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A domain decomposition method derived from the Primal Hybrid Formulations for 2nd order elliptic problems

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Resumen

We consider the primal hybrid formulation for second order elliptic problems introduced by Raviart-Thomas [9] and apply the classical iterative method of Uzawa to obtain a non overlapping domain decomposition method that converges geometrically with a mesh independent ratio. The proposed method connects with the Finite Element Tearing and Interconnecting (FETI) method proposed by Farhat-Roux and collaborators [7]-[8]. In this research work we use the detailed work on domains with corners developed by Grisvard [6], which clarifies the situation of cross-points, and the direct computation of the duality $H^{-1/2} - H^{1/2}$ using the $H^{1/2}$ scalar product; therefore no consistency error appears.

1. Introduction

The primal hybrid formulation for second order elliptic problems inforce via the Lagrange multipliers the continuity of the approximations across interfaces, and this is expressed via the duality $H^{-1/2} - H^{1/2}$, see Raviart-Thomas [9]. Usually, for numerical discretizations, this duality is worked out by means of some projection operator onto the L^2 space on the interfaces, see Ben Belgacem [3]. In our approach we use Riesz representation and replace the duality with the $H^{1/2}$ scalar product that is explicitly computed. As a consequence, we have a formulation in terms of a saddle point problem suitable for iterative techniques, see Bacuta [2].

The method presented is similar to the classical Lagrange Finite Element Tearing and Interconnecting (FETI) method proposed by Farhat-Roux and collaborators [7]-[8]. In this research work we use the detailed work on domains with corners developed by Grisvard

[6] which clarifies the situation of cross-points and the direct computation of the duality $H^{-1/2} - H^{1/2}$ using the $H^{1/2}$ scalar product; therefore no consistency error appears.

2. Formulation with Lagrange Multipliers

Let $\Omega \subset \mathbb{R}^2$ be a poligonal domain and consider $f \in L^2(\Omega)$. Then, our departure problem looks for $u \in H_0^1(\Omega)$ such that

$$(\nabla u, \nabla v)_\Omega + (u, v)_\Omega \quad = \quad (f, v)_\Omega \quad \forall v \in H_0^1(\Omega). \tag{1}$$

Assume that Ω is a polygonal bounded domain in $\mathbb{R}^2$ with a Lipschitz–continuous boundary and consider a decomposition without overlapping in polygonal subdomains

$$\overline{\Omega} = \cup_{r=1}^R \overline{\Omega}_r \qquad \text{and} \qquad \Omega_r \cap \Omega_{r'} = \emptyset, \quad 1 \leq r < r' \leq R \tag{2}$$

where each Ω_r has a Lipschitz–continuous boundary. We describe $\partial\Omega_r$ in terms of its edges via

$$\partial\Omega_r = \Gamma_{r,0} \cup \Gamma_{r,1} \cup ... \cup \Gamma_{r,J_r} \tag{3}$$

where $\Gamma_{r,0} = \partial\Omega_r \cap \partial\Omega$ such that $\partial\Omega = \cup_{r=1}^R \Gamma_{r,0}$ and assume that $\overline{\Omega_r} \cap \overline{\Omega_s}$ is either empty, a single point or a full edge $\Gamma_{r,s}$. On each $\Gamma_{r,j}$ we consider the classical Hilbert space of traces $H_{00}^{1/2}(\Gamma_{r,j})$ and its dual space $H_{00}^{-1/2}(\Gamma_{r,j})$, see Adams [1]. We call **skeleton** of Ω, and denote it by $\mathcal{E}$, the set of all interfaces in $\overline{\Omega}$

$$\mathcal{E} = \cup_{i=1}^I \Gamma_i \tag{4}$$

where $\Gamma_i = \Gamma_{i,0}$ for $i = 1, ..., R$ describe the boundary $\partial\Omega$, and for $i \geq R+1$ we set $\Gamma_i = \Gamma_{r,j}$ for some $r, j \geq 1$. Green's formulae on polygonal domains will be used

Lemma 1 (Grisvard [6]) *When $\mathcal{O} \subset \mathbb{R}^2$ is a poligonal domain and $\partial\mathcal{O} = \cup_{j=1}^J \Gamma_j$, then $H^2(\mathcal{O})$ is dense on $E = \{u \in H^1(\mathcal{O});\ \Delta u \in L^2(\Omega)\}$. The mapping $u \mapsto \partial_{\mathbf{n}_j} u_{|\Gamma_j}$ has a unique continuous extension from E to $H_{00}^{-1/2}(\Gamma_j)$ dual space of $H_{00}^{1/2}(\Gamma_j)$. Moreover, for each $u \in E$ and $v \in H^1(\mathcal{O})$ such that $v_{|\Gamma_j} \in H_{00}^{1/2}(\Gamma_j)$ we have*

$$-(\Delta u, v)_\mathcal{O} = (\nabla u, \nabla v)_\mathcal{O} - \sum_{j=1}^J < \partial_{\mathbf{n}_j} u, v >_{-1/2,00,\Gamma_j} . \tag{5}$$

Also, using that $\mathcal{D}(\overline{\mathcal{O}})^d$ is dense on $H(div;\mathcal{O})$, for any $\vec{q} \in H(div;\mathcal{O})$, we have $\mathbf{n}_j \cdot \vec{q} \in H_{00}^{-1/2}(\Gamma_j)$ and for any $v \in H^1(\Omega)$ with $v_{|\Gamma_j} \in H_{00}^{1/2}(\Gamma_j)$

$$(\vec{q}, \nabla v)_\mathcal{O} + (div(\vec{q}), v)_\mathcal{O} = \sum_{j=1}^J < \mathbf{n}_j \cdot \vec{q}, v >_{-1/2,00,\Gamma_j} . \tag{6}$$

Next, on each Ω_r we consider the classical Hilbert space

$$H_b^1(\Omega_r) = \{v_r \in H^1(\Omega_r); v_r = 0 \text{ on } \partial\Omega_r \cap \partial\Omega\} \tag{7}$$

with scalar product $(u_r, v_r)_{1,\Omega_r} = (u_r, v_r)_{\Omega_r} + (\nabla u_r, \nabla v_r)_{\Omega_r}$ the dense subspace W_r of $H_b^1(\Omega_r)$ given by

$$W_r = \{u \in H_b^1(\Omega_r);\ u_{|\Gamma_{r,j}} \in H_{00}^{1/2}(\Gamma_{r,j}),\ j = 1, ..., K_r\},$$

needed to apply Green's formulae, the global, defined on Ω, Hilbert space

$$X = \{v \in L^2(\Omega); v_r = v_{|\Omega_r} \in H_b^1(\Omega_r), r = 1, ..., R\} \approx \prod_{r=1}^{R} H_b^1(\Omega_r) \tag{8}$$

with scalar product and norm given by

$$(u, v)_X = \sum_{r=1}^{R} (u_r, v_r)_{1,\Omega_r}, \quad \|v\|_X^2 = (v, v)_X = \sum_{r=1}^{R} \|v_r\|_{1,\Omega_r}^2, \forall v, u \in X \tag{9}$$

and finally $X_0 \approx \prod_{r=1}^{R} W_r$ that is also a dense subspace of X. We also need the Hilbert space

$$H_0(div; \Omega) = \{\vec{q} \in L^2(\Omega)^d;\ div(\vec{q}) \in L^2(\Omega),\ \mathbf{n}_{r,0} \cdot \vec{q} = 0 \text{ en } \Gamma_{r,0},\ 1 \le r \le R\} \tag{10}$$

where $\mathbf{n}_{r,j} \cdot \vec{q} \in H_{00}^{-1/2}(\Gamma_{r,j}),\ j = 1, ..., K_r$, consider $T_r = \prod_{j=1}^{K_r} H_{00}^{-1/2}(\Gamma_{r,j})$ and M given by

$$M = \{\vec{\mu} \in \prod_{r=1}^{R} T_r; \mu_{r,j} = \mathbf{n}_{r,j} \cdot \vec{q},\ \text{for some } \vec{q} \in H_0(div; \Omega)\}. \tag{11}$$

Let $b : M \times X \mapsto \mathbb{R}$ given for $v \in X_0,\ \vec{\lambda} \in M$ by

$$b(\vec{\lambda}, v) \ = \ \sum_{i=R+1}^{I} < \lambda_i, v_s - v_t >_{-1/2,00,\Gamma_i} \tag{12}$$

when $\overline{\Omega_s} \cap \overline{\Omega_t} = \Gamma_i$ and extended by density to all $v \in X$. Then

$$H_0^1(\Omega) = \{v \in X;\ b(\vec{\lambda}, v) = 0, \quad \forall \vec{\lambda} \in M\}.$$

Define the bilinear form $a : X \times X \mapsto \mathbb{R}$ given by

$$a(u, v) \ = \ \sum_{r=1}^{R} \{(\nabla u_r, \nabla v_r)_{\Omega_r} + (u_r, v_r)_{\Omega_r}\} = \sum_{r=1}^{R} \int_{\Omega_r} \{\nabla u_r \cdot \nabla v_r + u_r\, v_r\}\, dx. \tag{13}$$

Then, the **primal hybrid formulation** for Poisson problem (1) consists in looking for a pair $(u, \vec{\lambda}) \in X \times M$ such that

$$a(u, v) + b(\vec{\lambda}, v) \ = \ \sum_{r=1}^{R} (f, v_r)_{\Omega_r},\ \forall v \in X\ (v_{|\Omega_r} = v_r) \tag{14}$$

$$b(\vec{\mu}, u) \ = \ 0, \quad \forall \vec{\mu} \in M. \tag{15}$$

Theorem 1 *If $u \in H_0^1(\Omega)$ solves the Dirichlet problem (1) then there exists a unique $(u, \vec{\lambda}) \in X \times M$ that solves problem (14)-(15). If $(u, \vec{\lambda}) \in X \times M$ solves (14)-(15) then $u \in H_0^1(\Omega)$ and solves the Dirichlet problem (1). Moreover, for $i = R+1, ..., I$*

$$\lambda_i = -\partial_{\mathbf{n}_i} u \in H_{00}^{-1/2}(\Gamma_i). \tag{16}$$

Next, via Riesz representation we identify $H_{00}^{-1/2}(\Gamma_i)$ (dual space of $H_{00}^{1/2}(\Gamma_i)$) with $H_{00}^{1/2}(\Gamma_i)$, write the duality in terms of the scalar product in $H_{00}^{1/2}(\Gamma_i)$, identify M with its dual space M' and define $b : M \times X \mapsto \mathbb{R}$ given for any $v \in X_0$, $\vec{\lambda} \in M$ by

$$b(\vec{\lambda}, v) \quad = \quad \sum_{i=R+1}^{I} (\lambda_i, v_s - v_t)_{1/2,00,\Gamma_i} \tag{17}$$

when $\overline{\Omega_s} \cap \overline{\Omega_t} = \Gamma_i$ and extended by density to all $v \in X$. Then, the formulation of Poisson problem (1) that we shall use is: Find a pair $(u, \vec{\lambda}) \in X \times M$ such that

$$a(u, v) + b(v, \vec{\lambda}) \quad = \quad \sum_{r=1}^{R} (f, v_r)_{\Omega_r}, \ \forall \ v \in X, \tag{18}$$

$$b(u, \vec{\mu}) \quad = \quad 0, \ \forall \ \vec{\mu} \in M. \tag{19}$$

Thanks to **Theorem 1** this is equivalent to (1) but it also is whithin the saddle point problems framework, see Girault-Raviart [5], which allows the use of different methods for computing the solution. Also, the analysis at the continuous level is reproduced in the discrete version of the saddle point problems as a simple consequence of the finite element extension theorems, see for instance Bernardi-Maday-Rapetti [4].

3. Domain decomposition methods

A rephrasing of the problem in terms of functional operators will clarify what we do. To fix ideas we work with Ω split up slicewise into two subdomains. Let $B : X \mapsto M$ given by $Bv = (v_1)_{|\Gamma} - (v_2)_{|\Gamma}$, i.e., the jump of v across the interface Γ, set $R : X' \mapsto X$ as the Riesz isomorphism associated with the scalar product $a(\cdot, \cdot)$ on X and $F : X \mapsto \mathbb{R}$ given by $< F, v >= \sum_{r=1}^{2} (f, v_r)_{\Omega_r}$. Then, our saddle point problem looks for $(u, \lambda) \in X \times M$ such that

$$R^{-1} \, u + B'\lambda \quad = \quad F \quad \text{ on } X' \tag{20}$$

$$B \, u \quad = \quad 0 \quad \text{ on } M, \tag{21}$$

where B' is the transpose operator to B. Then, $u = R(F - B'\lambda) \Rightarrow Bu = BRF - BRB'\lambda$ and (using $Bu = 0$) from here we have the **dual problem associated to the saddle point problem**

$$(BRB')\lambda = BRF \ \text{ on } M. \tag{22}$$

Thanks to the inf-sup condition **the operator BRB' is symmetric positive definite**, see Bacuta [2]. Now the resolution of (22) via an iterative method is possible; we propose

the use of the **iterative method of Richardson, which amounts to the classical Uzawa's Method**.
Given $\rho > 0$ and $\lambda_0 \in M$, for $m = 0, 1, 2, 3, ...$ set

$$r_m = BRF - (BRB')\lambda_m = B\,u_m, \text{ using (??)} \tag{23}$$
$$\lambda_{m+1} = \lambda_m + \rho\, r_m \tag{24}$$

which unfolds from (20)-(21) as

$$\sum_{r=1}^{2}(u_{m,r}, v_r)_{1,\Omega_r} = \sum_{r=1}^{2}(f, v_r)_{\Omega_r} - (\lambda_m, v_1 - v_2,)_{1/2,00,\Gamma}, \quad \forall v \in X, \tag{25}$$
$$\text{and update} \quad \lambda_{m+1} = \lambda_m + \rho(u_{m,1} - u_{m,2}). \tag{26}$$

Following standard convergence results, see Bacuta [2] and references therein, we have **geometric convergence** for this iterative process by simply blocking to zero the test functions alternatively on each subdomain

Theorem 2 *The iterative process:*
Given $\rho > 0$ and $\lambda_0 \in M$, find for $m \geq 0$ $u_m \in X$ via

$$\begin{aligned}
(\nabla u_{m,1}, \nabla v_1)_{\Omega_1} + (u_{m,1}, v_1)_{\Omega_1} &= (f, v_1)_{\Omega_1} - (\lambda_m, v_1)_{1/2,00,\Gamma}, \quad \forall v_1 \in X_1,\\
(\nabla w_{m,2}, \nabla v_2)_{\Omega_2} + (u_{m,2}, v_2)_{\Omega_2} &= (f, v_2)_{\Omega_2} + (\lambda_m, v_2)_{1/2,00,\Gamma}, \quad \forall v_2 \in X_2,\\
\textit{and update} \quad \lambda_{m+1} &= \lambda_m + \rho(u_{m,1} - u_{m,2}) \;\; \textit{on} \;\; \Gamma
\end{aligned}$$

is a ***non overlapping domain decomposition method*** *geometrically convergent with a ratio of convergence independent of the mesh size.*

The drawback that this method presents is how to fix the optimal parameter $\rho > 0$. In the numerical experiments that we present the value of ρ has been tuned easily by hand thanks to the great speed of convergence that the method exhibits.

For a method that has no need of fixing any parameter we could use the application of the Conjugate Gradient Method which is the core of the FETI methods.

4. Numerical experiments

We compute on a non convex domain with three subdomains. We use a Galerkin approximation with $\mathbb{P}_1$ Lagrange finite elements on a uniform triangular mesh size h of $\overline{\Omega}$ and its restriction to each of the $\overline{\Omega_i}$ for $i = 1, 2, 3$. The numerical results show a geometric rate of convergence with a mesh independent ratio as the theory predicts.

We set $\Omega = (-1, 1)^2 \setminus \{(-1, 0) \times (-1, 0)\}$ and decompose it into three squares so that our interfaces are $\Gamma_1 = \{0\} \times (0, 1)$ and $\Gamma_2 = (0, 1) \times \{0\}$. Then we solve

$$-\Delta u = 1, \quad \text{on } \Omega, \quad u = 0 \text{ on } \partial\Omega.$$

We take $\lambda_0 = (0, 0)$ and stop iterating for $m = niter(h)$ such that

$$\frac{\|u_h^{m+1} - u_h^m\|_X}{\|u_h^m\|_X} = \frac{(\sum_{i=1}^{3}\int_{\Omega_i} |\nabla(u_{i,h}^{m+1} - u_{i,h}^m)|^2\,dx)^{1/2}}{(\sum_{i=1}^{3}\int_{\Omega_i} |\nabla u_{i,h}^m|^2\,dx)^{1/2}} \leq 10^{-7}; \tag{27}$$

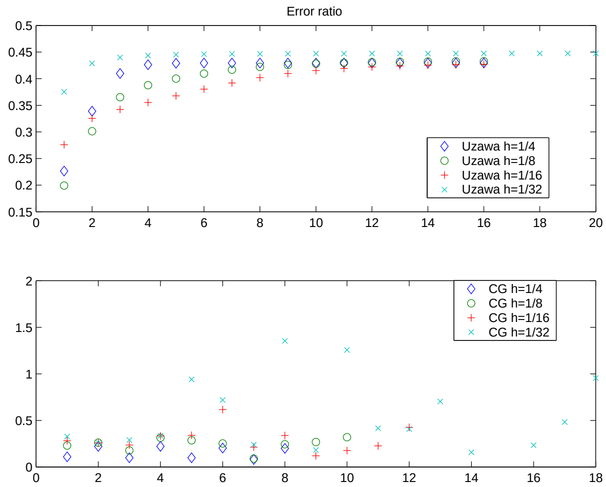

Figura 1: *Decrease error ratio given by (29) on the L-shaped domain test using Uzawa's Method and Conjugate Gradient Method (CG).*

we compute the errors and their decrease ratio given for $m \geq 0$ by

$$euh(h,m) \;=\; \|u_h - u_h^m\|_X = (\sum_{i=1}^{3} \int_{\Omega_i} |\nabla(u_h - u_{i,h}^m)|^2 \, dx)^{1/2} \tag{28}$$

$$r(h,m) \;=\; \frac{euh(h,m+1)}{euh(h,m)}, \quad \overline{r}(h) \approx \lim_m r(h,m) \tag{29}$$

where u_h is the $\mathbb{P}_1$ solution computed on the whole domain Ω. For Uzawa's Method we found by performing few several tests that $\rho \approx 0.12$ seems to be the closest value to the optimal one. The results are shown in Table 1.

$1/h$	4	8	16	32
$\#iterations$	17	17	17	21
ρ	≈ 0.12	≈ 0.12	≈ 0.12	≈ 0.12
$\overline{r}(h)$	$\approx 0.43..$	$\approx 0.43...$	$\approx 0.43...$	$\approx 0.44...$

Table 1: Uzawa's Method: Number of iterations, values of ρ and of $\overline{r}(h)$ obtained with the inverted L-shape domain for different values of h.

We also computed the solution using the Conjugate Gradient Method. In Figure 1, For both of the iterative methods proposed, we show the ratio between consecutive errors.

Figure 2 shows the Galerkin solution computed in Ω and Figure 3 shows the solution computed right after the first iteration of Uzawa's Method. We see the lack of jumps on the two interfaces.

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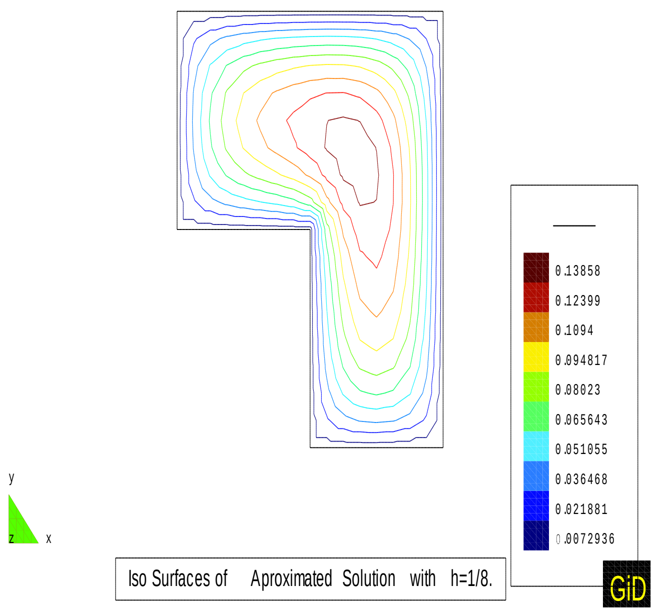

Figura 2: *Approximated solution computed with standard Galerkin* $\mathbb{P}_1$ *finite elements on the whole domain and with* $h = 1/8$.

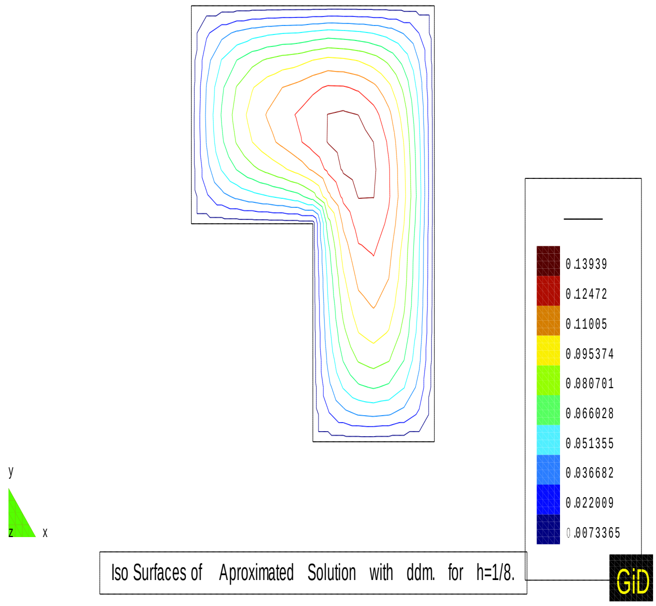

Figura 3: *Computed solution after the first iteration with $h = 1/8$ using Uzawa's Method.*

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