^{\frac{1}{2}J} t^{\frac{1}{2}J^*} e^{A^*t}, \quad t > 0, \quad \alpha > -1, \tag{8}$$

where A and J are the following matrices

$$A = \sum_{i=1}^{N-1} \nu_i E_{i,i+1}, \quad \nu_i \in \mathbb{C}, \quad i = 1, \cdots, N-1, \quad J = \sum_{i=1}^{N} (N-i) E_{ii}.$$

(We use E_{ij} to denote the matrix with entry (i,j) equal to 1 and 0 otherwise).

Then, there exist two symmetric linearly independent second order differential operators. A first one is given by

$$D_1 = \partial^2 tI + \partial^1[(\alpha+1)I + J + t(A-I)] + \partial^0[(J+\alpha I)A - J].$$

In order for (8) to generate another second order differential operator one has to assume the following conditions on the parameters in A:

$$i(N-i)\nu_{N-1}^2 = (N-1)\nu_i^2 + (N-i-1)\nu_i^2\nu_{N-1}^2, \; i = 1, \cdots, N-2.$$

Then, we can define another differential operator

$$D_2 = \partial^2 F_2(t) + \partial^1 F_1(t) + \partial^0 F_0,$$

where the coefficients are defined by

$$\begin{aligned} F_2 &= t(J - At), \\ F_1 &= (1+\alpha+J)J + Y - t(J + (\alpha+2)A + Y^* - AY + YA), \\ F_0 &= \frac{N-1}{\nu_{N-1}^2}[J - (\alpha+J)A], \end{aligned}$$

where $Y = \sum_{i=1}^{N-1} \frac{i(N-i)}{\nu_i} E_{i+1,1}$.

b) There are families of matrix valued orthogonal polynomials satisfying odd order differential equations. One can find examples satisfying first order differential operators in [1]. The weight matrix introduced below in (8) for $N = 2$ does not have symmetric first order differential operators, but *two linearly independent third* order differential operators. One is given by (putting $\nu_1 = a$):

$$\begin{aligned} D = & D^3 \begin{pmatrix} -|a|^2t^2 & at^2(1+|a|^2t) \\ -\bar{a}t & |a|^2t^2 \end{pmatrix} \\ & + D^2 \begin{pmatrix} -t(2+|a|^2(\alpha+5)) & at(2\alpha+4+t(1+|a|^2(\alpha+5))) \\ -\bar{a}(\alpha+2) & t(2+|a|^2(\alpha+2)) \end{pmatrix} \\ & + D^1 \begin{pmatrix} -2\alpha-4-2|a|^2(\alpha+2)+t & a(\alpha+1)(\alpha+2)+t\left(\frac{1+2|a|^2(1+|a|^2(\alpha+2))}{\bar{a}}\right) \\ -\frac{1}{a} & 2\alpha+2-t \end{pmatrix} \\ & + D^0 \begin{pmatrix} 1+\alpha & -\frac{1}{\bar{a}}(1+\alpha)(|a|^2\alpha-1) \\ \frac{1}{a} & -(1+\alpha) \end{pmatrix}. \end{aligned}$$

As more and more examples are discovered, more new phenomena are appearing. This allows us to think that the matrix case is much richer than the scalar one.

Finally, we would like to remark that as the examples of Hermite, Laguerre and Jacobi polynomials, which arose from concrete problems in the eighteen and nineteen centuries, have a extensive list of applications, we hope the matrix orthogonality to be more fruitful than the scalar situation. In this direction some progress has been made in fields like quantum mechanics, time-and-band limiting, random walks or birth-and-death processes.

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