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Miscellaneous properties of embeddings of line, total and middle graphs

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Abstract

Chartrand et al. (J. Combin. Theory Ser. B 10 (1971) 12–41) proved that the line graph of a graph G is outerplanar if and only if the total graph of G is planar. In this paper, we prove that these two conditions are equivalent to the middle graph of G been generalized outerplanar. Also, we show that a total graph is generalized outerplanar if and only if it is outerplanar. Later on, we characterize the graphs G such that $\mathscr{R}(G)$ is planar, where $\mathscr{R}$ is a composition of the operations line, middle and total graphs. Also, we give an algorithm which decides whether or not $\mathscr{R}(G)$ is planar in an $\mathcal{O}(n)$ time, where n is the number of vertices of G. Finally, we give two characterizations of graphs so that their total and middle graphs admit an embedding in the projective plane. The first characterization shows the properties that a graph must verify in order to have a projective total and middle graph. The second one is in terms of forbidden subgraphs.

1. Introduction

Sedláček defined in [14] a *generalized outerplanar* graph as a graph with a planar embedding such that at least one endvertex of each edge lies on the boundary of the outer face. Also, Sedláček proved in the same paper that a graph is generalized outerplanar if and only if it has no subgraph isomorphic to a subdivision of one of the graphs of Fig. 1.

The line graph of a graph G, denoted by $L(G)$, is defined by Whitney [17] as the graph whose vertices set is the edges set of G and two vertices are adjacent if they come from two incident edges. Akiyama et al. define in [2] the middle graph of G, $M(G)$, as the graph whose vertices set is the union of the sets of vertices and edges of G and two vertices are adjacent if they come from two incident edges or an incident vertex with an edge. The total graph of a graph G, $T(G)$, is a new graph

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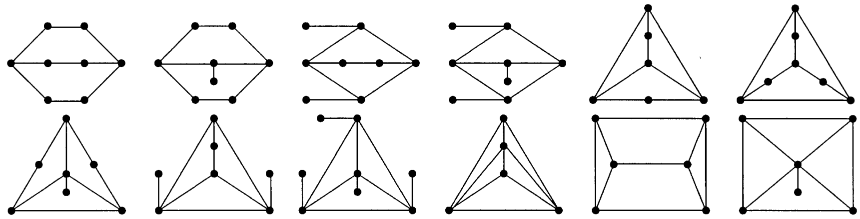


Fig. 1. Sedláček's list of forbidden generalized outerplanar subgraphs.

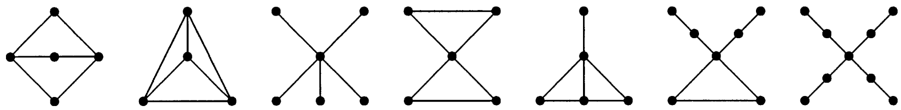


Fig. 2. Graphs with non-generalized outerplanar line graphs.

whose vertices set is the union of the vertices and edges sets of G. Two vertices of $T(G)$ are adjacent if they come from two adjacent vertices, two incident edges or an incident vertex with an edge [4].

In the literature some authors [1,2,4,6,8,14,16] have related line, middle and total graphs to planar or outerplanar graphs, giving characterizations of the graphs with a planar or outerplanar line, middle and total graphs. Furthermore, Sedláček characterized in [13] the graphs with generalized outerplanar line graphs in terms of forbidden subgraphs (see Fig. 2).

Chartrand et al. proved in [6] for any graph G, that $L(G)$ is outerplanar if and only if $T(G)$ is planar. In this paper, we show that these two conditions are equivalent to '$M(G)$ is generalized outerplanar'. Also, for the sake of completeness, we characterize the graphs with generalized outerplanar total graphs, proving that $T(G)$ is generalized outerplanar if and only if $T(G)$ is outerplanar.

In addition, we may consider the elemental operators L, M and T so that $L:G\mapsto L(G)$, $M:G\mapsto M(G)$ and $T:G\mapsto T(G)$. In a natural way, we have $L^n = L\circ L^{(n-1)}$, $M^n = M\circ M^{(n-1)}$ and $T^n = T\circ T^{(n-1)}$ for $n\geqslant 1$, where $L^1 = L$, $M^1 = M$ and $T^1 = T$. The planarity of $L^n(G)$, $M^n(G)$ and $T^n(G)$, $n\geqslant 1$, has been studied by several authors, in terms of the properties that the graphs must verify [2,4,11,13] and in terms of forbidden subgraphs [1,8]. However, the planarity of a composition of elemental operators, mixed among them, has not been specifically studied. In this paper, we characterize the graphs G such that $\mathscr{R}(G)$ is planar, where $\mathscr{R}$ is a composition of the operations line, middle and total graphs and we give an algorithm that decides if $\mathscr{R}(G)$ is planar in an $\mathcal{O}(n)$ time, where n is the number of vertices of G.

Embeddings of line, total and middle graphs in other surfaces have not been characterized, except in [5] where graphs with an embedable line graph in the projective plane are characterized. In this paper, we give a characterization of graphs such that their total and middle graphs admit an embedding in the projective plane. We consider the projective plane because it is the most simple surface after the plane and the sphere. Furthermore, embedable graphs in the projective plane are characterized [3,7].

2. Preliminaries

In this paper all graphs are finite, undirected and without loops and multiple edges. We follow the standard graph–theoretic terminology (for instance see [9]).

We denote the degree of a vertex v by $\delta(v)$ and the maximum degree of a vertex of a graph G by $\Delta(G)$. We say that a graph is projective if it admits an embedding into the projective plane.

Let G be a graph. A graph G' is a topological minor of G if a subdivision of G' is a subgraph of G.

Let G be a graph. A graph G' is a minor of G if G' is obtained from G by contracting or deleting edges or by deleting isolated vertices.

Clearly, if G_1 is a minor of G_2 and G_2 is generalized outerplanar, then G_1 is generalized outerplanar. Thus, generalized outerplanar graphs can be characterized in terms of forbidden minors. Using Sedláček's characterization of generalized outerplanar graphs, it is easy to check that a graph G is generalized outerplanar if and only if no graph of Sedláček's list is a minor of G.

It is not always true that if G_1 is a minor of G_2 then $T(G_1)$ is a minor of $T(G_2)$ and $M(G_1)$ is a minor of $M(G_2)$. However, we have the following result:

Lemma 1. *Let G_1 and G_2 be two graphs. If G_1 is a topological minor of G_2 then $T(G_1)$ is a minor of $T(G_2)$ and $M(G_1)$ is a minor of $M(G_2)$.*

Proof. Let G_3 be the subgraph of G_2 such that it is a subdivision of G_1. Clearly, $T(G_3)$ is a subgraph of $T(G_2)$ and $M(G_3)$ is a subgraph of $M(G_2)$. Thus, it is enough to prove that $T(G_1)$ is a minor of $T(G_3)$ and $M(G_1)$ is a minor of $M(G_3)$.

We can suppose, without losing generality, that G_3 is obtained from G_1 replacing only one edge, for instance the edge $\{u,w\}$, by the edges $x_1=\{u,v\}$ and $x_2=\{v,w\}$, where v is not a vertex of G_1.

We conclude the proof checking that $T(G_1)$ (resp. $M(G_1)$) is isomorphic to the graph obtained from $T(G_3)$ (resp. $M(G_3)$) contracting the edges $\{v,x_1\}$ and $\{v,x_2\}$. $\square$

Behzad characterized in [4] the graphs with a planar total graph with the following result:

Theorem 2. *A graph G has a planar total graph if and only if $\Delta(G)\leqslant 3$ and if $\delta(v)=3$ for some vertex v of G then v is a cutpoint.*

Let us follow with some auxiliary results.

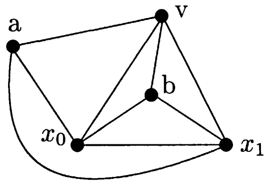


Fig. 3.

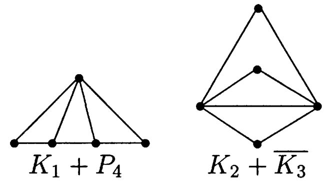


Fig. 4. $K_1 + P_4$ and $K_2 + \overline{K_3}$.

Lemma 3. *Let G be a graph such that $T(G)$ is planar. Let v be a vertex with $\delta(v)=2$ and such that $G - v$ is connected. Let x_0 and x_1 be the edges incident with v. Then there exists an embedding of $T(G)$ in the plane with v, x_0 and x_1 in the same face and, moreover, this face is a triangle.*

Proof. Let Γ be a plane embedding of $T(G)$. If v, x_0 and x_1 were not in the same face in Γ, we would have a similar situation to the one that Fig. 3 shows, where a and b are two elements of $T(G)$.

Since $G-v$ is connected, there exists a path in $T(G)$ between a and b not containing v, x_0 and x_1, but this path has to go through the triangle of vertices $\{v,x_0,x_1\}$ in a certain point so Γ is not an embedding. □

Greenwell and Hemminger proved in [9] the following result.

Lemma 4. *Let G be a finite graph and let v be a non-cutpoint vertex of G with $\delta(v) = 4$. Then, a subdivision of $K_1 + P_4$ or $K_2 + \overline{K_3}$ is a subgraph of G* (Fig. 4).

We follow with the last result of this section.

Lemma 5. *Let G be a planar graph with $\Delta(G)\leqslant 3$. Let $v_1,\dots,v_n$ be n vertices of G with $\delta(v_i) = 1$, $i = 1,\dots,n$ and such that there is an embedding of G in the plane with $v_1,\dots,v_n$ in the same face. Let $x_1,\dots,x_n$ be the edges incident with $v_1,\dots,v_n$ respectively. Then $M(G)$ is planar and there is an embedding of $M(G)$ in the plane such that $x_1,\dots,x_n$ are in the same face.*

Proof. Let Γ be a plane embedding of G with $v_1,\dots,v_n$ in the same face. We can build a plane embedding of $M(G)$ from Γ by drawing the middle point of all its edges and linking those ones which belong to incident edges. Since $\Delta(G)\leqslant 3$, every two incident edges share a face and so, there are not crossings among the new edges except in their vertices. It is obvious that $x_1,\dots,x_n$ are in the same face in the embedding of $M(G)$ from the construction we have done. □

3. Graphs with generalized outerplanar middle graphs

In this section we prove the following result:

Theorem 6. *Let G be a graph. The next three conditions are equivalent*:

(1) $L(G)$ *is outerplanar.*
(2) $T(G)$ *is planar.*
(3) $M(G)$ *is generalized outerplanar.*

Akiyama proved in [1] that the first two conditions are equivalent to G not having a subgraph homeomorphic to P_3+K_1 or $K_{1,4}$, thus, it is enough to prove the following theorem:

Theorem 7. *Let G be a graph. $M(G)$ is generalized outerplanar if and only if G has no subgraphs homeomorphic to P_3+K_1 or $K_{1,4}$.*

A graph has a generalized outerplanar middle graph if and only if each component has a generalized outerplanar middle graph. Also a graph has not a subgraph homeomorphic to P_3+K_1 or $K_{1,4}$ if and only if each component has not a subgraph homeomorphic to P_3+K_1 or $K_{1,4}$. Thus, we can suppose, without losing generality, that the graph G of Theorem 7 is connected.

In order to prove Theorem 7 we need the next result:

Lemma 8. *Let G be a connected graph such that $M(G)$ is generalized outerplanar and let v be a non-cutpoint vertex of G of degree 1 or 2. Then there exists a generalized outerplanar embedding of $M(G)$ with v on the boundary of the outer face.*

Proof. The result is obvious if the degree of v is 1. We suppose that the degree of v is 2 and there is a generalized outerplanar embedding Γ of G without v on the boundary of the outer face. Let x_1 and x_2 be the edges incident with v in G. As an endvertex of the edge $\{v,x_1\}$ of $M(G)$ and an endvertex of $\{v,x_2\}$ lie on the boundary of the outer face of Γ, the vertices x_1 and x_2 of $M(G)$ lie on the boundary of the outer face of Γ.

One of the components of $M(G)-\{x_1,x_2\}$ has only the vertex v. If $M(G)-\{x_1,x_2\}$ has only another component then it is easy to build from Γ a generalized outerplanar embedding of G with v in the outer face. If $M(G)-\{x_1,x_2\}$ has at least other two components G' and G'', let a_1 be a vertex of G' and let a_2 be a vertex of G''. a_1 and a_2 come from two elements (vertices or edges) of G and there is a path in G between a_1 and a_2 such that v, x_1 and x_2 are not in it, because v is not a cutpoint of G. This path induces a path in $M(G)$ between a_1 and a_2 such that x_1 and x_2 are not in it. Thus, we have a contradiction with the fact of a_1 and a_2 being in different components of $M(G)-\{x_1,x_2\}$ and we obtain the result. □

Now, we can prove Theorem 7.

Proof (Necessity). If G has a subgraph H homeomorphic to P_3+K_1 or $K_{1,4}$ then H is a subdivision of P_3+K_1 or $K_{1,4}$ and, by Lemma 1, $M(P_3+K_1)$ or $M(K_{1,4})$ is a

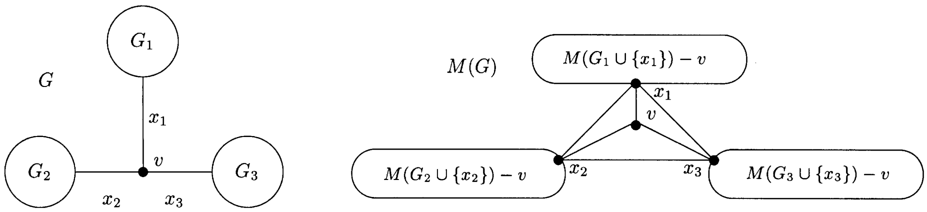


Fig. 5.

minor of $M(G)$. As $M(P_3+K_1)$ and $M(K_{1,4})$ have a subgraph isomorphic to the tenth graph of Fig. 1 then $M(P_3+K_1)$ and $M(K_{1,4})$ are non-generalized outerplanar. Thus, $M(G)$ is non-generalized outerplanar.

(*Sufficiency*). If G has not a subgraph homeomorphic to P_3+K_1 or $K_{1,4}$ then the maximum degree of the vertices of G is less than 4 and if a vertex has degree 3 then it is a cutpoint [1].

If G has not vertices with degree 3 then $M(G)$ is outerplanar [1]. Thus, $M(G)$ is generalized outerplanar and we can suppose that G has a vertex v with degree 3.

We prove the result by induction on the number of vertices of G. If the number of vertices of G is 4, the result is trivial. If the number of vertices of G is more than 3, then we consider the following two cases.

Case 1: *$G-v$ has* 3 *components.*

Let G_1, G_2 and G_3 be the 3 components of $G-v$ and let x_1, x_2 and x_3 be the edges incident with v and G_1, G_2 and G_3, respectively. By hypothesis of induction, $M(G_i \cup \{x_i\})$ is generalized outerplanar for $i=1,2,3$, and, by Lemma 8, there exist generalized outerplanar embeddings of $M(G_1 \cup \{x_1\})$, $M(G_2 \cup \{x_2\})$ and $M(G_3 \cup \{x_3\})$ with v on the boundary of the outer face. From these three embeddings, we can build a generalized outerplanar embedding of $M(G)$ (see Fig. 5).

Case 2: *$G-v$ has* 2 *components.*

Let G_1 be the component of $G-v$ with two vertices adjacent to v and G_2 another component. Let x_1 and x_2 be the two edges incident with v and G_1 and let x_3 be the edge incident with v and G_2. By hypothesis of induction, $M(G_1 \cup \{x_1,x_2\})$ and $M(G_2 \cup \{x_3\})$ are generalized outerplanar and, by Lemma 8, there exist generalized outerplanar embeddings of $M(G_1 \cup \{x_1,x_2\})$ and $M(G_2 \cup \{x_3\})$ with v on the boundary of the outer face. We can build a generalized outerplanar embedding of $M(G)$ from these two embeddings (see Fig. 6). $\square$

4. Graphs with generalized outerplanar total graphs

For the sake of completeness, we characterize the graphs with generalized outerplanar total graphs in terms of forbidden subgraphs.

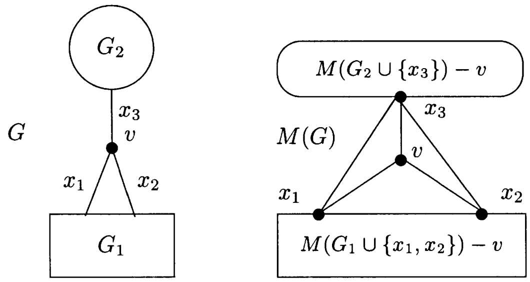


Fig. 6.

Theorem 9. *Let G be a graph. $T(G)$ is generalized outerplanar if and only if G has not a subgraph homeomorphic to K_3 or $K_{1,3}$.*

Proof (Necessity). If G has a subgraph H homeomorphic to K_3 or $K_{1,3}$ then H is a subdivision of K_3 or $K_{1,3}$ and, by Lemma 1, $T(K_3)$ or $T(K_{1,3})$ is a minor of $T(G)$. Since $T(K_{1,3})$ has a subgraph isomorphic to the eighth graph of Fig. 1 and $T(K_3)$ has a subgraph isomorphic to the eleventh graph of Fig. 1, $T(K_{1,3})$ and $T(K_3)$ are non-generalized outerplanar. Thus, $T(G)$ is non-generalized outerplanar.

(*Sufficiency*). If G has not a subgraph homeomorphic to K_3 or $K_{1,3}$ then $T(G)$ is outerplanar [1]. Thus, $T(G)$ is generalized outerplanar. □

Thus, we have the following result:

Corollary 10. *A total graph is generalized outerplanar if and only if it is outerplanar.*

Finally, we can complete the next table of forbidden subgraphs:

	Planar	Generalized outerplanar	Outerplanar
Line	$K_{3,3}$, $K_{1,5}$, P_4+K_1, $K_2+\overline{K_3}$ [9]	Seven graphs of Fig. 2 [16]	P_3+K_1, $K_{1,4}$ [1]
Middle	$K_{3,3}$, $K_{1,4}$ [1]	P_3+K_1, $K_{1,4}$	$K_{1,3}$ [1]
Total	P_3+K_1, $K_{1,4}$ [1]	$K_{1,3}$, K_3	$K_{1,3}$, K_3 [1]

5. A relation between E_n and the elemental operators L, M and T

The characterization of graphs with a planar iterated line, middle and total graphs is well known. In this section we characterize the graphs G such that $\mathscr{R}(G)$ is planar, where $\mathscr{R}$ is a composition of the operators line, middle and total graphs. Also, we give an algorithm that checks if $\mathscr{R}(G)$ is planar in an $\mathcal{O}(n)$ time, where n is the number of vertices of G.

Given a graph family F, let us denote $\mathscr{L}(F)=\{L(G)/G\in F\}$. In the same way, $\mathscr{M}(F)=\{M(G)/G\in F\}$ and $\mathscr{T}(F)=\{T(G)/G\in F\}$.

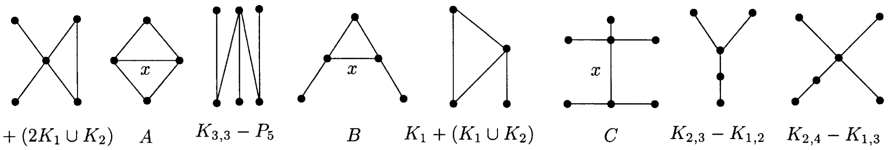


Fig. 7.

Let x be an edge of G. A graph G' is x-skew homeomorphic to G if G' is obtained from G by a finite sequence of subdivisions of some edges of G except x [1].

Given two graphs K and G, we denote $K \angle G$ if G contains a subdivision of K as a subgraph. We also denote $K \angle_x G$ if G has a subgraph x-skew homeomorphic to K.

We are going to define a partition $\{E_n,\ n \geqslant 0\}$ of the set of finite graphs so that $\bigcup_{n=1}^{\infty} E_n$ is the set of finite planar graphs and if N is one of the elemental operators, L, M or T, and G_1, G_2 are two graphs in the same family, then $N(G_1)$ and $N(G_2)$ are also in the same family. Furthermore, if $G \in E_k$ for some k and R is a composition of elemental operators, we will find the only n with $n \geqslant 0$ so that $R(G) \in E_n$.

Let us define some auxiliary families before defining the E_n families above mentioned. The graphs used in these definitions can be seen in Fig. 7.

$F_0 = \{G/K_{1,4} \angle G\}$, $F_1 = \{G/K_1 + (2K_1 \cup K_2) \angle G \textit{ or } A, B \textit{ or } C \angle_x G\}$, $F_2 = \{G/K_{2,4} - K_{1,3} \angle G\}$, $F_3 = \{G/A \textit{ or } B \angle_x G\}$, $F_4 = \{G/K_1 + P_3 \angle G\}$, $F_5 = \{G/B \angle_x G\}$, $F_6 = \{G/K_1 + (K_1 \cup K_2) \angle G\}$, $F_7 = \{G/K_{3,3} - P_5 \angle G\}$, $F_8 = \{G/K_{2,3} - K_{1,2} \angle G\}$, $F_9 = \{G/K_3 \angle G\}$, $F_{10} = \{G/K_{1,3} \angle G\}$.

Using the F_m families, for $m = 0, \ldots, 10$, let us define the E_n families:

$E_0 = \{G/K_5 \angle G\} \cup \{K_{3,3} \angle G\}$, $E_1 = (\{G/K_{1,5} \angle G\} \cup \{G/K_1 + P_4 \angle G\} \cup \{G/\overline{K_2 + K_3} \angle G\}) - E_0$, $E_2 = (F_0 \cap F_1) - (E_0 \cup E_1)$, $E_3 = F_2 - (E_0 \cup E_1 \cup F_1)$, $E_4 = F_0 - (E_0 \cup E_1 \cup F_1 \cup F_2)$, $E_5 = (F_3 \cap F_4) - (E_0 \cup F_0)$, $E_6 = F_4 - (E_0 \cup F_0 \cup F_3)$, $E_7 = F_5 - (F_0 \cup F_4)$, $E_8 = F_6 - (F_0 \cup F_4 \cup F_5)$, $E_9 = F_7 - (F_0 \cup F_6)$, $E_{10} = F_8 - (F_0 \cup F_6 \cup F_7)$, $E_{11} = F_9 - (F_0 \cup F_6 \cup F_8)$, $E_{12} = F_{10} - (F_0 \cup F_8 \cup F_9)$, $E_{n+12} = (\{G/P_n \subset G\} \cap \{G/P_{n+1} \not\subset G\}) - (F_9 \cup F_{10})$ *for* $n \geqslant 1$.

Notice that $\{E_n,\ n \geqslant 0\}$ is a partition of the set of finite graphs by the way they have been defined. Once we have defined the families E_n, $n \geqslant 0$ let us enunciate and prove the main result of this section.

Proposition 11. *In the above-mentioned conditions, we have the following statements*:

(1) $\mathscr{L}(E_0) \subset E_0$, $\mathscr{M}(E_0) \subset E_0$, $\mathscr{T}(E_0) \subset E_0$.
(2) $\mathscr{L}(E_1) \subset E_0$, $\mathscr{M}(E_1) \subset E_0$, $\mathscr{T}(E_1) \subset E_0$.
(3) $\mathscr{L}(E_2) \subset E_1$, $\mathscr{M}(E_2) \subset E_0$, $\mathscr{T}(E_2) \subset E_0$.
(4) $\mathscr{L}(E_3) \subset E_2$, $\mathscr{M}(E_3) \subset E_0$, $\mathscr{T}(E_3) \subset E_0$.
(5) $\mathscr{L}(E_4) \subset E_5$, $\mathscr{M}(E_4) \subset E_0$, $\mathscr{T}(E_4) \subset E_0$.
(6) $\mathscr{L}(E_5) \subset E_1$, $\mathscr{M}(E_5) \subset E_1$, $\mathscr{T}(E_5) \subset E_0$.
(7) $\mathscr{L}(E_6) \subset E_5$, $\mathscr{M}(E_6) \subset E_1$, $\mathscr{T}(E_6) \subset E_0$.
(8) $\mathscr{L}(E_7) \subset E_1$, $\mathscr{M}(E_7) \subset E_1$, $\mathscr{T}(E_7) \subset E_1$.

(9) $\mathscr{L}(E_8) \subset E_5$, $\mathscr{M}(E_8) \subset E_1$, $\mathscr{T}(E_8) \subset E_1$.
(10) $\mathscr{L}(E_9) \subset E_7$, $\mathscr{M}(E_9) \subset E_1$, $\mathscr{T}(E_9) \subset E_1$.
(11) $\mathscr{L}(E_{10}) \subset E_8$, $\mathscr{M}(E_{10}) \subset E_1$, $\mathscr{T}(E_{10}) \subset E_1$.
(12) $\mathscr{L}(E_{11}) \subset E_{11}$, $\mathscr{M}(E_{11}) \subset E_1$, $\mathscr{T}(E_{11}) \subset E_1$.
(13) $\mathscr{L}(E_{12}) \subset E_{11}$, $\mathscr{M}(E_{12}) \subset E_1$, $\mathscr{T}(E_{12}) \subset E_1$.
(14) $\mathscr{L}(E_{13})$, $\mathscr{M}(E_{13})$, $\mathscr{T}(E_{13}) \subset E_{13}$.
(15) $\mathscr{L}(E_{14}) \subset E_{13}$, $\mathscr{M}(E_{14}) \subset E_{15}$, $\mathscr{T}(E_{14}) \subset E_{11}$.
(16) $\mathscr{L}(E_{15}) \subset E_{14}$, $\mathscr{M}(E_{15}) \subset E_7$, $\mathscr{T}(E_{15}) \subset E_1$.
(17) $\mathscr{L}(E_{12+n}) \subset E_{11+n}$, $\mathscr{M}(E_{12+n}) \subset E_2$, $\mathscr{T}(E_{12+n}) \subset E_1$ *for* $n \geqslant 4$.

Proof.

(1) Notice that $K_2 + \overline{K_3}$ and $K_{1,4}$ are subgraphs of K_5 and $K_1 + P_3 \angle K_{3,3}$. So if K_5 or $K_{3,3} \angle G$ then $L(G)$, $M(G)$ and $T(G)$ are not planar by [1,9]. Then, they contain a subdivision of K_5 or $K_{3,3}$ by [12].
(2) It is evident by [1,8].
(3) Notice that $L^2(G)$ is not planar by [1]. Thus, by [9] $K_1 + P_4$, $K_{1,5}$ or $K_2 + \overline{K_3} \angle L^1(G)$, but not $K_{3,3}$ because it is not contained in G. It is evident for $M(G)$ and $T(G)$ by [1].
(4) $K_{1,4} \angle L(K_{2,4} - K_{1,3})$ and $A \angle_x L(K_{2,4} - K_{1,3})$. Furthermore, $K_{1,4} \angle K_{2,4} - K_{1,3}$ so if $G \in E_3$ then $M(G)$ and $T(G)$ contain K_5.
(5) Notice that $K_1 + P_3 \angle L(K_{1,4})$, $A \angle_x L(K_{1,4})$ and $L(G)$ does not contain $K_{1,4}$ for any $G \in E_4$. It is evident for $\mathscr{M}$ and $\mathscr{T}$ by [1].
(6) It is evident by [1,8].
(7) It is evident for $\mathscr{M}$ and $\mathscr{T}$ by [1]. Furthermore, if A or $B \angle G$, $G \in E_6$ the x edge must be subdivided, so $L(G)$ contains B and it does not contain $K_{1,4}$.
To finish the proof, notice that the statements (8) to (17) follow from [1] and the way in which E_n has been defined, with $n \geqslant 7$. □

In Fig. 8 can be seen a graphic representation of the statement of Proposition 11.

As a consequence of this result, we have the following theorem.

Theorem 12. *Let $G \in E_n$ and let $\mathscr{R}$ be a composition of the L, M and T operators. Then $\mathscr{R}(G)$ is at the end of the walk in Fig.* 8 *starting in E_n and containing the operations of $\mathscr{R}$.*

From Theorem 12, we obtain the result about the planarity of $\mathscr{R}(G)$ that we mean above.

Corollary 13. *Let $G \in E_n$ and let $\mathscr{R}$ be a composition of the L, M and T operators. $\mathscr{R}(G)$ is planar if and only if the walk in Fig.* 8 *starting in E_n and containing the operations of $\mathscr{R}$ ends at E_0.*

Using this result, we can test the planarity of $\mathscr{R}(G)$ according to the following theorem.

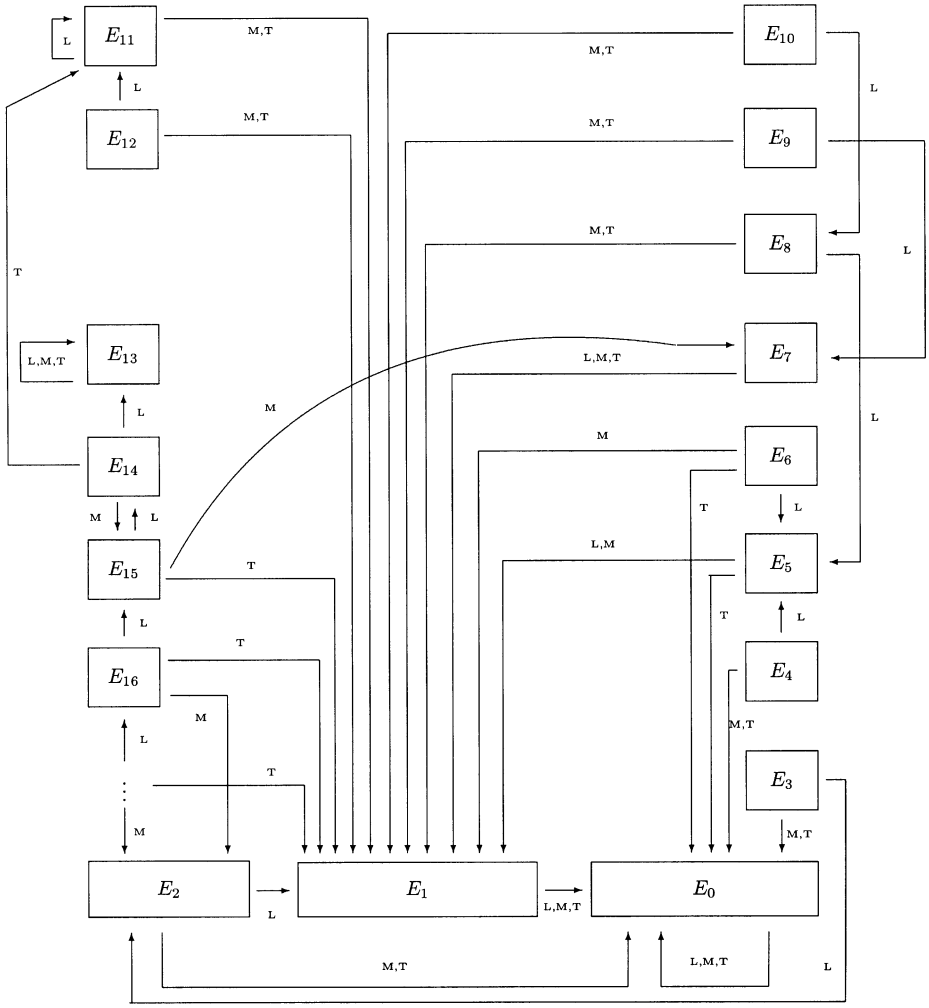


Fig. 8.

Theorem 14. *Let $G \in E_n$ then the planarity of $\mathscr{R}(G)$ can be tested in linear time with respect to the number of operations of $\mathscr{R}$.*

Note: This time does not depend on the number of vertices and edges of G.

6. An algorithm to test E_n

Let G be a graph. If we are able to check the E_n such that $G \in E_n$ in linear time then we also have the planarity of $\mathscr{R}(G)$ in linear time, according to the results in the last section.

For the sake of completeness, let us show an algorithm to check the E_n such that $G \in E_n$, for a given graph G. This algorithm uses a planarity algorithm in linear time. There are some algorithms for testing planarity in linear time (see, for instance, [10]).

Description of the algorithm

Step 1: If $v \in V(G)$ exists so that $\delta(v) \geqslant 4$ go to **Step 6**.
Step 2: If $v \in V(G)$ exists so that $\delta(v) = 3$ then go to **Step 4**.
Step 3: If G contains a cycle then $\boxed{G \in E_{11}}$. **END**.
Else let N be the number of vertices of the longest path of G. Then $\boxed{G \in E_{N+12}}$. **END**.
Step 4: If $v \in V(G)$ exists so that $\delta(v) = 3$ and $\delta(v_1) + \delta(v_2) + \delta(v_3) > 3$ where v_1, v_2 and v_3 are the vertices adjacent to v then go to **Step 6**.
Step 5: If G contains a cycle then $\boxed{G \in E_{11}}$. **END**.
Else $\boxed{G \in E_{12}}$. **END.**
Step 6: Check the planarity of G. If G is not planar then $\boxed{G \in E_0}$. **END**.
Step 7: Check the planarity of $L(G)$. If $L(G)$ is not planar then $\boxed{G \in E_1}$. **END**.
Step 8: Check the planarity of $L^2(G)$. If $L^2(G)$ is not planar then:
(a) Check the planarity of $M(G)$. If $M(G)$ is not planar then $\boxed{G \in E_2}$. **END**.
(b) Check the planarity of $T(G)$. If $T(G)$ is not planar then $\boxed{G \in E_5}$. **END**.
else $\boxed{G \in E_7}$. **END.**
Step 9: Check the planarity of $L^3(G)$. If $L^3(G)$ is planar then $\boxed{G \in E_{10}}$. **END**.
Step 10: Check the planarity of $M(G)$. If $M(G)$ is not planar then check the planarity of $M(L(G))$. If $M(L(G))$ is not planar then $\boxed{G \in E_3}$. **END**.
else $\boxed{G \in E_4}$. **END**.
else:
(a) Check the planarity of $T(G)$. If $T(G)$ is not planar then $\boxed{G \in E_6}$. **END**.
(b) Check the planarity of $T(L(G))$. If $T(L(G))$ is not planar then $\boxed{G \in E_8}$. **END**.
else $\boxed{G \in E_9}$. **END.**

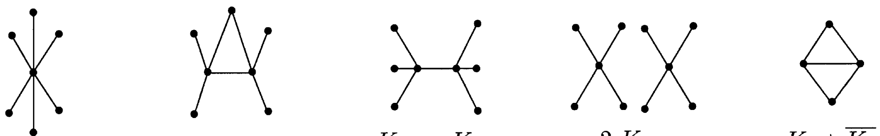
Notice that the algorithm works in an $\mathcal{O}(n)$ time, where n is the number of vertices of G. We have already remarked that testing planarity has a linear time complexity [10]. To check the planarity of $L(G)$, G must be planar. If G is a graph with n vertices and m edges, then $m \leqslant 3n - 6$. So we can test the planarity of $L(G)$ in an $\mathcal{O}(n)$ time.

Testing $L^2(G)$, $L^3(G)$, $M(G)$, $T(G)$, $M(L(G))$ and $T(L(G))$ can be done in an $\mathcal{O}(n)$ time by a reasoning similar to the one given above.

7. Graphs with a projective total graph

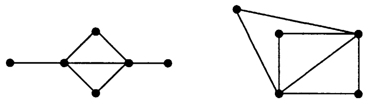
Up to now, works about characterizations of embeddings of line, total and middle graphs are referred to the plane, except [5], where graphs with a projective line graph are characterized. In this section, we characterize the graphs with a projective total graph.

Now let us prove the main result of this section.



$K_{1,6}$ $\quad (K_2 + 5K_1) - 2K_{1,2} \quad$ $K_{4,4} - K_{3,3}$ $\quad 2\,K_{1,4} \quad$ $K_2 + \overline{K_2}$

Fig. 9. List $\mathscr{L}$ of forbidden topological minors for a projective total graph.



G_1 $\quad G_2$

Fig. 10.

Theorem 15. *Let G be a finite graph. The following conditions are equivalent*:

(1) *$T(G)$ is projective.*
(2) *G satisfies the following conditions*:
  (a) *$\Delta(G) \leqslant 5$.*
  (a) *G has, at most, a vertex of degree more than or equal to* 4.
  (c) *If $\delta(v) = 3$ for some vertex v of G then v is a cutpoint.*
(3) *No graph in list $\mathscr{L}$ is a topological minor of G* (*see Fig. 9*).

Proof. Let us see that the third condition implies the second one.

If $\Delta(G) > 5$ then G contains to $K_{1,6}$ as a subgraph.

If G has, at least, two vertices of degree more than or equal to 4, then $(K_2 + 5K_1) - 2K_{1,2}$, $K_{4,4} - K_{3,3}$, $2\,K_{1,4}$ or one of the graphs in Fig. 10 is a topological minor of G, where $K_2 + \overline{K_2}$ is a subgraph of the last two. Finally, if there is a vertex of degree 3 which is not a cutpoint then $K_2 + \overline{K_2}$ is a topological minor of G.

Now, let us prove that the first condition implies the third one. Using Lemma 1, we only need to prove that $T(F)$ is not projective for every $F \in \mathscr{L}$. Archdeacon, Glover, Huneke and Wang, in [3,7], gave the list of 35 forbidden minors for the projective plane. Twelve of them are shown in Fig. 11.

Notice that B_1 is a minor of $T(K_{1,6})$; A_1 is a minor of $T((K_2 + 5K_1) - 2K_{1,2})$ and $T(K_{4,4} - K_{3,3})$; A_5 is a minor of $T(K_{1,4})$, and E_{22} is a minor of $T(K_2 + \overline{K_2})$.

To complete the proof, let us see that the second condition implies the first one.

Suppose that a vertex v of G of degree 5 exists. Since $\Delta(G - v) \leqslant 3$ and every vertex of degree 3 is a cutpoint, we have v is a cutpoint and $K_2 + \overline{K_2}$ cannot be a topological minor of G. So G must be one like those shown in Fig. 12, where $\Delta(G_i) \leqslant 3$, $i = 1, \ldots, 5$.

Let us consider the first case. Notice that $T(G_i)$, $i = 1, 2, 3$, are planar according to [4]. Moreover, by Lemma 3, a plane embedding of $T(G_1 \cup \{x_1, x_2\})$ and $T(G_2 \cup \{x_3, x_4\})$ with

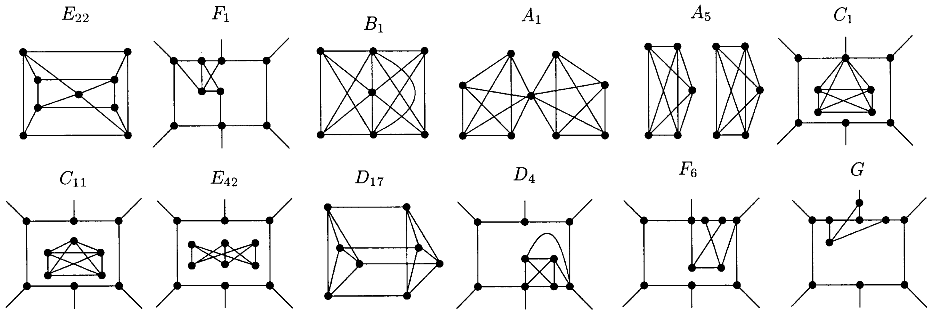


Fig. 11. Twelve forbidden minors for the projective plane.

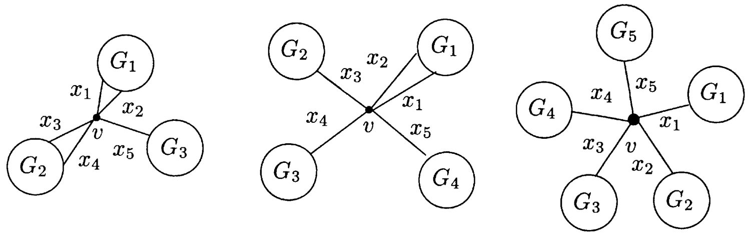


Fig. 12.

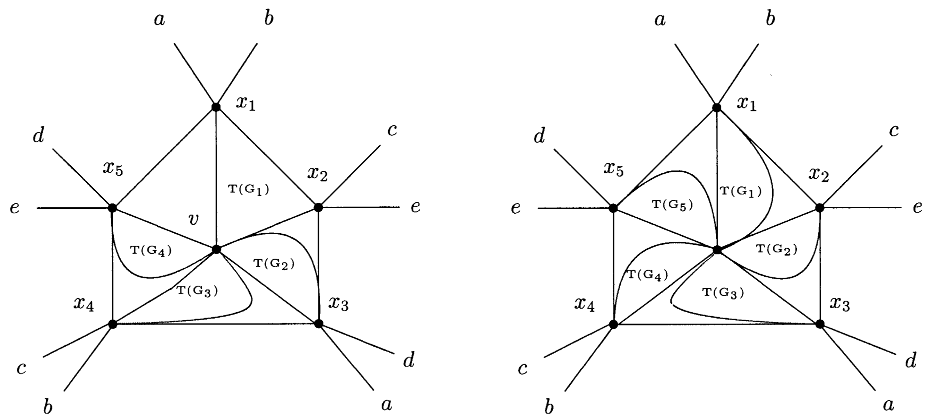


Fig. 13.

v, x_1, x_2 in a triangular face and v, x_3, x_4 in another one exists. Also a plane embedding of $T(G_3 \cup \{x_5\})$ with v, x_5 in a same face exists.

From this point we can give an embedding of $T(G)$ in the projective plane according to Fig. 13, where $T(G_1)$, $T(G_2)$ and $T(G_3)$ can be drawn in a plane way in the faces vx_1x_2, vx_3x_4 and vx_5x_2 respectively. So $T(G)$ is projective.

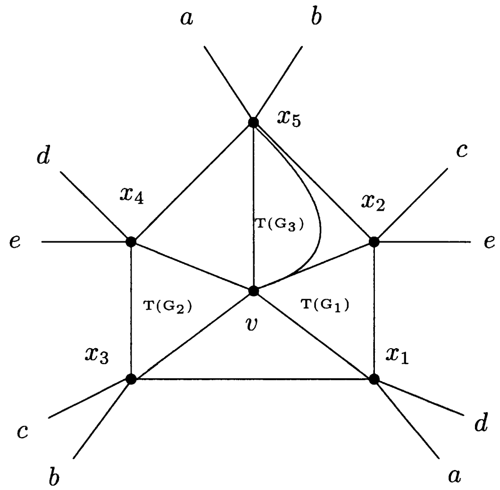


Fig. 14.

An embedding of the total graph of the second and third graphs on Fig. 12 into the projective plane can be built in a similar way as Fig. 14 shows.

Suppose that G has a vertex v of degree 4 and let us consider $G' = G \cup \{x\}$ where x is a new edge incident with v. The graph G' satisfies the third condition of Theorem 15. We have $\delta(v) = 5$ in G' so, according to the previous case, $T(G')$ is projective. $T(G)$ is a subgraph of $T(G')$ so $T(G)$ is projective.

Finally, if $\Delta(G) \leqslant 3$ we have $T(G)$ is plane by Theorem 2, which implies that $T(G)$ is projective, and the proof is complete. □

8. Graphs with a projective middle graph

In this section, we give the next characterization of graphs with a projective middle graph.

Theorem 16. *Let G be a graph. The following conditions are equivalent.*

(1) *$M(G)$ is projective.*
(2) *G verifies the following conditions*:
  (a) *$\Delta(G) \leqslant 5$.*
  (b) *G has, at most, a vertex of degree more than or equal to* 4.
  (c) *If $\delta(v) = 5$ for some vertex v of G, then there are four edges not incident with it in the same block.*
  (d) *If $\Delta(G) \geqslant 4$ then G is planar.*
  (e) *G is projective.*
(3) *G does not contain any subdivision of a graph in list $\mathscr{D}$* (*see Fig. 15*).

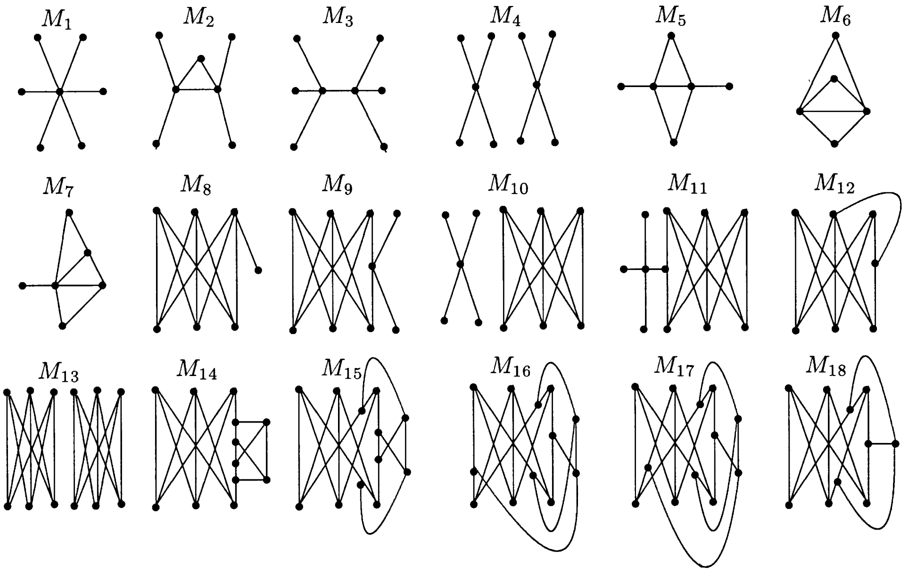


Fig. 15. List $\mathscr{D}$ of graphs with a non-projective middle graph.

Proof. Let us prove that the first condition implies the third one.

No graph of $\mathscr{D}$ has a projective middle graph because every one of them contains a forbidden minor for the projective plane.

Using the nomenclature given in [7], we can see that B_1 is a minor of $M(M_1)$, A_1 of $M(M_2)$, $M(M_3)$, $M(M_5)$; A_5 of $M(M_4)$, $M(M_6)$; G of $M(M_8)$, $M(M_{12})$; D_4 of $M(M_9)$; C_{11} of $M(M_{10})$; C_1 of $M(M_{11})$; D_{17} of $M(M_7)$; E_{42} of $M(M_{13})$; F_6 of $M(M_{14})$, $M(M_{15})$ and F_1 of $M(M_{16})$, $M(M_{17})$, $M(M_{18})$. Therefore, by Lemma 1, if G contains a subdivision of a graph of $\mathscr{D}$, its middle graph is a minor of $M(G)$ so $M(G)$ is not projective.

Now, let us prove that the third condition implies the second one. If $\Delta(G) > 5$ then G contains M_1. If G has, at least, two vertices of degree more than or equal to 4 then M_2, M_3, M_4, M_5 or M_6 is a topological minor of G.

Let v be a vertex of G so that $\delta(v) = 5$ and there are four edges incident with v in the same block. Then, by Lemma 4, we have G contains a subdivision of M_7 or M_6.

If $\delta(v) \geqslant 4$ for some $v \in V(G)$ and G is not planar then G contains a subdivision S of $K_{3,3}$ because K_5 has more than one vertex of degree 4.

Given a subdivision S of $K_{3,3}$, we say that $w \in V(S)$ is a $K_{3,3}$-main vertex of S if $\delta(w) = 3$ in S.

We distinguish several cases:

(1) v is a $K_{3,3}$-main vertex of S. Then G must contain M_8 or M_{12}.
(2) $v \in V(S)$ but v is not a $K_{3,3}$-main vertex of S. Then, G must contain M_9 or another subdivision of $K_{3,3}$ where v is a $K_{3,3}$-main vertex, which has been studied in the last case.

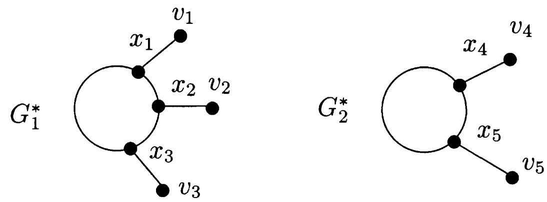


Fig. 16.

(3) v and S are not in the same connected component of G. Then, G contains a subdivision of M_{10}.
(4) v and S are in the same component but $v \notin V(S)$. If there is only a path joining v with S, then G contains a subdivision of M_{11} or M_8.
If there are two distinct paths joining v with S, then G contains a subdivision S' of $K_{3,3}$ v being a $K_{3,3}$-main vertex of S' which has been already studied.

And these are all the possibilities we have.

Finally, let us suppose that G is not projective. In this case G contains a subdivision of one of the 103 forbidden subgraphs for the projective plane given in [7]. But these forbidden subgraphs are M_{13}, M_{14}, M_{15}, M_{16}, M_{17} and M_{18} in the case of degree 3 and they must contain a subdivision of M_i, with $i=1,\dots,12$, in another case.

To complete the proof, let us see that the second condition implies the first one.

Let us suppose that $\Delta(G)=5$, let v be the only vertex of degree 5 and let x_i, with $i=1,\dots,5$, being the edges incident with v. We have G is planar from (d).

Since, there are not four edges incident with v in the same block, v is a cutpoint. Let us suppose that x_1, x_2 and x_3 are not in the same block as x_4 and x_5. Let G_1 be the component of $G-\{x_4,x_5\}$ incident with x_i, $i=1,2,3$, and let G_2 be the component of $G-\{x_1,x_2,x_3\}$ incident with x_4 and x_5.

Let G_1^* be the graph obtained from G_1 by replacing v with v_1,v_2,v_3 where $\delta(v_i)=1$ and x_i is incident with v_i for $i=1,2,3$, and let G_2^* be the graph obtained from G_2 by replacing v with v_4,v_5 where $\delta(v_i)=1$ and x_i is incident with v_i for $i=4,5$ (see Fig. 16).

G_1^* is planar and $\Delta(G_1^*)\leqslant 3$, then $M(G_1^*)$ is planar and it can be embedded into the plane so that x_i with $i=1,2,3$ will be in the same face according to Lemma 5.

In the same way, G_2^* is planar and $\Delta(G_2^*)\leqslant 3$, so $M(G_2^*)$ is planar.

Thus, we can build an embedding of $M(G)$ in the projective plane by drawing the K_6 graph formed by v,x_1,x_2,x_3,x_4,x_5 with x_1,x_2,x_3 in the same face and putting G_1^* into this face and G_2^* into a plane face with x_4,x_5 in its border, as Fig. 17 shows. So $M(G)$ is projective.

If $\Delta(G)=4$ then G is planar. Let v be the only vertex of degree 4, let x_i, $i=1,\dots,4$ be the edges incident with v in G and let G^* be the graph obtained from G replacing v by $v_i \setminus \delta(v_i)=1$ and each v_i is incident with x_i, $i=1,\dots,4$.

Then G^* is plane and $\Delta(G^*)\leqslant 3$, so $M(G)$ is planar and it can be embedded into the plane with x_1, x_2, x_3 and x_4 in the outer face. Thus, we can build an embedding

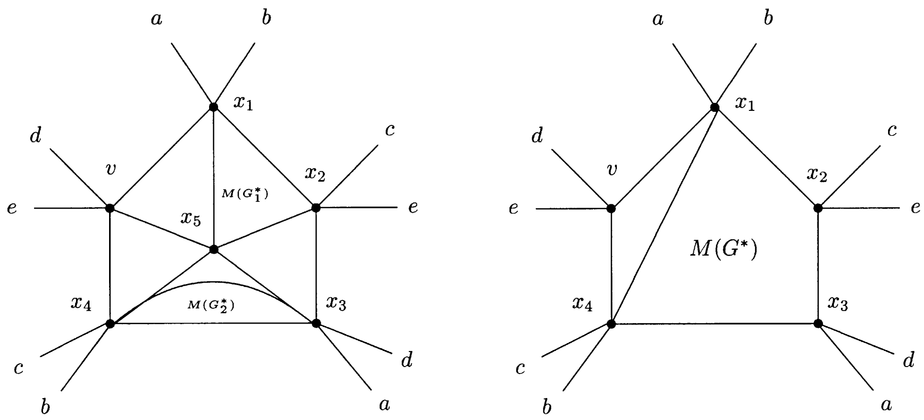


Fig. 17.

of $M(G)$ in the projective plane by drawing the K_5 graph formed by v, x_1, x_2, x_3, x_4 and putting G^* into the face bounded by x_1, x_2, x_3, x_4, as Fig. 17 shows.

So $M(G)$ is projective.

Finally, if $\Delta(G) \leqslant 3$ then G we can build an embedding of $M(G)$ into the projective plane as the proof of the Lemma 5 shows. □

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