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The homological reduction method for computing cocyclic Hadamard matrices ☆

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ABSTRACT

An alternate method for constructing (Hadamard) cocyclic matrices over a finite group G is described. Provided that a *homological model* $\phi: \bar{B}(\mathbb{Z}[G]) \underset{H}{\overset{F}{\rightleftharpoons}} hG$ for G is known, the *homological reduction method* automatically generates a full basis for 2-cocycles over G (including 2-coboundaries). From these data, either an exhaustive or a heuristic search for Hadamard cocyclic matrices is then developed. The knowledge of an explicit basis for 2-cocycles which includes 2-coboundaries is a key point for the designing of the heuristic search. It is worth noting that some Hadamard cocyclic matrices have been obtained over groups G for which the exhaustive searching techniques are not feasible. From the computational-cost point of view, even in the case that the calculation of such a homological model is also included, comparison with other methods in the literature shows that the homological reduction method drastically reduces the required computing time of the operations involved, so that even exhaustive searches succeeded at orders for which previous calculations could not be completed. With aid of an implementation of the method in Mathematica, some examples are discussed, including the case of very well-known groups (finite abelian groups, dihedral groups) for clarity.

1. Introduction

Since the *cocyclic* Hadamard conjecture was stated (Horadam and de Launey, 1995), interest in calculating cocyclic Hadamard matrices over finite groups G has increased considerably. Taking into account that only 2×2 Hadamard matrices exist whose sizes are not multiple of 4, we may assume in the sequel that $|G| = 4t$.

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Basically, two methods have been proposed in order to compute the whole set of cocyclic Hadamard matrices over a group G. In both cases, once a set of generators for representative 2-cocycles is determined, it suffices to add a basis for 2-coboundaries of G, so that a whole basis for 2-cocycles is finally achieved.

The first method constitutes the foundational work on the subject Horadam and de Launey (1993, 1995), and is applied over abelian groups. Attending to the Universal Coefficient Theorem, a basis for representative 2-cocycles may be obtained from the relation

$$H^2(G, \mathbb{Z}_2) \cong Ext(G/[G, G], \mathbb{Z}_2) \oplus Hom(H_2(G), \mathbb{Z}_2).$$

Generators coming from the first factor are uniquely determined (up to the internal ordering in G) as the Kronecker product of back negacyclic matrices, accordingly to the primary invariant decomposition of the abelianization $G/[G, G]$ of G, so that each 2^t factor on $G/[G, G]$ contributes a $2^t \times 2^t$ back negacyclic matrix. Generators coming from the factor $Hom(H_2(G), \mathbb{Z}_2)$ may be computed as soon as a basis of 2-cycles in $H_2(G)$ is described. Unfortunately, in general, this is a difficult task.

The method in Flannery (1996) applies over groups G for which the word problem is solvable, and uses the inflation and transgression maps. The inflation map generates the representative 2-cocycles of $Ext(G/[G, G], \mathbb{Z}_2)$, once again in terms of back negacyclic matrices. However, the whole description of $H^2(G; \mathbb{Z}_2)$ depends on the choice of a Schur complement of the image of inflation, which is no longer canonical and could reveal itself as a computationally hard task. The case of dihedral groups and central extensions is described in Flannery (1996, 1997) and Flannery and O'Brien (2000).

We describe here, a third approximation to this question, which we term the *homological reduction method*. It is the crystallization of a previous work of the authors in Álvarez et al. (2001). Origins of the method may be located in Grabmeier and Lambe (2000), which includes the construction of a basis for cocyclic matrices over p-groups from a cohomological model for these groups.

Provided a *homological model hG* for G is known (that is, a differential graded module of finite type which shares the homology groups with G), we explicitly describe an algorithm for constructing a basis for 2-cocycles over G in a straightforward manner. In fact, the goodness of this approximation is supported by the efficiency in which both $H_1(G) \simeq G/[G, G]$ and $H_2(G)$ are computed from the homological model hG.

From such a basis, a search for Hadamard cocyclic matrices may be developed at once. At this point, it is remarkable that an exhaustive search is feasible only for low orders (limited up to $|G| \leq 28$, attending to the computing capability of today more common processors, as it has been experimentally checked), since the search space often grows exponentially depending on the order of the group (e.g. see the examples described in Section 3). In case of higher orders, heuristic searches such as Álvarez et al. (2006a) are a better choice. The knowledge of an explicit basis for 2-cocycles which includes 2-coboundaries is a key point for the designing of such an heuristic search.

From the computational-cost point of view, even in the case that the calculation of such a homological model is also included, comparison with other methods in the literature shows that the homological reduction method drastically reduces the required computing time of the operations involved, so that even exhaustive searches succeeded at orders for which previous calculations could not be completed (see Table 1 in page 20 for details).

We organize the paper as follows. In Section 2 we describe the homological reduction method itself, that is, how to construct a full basis for 2-cocycles over G from a homological model hG of G. Section 3 is devoted to showing several examples, including the well-known cases of dihedral groups D_{4t} and abelian groups $\mathbb{Z}_t \times \mathbb{Z}_2^2$ for clarity. From these data, we construct Table 1, which completes that in Horadam (1996) about the total number of cocyclic Hadamard matrices over these groups for small orders. All the calculations have been made with aid of packages in Mathematica (Álvarez et al., 2006b,c,d,e,f).

2. Describing the homological reduction method

Consider a multiplicative group $G = \{g_1 = 1, g_2, \ldots, g_{4t}\}$, not necessarily abelian. A cocyclic matrix M_f on G consists in a binary matrix $M_f = (f(g_i, g_j))$ coming from a 2-cocycle f, that is, a map $f : G \times G \to \{1, -1\}$ such that

$$f(g_i, g_j)f(g_ig_j, g_k) = f(g_j, g_k)f(g_i, g_jg_k), \quad \forall g_i, g_j, g_k \in G.$$

We note $Z(G)$ the group of 2-cocycles, with regards to the pointwise (also termed Hadamard) product. Let $B(G)$ be the group of 2-coboundaries, which consist in the functions

$$\partial_\alpha(g_i, g_j) = \alpha(g_i)\alpha(g_j)\alpha(g_i g_j)^{-1}, \quad g_i, g_j \in G,$$

for set maps $\alpha : G \to \{1, -1\}$.

It is a well-known fact that $Z(G)/B(G) \cong H^2(G; \mathbb{Z}_2)$. This way, the Universal Coefficient Theorem provides at once a first chance for computing cocyclic matrices,

$$Z(G)/B(G) \cong H^2(G; \mathbb{Z}_2) \cong \mathrm{Ext}\,(H_1(G), \mathbb{Z}_2) \oplus \mathrm{Hom}\,(H_2(G), \mathbb{Z}_2)$$

where $H_1(G) \cong G/[G, G]$.

This way, a basis for 2-cocycles is performed by joining three different bases: a basis for 2-coboundaries, a basis for representative symmetric 2-cocycles coming from inflation (i.e. from the $\mathrm{Ext}\,(H_1(G), \mathbb{Z}_2)$ factor), and a basis for 2-cocycles coming from transgression (i.e. the $\mathrm{Hom}\,(H_2(G), \mathbb{Z}_2)$ factor). Such a basis consists, then, in a set $\mathcal{B} = \{\partial_1, \ldots, \partial_b, \beta_1, \ldots, \beta_s, \gamma_1, \ldots, \gamma_r\}$, for some 2-coboundaries ∂_i, inflation 2-cocycles β_j and transgression cocycles γ_k.

As pointed out in Horadam and de Launey (1995), we may reduce to the case of *normalized* 2-cocycles f (such that $f(1, 1) = 1$), as well as the related *normalized* cocyclic matrices M_f. The term "normalized" means that the first row (and column) in M_f is formed all by 1. From now on, cocycle will mean normalized 2-cocycle.

A basis for 2-coboundaries may be obtained by Linear Algebra. More concretely, denote ∂_i the 2-coboundary associated with the characteristic map of the element g_i, $\delta_i : G \to \mathbb{Z}$,

$$\delta_i(g_j) = \begin{cases} -1, & \text{if } i = j \\ \ \ 1, & \text{otherwise.} \end{cases}$$

Take the matrices M_{∂_i} related to ∂_i as vectors of length $16t^2$. Moreover, consider the $4t \times 16t^2$ matrix C whose rows are the vectors M_{∂_i}. Then a row reduction on C leads to a basis for 2-coboundaries. It suffices to keep trace of those coboundaries ∂_i whose transformed rows in M_{∂_i} after the row reduction are not zero.

Lemma 1. *The morphisms ∂_j above define a basis for 2-coboundaries.*

The homological reduction method intends that the calculation of $H_1(G)$ and $H_2(G)$ is as economical as possible.

Roughly speaking, the idea consists of determining a homological model hG for G, and then project the (co)homological information from hG to G.

The term *homological model* refers to a *contraction* $\phi{:}\bar{B}(\mathbb{Z}[G]) \underset{H}{\overset{F}{\rightleftharpoons}} hG$ from the *reduced bar construction* of the group G (i.e. the reduced complex associated with the standard bar resolution (Mac Lane, 1995)) to a differential graded module of finite type hG, so that

$$H_*(G) = H_*(\bar{B}(\mathbb{Z}[G])) \cong H_*(hG)$$

and the homology of hG may be effectively computed by means of Veblen's algorithm (Veblen, 1931) (involving the Smith's normal forms of the matrices representing the differential operator).

Concerning to the inflation and transgression generators, the use of a homological model will often simplify the calculation of $G/[G, G] \cong H_1(G) \cong H_1(hG)$ and $H_2(G) \cong H_2(hG)$. More concretely, the simplification depends on the decrease of the number of generators at each degree, as it will be clear from the description of Veblen's algorithm below.

However, we need to lift the (co)homological information from hG to $\bar{B}(\mathbb{Z}[G])$ in order to explicitly generate a full set of representative 2-cocycles in G. The projection morphism

$$F : \bar{B}(\mathbb{Z}[G]) \longrightarrow hG$$

helps in this task.

From Horadam and de Launey (1995), we know that there are as many generators coming from inflation as factors $\mathbb{Z}_{2^{t_j}}$ in the primary decomposition $\oplus_{i=1}^{n} \mathbb{Z}_{p_i^{t_i}}$ of $G/[G, G]$. All of them are $2^{t_j} \times 2^{t_j}$ back negacyclic matrices of the type

$$BN_{2^{t_j}} = \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & \cdots & 1 & - \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & 1 & \cdots & - & - \\ 1 & - & \cdots & - & - \end{pmatrix}$$

for a suitable ordering of the elements in G. We use here "$-$" instead of "-1" for short.

The problem is that the initial ordering $G = \{1, g_2, \ldots, g_{4t}\}$ will differ in general from the above one. The difficulty lies in how to link such two orderings.

Let $d : hG \to hG$ be the differential on hG and $\mathcal{B}_1 = \{\mathbf{u_1}, \ldots, \mathbf{u_m}\}$ and $\mathcal{B}_2 = \{\mathbf{e_1}, \ldots, \mathbf{e_n}\}$ be some basis of hG on dimensions 1 and 2, respectively.

We compute $G/[G, G]$ as $H_1(hG)$, which consists only of torsion part, as G is a finite group. So Veblen's algorithm reduces to compute the Smith's normal form D_2 of $M_2(d)$,

$$M_2(d) = \begin{pmatrix} d(\mathbf{e_1}) \\ \vdots \\ d(\mathbf{e_n}) \end{pmatrix}_{n\times m} \qquad D_2 = \left(\begin{array}{ccc|c} b_1 & & & \\ & \ddots & & 0 \\ & & b_l & \\ \hline & 0 & & 0 \end{array}\right)_{n\times m}$$

so that we get the torsion-invariant decomposition $G/[G, G] \cong H_1(G) \cong H_1(hG) \cong \mathbb{Z}_{b_1} \oplus \cdots \oplus \mathbb{Z}_{b_l}$, $1 \le b_1 | b_2 | \cdots | b_l$, which is by no means a primary decomposition of $G/[G, G]$.

Moreover, some (not uniquely determined) change basis matrices P and Q exist such that

$$\begin{array}{ccc} \mathcal{B}_2 & \xrightarrow{M_2(d)} & \mathcal{B}_1 \\ {\scriptstyle P}\uparrow & \# & \downarrow {\scriptstyle Q} \\ \bar{\mathcal{B}}_2 & \xrightarrow{D_2} & \bar{\mathcal{B}}_1 \end{array} \qquad D_2 = P \cdot M_2(d) \cdot Q$$

Now we proceed according to the following steps:

(1) We select the columns j of D_2 with an even entry at the diagonal position, precisely the ones corresponding to those $\mathbb{Z}_{b_j}$ in $H_1(hG)$ which contribute a factor $\mathbb{Z}_{2^t}$ to the primary decomposition of $G/[G, G]$. There will be as many inflation generators as columns of this kind.

(2) Choose one of these columns, say the j-th for instance. Furthermore, assume that $b_j = 2^{t_j} q_j$, for q_j odd. The symmetric matrix $M_j = (\beta_j(g, h))$ which corresponds to the generator β_j of this j-th column is constructed by lifting the map $w_j : \mathbb{Z}_{2^{t_j}} \times \mathbb{Z}_{2^{t_j}} \to \{1, -1\}$, $w_j(k, l) = (-1)^{\lfloor \frac{k+l}{2^{t_j}} \rfloor}$, from $\mathbb{Z}_{2^{t_j}} \times \mathbb{Z}_{2^{t_j}}$ to the whole $G \times G$. For $g, h \in G$ we define $\beta_j(g, h) = w_j([g]_j, [h]_j)$, where $[g]_j$ is the j-th coordinate of the coset of g in $G/[G, G]$ regarding to the basis of $H_1(hG)$ canonically associated to $\bar{\mathcal{B}}_1$. Explicitly, $\beta_j(g, h)$ is determined from $F : \bar{B}_1(\mathbb{Z}[G]) \to hG$, since $[g]_j$ is the j-th coordinate of $F(g)$ with regards to $\bar{\mathcal{B}}_1$; that is, the j-th coordinate of the vector $F(g) \cdot Q$ modulo 2^{t_j}.

Graphically,

$$\begin{array}{ccc} \bar{B}_1(\mathbb{Z}[G]) & \xrightarrow{F} & \mathcal{B}_1 \\ & & \downarrow {\scriptstyle Q} \\ & & \bar{\mathcal{B}}_1 \end{array}$$

Proposition 2. *The morphisms β_j above define a basis for 2-cocycles coming from inflation.*

We proceed in an analogous way in order to construct the generators coming from transgression.

Let $\mathcal{B}_3 = \{\mathbf{v_1}, \ldots, \mathbf{v_s}\}$ be a basis for hG at dimension 3. Since G is a finite group, again $H_2(G) \cong H_2(hG)$ consists only of torsion part, so that Veblen's algorithm reduces to compute the Smith's normal form D_3 of $M_3(d)$,

$$M_3(d) = \begin{pmatrix} d(\mathbf{v_1}) \\ \vdots \\ d(\mathbf{v_s}) \end{pmatrix}_{s\times n} \qquad D_3 = \left(\begin{array}{ccc|c} b_1 & & & \\ & \ddots & & 0 \\ & & b_l & \\ \hline & 0 & & 0 \end{array}\right)_{s\times n}$$

where $H_2(G) \cong H_2(hG) \cong \mathbb{Z}_{b_1} \oplus \cdots \oplus \mathbb{Z}_{b_l}$, and $1 \leq b_1|b_2|\cdots|b_l$.

Furthermore, some change basis matrices P and Q exist such that

$$\begin{array}{ccc} \mathcal{B}_3 & \xrightarrow{M_3(d)} & \mathcal{B}_2 \\ {\scriptstyle P}\uparrow & \# & \downarrow {\scriptstyle Q} \\ \bar{\mathcal{B}}_3 & \xrightarrow{D_3} & \bar{\mathcal{B}}_2 \end{array} \qquad D_3 = P \cdot M_3(d) \cdot Q$$

Now we proceed according to the following steps:

(1) We select those columns j of D_3 with an even entry b_j at the diagonal position: since $\mathrm{Hom}\,(\mathbb{Z}_n, \mathbb{Z}_2) \cong \mathbb{Z}_{gcd(n,2)}$ we are only interested in factors $\mathbb{Z}_{b_j}$ with b_j even. There will be as many generators coming from transgression as columns of this kind.

(2) Set one of these columns, say the j-th for instance. The cocyclic matrix $M_j = (\gamma_j(g, h))$ which corresponds to the generator γ_j of this j-th column is constructed by projecting the elements $(g, h) \in G \times G$ onto $\bar{\mathcal{B}}_2$ by means of the composition of F and Q. For $g, h \in G$ we define $\gamma_j(g, h) = F([g, h]) \cdot Q \mod 2$.

Graphically,

$$\begin{array}{ccc} \bar{B}_2(\mathbb{Z}[G]) & \xrightarrow{F} & \mathcal{B}_2 \\ & & \downarrow {\scriptstyle Q} \\ & & \bar{\mathcal{B}}_2 \end{array}$$

Proposition 3. *The morphisms γ_j above define a basis for 2-cocycles coming from transgression.*

The homological reduction method provides then the following algorithm for computing the whole set of Hadamard cocyclic matrices over G.

Algorithm 1 (*Homological Reduction Method*). INPUT: group with homological model $\{G, hG, F, H, \phi\}$
OUTPUT: Some (eventually the full set of) cocyclic Hadamard matrices over G.

P1 Construct a basis for 2-coboundaries (Lemma 1)
P2 Construct a basis for inflation 2-cocycles (Proposition 2)
P3 Construct a basis for transgression 2-cocycles (Proposition 3)
P4 Construct a basis $\mathcal{B}$ for 2-cocycles over G
P5 Develop from $\mathcal{B}$ an exhaustive or heuristic search for Hadamard cocyclic matrices, depending on the size of $|G|$.

Knowledge of such a basis $\mathcal{B}$ for 2-cocycles, which includes 2-coboundaries, is a key point for the designing of the genetic algorithm in Álvarez et al. (2006a) searching for Hadamard cocyclic matrices over G. The individuals of this genetic algorithm consist of binary tuples, which are to be understood as the coordinates of a 2-cocycle with regards to the basis $\mathcal{B}$. The interested reader is referred to Álvarez et al. (2006a) for details.

We want to emphasize that such a heuristic search is only possible since a basis for 2-coboundaries is explicitly used. It seems that other methods in the literature ignore the issue of finding a basis for 2-coboundaries. In the opinion of the authors, a deeper analysis on the way in which 2-coboundaries and representative 2-cocycles have to be combined in order to get a Hadamard cocyclic matrix becomes of capital interest (see Álvarez et al. (2008) for instance).

3. Examples

All the executions and examples of this section have been worked out with aid of the *Mathematica 4.0* notebooks (Álvarez et al., 2006d,e) described in Álvarez et al. (2006c, submitted for publication) (for constructing homological models) and Álvarez et al. (2006b) (in order to form up basis for 2-cocycles from which the search for Hadamard cocyclic matrices is then developed), running on a *Pentium IV* 2.400 MHz *DIMM DDR266* 512 MB.

In the sequel, the elements of a product $A \times B$ are ordered as the rows of a matrix indexed in $|A| \times |B|$. For instance, if $|A| = r$ and $|B| = c$, the ordering is

$$\langle a_1b_1, a_1b_2, \ldots, a_1b_c, a_2b_1, a_2b_2, \ldots, a_2b_c, \ldots, a_rb_1, \ldots, a_rb_c \rangle.$$

The elements in the group are labeled from 1 to $|G|$, according to this ordering.

Let consider the families of groups below (assume $\mathbb{Z}_k = \{0, 1, \ldots, k-1\}$ with additive law).

(1) $G_1^t = \mathbb{Z}_{4t}$.
(2) $G_2^t = \mathbb{Z}_{2t} \times \mathbb{Z}_2$.
(3) $G_3^t = \mathbb{Z}_t \times \mathbb{Z}_4$. Note that $G_3^2 \simeq G_2^2$, and $G_3^t \simeq G_1^t$ for odd t.
(4) $G_4^t = \mathbb{Z}_t \times \mathbb{Z}_2^2 = \mathbb{Z}_t \times (\mathbb{Z}_2 \times \mathbb{Z}_2)$. Note that $G_4^t \simeq G_2^t$ for odd t.
(5) $G_5^t = D_{4t} = \mathbb{Z}_2 \,{}_\chi\!\ltimes \mathbb{Z}_{2t}$, $\chi : \mathbb{Z}_2 \times \mathbb{Z}_{2t} \to \mathbb{Z}_{2t}$ such that $\chi(1, x) = -x$ and $\chi(0, x) = x$. Note that $G_5^1 \simeq G_2^1$ is abelian, but G_5^t is not abelian, for $t > 1$.
(6) $G_6^t = \mathbb{Z}_{2t} \ltimes_f \mathbb{Z}_2$, for $f : \mathbb{Z}_2 \times \mathbb{Z}_2 \to \mathbb{Z}_{2t}$ being the 2-cocycle

$$f(g_i, g_j) = \begin{cases} \lceil \frac{t}{2} \rceil + 1 & \text{if} \quad g_i = g_j = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Note that G_6^t is abelian, since f is symmetric. Furthermore, $G_6^t \simeq G_2^t$ for even $\lceil \frac{t}{2} \rceil + 1$ (that is, for $t \equiv 1, 2 \pmod 4$), since f is a 2-coboundary in these circumstances (i.e. the extension is trivial). In fact, $f = \partial_\alpha$, for $\alpha : \mathbb{Z}_2 \to \mathbb{Z}_{2t}$ such that $\alpha(0) = 0$, $\alpha(1) = \lfloor \frac{t}{4} \rfloor + 1$. The extension is not trivial for $t \equiv 0, 3 \bmod 4$.

(7) $G_7^t = (\mathbb{Z}_t \ltimes_f \mathbb{Z}_2) \rtimes_\chi \mathbb{Z}_2$, for $f : \mathbb{Z}_2 \times \mathbb{Z}_2 \to \mathbb{Z}_t$ being the 2-cocycle

$$f(g_i, g_j) = \begin{cases} \lceil \frac{t}{2} \rceil + 1 & \text{if} \quad g_i = g_j = 1 \\ 0 & \text{otherwise} \end{cases}$$

and χ being the dihedral action $\chi(a, b) = \begin{cases} -b & \text{if} \quad a = 1 \\ b & \text{if} \quad a = 0. \end{cases}$

Note that $\mathbb{Z}_t \ltimes_f \mathbb{Z}_2$ is abelian (since f is symmetric), but G_7^t is not for $t \neq 2$ (because of the dihedral action). Furthermore, $G_7^t \simeq G_5^t$ for odd t, since f is a 2-coboundary in these circumstances: $f = \partial_\alpha$, for $\alpha : \mathbb{Z}_2 \to \mathbb{Z}_t$ such that $\alpha(0) = 0$, $\alpha(1) = \dfrac{t^2+3}{4} \bmod t$. Analogously, the extension is also trivial for $t \equiv 2 \bmod 4$, since $f = \partial_\alpha$, for $\alpha(0) = 0$, $\alpha(1) = \lfloor \frac{t}{4} \rfloor + 1$, so that $G_7^t \simeq (\mathbb{Z}_t \times \mathbb{Z}_2) \rtimes_\chi \mathbb{Z}_2$. In particular, $G_7^2 \equiv G_4^2$.

We may assume that $t > 1$, since there are only two abelian groups of order 4, which are $\mathbb{Z}_2 \times \mathbb{Z}_2$ (this is the case of G_i^1, for $i = 2, 4, 5, 6, 7$) and $\mathbb{Z}_4$ (the case of G_i^1, for $i = 1, 3$). There are six Hadamard cocyclic matrices over $\mathbb{Z}_2 \times \mathbb{Z}_2$ and two Hadamard cocyclic matrices over $\mathbb{Z}_4$. So we are interested in describing cocyclic Hadamard matrices for $t > 1$.

In this section, we will first use the homological reduction method for calculating the basis for 2-cocycles for these families of groups G_i^t. For brevity, we will only characterize the homological models (hG_k^t, F, d) for G_k^t, in terms of some basis $\mathcal{B}_i$ for hG_k^t on degree $1 \le i \le 3$, differential operators $d_j : B_{j+1} \to B_j$ and projections $F_j : \bar{B}_j(\mathbb{Z}[G_k^t]) \to B_j$ for $1 \le j \le 2$. Notice that $\bar{B}_1(\mathbb{Z}[G]) = \langle [g] : g \in G \rangle$ and $\bar{B}_2(\mathbb{Z}[G]) = \langle [g, h] : g, h \in G \rangle$.

From these data, a basis for representative 2-cocycles may be formed in a straightforward manner. Afterwards, we will develop an exhaustive search for $2 \le t \le 5$ and a heuristic search for $2 \le t \le 8$ (notice that the cocyclic Hadamard matrices listed here are new, different from those of Álvarez et al. (2006b)).

Note that the matrices P and Q involved in the calculation of the Smith Normal Form, D, for A (so that $D = P \cdot A \cdot Q$) are not uniquely determined, in general. In the sequel we will use the matrices coming from the SmithNormalForm package programmed by the authors in Álvarez et al. (2006f).

Due to space restrictions, we will include here only the cases of G_4^t, G_5^t (for comparison with other calculations available in the literature) and G_7^t (as far as we know, these calculations are new. Furthermore, this family of groups seems to provide a large amount of Hadamard cocyclic matrices). The remaining calculations are available at the following web address, http://ma1.eii.us.es/miembros/armario/cvarmario.htm.

Finally, we show in Table 1 the number of all cocyclic Hadamard matrices over G_i^t for $1 \leq i \leq 7$, $2 \leq t \leq 5$. This table corrects and completes that in Horadam (1996).

3.1. Construction basis for 2-cocycles

In the sequel, we use the following notation. We define the set map $\lambda^n : \mathbb{Z} \to \mathbb{Z}_2$, so that $\lambda^n(j) = \lambda^n_j = 1$ if $j \geq n$ and 0 otherwise. The back negacyclic matrix of order j is denoted by BN_j, as before. The square matrix of order n formed up all of 1s is denoted by 1_n. The Kronecker product of matrices is denoted by $\otimes$, so that $A \otimes B$ is the block matrix $\begin{pmatrix} a_{11}B & \dots & a_{1n}B \\ \vdots & & \vdots \\ a_{n1}B & \dots & a_{nn}B \end{pmatrix}$. The Hadamard (pointwise) product of matrices is simply denoted as $A \cdot B$. Finally, the notation $[x]_m$ refers to x mod m.

3.1.1. Basis for $G_4^t = \mathbb{Z}_t \times \mathbb{Z}_2^2 = \mathbb{Z}_t \times (\mathbb{Z}_2 \times \mathbb{Z}_2)$

$\mathcal{B}_1 = \{u_1, u_2, u_3\}$, $\mathcal{B}_2 = \{e_1, \dots, e_6\}$, $\mathcal{B}_3 = \{v_1, \dots, v_{10}\}$,
$d_2(e_1) = t \cdot u_1$, $d_2(e_4) = 2 \cdot u_2$, $d_2(e_6) = 2 \cdot u_3$, $d_3(v_2) = te_2$, $d_3(v_3) = te_3$,
$d_3(v_4) = -2e_2$, $d_3(v_6) = -2e_3$, $d_3(v_8) = 2e_5$, $d_3(v_9) = -2e_5$,
$F[(g, h, i)] = g \cdot u_1 + h \cdot u_2 + i \cdot u_3$,
$F[(g, h, i)|(a, b, c)] = \lambda^t_{g+a} \cdot e_1 + gb \cdot e_2 + gc \cdot e_3 + \lambda^2_{h+b} \cdot e_4 + hc \cdot e_5 + \lambda^2_{i+c} \cdot e_6$.

From these data, the matrices D_i and Q_i may be described, in terms of the the coset of t modulo 2:

t	D_2	Q_2		D_3	Q_3
$[t]_2 = 0$	$\begin{pmatrix} 2&0&0\\0&2&0\\0&0&t\\0&0&0\\0&0&0\\0&0&0 \end{pmatrix}$	$t=2$ I_3	$t \neq 2$ $\begin{pmatrix} 0&0&1\\1&0&0\\0&1&0 \end{pmatrix}$	$\begin{pmatrix} 2&0&0&0&0&0\\0&2&0&0&0&0\\0&0&2&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0 \end{pmatrix}$	$\begin{pmatrix} 0&0&0&0&1&0\\1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&0&1&0&0\\0&0&1&0&0&0\\0&0&0&0&0&1 \end{pmatrix}$
$[t]_2 = 1$	$\begin{pmatrix} 1&0&0\\0&2&0\\0&0&2t\\0&0&0\\0&0&0\\0&0&0 \end{pmatrix}$	$\begin{pmatrix} 1&0&-2\\-\lfloor\frac{t}{2}\rfloor&0&t\\0&1&0 \end{pmatrix}$		$\begin{pmatrix} 1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&2&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0 \end{pmatrix}$	$\begin{pmatrix} 0&0&0&0&1&0\\1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&0&1&0&0\\0&0&1&0&0&0\\0&0&0&0&0&1 \end{pmatrix}$

This way, a basis for 2-cocycles is given by $\mathcal{B}_4^t$, for $t = 2^r q$, q odd:

t	$\mathcal{B}_4^t = \langle \partial_2, \ldots, \partial_{4t-2}, \beta_1, \beta_2, \gamma_1 \rangle$ for odd t $\mathcal{B}_4^t = \langle \partial_2, \ldots, \partial_{4t-3}, \beta_1, \beta_2, \beta_3, \gamma_1, \gamma_2, \gamma_3 \rangle$ for even t
2	$\langle \partial_2, \ldots, \partial_5, BN_2 \otimes 1_4, 1_2 \otimes BN_2 \otimes 1_2, 1_4 \otimes BN_2, K_2, K_3, 1_2 \otimes K_1 \rangle$
$[t]_2 = 1$	$\langle \partial_2, \ldots, \partial_{4t-2}, 1_{2t} \otimes BN_2, 1_t \otimes BN_2 \otimes 1_2, 1_t \otimes K_1 \rangle$
$[t]_2 = 0$	$\langle \partial_2, \ldots, \partial_{4t-3}, 1_t \otimes BN_2 \otimes 1_2, 1_{2t} \otimes BN_2, 1_q \otimes BN_{2^r} \otimes 1_4, 1_{\frac{t}{2}} \otimes K_2, 1_{\frac{t}{2}} \otimes K_3, 1_t \otimes K_1 \rangle$

The matrices K_1, K_2, K_3 are given by

K_1	K_2	K_3
$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & - & - & 1 & 1 & - & - \\ 1 & 1 & - & - & 1 & 1 & - & - \\ 1 & 1 & - & - & 1 & 1 & - & - \\ 1 & 1 & - & - & 1 & 1 & - & - \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & - & 1 & - & 1 & - & 1 & - \\ 1 & - & 1 & - & 1 & - & 1 & - \\ 1 & - & 1 & - & 1 & - & 1 & - \\ 1 & - & 1 & - & 1 & - & 1 & - \end{pmatrix}$

3.1.2. Basis for $G_5^t = D_{4t} = \mathbb{Z}_2 \ {}_\chi\!\ltimes \mathbb{Z}_{2t}$

$\mathcal{B}_1 = \{u_1, u_2\}$, $\mathcal{B}_2 = \{e_1, e_2, e_3\}$, $\mathcal{B}_3 = \{v_1, v_2, v_3, v_4\}$,
$d_2(e_1) = 2 \cdot u_1$, $d_2(e_2) = (2 - 2t) \cdot u_2$, $d_2(e_3) = 2t \cdot u_2$,
$d_3(v_2) = 2t \cdot e_2 + (2t - 2) \cdot e_3$, $d_3(v_3) = -2t \cdot e_2 - (2t - 2) \cdot e_3$,
$F[(g, h)] = g \cdot u_1 + [(-1)^g h]_{2t} \cdot u_2$,

$$F[(g, h)|(a, b)] = ag \cdot e_1 + [-a(-1)^g h]_{2t} \cdot e_2 + \lambda^{2t}_{[(-1)^a b]_{2t} + [(-1)^{g+a} h]_{2t}} \cdot e_3 + a\lambda^1_h([(-1)^{g+a} h]_{2t} - 1) \cdot e_3.$$

From these data, it may be checked that

D_2	Q_2	D_3	Q_3: $t = 2$	Q_3: $t > 2$
$\begin{pmatrix} 2 & 0 \\ 0 & 2 \\ 0 & 0 \end{pmatrix}$	I_2	$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & -2 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 1-t & 0 \\ -1 & t & 0 \end{pmatrix}$

This way, a basis for 2-cocycles is given by $\mathcal{B}_5^t$

t	$\mathcal{B}_5^t = \langle \partial_2, \ldots, \partial_{4t-2}, \beta_1, \beta_2, \gamma \rangle$
2	$\langle \partial_2, \ldots, \partial_6, BN_2 \otimes 1_{2t}, 1_{2t} \otimes BN_2, K_1 \rangle$
$t > 2$	$\langle \partial_2, \ldots, \partial_{4t-2}, BN_2 \otimes 1_{2t}, 1_{2t} \otimes BN_2, K_2 \rangle$

The matrices K_1 and K_2 are given by

K_1	K_2
$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & - & 1 & - & - & - \\ 1 & 1 & - & - & - & 1 & 1 & - \\ 1 & - & - & - & 1 & - & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & - & - & - & 1 & - & 1 & 1 \\ 1 & 1 & - & - & - & 1 & 1 & - \\ 1 & 1 & 1 & - & 1 & - & - & - \end{pmatrix}$	$\left(\begin{array}{c\|ccc} & 1 & \cdots & 1 \\ & - & & \\ BN_{2t} & \vdots & BN_{2t-1} & \\ & - & & \\ \hline & 1 & & \\ FN_{2t} & - & FN_{2t-1} & \\ & \vdots & & \\ & - & 1 \ \cdots & 1 \end{array}\right)$

Here FN_k denotes the forward negacyclic matrix of size $k \times k$,

$$FN_k = \begin{pmatrix} 1 & \cdots & \cdots & 1 \\ 1 & - & \cdots & - \\ \vdots & \ddots & \ddots & \vdots \\ 1 & \cdots & 1 & - \end{pmatrix}_{k\times k}.$$

3.1.3. Basis for $G_7^t = (\mathbb{Z}_t \ltimes_f \mathbb{Z}_2) \rtimes_\chi \mathbb{Z}_2$

$\mathcal{B}_1 = \{u_1, u_2, u_3\}$, $\mathcal{B}_2 = \{e_1, \ldots, e_6\}$, $\mathcal{B}_3 = \{v_1, \ldots, v_{10}\}$,
$d_2(e_1) = t \cdot u_1, d_2(e_3) = (2-t) \cdot u_1, d_2(e_4) = (\lfloor \frac{t}{2} \rfloor - 1) \cdot u_1 + 2 \cdot u_2,$
$d_2(e_5) = (1 - \lfloor \frac{t}{2} \rfloor) \cdot u_1, d_2(e_6) = 2 \cdot u_3,$
$d_3(v_2) = t \cdot e_2, d_3(v_3) = (t-2) \cdot e_1 + t \cdot e_3, d_3(v_4) = -2 \cdot e_2, d_3(v_5) = (t-2) \cdot e_2,$
$d_3(v_6) = (2-t) \cdot e_1 - t \cdot e_3, d_3(v_7) = (\lfloor \frac{t}{2} \rfloor - 1) \cdot e_2,$
$d_3(v_8) = (\lfloor \frac{t}{2} \rfloor - 1) \cdot e_1 + (\lfloor \frac{t}{2} \rfloor - 1) \cdot e_2 + (\lfloor \frac{t}{2} \rfloor - 1) \cdot e_3 + 2 \cdot e_5,$
$d_3(v_9) = (1 - \lfloor \frac{t}{2} \rfloor) \cdot e_1 + (1 - \lfloor \frac{t}{2} \rfloor) \cdot e_3 - 2 \cdot e_5,$
$F[(j, g, h)] = j \cdot u_1 + g \cdot u_2 + h \cdot u_3,$

$$\begin{aligned} F[(j,g,h)|(a,b,c)] = {} & (\lambda^t_{[(-1)^h(a+hf(b,b))]_t + [j+f(g,b)]_t} - \lambda^t_{[-f(g,b)]_t + [j+f(g,b)]_t}) \cdot e_1 \\ & + h(a-1)\lambda^2_a \cdot e_1 + \lambda^t_{[(-1)^h hf(b,b)]_t + [(-1)^h a]_t} \cdot e_1 + [(-1)^h(a + hf(b,b))]_t g \cdot e_2 \\ & + ah \cdot e_3 + \lambda^2_{h+b} \cdot e_4 + bh \cdot e_5 + \lambda^2_{h+c} \cdot e_6. \end{aligned}$$

From these data, the matrices D_i and Q_i may be described, in terms of the the coset of t modulo 4:

t	D_2	Q_2	D_3	Q_3
2	$\begin{pmatrix} 2&0&0\\0&2&0\\0&0&2\\0&0&0\\0&0&0\\0&0&0 \end{pmatrix}$	$\begin{pmatrix} 1&0&0\\0&1&0\\0&0&1 \end{pmatrix}$	$\begin{pmatrix} 2&0&0&0&0&0\\0&2&0&0&0&0\\0&0&2&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0 \end{pmatrix}$	$\begin{pmatrix} 0&0&0&0&1&0\\1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&0&1&0&0\\0&0&1&0&0&0\\0&0&0&0&0&1 \end{pmatrix}$
3	$\begin{pmatrix} 1&0&0\\0&2&0\\0&0&2\\0&0&0\\0&0&0\\0&0&0 \end{pmatrix}$	$\begin{pmatrix} 1&0&0\\0&1&0\\0&0&1 \end{pmatrix}$	$\begin{pmatrix} 1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&2&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0 \end{pmatrix}$	$\begin{pmatrix} 1&0&0&0&-3&0\\0&1&0&0&0&0\\0&0&0&0&1&0\\0&0&0&1&0&0\\0&0&1&0&0&0\\0&0&0&0&0&1 \end{pmatrix}$
4, 5	$\begin{pmatrix} 1&0&0\\0&2&0\\0&0&2\\0&0&0\\0&0&0\\0&0&0 \end{pmatrix}$	$\begin{pmatrix} 1&-2&0\\0&1&0\\0&0&1 \end{pmatrix}$	$\begin{pmatrix} 1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&2&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0 \end{pmatrix}$	$\begin{pmatrix} 0&1&-1&0&-t&0\\1&0&0&0&0&0\\0&0&1&0&t-2&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1 \end{pmatrix}$

t	D_2	Q_2	D_3	Q_3
6	$\begin{pmatrix}2&0&0\\0&2&0\\0&0&2\\0&0&0\\0&0&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}1&-1&0\\0&1&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}2&0&0&0&0&0\\0&2&0&0&0&0\\0&0&2&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix}$	$\begin{pmatrix}0&1&-1&0&-3&0\\1&0&0&0&0&0\\0&0&1&0&2&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{pmatrix}$
7	$\begin{pmatrix}1&0&0\\0&2&0\\0&0&2\\0&0&0\\0&0&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}1&0&-6\\0&0&1\\0&1&0\end{pmatrix}$	$\begin{pmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&2&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix}$	$\begin{pmatrix}0&1&-5&0&7&0\\1&0&0&0&0&0\\0&0&3&0&-5&0\\0&0&0&1&0&0\\0&0&1&0&-2&0\\0&0&0&0&0&1\end{pmatrix}$
$[t]_4 = 0, 1$	$\begin{pmatrix}1&0&0\\0&2&0\\0&0&2\\0&0&0\\0&0&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}1&0&-2\\1-\lfloor\frac{t}{4}\rfloor&0&\lfloor\frac{t}{2}\rfloor-1\\0&1&0\end{pmatrix}$	$\begin{pmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&2&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix}$	$\begin{pmatrix}0&0&1&0&-t&0\\1&0&0&0&0&0\\0&1&-1&0&t-2&0\\0&0&0&1&0&0\\0&1-\lfloor\frac{t}{4}\rfloor&0&0&\lfloor\frac{t}{2}\rfloor-1&0\\0&0&0&0&0&1\end{pmatrix}$

t	D_2	Q_2	D_3	Q_3
$[t]_4 = 2$	$\begin{pmatrix}2&0&0\\0&2&0\\0&0&2\\0&0&0\\0&0&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}0&0&1\\1&0&-\lfloor\frac{t}{4}\rfloor\\0&1&0\end{pmatrix}$	$\begin{pmatrix}2&0&0&0&0&0\\0&2&0&0&0&0\\0&0&2&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix}$	$\begin{pmatrix}0&0&-1&0&\frac{t}{2}&0\\1&0&0&0&0&0\\0&0&1&0&-2\lfloor\frac{t}{4}\rfloor&0\\0&0&0&1&0&0\\0&1&0&0&-\lfloor\frac{t}{4}\rfloor&0\\0&0&0&0&0&1\end{pmatrix}$
$[t]_4 = 3$	$\begin{pmatrix}1&0&0\\0&2&0\\0&0&2\\0&0&0\\0&0&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}1&0&-2\\\lceil\frac{t}{4}\rceil&0&-\lfloor\frac{t}{2}\rfloor\\0&1&0\end{pmatrix}$	$\begin{pmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&2&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix}$	$\begin{pmatrix}0&0&1&0&-t&0\\1&0&0&0&0&0\\0&1&-1&0&t-2&0\\0&0&0&1&0&0\\0&-1-3\lfloor\frac{t}{4}\rfloor&1&0&2\lfloor\frac{t}{4}\rfloor&0\\0&0&0&0&0&1\end{pmatrix}$

This way, a basis for 2-cocycles is given by $\mathcal{B}_7^t$

t	$\mathcal{B}_7^t = \langle \partial_2, \ldots, \partial_{4t-3}, \beta_1, \ldots, \beta_3, \gamma_1, \ldots, \gamma_3 \rangle$ for $[t]_4 = 2$ $\mathcal{B}_7^t = \langle \partial_2, \ldots, \partial_{4t-2}, \beta_1, \beta_2, \gamma_1 \rangle$ for $[t]_4 = 0, 1, 3$
2	$\langle \partial_2, \ldots, \partial_5, BN_2 \otimes 1_4, 1_2 \otimes BN_2 \otimes 1_2, 1_4 \otimes BN_2, K_1^T, K_2^T, 1_2 \otimes K_3^T \rangle$
3	$\langle \partial_2, \ldots, \partial_{10}, 1_3 \otimes BN_2 \otimes 1_2, 1_6 \otimes BN_2, 1_3 \otimes K_3 \rangle$
4, 5	$\langle \partial_2, \ldots, \partial_{4t-2}, 1_t \otimes BN_2 \otimes 1_2, 1_{2t} \otimes BN_2, K_4^t \rangle$
6	$\langle \partial_2, \ldots, \partial_{21}, 1_3 \otimes BN_2 \otimes 1_4, 1_3 \otimes H_1, 1_{12} \otimes BN_2, 1_3 \otimes K_1, (1_3 \otimes K_2) \cdot K_4^6, K_4^t \rangle$
7	$\langle \partial_2, \ldots, \partial_{26}, 1_{14} \otimes BN_2, 1_7 \otimes BN_2 \otimes 1_2, (1_7 \otimes K_3) \cdot K_4^7 \rangle$
$[t]_4 = 0, 1$	$\langle \partial_2, \ldots, \partial_{4t-2}, 1_{2t} \otimes BN_2, 1_t \otimes BN_2 \otimes 1_2, K_4^t \rangle$
$[t]_8 = 2$	$\langle \partial_2, \ldots, \partial_{4t-3}, 1_t \otimes BN_2 \otimes 1_2, 1_{2t} \otimes BN_2, 1_{\frac{t}{2}} \otimes BN_2 \otimes 1_4, 1_{\frac{t}{2}} \otimes K_1, 1_t \otimes K_3, K_4^t \rangle$
$[t]_8 = 6$	$\langle \partial_2, \ldots, \partial_{4t-3}, 1_t \otimes BN_2 \otimes 1_2, 1_{2t} \otimes BN_2, 1_{\frac{t}{2}} \otimes H_1, 1_{\frac{t}{2}} \otimes K_1, 1_t \otimes K_3, K_4^t \rangle$
$[t]_4 = 3$	$\langle \partial_2, \ldots, \partial_{4t-2}, 1_{2t} \otimes BN_2, 1_t \otimes BN_2 \otimes 1_2, (1_t \otimes K_3) \cdot K_4^t \rangle$

The matrices K_1, K_2, K_3 and H_1 are given by

K_1	K_2	K_3	H_1
$\begin{pmatrix} 1&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&1\\ 1&1&1&1&-&-&-&-\\ 1&1&1&1&-&-&-&-\\ 1&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&1\\ 1&1&1&1&-&-&-&-\\ 1&1&1&1&-&-&-&- \end{pmatrix}$	$\begin{pmatrix} 1&1&1&1&1&1&1&1\\ 1&1&1&1&-&-&-&-\\ 1&1&1&1&1&1&1&1\\ 1&1&1&1&-&-&-&-\\ 1&1&1&1&1&1&1&1\\ 1&1&1&1&-&-&-&-\\ 1&1&1&1&1&1&1&1\\ 1&1&1&1&-&-&-&- \end{pmatrix}$	$\begin{pmatrix} 1&1&1&1\\ 1&1&-&-\\ 1&1&1&1\\ 1&1&-&- \end{pmatrix}$	$\begin{pmatrix} 1&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&1\\ 1&1&-&-&-&-&1&1\\ 1&1&-&-&-&-&1&1\\ 1&1&-&-&-&-&1&1\\ 1&1&-&-&-&-&1&1\\ 1&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&1 \end{pmatrix}$

The matrix K_4^t consists in a "reflected" matrix $\begin{pmatrix} \cup \\ \cap \end{pmatrix}$, which may be described row by row in terms of blocks of length 4:

- Rows $4k+1$, for $0 \le k \le \frac{t}{2} - 1$, consists in

$$(\overbrace{1111}\stackrel{t-k}{\cdots}\overbrace{1111}\overbrace{-\,-\,-\,-}\stackrel{k}{\cdots}\overbrace{-\,-\,-\,-})$$

- Rows $4k+2$, for $0 \le k \le \frac{t}{2} - 1$, consists in

$$(\overbrace{1111}\stackrel{k+1}{\cdots}\overbrace{1111}\overbrace{-\,-\,11}\stackrel{\frac{t}{2}-1}{\cdots}\overbrace{-\,-\,11}\overbrace{-\,-\,-\,-}\stackrel{\frac{t}{2}-k}{\cdots}\overbrace{-\,-\,-\,-})$$

- Rows $4k+3$, for $0 \le k \le \frac{t}{2} - 1$, consists in

$$(\overbrace{11\,-\,-}\stackrel{\frac{t}{2}-k-1}{\cdots}\overbrace{11\,-\,-}\overbrace{1111}\stackrel{\frac{t}{2}+1}{\cdots}\overbrace{1111}\overbrace{-\,-\,11}\stackrel{k}{\cdots}\overbrace{-\,-\,11})$$

- Rows $4k+4$, for $0 \le k \le \frac{t}{2} - 1$, consists in

$$(\overbrace{1111}\stackrel{k+1}{\cdots}\overbrace{1111}\overbrace{-\,-\,-\,-}\stackrel{t-k-1}{\cdots}\overbrace{-\,-\,-\,-})$$

3.2. Exhaustive search

Next, we include a table with the number of cocyclic Hadamard matrices that we have found in each case, for $1 \le t \le 5$, as well as the required computing time.

The black entries correspond to new results, as far as we know (the case of $G_5^5 = D_{4\cdot 5}$ is included, since the computation of Flannery in Horadam (1996), 2380, differs from ours, 2200).

Table 1

t	$\mathbb{Z}_{4t}$	$\mathbb{Z}_2 \times \mathbb{Z}_{2t}$	$\mathbb{Z}_4 \times \mathbb{Z}_t$	$\mathbb{Z}_2^2 \times \mathbb{Z}_t$	D_{4t}	$\mathbb{Z}_{2t} \ltimes_f \mathbb{Z}_2$	$(\mathbb{Z}_t \ltimes_f \mathbb{Z}_2) \rtimes_\chi \mathbb{Z}_2$	Time
1	2	6	2	6	6	6	6	0″
2	0	**16**	**16**	168	32	**16**	168	0.28″
3	0	24	0	24	72	**0**	72	12.25″
4	0	**96**	**192**	**1984**	768	**0**	**768**	7′20″
5	0	120	0	120	**2200**	120	**2200**	3h29′10″

3.3. Heuristic search

The search space for cocyclic Hadamard matrices over the families G_i^t above grows exponentially with t (according to the dimensions of the basis $\mathcal{B}_i^t$ for 2-cocycles), so that an exhaustive search is only possible in low orders (up to $t = 5$). Each of the matrices M_f is represented as a binary tuple $(x_1, \ldots, x_{r+s+d})$, which corresponds to the coordinates of the related 2-cocycle f with regards to the basis $\mathcal{B}_i^t = \{\partial_r | \beta_s | \gamma_d\}$ for 2-cocycles over G_i^t described in the subsection above. So that precisely those cocycles corresponding to non zero entries x_i come into play in practise,

$$f = \partial_{i_1}^{x_1} \cdot \cdots \cdot \partial_{i_r}^{x_r} \cdot \beta_{j_1}^{x_{r+1}} \cdot \cdots \cdot \beta_{j_s}^{x_{r+s}} \cdot \gamma_{k_1}^{x_{r+s+1}} \cdot \cdots \cdot \gamma_{k_d}^{x_{r+s+d}}.$$

Apparently, the genetic algorithm described in Álvarez et al. (2006a) seems to provide some cocyclic Hadamard matrices of larger order than those previously obtained with other algorithms. It is worth noting that this heuristic algorithm may be performed provided that an explicit basis for 2-cocycles (both representative 2-cocycles and 2-coboundaries) is known.

Calculations in Baliga and Horadam (1995), Flannery (1997) and Álvarez et al. (2006a) suggest that $G_4^t = \mathbb{Z}_t \times \mathbb{Z}_2^2$ and $G_5^t = D_{4t}$ give rise to a large number of Hadamard cocyclic matrices. The authors have observed this behavior on a third family of groups, $G_7^t = (\mathbb{Z}_t \ltimes_f \mathbb{Z}_2) \rtimes_\chi \mathbb{Z}_2$.

Now we include some executions of the genetic algorithm running on these families. The tables below show some Hadamard cocyclic matrices over G_i^t, and the number of generations (i.e. iterations) and time required (in seconds) as well. Note that the number of generations is not directly related to the size of the matrices. Do not forget about randomness of the genetic algorithm.

t	iter.	time	product of generators of 2-cocycles over G_2^t
2	2	0.05″	(1, 1, 1, 0, 1, 1, 1, 0)
3	3	0.23″	(0, 1, 0, 0, 0, 0, 1, 1, 1, 1, 1, 1)
4	25	5.13″	(1, 0, 1, 0, 1, 1, 0, 0, 0, 1, 1, 0, 0, 0, 0, 0)
5	72	31.95″	(1, 0, 0, 0, 1, 0, 0, 1, 0, 0, 0, 1, 1, 1, 0, 0, 0, 1, 1, 1)
6	9000	10h	No matrix found! It seems that no Hadamard cocyclic matrix exist!
7	128	3′38″	(0, 0, 1, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 0, 1, 1, 0, 1, 1, 0, 0, 0, 1, 1, 1, 1, 1, 1)
8	260	11′30″	(0, 1, 0, 1, 1, 1, 1, 0, 1, 1, 1, 1, 0, 1, 0, 0, 0, 0, 1, 1, 0, 0, 1, 0, 1, 0, 0, 1, 1, 1, 0, 1)

t	iter.	time	product of generators of 2-cocycles over G_4^t
2	1	0.03″	(0, 0, 0, 0, 1, 1, 1, 0, 0, 0)
3	15	1.13″	(0, 1, 0, 0, 0, 1, 0, 1, 1, 1, 1, 1)
4	6	1.28″	(1, 1, 1, 1, 0, 0, 0, 1, 1, 1, 0, 1, 0, 0, 1, 1, 0, 0)
5	316	2′48″	(1, 1, 0, 0, 1, 1, 0, 1, 1, 0, 0, 0, 0, 0, 1, 1, 1, 1, 1, 1)
6	40	39.39″	(0, 1, 1, 0, 1, 0, 1, 1, 0, 1, 0, 1, 1, 0, 0, 1, 1, 1, 1, 0, 1, 1, 1, 0, 0, 1)
7	94	2′45″	(1, 0, 1, 1, 0, 0, 0, 0, 0, 1, 0, 1, 0, 1, 1, 1, 0, 1, 1, 0, 0, 1, 0, 1, 0, 1, 1, 1)
8	373	30′03″	(1, 1, 1, 1, 0, 0, 0, 0, 1, 1, 1, 0, 1, 1, 1, 0, 1, 1, 1, 1, 0, 0, 0, 1, 1, 0, 1, 1, 0, 1, 1, 1, 1, 0)

t	iter.	time	product of generators of 2-cocycles over G_5^t
2	0	0.02″	(0, 1, 0, 0, 1, 1, 1, 1)
3	4	0.27″	(1, 0, 0, 0, 0, 1, 1, 1, 0, 1, 0, 1)
4	2	0.5″	(0, 0, 0, 0, 0, 1, 1, 1, 1, 0, 1, 0, 1, 1, 0, 0)
5	6	2.83″	(0, 0, 1, 1, 0, 1, 0, 1, 1, 1, 0, 0, 1, 0, 0, 1, 0, 1, 0, 1)
6	3	3.03″	(1, 0, 0, 1, 1, 0, 1, 0, 1, 1, 1, 0, 1, 1, 0, 0, 1, 1, 1, 0, 1, 1, 0, 1)
7	37	1′08″	(1, 1, 0, 0, 1, 1, 1, 1, 1, 0, 0, 1, 0, 1, 0, 1, 1, 1, 1, 1, 1, 0, 1, 0, 1, 1, 0, 1)
8	30	1′24″	(1, 1, 1, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 1, 1, 1, 0, 0, 1, 1, 0, 1, 1, 1, 0, 1, 0, 1, 1, 0, 1)

t	iter.	time	product of generators of 2-cocycles over G_7^t
2	0	0.02″	(1, 1, 0, 0, 0, 1, 1, 1, 0, 1)
3	0	0.05″	(0, 0, 1, 0, 1, 1, 0, 0, 1, 1, 1, 1)
4	0	0.1″	(1, 1, 1, 1, 0, 1, 0, 1, 1, 1, 0, 1, 1, 0, 1, 0)
5	2	1.06″	(0, 1, 1, 1, 1, 0, 1, 1, 1, 1, 0, 1, 1, 1, 1, 1, 0, 0, 1, 1)
6	27	28.6″	(1, 0, 1, 1, 0, 1, 1, 0, 1, 1, 1, 1, 1, 1, 1, 0, 1, 1, 0, 0, 1, 0, 1, 0, 1, 1)
7	59	1′45″	(1, 1, 1, 1, 0, 1, 1, 0, 1, 0, 1, 1, 0, 0, 1, 0, 1, 0, 1, 0, 1, 1, 1, 1, 0, 1, 1, 1)
8	10	31.48″	(0, 1, 1, 1, 0, 1, 0, 0, 1, 0, 0, 0, 0, 1, 0, 1, 1, 0, 1, 0, 1, 1, 1, 0, 1, 0, 0, 1, 1, 1, 0, 1)

There is no doubt that an exhaustive search is only possible for small $|G|$. In the light of the tables above, it seems that the heuristic search may not overcome this difficulty as desired. The study of some local properties on a particular group may lead to improved versions of the genetic algorithm. This has been the case of dihedral groups (Álvarez et al., 2006a). In spite of this fact, the exhaustive search may be improved with a deeper analysis of the way in which the elements in a basis $\mathcal{B}$ for 2-cocycles have to be combined so that a Hadamard cocyclic matrix is obtained (see Álvarez et al. (2008) for details).

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