WELL–POSEDNESS AND ASYMPTOTIC BEHAVIOUR FOR A NON-CLASSICAL AND NON-AUTONOMOUS DIFFUSION EQUATION WITH DELAY

TOMÁS CARABALLO
Departamento de Ecuaciones Diferenciales y Análisis Numérico, Universidad de Sevilla, Apdo. de Correos 1160, 41080-Sevilla Spain
E-mail: caraball@us.es

ANTONIO M. MÁRQUEZ-DURÁN
Departamento de Economía, Métodos Cuantitativos e Historia Econoómica, Universidad Pablo de Olavide, Ctra. de Utrera, Km. 1, 41013Sevilla, Spain
E-mail: ammardur@upo.es

FELIPE RIVERO
Departamento de Análise, Instituto de Matemática e Estatística, Universidade Federal Fluminense, Rua Mário Santos Braga S/N Valonguinho, 24.020-140 Niteró, RJ, Brazil
E-mail: lf_rivero@id.uff.br

February 27, 2015

In this paper, it is analyzed a non-classical non-autonomous diffusion equation with delay. First, the well-posedness and the existence of a local solution is proved by using a fixed point theorem. Then, the existence of solutions defined globally in future is ensured. The asymptotic behaviour of solutions is analyzed within the framework of pullback attractors as it has revealed a powerful theory to describe the dynamics of non-autonomous dynamical systems. One difficulty in the case of delays concerns the phase space that one needs to consider to construct the evolution process. This yields to the necessity of using a version of the Ascoli-Arzelà theorem to prove the compactness.

Keywords: Delay equations, pullback attractors, non-autonomous problems, evolution processes, non-classical diffusion equations.

Mathematics Subject Classifications (2010): 35B40, 35B41, 35K55, 37B55

In this work we are interested in the analysis of a non-classical diffusion equation with delays, written in the following abstract functional formulation,

$$\begin{cases} \dfrac{\partial u}{\partial t} - \gamma(t)\Delta\dfrac{\partial u}{\partial t} - \Delta u = g(u) \\ \qquad\qquad\qquad + f(t,u_t) \text{ in } (s,+\infty)\times\Omega, \\ u = 0 \text{ on } (s,+\infty)\times\partial\Omega \\ u(t,x) = \phi(t-s,x), t\in[s-h,s], x\in\Omega, \end{cases} \tag{1}$$

where $s \in \mathbb{R}$ is the initial time, $\Omega \subset \mathbb{R}^n$ is a smooth bounded domain, Δ represents the Laplacian operator with respect to the spatial variables, i.e. $\Delta = \sum_{i=1}^n \partial/\partial x_i$,

$$g \in C^1(\mathbb{R}), \qquad \limsup_{|a|\to+\infty} \frac{g(a)}{a} \le 0 \tag{2}$$

$$|g(a)-g(b)| \le c|a-b|(1+|a|^{\rho-1}+|b|^{\rho-1}), \tag{3}$$

with $1<\rho<\frac{n+2}{n-2}$. The time-dependent delay term $f(t,u_t)$ represents, for instance, the influence of an external force with some kind of delay, memory or hereditary characteristics, although can also model some kind of feedback controls. Here, u_t denotes a segment of the solution, in other words, given $h>0$ and a function $u:[s-h,+\infty)\times\Omega\to\mathbb{R}$, for each $t\ge s$ we define the mapping $u_t:[-h,0]\times\Omega\to\mathbb{R}$ by

$$u_t(\theta,x) = u(t+\theta,x), \text{ for } \theta\in[-h,0], x\in\Omega.$$

In this way, this abstract formulation includes several types of delay terms in a unified way. For instance, terms like

$$\begin{aligned} &F_1(t,u(t-h)), \\ &F_2(u(t-\tau(t))), \\ &\int_{-h}^0 F_3(t,\theta,u(t+\theta))\,d\theta, \end{aligned} \tag{4}$$

where F_i $(i=1,2,3)$ are suitable functions, and $\tau:\mathbb{R}\to[0,h]$, can all be described by the following corresponding f_i defined as

$$\begin{aligned} f_1(t,\phi) &= F_1(t,\phi(-h)), \\ f_2(t,\phi) &= F_2(\phi(-\tau(t))), \\ f_3(t,\phi) &= \int_{-h}^0 F_3(t,\theta,\phi(\theta))\,d\theta, \end{aligned} \tag{5}$$

where $\phi:[-h,0]\to X$ (X denotes certain Banach or Hilbert space concerning the spatial variable). Then, when we replace ϕ by u_t in (5), we obtain (4).

Non-classical parabolic equations arise as models to describe physical phenomena such as non-Newtonian flow, soil mechanics, heat conduction, etc (see [Aifantis, 1980], [Kuttlerand & Aifantis, 1987], [Kuttlerand & Aifantis, 1988], [Anh & Bao, 2010], [Camassa & Holm, 1993], [Ma *et al.*, 2012], [Peter & Gurtin], [Sun *et al.*, 2007], [Sun *et al.*, 2008] and references therein). In the work of Aifantis et al., [Aifantis, 1980], [Kuttlerand & Aifantis, 1987], [Kuttlerand & Aifantis, 1988] we can find a quite general approach to deduce these equations in the autonomous case without delay. In the aforementioned papers, it is pointed out that the classical reaction-diffusion equation

$$u_t - \Delta u = g(u)$$

does not contain each aspect of the reaction-diffusion problem, and it neglects viscidity, elasticity, and pressure of medium in the process of solid diffusion. The authors obtained a diffusion theory similar to Fick's classical model for solute in an undisturbed solid matrix, obtaining a hyperbolic equation

$$u_t + D_1 u_{tt} = D_2\Delta u,$$

where D_1 and D_2 are positive constants. Assigning viscosity to the diffusing substance, they arrived to que following equation

$$u_t + D_1 u_{tt} = D_2\Delta u + D_3\Delta u_t,$$

and neglecting the inertia term, finally obtained the non-classical parabolic equation

$$u_t = D_2\Delta u + D_3\Delta u_t,$$

where D_3 is also a positive constant.

The asymptotic behaviour of the model without delay terms and with constant coefficients

$$u_t - \mu\Delta u_t - \Delta u + g(u) = f(x), \qquad \mu\in[0,1]$$

is studied in [Wang *et al.*, 2006], where, in particular, it is shown the well-posedness of the problem and the existence of the global attractor either in $H_0^1(\Omega)$ or in $H^2(\Omega)$, depending on the regularity of the initial data. They also showed the continuity of the global attractor in Housdorff semidistance when $\mu \to 0$ in $H_0^1(\Omega)$.

The introduction of a time dependence in coefficient $\gamma(t)$ represents the variability of viscosity in time due to, for example, external environment temperatures. This time dependence provides the system with a non-autonomous nature. Although there exist several possibilities to analyze non-autonomous dynamical systems (e.g. skew-product flows, uniform attractors, kernel section, etc), we have preferred to consider the theory of evolution processes and pullback attractors. This is a general theory which studies the asymptotic behaviour of non-autonomous problems in the pullback sense, that is, when we fix the final time and study the behaviour when we start earlier and earlier in the past. Therefore, we have two different kinds of dynamics in a non-autonomous problem: the forward dynamics, when final time goes to infinity, and the pullback dynamics, when the initial times go to minus infinity. This theory has been widely developed over the last thirty years (see [Caraballo *et al.*, 2010], [Carvalho *et al.*, 2013], [Caraballo *et al.*, 2006a], [Chepyzhov & Vishik, 2002], [Marín-Rubio & Real, 2009]) and it can be seen as a natural generalization of the classical theory of semigroups and global attractors.

In [Rivero, 2013], Rivero studied the existence of the pullback attractor and its continuity under non-autonomous perturbations without delay, showing the existence of a concrete structure under some assumptions on the non-linearity. The study of a non-autonomous case with delay appeared in [Caraballo & Márquez-Durán, 2013] for the first time, where it was established the well-posedness of the problem when $\gamma(t) \equiv \gamma$ is constant. Also, it was proved in [Caraballo & Márquez-Durán, 2013] the stability of the stationary solutions under some appropriate hypotheses on the delay term. In [Hu & Wang, 2012], Hu and Wang studied this equation with a specific variable delay term with bounded derivative, showing the existence of the pullback attractor in H_0^1 and H^2 without neither non-linearity nor variable coefficients.

The presence of delays in the models of certain phenomena implies the necessity of working in a quite different phase space. Indeed, our initial data needs now to be a function defined in an interval of certain length $h > 0$, prior the initial time, in other words, we need to know the values of a mapping defined in $[s-h, s]$ if s is such an initial time. In particular, this fact implies that the compactness property of our non-autonomous dynamical system has to be proved in more complicated functional spaces, for instance, in a Banach space of continuous functions. This requires the use of Ascoli-Arzelà's arguments.

The content of this paper is as follows. In order to show the existence and uniqueness of solution for (1) and the continuous dependence on the initial data, in Sec. 1 we establish the notation for delay equations, stating the conditions and results ensuring the well-posedness of (1). Sec. 2 is devoted to the study of the global existence of solutions and the existence of a pullback absorbing family within the universe of global bounded families. In Sec. 3 we recall the basic definitions and results for pullback attractors, setting the framework for our analysis. Finally, in Sec. 4 the existence of the pullback attractor for our model is proved.

1. Existence of solution

We will first analyze the existence and uniqueness of solution to our model. Let us denote $C_L = C([-h,0], L^2(\Omega))$ with norm $\|\psi\|_{C_L} = \sup_{s\in[-h,0]}\|\psi(s)\|_{L^2}$. In this way, given $\psi \in C([-h,T], L^2(\Omega))$, for any $t \in [0,T]$ we can define $\psi_t : [-h,0] \to L^2(\Omega)$ as $\psi_t(\theta) = \psi(t+\theta)$ for $\theta \in [-h,0]$. Obviously $\psi_t \in C_L$. Analogously, it can be defined $C_H = C([-h,0], H_0^1(\Omega))$ with norm $\|\psi\|_{C_H} = \sup_{s\in[-h,0]}\|\psi(s)\|_{H_0^1}$.

We will pick up an initial value $\phi \in C_L$ such that $\phi(0) \in H_0^1(\Omega)$. Assume also that $f : (s,+\infty) \times C_H \to L^2(\Omega)$ is continuous in t and locally Lipschitz in C_H uniformly in time, that is, for all $R > 0$ there exists a constant $C(R) > 0$ such that, for any $t \in \mathbb{R}$,

$$\|f(t,\xi) - f(t,\eta)\|_{L^2(\Omega)} \le C(R)\|\xi - \eta\|_{C_H}, \tag{6}$$

for all $\xi, \eta \in C_H$ with $\|\xi\|_{C_H}, \|\eta\|_{C_H} \le R$.

Proceeding as in [Rivero, 2013], we can define operators $B(t) = (I + \gamma(t)A)^{-1}$ and $\tilde{A}(t) = AB(t)$, where $A = -\Delta$ with Dirichlet boundary conditions and the functions $\tilde{g}(t,u) = B(t)g(u)$ and $\tilde{f}(t,\phi) = B(t)f(t,\phi)$, $\forall t \ge s$, $\forall \phi \in C_H$.

Then, we can write problem (1) as

$$\frac{du}{dt} = h(t, u_t), \tag{7}$$

with $h : (s,+\infty)\times C_H \to L^2(\Omega)$ defined as $h(t,\phi) = \tilde{A}(t)\phi(0) + \tilde{g}(t,\phi(0)) + \tilde{f}(t,\phi)$, $\forall t \geq s$, $\forall \phi \in C_H$.

The domain of the operator $\tilde{A}(t)$ does not depend on time. In fact, if we define our problem in $H_0^1(\Omega)$, then $D(\tilde{A}(t)) = H_0^1(\Omega)$. This operator is uniformly bounded in time and

$$\tilde{A}(t) = \frac{1}{\gamma(t)}\left[I - (1+\gamma(t)A)^{-1}\right], \tag{8}$$

for any $t \in \mathbb{R}$. Also, for any $\alpha > 0$ and $x \in D(A^\alpha)$, $A^\alpha \tilde{A}(t)x = \tilde{A}(t)A^\alpha x$.

Thanks to the continuity of the function $\mathbb{R} \ni t \mapsto B(t) \in \mathcal{L}(H_0^1(\Omega))$, we obtain the following estimate (see [Rivero, 2013])

$$\|\tilde{A}(t) - \tilde{A}(s)\|_{\mathcal{L}(H_0^1(\Omega))} \leq C|\gamma(t)-\gamma(s)|,$$

for a constant $C \in \mathbb{R}$.

We can now state and prove the existence of solution to our problem.

Theorem 1.1. *For each $\phi \in C_H$ and under the assumptions* (2), (3) *and* (6), *there exists $\delta > 0$ such that in the interval $[s-h, s+\delta)$ there is a unique solution of problem* (1). *In other words, there exists a function $u \in C([s-h,s+\delta), H_0^1(\Omega))$ with $u(t,s;\phi) = \phi(t-s)$ for all $t \in [s-h,s]$ which satisfies*

$$u(t,s;\phi) = \phi(0) + \int_s^t h(r,u_r)dr,$$

for all $t \in [s,s+\delta)$, Moreover, if f is continuous in time, then $u \in C([s-h,s+\delta), H_0^1(\Omega)) \cap C^1((s,s+\delta), H_0^1(\Omega))$, i.e. u is a strong solution.

Proof. The proof is based on the contraction mapping theorem. To this end, for the given initial datum $\phi \in C_H$, and for a positive T to be determined later on, we define the following space

$$\begin{aligned} X_\phi^T = \Big\{ u \in C\left([s-h,T), H_0^1(\Omega)\right) : u(t) = \phi(t-s) \\ \text{for all} \quad t \in [s-h,s], \\ \text{and } \|u\|_{X_\phi^T} \leq 2\|\phi\|_{C_H} \Big\}, \end{aligned} \tag{9}$$

where $\|u\|_{X_\phi^T} = \sup_{\sigma\in[s-h,T)} \|u(\sigma)\|_{H_0^1}$.

This space X_ϕ^T is a complete metric space (since it is a closed subset of a Banach space).

Now we consider the operator $\Phi : X_\phi^T \to X_\phi^T$ given by

$$\Phi(u)(t) = \begin{cases} \phi(t-s), & t \in [s-h,s] \\ \phi(0) + \displaystyle\int_s^t h(r,u_r)dr, & t \in (s,T). \end{cases}$$

Let us first check that Φ is well defined, i.e. $\Phi(u) \in X_\phi^T$ for all $u \in X_\phi^T$.

Given $u \in X_\phi^T$, the mapping

$$\Phi(u)(\cdot) : [s-h,T) \to L^2(\Omega)$$

is continuous thanks to the continuity of ϕ and the mapping $t \in (s,T) \to h(t,u_t)$. Moreover, for all $t \geq s$ we have that

$$\begin{aligned} \|\Phi(u)(t)\|_{H_0^1} &= \left\|\phi(0) + \int_s^t h(r,u_r)dr\right\|_{H_0^1} \\ &\leq \|\phi(0)\|_{H_0^1} + \int_s^t \|h(r,u_r)\|_{H_0^1}dr \\ &\leq \|\phi(0)\|_{H_0^1} + \int_s^t \|\tilde{A}(r)\|_{\mathcal{L}(H_0^1)}\|u(r)\|_{H_0^1}dr \\ &\quad + \int_s^t \|B(r)g(u(r))\|_{H_0^1}dr \\ &\quad + \int_s^t \|B(r)f(r,u_r)\|_{H_0^1}dr. \end{aligned} \tag{10}$$

But, $\|\tilde{A}(r)\|_{\mathcal{L}(H_0^1)} \leq a$, for all $r \geq s$, and proceeding as in [Rivero, 2013], we can prove that $B(t)\circ g$ is locally lipschitz in $H_0^1(\Omega)$ uniformly in t. Indeed, by (3),

$$\begin{aligned} &\|g(u)-g(v)\|_{L^{\frac{2n}{n+2}}} \\ &\leq c\left[\int_\Omega \left[|u-v|(1+|u|^{\rho-1}+|v|^{\rho-1})\right]^{\frac{2n}{n+2}}\right]^{\frac{n+2}{2n}} \\ &\leq \tilde{c}\|u-v\|_{L^{\frac{2n}{n-2}}}\left(1+\|u\|^{\rho-1}_{L^{\frac{n(\rho-1)}{2}}} + \|v\|^{\rho-1}_{L^{\frac{n(\rho-1)}{2}}})\right). \end{aligned} \tag{11}$$

Since

$$\frac{2n(\rho-1)}{4} \leq \frac{2n}{n-2},$$

we have that

$$H_0^1(\Omega) \subset L^{\frac{n(\rho-1)}{2}}.$$

Due to

$$H_0^1(\Omega) \xrightarrow{g} H^{-1}(\Omega) \xrightarrow{B(t)} H_0^1(\Omega),$$

function $B(t) \circ g$ is locally Lipschitz in $H_0^1(\Omega)$.

Therefore, for $R = 2\|\phi\|_{C_H}$ there exists $L_g(R)$ such that

$$\begin{aligned}
&\|B(r)g(u(r))\|_{H_0^1} \\
&\qquad \leq \|B(r)\left(g(u(r)) - g(0)\right) + B(r)g(0)\|_{H_0^1} \\
&\qquad \leq L_g(R)\|u(r)\|_{H_0^1} + b_0|g(0)|,
\end{aligned} \tag{12}$$

where $b_0 > 0$ is a constant such that $\|B(t)\|_{\mathcal{L}(H_0^1)} \leq b_0$, for all t.

On the other hand, as f is locally Lipschitz,

$$\begin{aligned}
&\|B(r)f(r,u_r)\|_{H_0^1} \\
&\quad \leq \|B(r)\left(f(r,u_r) - f(r,0)\right) + B(r)f(r,0)\|_{H_0^1} \\
&\quad \leq b_0C(R)\|u_r\|_{C_H} + b_0\|f(r,0)\|_{L^2(\Omega)}.
\end{aligned} \tag{13}$$

Then, for all $t \geq s$ we have that

$$\begin{aligned}
\|\Phi(u)(t)\|_{H_0^1} &\leq \|\phi(0)\|_{H_0^1} + a\int_s^t \|u(r)\|_{H_0^1}dr \\
&+ L_g(R)\int_s^t \|u(r)\|_{H_0^1}dr + b_0|g(0)|(t-s) \\
&+ b_0C(R)\int_s^t \|u_r\|_{C_H}dr + b_0\int_s^t \|f(r,0)\|_{L^2(\Omega)}dr.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|\Phi(u)(t)\|_{H_0^1} &\leq \|\phi(0)\|_{H_0^1} \\
&+ (a + L_g(R) + b_0C(R))\int_s^t \|u_r\|_{C_H}dr \\
&+ b_0|g(0)|(t-s) + b_0\int_s^t \|f(r,0)\|_{L^2(\Omega)}dr.
\end{aligned}$$

Taking into account that

$$\begin{aligned}
\|u_r\|_{C_H} &= \sup_{\theta\in[-h,0]} \|u(r+\theta)\|_{H_0^1} \\
&\leq \sup_{\sigma\in[s-h,T)} \|u(\sigma)\|_{H_0^1} \\
&= \|u\|_{X_\phi^T} \\
&\leq R,
\end{aligned}$$

for all $s \leq r \leq t < T$.
Then, we obtain that

$$\begin{aligned}
\|\Phi(u)(t)\|_{H_0^1} \leq \|\phi\|_{C_H} & \\
&+ R(t-s)\left(a + L_g(R) + b_0C(R)\right) \\
&+ b_0|g(0)|(t-s) \\
&+ b_0\int_s^t \|f(r,0)\|_{L^2(\Omega)}dr,
\end{aligned}$$

for all $t \in (s,T)$. If we write $T = s+\delta$, we then have

$$\begin{aligned}
\|\Phi(u)(t)\|_{H_0^1} &\leq \|\phi\|_{C_H} + R\delta\left(a + L_g(R) + b_0C(R)\right) \\
&+ b_0|g(0)|\delta + b_0\int_s^{s+\delta} \|f(r,0)\|_{L^2(\Omega)}dr,
\end{aligned}$$

for all $t \in (s,T)$, and considering $\delta > 0$ small enough, we can ensure that

$$\|\Phi(u)(t)\|_{H_0^1} \leq 2\|\phi\|_{C_H}, \quad \forall t \in (s,T).$$

We also have the same conclusion for $t \in [s-h,s]$, in fact $\|\Phi(u)(t)\|_{H_0^1} \leq \|\phi\|_{C_H}, \forall t \in [s-h,s]$, and therefore, we can conclude that the operator Φ is well defined.

Now, by using the contraction mapping theorem, we prove the existence of a fixed point for $\Phi(\cdot)$, which will be the solution of our problem. To this end, we need to prove that Φ is a contracting mapping. Let us take $u, v \in X_\phi^T$. Then $\|u(t)\|_{H_0^1}, \|v(t)\|_{H_0^1} \leq R$ for all $t \in [s,T)$ and

$$\begin{aligned}
&\|\Phi(u)(t) - \Phi(v)(t)\|_{H_0^1} \\
&\leq \int_s^t \|h(r,u_r) - h(r,v_r)\|dr \\
&\leq \int_s^t \|\tilde{A}(r)\|_{\mathcal{L}(H_0^1)}\|u(r) - v(r)\|_{H_0^1}dr \\
&+ \int_s^t \|B(r)(g(u(r)) - g(v(r)) + f(r,u_r) - f(r,v_r))\|_{H_0^1}dr.
\end{aligned}$$

Using the uniform bound in time for $\tilde{A}(t)$ and $B(t)$, and (6)

$$\begin{aligned}
&\|\Phi(u)(t) - \Phi(v)(t)\|_{H_0^1} \\
&\quad \leq K_1\int_s^t \|u(r) - v(r)\|_{H_0^1}dr \\
&\quad + K_2\int_s^t \|f(r,u_r) - f(r,v_r)\|_{L^2}dr \\
&\quad \leq K_1\int_s^t \|u(r) - v(r)\|_{H_0^1}dr \\
&\quad + K(R)\int_s^t \sup_{\theta\in[-h,0]} \|u(r+\theta) - v(r+\theta)\|_{H_0^1}dr.
\end{aligned}$$

Taking supremum in $[s,T)$ with $T = s+\delta$

$$\begin{aligned}
&\|\Phi(u) - \Phi(v)\|_{X_\phi^T} \\
&\quad \leq K_1\delta\|u - v\|\|_{X_\phi^T} \\
&\quad + K(R)\delta\left(\sup_{r\in[s,T)}\sup_{\theta\in[-h,0]} \|u(r+\theta) - v(r+\theta)\|_{H_0^1}\right),
\end{aligned}$$

but, if $u, v \in X_\phi^T$,

$$\begin{aligned}\sup_{r\in[s,T)}\sup_{\theta\in[-h,0]}&\|u(r+\theta)-v(r+\theta)\|_{H_0^1}\\ &=\sup_{r\in[s-h,T)}\|u(r)-v(r)\|_{H_0^1}\\ &\le \|u-v\|_{X_\phi^T}.\end{aligned}$$

Therefore, for $\delta>0$ small enough, Φ is well defined and is a contraction in X_ϕ. The proof is therefore complete. ■

2. Global solution and pullback absorbing family

In this section we will prove that the local solution, whose existence has been proved in Theorem 1.1, is in fact a global one, i.e. it is defined in the whole future and not only in a small time interval. However, we will deduce this result after obtaining some a priori estimates which will be also useful to deduce the existence of absorbing sets for the process generated by our model.

For any $\varphi \in H_0^1(\Omega)$, taking into account (2) and arguing as in [Hale, 1989], for each $\delta>0$ there is a constant $K_\delta>0$ such that

$$\begin{aligned}&\int_\Omega g(u)u \le \delta\|u\|^2_{L^2(\Omega)} + K_\delta,\\ &\int_\Omega G(u)\le \delta\|u\|^2_{L^2(\Omega)}+K_\delta\end{aligned} \tag{14}$$

for all $u\in L^2(\Omega)$, where $G(r)=\int_0^r g(\theta)d\theta$.

Let $L_b(\varphi)$ be the following energy functional

$$L_b(\varphi)=\frac{1}{2}\left(\|\varphi\|^2_{L^2}+b\|\varphi\|^2_{H_0^1}\right)-b\int_\Omega G(\varphi), \tag{15}$$

with $b\ge 0$. It is easy to prove that for $\delta=\frac{\lambda_1}{6}$,

$$L_b(\varphi)\ge \frac{b}{3}\|\varphi\|^2_{H_0^1}-bK_{\frac{\lambda_1}{6}} \tag{16}$$

and for any $\delta>0$,

$$L_b(\varphi)\le \frac{1+b(\lambda_1+2\delta)}{2\lambda_1}\|\varphi\|^2_{H_0^1}+bK_\delta, \tag{17}$$

with λ_1 the first eingenvalue of A.

Taking a solution $u(t,s;\phi)$ of (1) and for $b>0$,

$$\begin{aligned}\frac{d}{dt}L_b(u)&=(u,\frac{du}{dt})_{L^2}+b(u,\frac{du}{dt})_{H_0^1}-b\int_\Omega g(u)\frac{du}{dt}\\ &\le -\left(1-\frac{\gamma_1\varepsilon_1}{2}-\frac{2\delta+\varepsilon_2}{2\lambda_1}\right)\|u\|^2_{H_0^1}\\ &\quad+\frac{\varepsilon_2+1}{2\varepsilon_2}\|f(t,u_t)\|^2_{L^2}\\ &\quad+\gamma(t)\left(\frac{1}{2\varepsilon_1}-b\right)\|\frac{du}{dt}\|^2_{H_0^1}+K_\delta,\end{aligned}$$

for $\varepsilon_1,\varepsilon_2,\delta>0$. Taking $\varepsilon_1=\frac{1}{2\gamma_1}$, $\varepsilon_2=\frac{\lambda_1}{4}$, $\delta=\frac{\lambda_1}{8}$ and $b\ge\frac{1}{2\varepsilon_1}=2\gamma_1$, we obtain

$$\begin{aligned}\frac{d}{dt}L_b(u)&\le -\frac{1}{2}\|u\|^2_{H_0^1}+\left(\frac{\lambda_1+4}{2\lambda_1}\right)\|f(t,u_t)\|^2_{L^2}+K_{\frac{\lambda_1}{8}}\\ &\le -\left(\frac{\lambda_1}{1+b(\lambda_1+2\tilde\delta)}\right)L_b(u)\\ &\quad+\left(\frac{\lambda_1+4}{2\lambda_1}\right)\|f(t,u_t)\|^2_{L^2}\\ &\quad+\left(\frac{\lambda_1}{1+b(\lambda_1+2\tilde\delta)}\right)K_{\tilde\delta}+K_{\frac{\lambda_1}{8}}.\end{aligned}$$

Denoting $C_b=\left(\frac{\lambda_1}{1+b(\lambda_1+2\tilde\delta)}\right)$, $C_{\lambda_1}=\left(\frac{\lambda_1+4}{2\lambda_1}\right)$ and $\tilde K_b=C_bK_{\tilde\delta}+K_{\frac{\lambda_1}{8}}$,

$$\begin{aligned}\frac{d}{dt}\left(e^{C_bt}L_b(u)\right)&=C_be^{C_bt}L_b(u)+e^{C_bt}\frac{d}{dt}L_b(u)\\ &\le e^{C_bt}(C_{\lambda_1}\|f(t,u_t)\|^2_{L^2}+\tilde K_b).\end{aligned}$$

Integrating between s and t and using (6),

$$\begin{aligned}&e^{C_bt}L_b(u(t))\\ &\quad\le e^{C_bs}L_b(\phi(0))+C_{\lambda_1}\int_s^t e^{C_br}\|f(r,u_r)\|^2_{L^2}dr\\ &\qquad+\frac{\tilde K_b}{C_b}\left(e^{C_bt}-e^{C_bs}\right)\\ &\quad\le e^{C_bs}L_b(\phi(0))+C_{\lambda_1}C(R)\int_s^t e^{C_br}\|u_r\|^2_{C_H}dr\\ &\qquad+C_{\lambda_1}\int_s^t e^{C_br}\|f(r,0)\|^2_{L^2}dr\\ &\qquad+\frac{\tilde K_b}{C_b}\left(e^{C_bt}-e^{C_bs}\right).\end{aligned}$$

Replacing t by $t+\theta$ with $\theta\in[-h,0]$, by (16) and (17),

$$\begin{aligned}\frac{b}{3}e^{C_bt}\|u(t+\theta)\|^2_{H^1_0}\\ &\le e^{C_b(s+h)}(\tilde{C}_b\|\phi(0)\|^2_{H^1_0}+bK_{\delta_2})\\ &+C_{\lambda_1}C(R)e^{C_bh}\int_s^{t+\theta}e^{C_br}\|u_r\|^2_{C_H}dr\\ &+C_{\lambda_1}e^{C_bh}\int_s^{t+\theta}e^{C_br}\|f(r,0)\|^2_{L^2}dr\\ &+e^{C_bt}\left(\frac{\tilde{K}_b}{C_b}+bK_{\frac{\lambda_1}{6}}\right),\end{aligned}$$

where δ_2 is chosen such that

$$\tilde{C}_b=\frac{1+b(\lambda_1+2\delta_2)}{2\lambda_1}.$$

If $t>s+h$, then

$$\begin{aligned}&e^{C_bt}\|u(t+\theta)\|^2_{H^1_0}\\ &\le\max\Bigg\{\sup_{\theta\in[-h,s-t]}e^{C_bt}\|\phi(t+\theta-s)\|^2_{H^1_0},\\ &\sup_{\theta\in[s-t,0]}\frac{3}{b}e^{C_b(s+h)}(\tilde{C}_b\|\phi(0)\|^2_{H^1_0}+bK_{\delta_2})\\ &+\frac{3}{b}C_{\lambda_1}C(R)e^{C_bh}\int_s^{t+\theta}e^{C_br}\|u_r\|^2_{C_H}dr\\ &+\frac{3}{b}C_{\lambda_1}e^{C_bh}\int_s^{t+\theta}e^{C_br}\|f(r,0)\|^2_{L^2}dr\\ &+\frac{3}{b}e^{C_bt}\left(\frac{\tilde{K}_b}{C_b}+bK_{\frac{\lambda_1}{6}}\right)\Bigg\},\end{aligned}$$

but,

$$\begin{aligned}\sup_{\theta\in[-h,s-t]}e^{C_bt}\|\phi(t+\theta-s)\|^2_{H^1_0}\\ &\le\sup_{\theta\in[-h,0]}e^{C_b(s+h)}\|\phi(\theta)\|^2_{H^1_0}\\ &=e^{C_b(s+h)}\|\phi\|^2_{C_H},\end{aligned}$$

and

$$\begin{aligned}\sup_{\theta\in[s-t,0]}\left(\int_s^{t+\theta}e^{C_br}\|u_r\|^2_{C_H}dr+\int_s^{t+\theta}e^{C_br}\|f(r,0)\|^2_{L^2}dr\right)\\ \le\int_s^te^{C_br}\|u_r\|^2_{C_H}dr+\int_s^te^{C_br}\|f(r,0)\|^2_{L^2}dr.\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{b}{3}e^{C_bt}\|u_t\|^2_{C_H}\le e^{C_b(s+h)}&\left(\tilde{C}_b\|\phi\|^2_{C_H}+bK_{\delta_2}\right)\\ &+C_{\lambda_1}e^{C_bh}\int_s^te^{C_br}\|f(r,0)\|^2_{L^2}dr\\ &+e^{C_bt}\left(\frac{\tilde{K}_b}{C_b}+bK_{\frac{\lambda_1}{6}}\right)\\ &+C_{\lambda_1}C(R)e^{C_bh}\int_s^te^{C_br}\|u_r\|^2_{C_H}dr.\end{aligned}$$

Assuming that

$$C(R)<\frac{bC_b}{3C_{\lambda_1}e^{C_bh}}\tag{18}$$

and calling $\beta=\frac{3}{b}C_{\lambda_1}C(R)e^{C_bh}$ (it means $\beta<C_b$) and

$$\begin{aligned}\alpha(t)=\frac{3}{b}e^{C_b(s+h)}&\left(\tilde{C}_b\|\phi\|^2_{C_H}+bK_{\delta_2}\right)\\ &+\frac{3}{b}C_{\lambda_1}e^{C_bh}\int_s^te^{C_br}\|f(r,0)\|^2_{L^2}dr\\ &+\frac{3}{b}e^{C_bt}\left(\frac{\tilde{K}_b}{C_b}+bK_{\frac{\lambda_1}{6}}\right),\end{aligned}$$

by the Gronwall Lemma we obtain that

$$e^{C_bt}\|u_t\|^2_{C_H}\le\alpha(t)+\beta\int_s^t\alpha(r)e^{\beta(t-r)}dr.$$

Now,

$$\begin{aligned}\beta\int_s^t&\alpha(r)e^{\beta(t-r)}dr\\ &\le\frac{3}{b}e^{C_b(s+h)}e^{\beta(t-s)}\left(\tilde{C}_b\|\phi\|^2_{C_H}+bK_{\delta_2}\right)\\ &+\frac{3}{b}\frac{\beta}{C_b-\beta}\left(\frac{\tilde{K}_b}{C_b}+bK_{\frac{\lambda_1}{6}}\right)e^{C_bt}\\ &+\frac{3}{b}\beta C_{\lambda_1}e^{(C_bh+\beta t)}\int_s^te^{(C_b-\beta)r}\|f(r,0)\|^2_{L^2}dr.\end{aligned}$$

Then,

$$\begin{aligned}\|u_t\|^2_{C_H}\\ &\le e^{-C_bt}\alpha(t)\\ &+\frac{3}{b}e^{C_bh}e^{(C_b-\beta)(s-t)}\left(\tilde{C}_b\|\phi\|^2_{C_H}+bK_{\delta_2}\right)\\ &+\frac{3}{b}\frac{\beta}{C_b-\beta}\left(\frac{\tilde{K}_b}{C_b}+bK_{\frac{\lambda_1}{6}}\right)\\ &+\frac{3}{b}\beta C_{\lambda_1}e^{(C_bh-(C_b-\beta)t)}\int_s^te^{(C_b-\beta)r}\|f(r,0)\|^2_{L^2}dr.\end{aligned}$$

Assuming that there exists a $\eta_0 \geq 0$ such that for any $\eta \in [0, \eta_0]$,

$$\int_{-\infty}^{t} e^{\eta r} \|f(r,0)\|_{L^2}^2 dr < +\infty, \tag{19}$$

we have

$$\|u_t\|_{C_H}^2 \xrightarrow{s \to -\infty} l(t), \tag{20}$$

where

$$l(t) = \frac{3}{b}\left(\frac{\tilde{K}_b}{C_b} + bK_{\frac{\lambda_1}{6}}\right)\left(1 + \frac{\beta}{C_b - \beta}\right) + \frac{3}{b}\beta C_{\lambda_1} e^{C_b h}\left(\int_{-\infty}^{t} e^{C_b r}\|f(r,0)\|_{L^2}^2 dr + e^{-(C_b-\beta)t}\int_{-\infty}^{t} e^{(C_b-\beta)r}\|f(r,0)\|_{L^2}^2 dr\right).$$

Then, we have the global existence of any solution $u(t,s;\phi)$ of (1), i.e. for each $\phi \in C_H$, $u(\cdot,s;\phi) \in C([s-h,+\infty), H_0^1(\Omega))$ in Theorem 1.1, and, once we justify in the next section that the solutions of our problem generates a non-autonomous dynamical system, this also ensures the existence of a family of closed subsets $\left\{\overline{B}_{C_H}(0, l^{1/2}(t)) : t \in \mathbb{R}\right\}$ which pullback attracts bounded subsets of C_H.

Also we need a result on the continuous dependence on the initial data.

Proposition 2.1. *Under the assumptions of Theorem 1.1, any solution $u(t,s;\phi)$ of (1) is continuous with respect to the initial condition $\phi \in C_H$. More precisely, if u^i, for $i = 1, 2$, are the corresponding solutions to the initial data $\phi^i \in C_H$, $i = 1, 2$, the following estimate holds:*

$$\|u_t^1 - u_t^2\|_{C_H} \leq \|\phi^1 - \phi^2\|_{C_H} e^{(a+L_g(R)+C(R))t}, \tag{21}$$

for all $t \in [s,T)$, where $R \geq 0$ is given by

$$R = \max(2\|\phi^1\|_{C_H}, 2\|\phi^2\|_{C_H}).$$

Proof. Let u^i, for $i = 1, 2$, be the corresponding solutions to the initial data $\phi^i \in C_H$, $i = 1, 2$, in the interval $[s-h,T)$, for a fixed $T > s$. Then we have that

$$u^1(t) - u^2(t) = \phi^1(0) - \phi^2(0) + \int_s^t (h(r,u_r^1) - h(r,u_r^2))dr,$$

for $t \in (s,T)$.

Now, taking into account (6), (9), that $\|\tilde{A}(r)\|_{\mathcal{L}(H_0^1)} \leq a$, and function $B(t)\circ g$ is locally Lipschitz in $H_0^1(\Omega)$, for $R = \max(2\|\phi^1\|_{C_H}, 2\|\phi^2\|_{C_H})$ it is not difficult deduce that

$$\|u^1(t) - u^2(t)\|_{H_0^1} \leq \|\phi^1(0) - \phi^2(0)\|_{H_0^1} + (a + L_g(R) + C(R))\int_s^t \|u_r^1 - u_r^2\|_{C_H} dr,$$

for $t \in (s-h,T)$.

Thus, replacing now t by $t+\theta$ with $\theta \in [-h,0]$ and taking supremum in θ,

$$\|u_t^1 - u_t^2\|_{C_H} \leq \|\phi^1 - \phi^2\|_{C_H} + (a + L_g(R) + C(R))\int_s^t \|u_r^1 - u_r^2\|_{C_H} dr,$$

por $t \in (s,T)$, and therefore, thanks to the Gronwall lemma, we deduce (21). ∎

3. Abstract results on the theory of attractors. Existence of pullback attractors

In this section we recall some abstract results on the theory of pullback attractors. We present a summary of some results on the existence of minimal pullback attractors obtained in [Carvalho *et al.*, 2013] (see also [Caraballo *et al.*, 2010, Caraballo *et al.*, 2011, Caraballo *et al.*, 2006a, Caraballo *et al.*, 2006b, García-Luengo *et al.*, 2012, Marín-Rubio & Real, 2009]).

Consider given a metric space $(\mathcal{X}, d_{\mathcal{X}})$. For A, B subsets of $\mathcal{X}$ let $\mathrm{dist}(A,B)$ denote the Hausdorff semidistance between A and B, i.e.

$$\mathrm{dist}(A,B) = \sup_{a\in A}\inf_{b\in B} d_{\mathcal{X}}(a,b).$$

Definition 3.1. An *evolution process* in a metric space $(\mathcal{X}, d_{\mathcal{X}})$ is a family of continuous maps $\{S(t,s) : t \geq s\}$ from $\mathcal{X}$ into itself with the following properties

(1) $S(t,t) = I$, for all $t \in \mathbb{R}$,
(2) $S(t,s) = S(t,\tau)S(\tau,s)$, for all $t \geq \tau \geq s$,
(3) $\{(t,s) \in \mathbb{R}^2 : t \geq s\} \times \mathcal{X} \ni (t,s,x) \mapsto S(t,s)x \in \mathcal{X}$ is continuous.

Let us denote $\mathcal{P}(\mathcal{X})$ the family of all nonempty subsets of $\mathcal{X}$, and consider a family of nonempty sets $\widehat{D}_0 = \{D_0(t) : t \in \mathbb{R}\} \subset \mathcal{P}(\mathcal{X})$. Observe that we do not require any additional condition on these sets as compactness or boundedness. Let $\mathcal{D}$ be a nonempty class of families parameterized in time $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$. The class $\mathcal{D}$ will be called a universe in $\mathcal{P}(X)$.

Definition 3.2. It is said that $\widehat{D}_0 = \{D_0(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ is pullback $\mathcal{D}-$absorbing for the process $\{S(t,s) : t \geq s\}$ on $\mathcal{X}$ if for any $t \in \mathbb{R}$ and any $\widehat{D} \in \mathcal{D}$, there exists a $s_0(t, \widehat{D}) \leq t$ such that

$$S(t,s)D(s) \subset D_0(t) \quad \text{for all } s \leq s_0(t, \widehat{D}).$$

Definition 3.3. The family $\mathcal{A}_\mathcal{D} = \{\mathcal{A}_\mathcal{D}(t) : t \in \mathbb{R}\}$ is the $\mathcal{D}$-pullback attractor for the process $\{S(t,s) : t \geq s\}$ in $\mathcal{X}$ if:

(a) for any $t \in \mathbb{R}$, the set $\mathcal{A}_\mathcal{D}(t)$ is a nonempty compact subset of $\mathcal{X}$.

(b) $\mathcal{A}_\mathcal{D}$ is pullback $\mathcal{D}-$attracting, i.e.,

$$\lim_{\tau \to -\infty} \operatorname{dist}(S(t,s)D(s), \mathcal{A}_\mathcal{D}(t)) = 0$$

for all $\widehat{D} \in \mathcal{D}$,for $t \in \mathbb{R}$,

(c) $\mathcal{A}_\mathcal{D}$ is invariant, i.e.,

$$S(t,s)\mathcal{A}_\mathcal{D}(s) = \mathcal{A}_\mathcal{D}(t) \quad \text{for all } s \leq t.$$

The family $\mathcal{A}_\mathcal{D}$ is minimal in the sense that if $\widehat{C} = \{C(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ is a family of closed sets such that for any $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \in \mathcal{D}$,

$$\lim_{\tau \to -\infty} \operatorname{dist}_X(S(t,s)D(s), C(t)) = 0,$$

then $\mathcal{A}_\mathcal{D}(t) \subset C(t)$.

Definition 3.4. Given a family parameterized in time, $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{P}(\mathcal{X})$, it is said that a process $\{S(t,s) : t \geq s\}$ on $\mathcal{X}$ is $\widehat{D}-$asymptotically compact if for any $t \in \mathbb{R}$ and any sequences $\{s_n\} \subset (-\infty, t]$ and $\{x_n\} \subset \mathcal{X}$ bounded satisfying $s_n \to -\infty$ and $x_n \in D(s_n)$ for all n, the sequence $\{S(t, s_n)x_n\}$ is relatively compact in $\mathcal{X}$.

Definition 3.5. A process $\{S(t,s) : t \geq s\}$ on $\mathcal{X}$ is said to be pullback $\mathcal{D}-$asymptotically compact if it is $\widehat{D}$-asymptotically compact for any $\widehat{D} \in \mathcal{D}$.

Denoting by

$$\omega(\widehat{D}_0, t) := \bigcap_{s \leq t} \overline{\bigcup_{\tau \leq s} S(t,\tau)D_0(\tau)}^{\mathcal{X}} \quad \text{for all } t \in \mathbb{R},$$

the natural generalization of the ω-limit set in the pullback sense, we have the following result on existence of pullback attractors.

Theorem 3.6. *Consider a process $\{S(t,s) : t \geq s\}$ in $\mathcal{X}$, a universe $\mathcal{D}$ in $\mathcal{P}(\mathcal{X})$, and a family $\widehat{D}_0 = \{D_0(t) : t \in \mathbb{R}\} \subset \mathcal{P}(\mathcal{X})$ which is pullback $\mathcal{D}-$absorbing and assume also that the process is pullback $\widehat{D}_0-$asymptotically compact.*

Then, the family $\mathcal{A}_\mathcal{D} = \{\mathcal{A}_\mathcal{D}(t) : t \in \mathbb{R}\}$ defined by

$$\mathcal{A}_\mathcal{D}(t) = \overline{\bigcup_{\widehat{D} \in \mathcal{D}} \omega(\widehat{D}, t)}^{\mathcal{X}} \quad t \in \mathbb{R},$$

is the $\mathcal{D}$-pullback attractor.

4. Existence of pullback attractors for the process associated to (1)

On account of the previous results, we will be able to construct a non-autonomous dynamical system. More precisely, we will be able to construct a process $S : C_H \to C_H$ associated to (1), and prove the existence of a pullback attractor for such a process.

Assume hypotheses (2), (3), (6), with $f(t,0) = 0$ for all $t \in \mathbb{R}$ (this assumption is not a real restriction as we can subtract such term to $f(t,\cdot)$ and add it to the term g), and suppose that $C(R)$ satisfies (18). For each $\phi \in C_H$ and $s \in \mathbb{R}$, Theorem 1.1 and the a priori estimates in Section 2 ensure the existence of a unique global solution $u(\cdot; s, \phi)$ of (1).

Then we can define the family of maps $S(t,s)$ in C_H as

$$S(t,s)\phi = u_t(\cdot; s, \phi) \quad \forall t \geq s.$$

According to the previous results, it is not difficult to prove that $S(\cdot,\cdot)$ is a process and we can also write

$$\begin{aligned}(S(t,s)\phi)(\theta) &= u_t(\theta; s, \phi) \\ &= u(t+\theta; s, \phi) \\ &= T(t+\theta, s)\phi(0) \\ &\quad + \int_s^{t+\theta} T(t+\theta,\tau)\tilde{f}(\tau, u_\tau)d\tau \\ &\quad + \int_s^{t+\theta} T(t+\theta,\tau)\tilde{g}(\tau, u)d\tau,\end{aligned} \tag{22}$$

for all $t \geq s$ and $\theta \in [-h, 0]$, where $T(t,s)$ is the evolution process associated to (1) with $f = 0$ and $g = 0$.

Our universe in this case is the universe $\mathcal{D}_b$ of all families with bounded union, that is, the family $\{D(t) : t \in \mathbb{R}\}$ is in $\mathcal{D}_b$ if and only if $\bigcup\{D(t) : t \in \mathbb{R}\}$ is bounded in C_H. Therefore, assuming that $f(t,0) = 0$ for all $t \in \mathbb{R}$, by estimates in Section 2, there exists a pullback $\mathcal{D}_b-$absorbing family $\widehat{B}_0 = \{B_0(t) : t \in \mathbb{R}\}$ in $\mathcal{D}_b$.

In order to prove the existence of the pullback attractor applying Theorem 3.6, we have the following result.

Theorem 4.1. *Let $\{S(t,s) : t \geq s\}$ be a process such that $S(t,s) = T(t,s) + U(t,s)$, where $U(t,s)$ is compact and there exists a non-increasing function*

$$k : \mathbb{R}^+ \times \mathbb{R}^+ \longrightarrow \mathbb{R}$$

with $k(\sigma, r) \to 0$ when $\sigma \to \infty$, and for all $s \leq t$ and $x \in C_H$ with $\|x\|_{C_H} \leq r$, $\|T(t,s)x\|_{C_H} \leq k(t-s, r)$. Then, $\{S(t,s) : t \geq s\}$ is $\mathcal{D}_b$-pullback asymptotically compact.

Proof. Using the fact that any family $\widehat{D}$ of $\mathcal{D}_b$ has bounded union, the result following for Theorem 2.8 in [Caraballo *et al.*, 2010]. ■

Therefore, on the one hand we need to prove that

$$T(t+\theta, s)\phi(0) + \int_s^{t+\theta} T(t+\theta, \tau)\tilde{f}(\tau, u_\tau)d\tau$$

tends to zero exponentially, in fact, in C_H. But this fact easily follows by arguing as in Section 3, taking into account that $f(t,0) = 0$ for all $t \geq s$.

On the other hand, we need to prove that $U(t,s)B$ is relatively compact for any bounded subset $B \subset C_H$, where $U(t,s) : C_H \to C_H$ is defined as

$$(U(t,s)\phi)(\theta) = \int_s^{t+\theta} T(t+\theta, \tau)\tilde{g}(\tau, u(\tau, s; \phi))d\tau.$$

To this end, it is enough to prove that for any bounded subset $D \subset C_H$, $U(t,s)D$ is pre-compact in C_H for any $t \geq s$, and to prove this we will apply the Azcoli-Arzelà theorem (see, for instance [Kelley & Namioka, 1982]), for which we need to check that

(i) $U(t,s)D$ is bounded, $\forall t \geq s$.

(ii) For each $\theta \in [-h, 0]$, $\overline{\bigcup_{\phi \in D} (U(t,s)\phi)(\theta)}$ is a compact subset of $H_0^1(\Omega)$.

(iii) The set $U(t,s)D$ is equicontinuous (i.e., fro all $\varepsilon > 0$, there exists a $\delta > 0$ such that if $|\theta_1 - \theta_2| \leq \delta$, then $|(U(t,s)\phi)(\theta_1) - (U(t,s)\phi)(\theta_2)| \leq \varepsilon$, for all $t \geq s$, and for all $\phi \in D$),

Assertion (i) follows from the same estimates obtained in the proof of the existence of the absorbing family.

Statement (ii) is a consequence of the same analysis carried out in [Rivero, 2013], just using the fact that $\rho < \frac{n+2}{n-2}$ and, for any $\eta \in \left(\frac{1}{2}, 1\right)$, we have the following chain of inclusions:

$$H_0^1 \hookrightarrow L^{\frac{2n}{n-2}} \xrightarrow{g} L^{\frac{2n}{n+2\eta}} \hookrightarrow H^{-\eta} \subset\subset H^{-1} \xrightarrow{B(t)} H_0^1.$$

Finally, to prove (iii) we need to estimate

$$\left| \int_s^{t+\theta_1} T(t+\theta_1, \tau)\tilde{g}(\tau, u(\tau, s; \phi))d\tau - \int_s^{t+\theta_2} T(t+\theta_2, \tau)\tilde{g}(\tau, u(\tau, s; \phi))d\tau \right|.$$

If we suppose $\theta_1 < \theta_2$, we have that

$$\begin{aligned}
&\left| \int_s^{t+\theta_1} T(t+\theta_1, \tau)\tilde{g}(\tau, u(\tau, s; \phi))d\tau - \int_s^{t+\theta_2} T(t+\theta_2, \tau)\tilde{g}(\tau, u(\tau, s; \phi))d\tau \right| \\
&\leq \int_s^{t+\theta_1} |(T(t+\theta_1, \tau) - T(t+\theta_2, \tau))\tilde{g}(\tau, u(\tau, s; \phi))|d\tau \\
&\quad + \int_{t+\theta_1}^{t+\theta_2} |T(t+\theta_2, \tau)\tilde{g}(\tau, u(\tau, s; \phi))|\, d\tau.
\end{aligned}$$

Taking into account (3),

$$\|T(t,s)\|_{\mathcal{L}(C_H)} \leq Ce^{-\alpha(t-s)},$$

for certain $\alpha > 0$ and as any solution of (1) is in $C([s-h, T), H_0^1(\Omega))$, we can obtain that

$$\begin{aligned}
&\int_{t+\theta_1}^{t+\theta_2} |T(t+\theta_2, \tau)\tilde{g}(\tau, u(\tau, s; \phi))|\, d\tau \\
&\qquad \leq \frac{1}{\alpha}(1 - e^{-\alpha(\theta_2 - \theta_1)}) \to 0
\end{aligned} \tag{23}$$

when $|\theta_2 - \theta_1| \to 0$.
On the other hand, taking into account that the operator $\tilde{A}(t)$ is uniformly bounded in time, $T(t,s)$ is the evolution process associated to (1) with $f = 0$ and $g = 0$, and the solutions of this problem are bounded, we can obtain that

$$|T(t+\theta_1, \tau) - T(t+\theta_2, \tau)| \leq C_1 |\theta_1 - \theta_2| ,$$

and therefore,

$$\int_s^{t+\theta_1} |(T(t+\theta_1,\tau) - T(t+\theta_2,\tau))\tilde{g}(\tau, u(\tau, s;\phi))| \, d\tau \leq C_2 |\theta_1 - \theta_2| .$$

Now, we can apply Theorem 4.1 and Theorem 3.6 and obtain the existence of the $\mathcal{D}_b$-pullback attractor.

Acknowledgement. We would like to thank the referees for their helpful suggestions and comments that allowed us to improve the presentation of the paper.

References

Aifantis, E.C. [1980] "On the problem of diffusion in solids," *Acta Mech.* **37**, 265–296.

Kuttlerand, K. & Aifantis, E.C. [1987] "Existence and uniqueness in non classical diffusion," *Quarterly of Applied Mathematics* **45, no. 3**, 549–560.

Kuttlerand, K. & Aifantis, E.C. [1988] "Quasilinear evolution equations in nonclassical diffusion," *SIAM Journal on Mathematical Analysis* **19, no. 1**, 110–120.

Anh, C.T. & Bao, T.Q. [2010] "Pullback attractors for a class of non-autonomous nonclassical diffusion equations," *Nonlinear Analysis* **73**, 399–412.

Camassa, R. & Holm, D. [1993] "An integrable shallow water equation with peaked solitons," *Phys. Rev. Lett.* **71**, 1661–1665.

Caraballo,T., Carvalho, A.N., Langa,J.A. & Rivero, F. [2010] "Existence of pullback attractors for pullback asymptotically compact processes," *Nonlinear Anal* **72**, 1967–1976.

Caraballo,T., Carvalho, A.N., Langa,J.A. & Rivero, F. [2011] "A non–autonomous strongly damped wave equation: Existence and continuity of the pullback attractor," *Nonlinear Anal.* **74**, 2272–2283.

Caraballo, T. & Márquez-Durán, A.M. [2013] "Existence, uniqueness and asymptotic behavior of solutions for a nonclassical difusion equation with delay," *Dynamics of Partial Differential Equations* **10 No. 3**, 267–281.

Caraballo, T., Łukaszewicz, G. & Real, J. [2006] "Pullback attractors for asymptotically compact non-autonomous dynamical systems," *Nonlinear Anal.* **64**, 484–498.

Caraballo, T., Łukaszewicz, G. & Real, J. [2006] "Pullback attractors for non-autonomous 2D Navier-Stokes equations in unbounded domains," *C. R. Acad. Sci. Paris* **342**, 263–268.

Carvalho, A.N., Langa, J.A. & Robinson, J.C. [2013] *Attractors for infinite-dimensional non-autonomous dynamical systems* (App. Math. Series 182, Springer, New York).

Chepyzhov, V.V. & Vishik, M.I. [2002] *Attractors for Equations of Mathematical Physics* (Colloquium Publications, Vol. 49. American Mathematical Society)

García-Luengo, J., Marín-Rubio, P. & Real, J. [2012] "Pullback attractors in V for non-autonomous 2D-Navier-Stokes equations and their tempered behaviour," *J. Differential Equations* **252, no. 8**, 4333–4356.

Hale, J.K., [1989] *Asymptotic Behavior of Dissipative System* (American Mathematical Society)

Henry, D. [1981] *Geometric Theory of Semilinear Parabolic Equations*, (Springer-Verlag, Berlin-New York)

Hu, Z. & Wang, Y. [2012] "Pullback attractors for a nonautonomous nonclassical diffusion equation with variable delay," *J. Math. Phys* **53**, 072702.

Kelley, J. L. & Namioka, I. [1982] *Linear Topological Spaces* (Springer-Verlag)

Ma, Q., Liu, Y. & Zhang, F. [2012] "Global attractors in $H^1(\mathbb{R}^N)$ for nonclassical diffusion equations," *Discrete Dyn. Nat. Soc.* **Art. ID 672762**, 16 pp.

Marín-Rubio, P. & Real, J. [2009] "On the relation between two different concepts of pullback attractors for non-autonomous dynamical systems," *Nonlinear Anal.* **71**, 3956–3963.

Pazy, A. [1983] *Semigroups of Linear Operators and Applications to Partial Differential Equations* (Springer-Verlag)

Peter, J.C. & Gurtin, M.E. [1968] "On a theory of heat conduction involving two temperatures," *Z. Angew. Math. Phys.* **19**, 614–627.

Rivero, F. [2013] "Pullback attractor for non-autonomous non-classical parabolic equation," *Discrete Contin. Dyn. Syst. Ser. B* **18, no.1**, 209–221.

Sun, C., Wang, S. & Zhong, Ch. [2007] "Global attractors for a nonclassical diffusion equation," *Acta Math. Appl. Sin. Engl.* **Ser 23**, 1271–1280.

Sun, C. & Yang, M. [2008] "Dynamics of the nonclassical diffusion equations," *Asymptotic Analysis* **59**, 51–81.

Teschl, G. [2012] *Ordinary Differential Equations and Dynamical Systems* (American Mathematical Society, Graduate Studies in Mathematics, vol. 140)

Wang, S., Li, D. & Zhong, Ch. [2006] "On the dynamics of a class of nonclassical parabolic equations," *J. Math. Anal. Appl.* **317**, 565–582.