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A SURVEY ON NAVIER-STOKES MODELS WITH DELAYS: EXISTENCE, UNIQUENESS AND ASYMPTOTIC BEHAVIOR OF SOLUTIONS

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ABSTRACT. In this survey paper we review several aspects related to Navier-Stokes models when some hereditary characteristics (constant, distributed or variable delay, memory, etc) appear in the formulation. First some results concerning existence and/or uniqueness of solutions are established. Next the local stability analysis of steady-state solutions is studied by using the theory of Lyapunov functions, the Razumikhin-Lyapunov technique and also by constructing appropriate Lyapunov functionals. A Gronwall-like lemma for delay equations is also exploited to provide some stability results. In the end we also include some comments concerning the global asymptotic analysis of the model, as well as some open questions and future lines for research.

1. **Introduction.**

Navier-Stokes equations have been studied extensively over the last decades, for their important contributions to understanding fluids motion and turbulence (see [2], [17], [18], [26], [34], [43], amongst others). In real world applications when we want to control one system by applying some type of external forces, it is natural to assume that these forces take into account not only the present state of the system but also its history, either the finite time history (bounded delay) or the whole past (unbounded or infinite delay). Motivated by this fact, in 2001 Caraballo and Real started an investigation related Navier-Stokes models containing some hereditary features in the forcing term in [11], in which the existence and uniqueness of solutions were established. Later some first results on the asymptotic behavior of those solutions were established in [12, 13].

The asymptotic behavior of dynamical systems has been a very important and challenging topic of study, as it provides crucial information on future evolution of the system. The analysis for asymptotic behaviors of dynamical systems can be carried out according to several different points of view. One is to consider the local asymptotic stability of constant (steady state or stationary) solutions. To this end, the Lyapunov theory has been successfully applied in several situations. In particular for 2D-Navier-Stokes delay models, a sufficient condition ensuring the exponential behavior of solutions was established in [12], via the existence of a suitable Lyapunov function for the problem (see also the interesting paper [41]).

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In some cases it is better to combine the Lyapunov theory with some techniques due to Razumikhin (see [38]), which requires some continuity on the coefficients of the model but allows more general types of delays. Another alternative to obtain local stability is to construct Lyapunov functionals rather than Lyapunov functions. Constructing such functionals can be, in general, more complicated, but the stability results can be sharper upon successful constructions (see [14]).

In terms of the parameters of Navier-Stokes models, it has been proved that when the viscosity is large, specific model possesses a unique stationary solution and this solution is exponentially stable. This can be interpreted as a global asymptotic analysis of the model, since it implies that the global attractor for the model becomes the unique stationary solution. While considering the global asymptotic behavior of the system, it is sensible to think that when the viscosity is small, the behavior of a model with delay may be similar to that of one without delay. In order words, there may exist a compact invariant attracting set for the model, i.e., a global attractor for the associated semigroup. It is worth mentioning that when the delay terms are assumed be general enough, special attention is needed for such analysis, since we have to consider the semigroup in a different phase space. In fact, the dynamical system needs to be defined in a phase space of trajectories (for a similar approach for non-delay models see [36]).

One can also consider an abstract functional model for the delay so that a wide range of hereditary characteristics such as constant, variable delay or distributed delay can be treated in a unified way. This implies that although for some particular cases the resulting abstract equation may be autonomous (e.g. for constant delays) and the well-known techniques for autonomous dynamical systems can be applied to solve the problem, most systems result in nonautonomous models, for which a nonautonomous context will be necessary to set up the problem accordingly.

Various techniques exist to deal with the problems of attractors for nonautonomous systems, e.g., kernel sections [17], skew-product formalism [40], etc. However, most results obtained in this direction have been proved within the theory of pullback attractors (see [15], [31], [32], [39]), which has been extremely fruitful, particularly in the case of random dynamical systems (see [19], [20], [39]). This is because that constructing the parameter set needed to construct a skew-product flow (or the symbols set in the theory of kernel sections) is possible when the explicit dependence on the delay (e.g. as in the variable or distributed delay cases) is available, but it is not known when one is interested in developing a general theory concerning abstract delay terms, i.e., when one wishes to do it under a general functional formulation (see [8] for more details). In this survey we will not elaborate the existence of non-autonomous attractors for Navier-Stokes models. Instead, we will emphasize more on different methods that can be used for the local stability analysis, and particularly in the cases where bounded variable delays are considered.

The rest of the paper is organized as follows. In Section 2 we will set up our aimed problems and will include the preliminary results on the Navier–Stokes model with delay. Section 3 will be devoted to the existence and uniqueness of solutions of the model. In particular we will sketch the major proofs by applying Galerkin's method. Local asymptotic behavior will be analyzed in Section 4, by proving the existence and uniqueness (under more restrictive conditions) of stationary solutions and their stability properties. Specifically we will review this topic by describing first the method of Lyapunov functions, second the so-called Razumihin-Lyapunov's method, third the method based on the constructions of Lyapunov functionals and last the application of a Gronwall-like lemma which allows the delay function to be only measurable. Finally, in Section 5 we will include several remarks about the existence of attractors, and possible generalizations and variations. In addition we will state some future open directions in which continuous investigation can be made within this challenging field of Navier-Stokes equations.

2. **Preliminaries.** We will start by describing the general formulation of the model that will be considered in our analysis.

Let $\Omega \subset \mathbb{R}^N$ ($N = 2$ or 3) be an open and bounded set with a regular boundary Γ. Given $T > 0$, we consider the following functional Navier-Stokes problem (for further details and notations see Lions [35] and Temam [42]):

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu\Delta u + \sum_{i=1}^{N} u_i \dfrac{\partial u}{\partial x_i} = f - \nabla p + g(t,u_t) \quad \text{in } (0,T)\times\Omega, \\ \text{div } u = 0 \quad \text{in } (0,T)\times\Omega, \\ u = 0 \ \text{ on } (0,T)\times\Gamma, \\ u(0,x) = u_0(x), \quad x\in\Omega, \\ u(t,x) = \phi(t,x), t\in(-h,0) \quad x\in\Omega, \end{cases}$$

where u is the velocity field of the fluid, $\nu > 0$ is the kinematic viscosity, p the pressure, u_0 the initial velocity field, ϕ the initial datum in the time interval $(-h,0)$ where h is a positive fixed number, f a nondelayed external force field, and g is the external force containing some hereditary characteristic.

Define the following abstract spaces:

- $\mathscr{V} = \left\{ u \in (C_0^\infty(\Omega))^N : \text{div}\, u = 0 \right\}$;
- $H =$ the closure of $\mathscr{V}$ in $(L^2(\Omega))^N$ with norm $|\cdot|$, and inner product $(\cdot,\cdot)$ defined by

$$(u,v) = \sum_{j=1}^{N} \int_\Omega u_j(x) v_j(x) dx \quad \text{for} \quad u,v \in (L^2(\Omega))^N;$$

- $V =$ the closure of $\mathscr{V}$ in $(H_0^1(\Omega))^N$ with norm $\|\cdot\|$, and associated scalar product $((\cdot,\cdot))$ defined by

$$((u,v)) = \sum_{i,j=1}^{N} \int_\Omega \frac{\partial u_j}{\partial x_i} \frac{\partial v_j}{\partial x_i} dx \quad \text{for} \quad u,v \in (H_0^1(\Omega))^N.$$

It follows that $V \subset H \equiv H' \subset V'$, where the injections are dense and compact. In the sequel we will use $\|\cdot\|_*$ to denote the norm in V', and $\langle\cdot,\cdot\rangle$ to denote the duality $\langle V', V\rangle$.

Denote $a(u,v) := ((u,v))$, and define the tri-linear form b on $V\times V\times V$ by

$$b(u,v,w) := \sum_{i,j=1}^{N} \int_\Omega u_i \frac{\partial v_j}{\partial x_i} w_j dx, \quad \forall u,v,w \in V.$$

Note that the tri-linear form b satisfies the following inequalities which will be used later in proofs (see Lions [35]):

(I1) there exists $\kappa_1 := \kappa_1(\Omega) > 0$ such that

$$|b(u,v,w)| \le \kappa_1 |u|^{1/2} \|u\|^{1/2} \|v\| |w|^{1/2} \|w\|^{1/2}, \quad \forall u,v,w \in V;$$

(I2) there exists $\kappa_2 := \kappa_2(\Omega) > 0$ such that

$$|b(u,v,w)| \le \kappa_2 |u|_{(L^4(\Omega))^2} \|u\| \|w\|, \quad \forall u,v,w \in V.$$

Let X be a Banach space and consider a fixed $T > 0$. Given $u : (-h,T) \to X$, for each $t \in (0,T)$ we denote by u_t the function defined on $(-h,0)$ via the relation

$$u_t(s) = u(t+s), \quad s \in (-h,0).$$

Before stating the problem in a suitable framework, we enumerate the assumptions on the term in which the delay is present. In a general way, let X and Y be two separable Banach spaces, and $g : [0,T] \times C^0([-h,0];X) \to Y$ satisfies

(I) for all $\xi \in C^0([-h,0];X)$, the mapping $t \in [0,T] \to g(t,\xi) \in Y$ is measurable;
(II) for each $t \in [0,T]$, $g(t,0) = 0$;
(III) there exists $L_g > 0$ such that $\forall t \in [0,T], \quad \forall \xi, \eta \in C^0([-h,0];X)$

$$\|g(t,\xi) - g(t,\eta)\|_Y \le L_g \|\xi - \eta\|_{C^0([-h,0];X)};$$

(IV) there exists $C_g > 0$ such that $\forall t \in [0,T]$, $\forall u, v \in C^0([-h,T];X)$

$$\int_0^t \|g(s,u_s) - g(s,v_s)\|_Y^2 \, ds \le C_g \int_{-h}^t \|u(s) - v(s)\|_X^2 \, ds.$$

Observe that assumptions (I)-(III) imply that given any $u \in C^0([-h,T];X)$, the function $g_u : t \in [0,T] \to Y$ defined by $g_u(t) = g(t,u_t)$ for any $t \in [0,T]$ is measurable (see Bensoussan et al. [4]) and belongs to $L^\infty(0,T;Y)$. Then, thanks to assumption (IV), the mapping

$$\mathscr{G} : u \in C^0([-h,T];X) \to g_u \in L^2(0,T;Y)$$

possesses a unique extension to a mapping $\widetilde{\mathscr{G}}$ which is uniformly continuous from $L^2(-h,T;X)$ into $L^2(0,T;Y)$.

From now on, we will denote $g(t,u_t) = \widetilde{\mathscr{G}}(u)(t)$ for each $u \in L^2(-h,T;X)$, and thus, $\forall t \in [0,T]$, $\forall u, v \in L^2(-h,T;X)$, we will have

$$\int_0^t \|g(s,u_s) - g(s,v_s)\|_Y^2 \, ds \le C_g \int_{-h}^t \|u(s) - v(s)\|_X^2 \, ds.$$

We are interested in the problem: to find $u \in L^2(-h,T;V) \cap L^\infty(0,T;H)$ such that for all $v \in V$,

$$\begin{cases} \dfrac{d}{dt}(u(t),v) + \nu a(u(t),v) + b(u(t),u(t),v) \\ \qquad = \langle f(t),v\rangle + (g_1(t,u_t),v) + \langle g_2(t,u_t),v\rangle, \\ u(0) = u_0, \;\; u(t) = \phi(t), \;\; t \in (-h,0), \end{cases} \tag{1}$$

which is understood in the distributional sense of $\mathscr{D}'(0,T)$.

Remark 1. Observe that the terms in (1) are well defined. In particular, by hypotheses (I)-(IV), if $u \in L^2(-h,T;V)$ the term $g_1(t,u_t)$ defines a function in $L^2(0,T;(L^2(\Omega)^N)$, and the term $g_2(t,u_t)$ defines a function in $L^2(0,T;V')$. Thus if $u \in L^2(-h,T;V) \cap L^\infty(0,T;H)$ satisfies the equation in (1), u is weakly continuous from $[0,T]$ into H (see Lions [35]), and therefore the initial condition $u(0) = u_0$ makes sense. Clearly for $N = 2$, if there exists a solution u to the problem (1), it then belongs to the space $C^0([0,T];H)$.

In the next section we will state the existence of solutions to (1), the uniqueness of solution to the problem in the case $N = 2$, as well as some regularity results. Additionally, we will include some other particular cases in which the initial data is a continuous function instead of a square integrable function.

3. **Existence, uniqueness and regularity of solutions.** We first prove a general result on the existence (when $N = 2$ or 3) and uniqueness (when $N = 2$ or 3) of solutions to system (1).

Theorem 3.1. *Consider $u_0 \in H$, $\phi \in L^2(-h,0;V)$, $f \in L^2(0,T;V')$, and assume that $g_1 : [0,T] \times C^0([-h,0];V) \to (L^2(\Omega))^N$ satisfies hypotheses (I)-(IV) with $X = V$, $Y = (L^2(\Omega))^N$, $L_{g_1} = L_1$ and $C_{g_1} = C_1$, and $g_2 : [0,T] \times C^0([-h,0];V) \to V'$ satisfies hypotheses (I)-(IV) with $X = V$, $Y = V'$, $L_{g_2} = L_2$ and $C_{g_2} = C_2$. Then:*

(a) If $N=2$ and $\nu^2>C_2$, there exists at most one solution to problem (1).

(b) If $N\in\{2,3\}$ and $\nu^2>C_2$, there exists a solution to (1) *if, in addition, the following assumption (V) holds:*

(V) If $v^{(m)}$ converges weakly to v in $L^2(-h,T;V)$ and strongly in $L^2(-h,T;H)$, then $g_i(\cdot,v^{(m)}_\cdot)$ converges weakly to $g_i(\cdot,v_\cdot)$ in $L^2(0,T;V')$ for $i=1,2$.

Proof. We only include a sketch of the proof (see Caraballo and Real [11] for more details).

(a) If $N=2$ and $\nu^2>C_2$, then the uniqueness of solutions follows from Gronwall's lemma. In fact, let u,v be two solutions to (1) and set $w=u-v$. Then, it follows from the energy equality and (I1) that for all $t\in(0,T)$

$$\begin{aligned}|w(t)|^2+2\nu\textstyle\int_0^t\|w(s)\|^2\,ds \quad\leq\quad & 2\kappa_1\textstyle\int_0^t|w(s)|\cdot\|w(s)\|\cdot\|u(s)\|\,ds\\ &+2\textstyle\int_0^t|g_1(s,u_s)-g_1(s,v_s)|\,|w(s)|\,ds\\ &+2\textstyle\int_0^t\|g_2(s,u_s)-g_2(s,v_s)\|_*\,\|w(s)\|\,ds.\end{aligned}$$

Then, from assumption (IV), taking into account that $w(s)=0$ for $s\in(-h,0)$, and denoting $2\varepsilon=\nu-\sqrt{C_2}>0$, we have for all $t\in(0,T)$

$$\begin{aligned}|w(t)|^2+2\nu\textstyle\int_0^t\|w(s)\|^2\,ds \quad\leq\quad & \frac{\kappa_1^2}{\varepsilon}\textstyle\int_0^t|w(s)|^2\|v(s)\|^2\,ds+\varepsilon\textstyle\int_0^t\|w(s)\|^2\,ds\\ &+\frac{C_1}{\varepsilon}\textstyle\int_0^t|w(s)|^2\,ds+\varepsilon\textstyle\int_0^t\|w(s)\|^2\,ds\\ &+2\sqrt{C_2}\textstyle\int_0^t\|w(s)\|^2\,ds.\end{aligned}$$

Hence we have

$$|w(t)|^2+2\varepsilon\int_0^t\|w(s)\|^2\,ds\leq\frac{\kappa_1^2}{\varepsilon}\int_0^t|w(s)|^2\|v(s)\|^2\,ds+\frac{C_1}{\varepsilon}\int_0^t|w(s)|^2\,ds,$$

from which uniqueness follows according to the Gronwall lemma.

(b) For $N=2$ or $N=3$, $\nu^2>C_2$ and assume that condition (V) holds, we will proceed by using a Galerkin scheme as in Constantin and Foias [18]. We only provide details concerning the delay terms g_i $(i=1,2)$.

Let us consider $\{w_j\}\subset V\cap\left(H^2(\Omega)\right)^N$, the orthonormal basis of H of all the eigenfunctions of the Stokes problem in Ω with homogeneous Dirichlet conditions. The subspace of V spanned by $w_1,\ldots,w_m$ will be denoted V_m. Consider the projector $P_m:H\to V_m$ given by $P_mu=\sum_{j=1}^m(u,w_j)w_j$, and define $u^{(m)}(t)=\sum_{j=1}^m\gamma_{mj}(t)w_j$, where $u^{(m)}\in L^2(-h,T;V_m)\cap C^0([0,T];V_m)$ satisfies

$$\left\{\begin{array}{l}\dfrac{d}{dt}(u^{(m)}(t),w_j)+\nu a(u^{(m)}(t),w_j)+b(u^{(m)}(t),u^{(m)}(t),w_j)\\ \quad=\left\langle f(t),w_j\right\rangle+(g_1(t,u_t^m),w_j)+\left\langle g_2(t,u_t^m),w_j\right\rangle \text{ in } \mathscr{D}'(0,T)\ \ 1\leq j\leq m,\\ u^{(m)}(0)=P_mu_0,\ \ u^{(m)}(t)=P_m\phi(t),\ \ t\in(-h,0).\end{array}\right. \tag{2}$$

(2) is a system of ordinary functional differential equations in the unknown, $\gamma^{(m)}(t)=(\gamma_{m1}(t),\ldots,\gamma_{mm}(t))$. The existence and uniqueness of solution follows from Theorem A1 in [11] and we can ensure that problem (2) possesses a solution defined in $[0,t_*]$ with $0<t_*\leq T$. However, thanks to the a priori estimates below, we can set $t_*=T$.

In fact, multiplying in (2) by $\gamma_{mj}(t)$ and summing in j, we have for all $t\in[0,t_*]$

$$\begin{aligned}|u^{(m)}(t)|^2+2\nu\textstyle\int_0^t\|u^{(m)}(s)\|^2\,ds \quad\leq\quad & |u_0|^2+2\textstyle\int_0^t\left\langle f(s),u^{(m)}(s)\right\rangle\\ &+2\textstyle\int_0^t(g_1(s,u_s^{(m)}),u^{(m)}(s))\,ds\\ &+2\textstyle\int_0^t\left\langle g_2(s,u_s^{(m)}),u^{(m)}(s)\right\rangle ds,\end{aligned}$$

and by the same argument as in the proof of uniqueness in the 2-dimensional case, we obtain two constants K_1 and K_2 (depending only on ϕ,ν,f,g_1,g_2,h,T, but not on m or t_*) such that

$$\sup_{t\in[0,t_*]}|u^{(m)}(t)|^2\le K_{1,}\quad \int_0^{t_*}\|u^{(m)}(s)\|^2\,ds\le K_2.$$

Thus we can take $t_*=T$ to obtain that $\{u^{(m)}\}$ is bounded in $L^2(0,T;V)\cap L^\infty(0,T;H)$. Moreover, observe that $u^{(m)}=P_m\phi$ in $(-h,0)$ and, by the choice of the basis $\{w_j\}$, the sequence $\{u^{(m)}\}$ converges to ϕ in $L^2(-h,0;V)$.

Note that $\{g_1(\cdot,u^{(m)}_\cdot)+g_2(\cdot,u^{(m)}_\cdot)\}$ is bounded in $L^2(0,T;V')$ and it is a straight forward to bound the nonlinear term $\{b(u^{(m)},u^{(m)},\cdot)\}$. By using the same argument as in Constantin and Foias [18, page 67], one can obtain that $\left\{\frac{du^{(m)}}{dt}\right\}$ is bounded in $L^{4/3}(0,T;V')$ (in fact, if $N=2$, $\left\{\frac{du^{(m)}}{dt}\right\}$ is bounded in $L^2(0,T;V')$). Using the compactness of the injection of the space $W=\{u\in L^2(0,T;V):\frac{du}{dt}\in L^{4/3}(0,T;V')\}$ into $L^2(0,T;H)$, from the preceding analysis and the assumptions on g_1 and g_2, we can deduce that there exist a subsequence (denoted again $\{u^{(m)}\}$) and $u\in L^2(-h,T;V)$ such that:

$$\begin{aligned}
&u^{(m)}\to u \ \text{ weakly in } \ L^2(-h,T;V),\\
&u^{(m)}\to u \ \text{ weakly star in } \ L^\infty(0,T;H),\\
&u^{(m)}\to u \ \text{ in } \ L^2(-h,T;H),\\
&g_i(\cdot,u^{(m)}_\cdot)\to g_i(\cdot,u_\cdot) \ \text{ weakly } \ \text{ in } \ L^2(0,T;V'),\ \ i=1,2.
\end{aligned}$$

Arguing now as in the non-delay case, we can take limits in (2) after integrating over the interval $(0,t)$ (for $t\in(0,T)$), and obtain that u is a solution to our problem (1) (see Constantin and Foias [18] for the complete details). □ □

In order to analyze the asymptotic behavior of our model and construct a dynamical system generated in the 2-dimensional case, it is necessary to state another result on the existence and uniqueness of solutions in different spaces and with non-zero initial time. The result below only requires assumptions (I)-(III), which allows more general forms of the delay term (for instance in the case of variable delay, only measurability of the delay is needed), while in general assumption (IV) requires more regularity, e.g., continuous differentiability and boundedness on the derivative of the delay). But on the other side, the initial values must be continuous and not only square integrable (see García-Luengo et al. [24]).

Consider the following version of Navier-Stokes for $\tau\in\mathbb{R}$:

$$\begin{cases}
\dfrac{\partial u}{\partial t}-\nu\Delta u+\sum_{i=1}^N u_i\dfrac{\partial u}{\partial x_i}=f-\nabla p+g(t,u_t) \quad \text{in } \ (\tau,+\infty)\times\Omega,\\
\operatorname{div} u\ =0 \quad \text{in } \ (\tau,+\infty)\times\Omega,\\
u=0 \ \text{ on } (\tau,+\infty)\times\Gamma,\\
u(\tau,x)=u_0(x),\quad x\in\Omega,\\
u(t,x)=\phi(t-\tau,x),\, t\in(\tau-h,\tau),\quad x\in\Omega.
\end{cases}\tag{3}$$

Theorem 3.2. ([24]) *Consider $\phi\in C^0([-h,0];H)$ with $u_0=\phi(0)$, $f\in L^2_{loc}(\mathbb{R};V')$, and assume that $g:\mathbb{R}\times C^0([-h,0];H)\to(L^2(\Omega))^2$ satisfies hypotheses (I)-(III) with $X=H$ and $Y=(L^2(\Omega))^N$. Then, for any $\tau\in\mathbb{R}$ there exists a unique solution $u=u(\cdot;\tau,\phi)$ of* (3)*, in other words, $u\in C^0([\tau-h,+\infty);H)$ and $u\in L^2(\tau,+\infty;V)$. Moreover, if $f\in L^2_{loc}(\mathbb{R};(L^2(\Omega))^2)$, then*

(a) $u\in C^0([\tau+\varepsilon,T];V)\cap L^2(\tau+\varepsilon,T;D(A))$ for all $T>\tau+\varepsilon>\tau$.

(b) *If $\phi(0) \in V$, in fact u is a strong solution of* (3), *i.e.* $u \in L^2(\tau,T;D(A)) \cap L^\infty(\tau,T;V)$ *for all* $T > \tau$.

Proof. The uniqueness follows from the same lines as the proof of Theorem 3.1. The existence requires now the use of the energy method (see García-Luengo et al. [24] for a detailed proof). □ □

3.1. **Examples of delay forcing terms.** Now, we will exhibit a few examples of delay forcing terms which can be set within our general set-up. Later on, to illustrate the different methods for the stability analysis, we will focus on the case of variable delays.

Example 1: forcing term with bounded variable delay

Let $G : [0,T] \times \mathbb{R}^N \to \mathbb{R}^N$ be a measurable function satisfying $G(t,0) = 0$ for all $t \in [0,T]$, and assume that there exists $M > 0$ such that

$$|G(t,u) - G(t,v)|_{\mathbb{R}^N} \leq M|u-v|_{\mathbb{R}^N}, \forall u,v \in \mathbb{R}^N.$$

Consider a function $\rho(t)$, which is going to play the role of the delay. Assume that $\rho(\cdot)$ is measurable and define $g_1(t,\xi)(x) = G(t,\xi(-\rho(t))(x))$ for each $\xi \in C^0([0,T];H)$, $x \in \Omega$ and $t \in [0,T]$. Notice that, in this case, the delayed term g_1 in our problem becomes

$$g_1(t,u_t) = G(t,u(t-\rho(t))).$$

Then, g_1 satisfies the hypotheses in Theorem 3.2 with $X = H$ and $Y = L^2(\Omega)^N$, since (I)-(III) follow immediately.

However, in order to apply Theorem 3.1 we need to impose stronger assumptions on the delay function. Indeed, we assume that $\rho \in C^1([0,T])$, $\rho(t) \geq 0$ for all $t \in [0,T]$, $h = \max_{t\in[0,T]} \rho(t) > 0$ and $\rho_* = \max_{t\in[0,T]} \rho'(t) < 1$. Then for $u,v \in L^2(-h,T;H)$, using the change of variable $\tau = s - \rho(s)$ gives immediately that

$$\int_0^t |g_1(s,u_s) - g_1(s,v_s)|^2\,ds \leq \int_{-h}^t |u(\tau)-v(\tau)|^2\,d\tau, \quad \forall t \in [0,T],$$

and consequently, (IV) and (V) are fulfilled.

Example 2: forcing term with distributed delay

Let $G : [0,T] \times [-h,0] \times \mathbb{R}^N \to \mathbb{R}^N$ be a measurable function satisfying $G(t,s,0) = 0$ for all $(t,s) \in [0,T] \times [-h,0]$ and there exists a function $\alpha \in L^2(-h,0)$ such that

$$|G(t,s,u) - G(t,s,v)|_{\mathbb{R}^N} \leq \alpha(s)|u-v|_{\mathbb{R}^N},\ \forall u,v \in \mathbb{R}^N,\ \forall (t,s) \in [0,T] \times [-h,0].$$

Define $g_1(t,\xi)(x) = \int_{-h}^0 G(t,s,\xi(s)(x))\,ds$ for each $\xi \in C^0([0,T];H)$, $t \in [0,T]$, and $x \in \Omega$. Then the delayed term g_1 in our problem becomes

$$g_1(t,u_t) = \int_{-h}^0 G(t,s,u(t+s))\,ds.$$

As in Example 1, g_1 satisfies the hypotheses in Theorem 3.2 with $X = H$ and $Y = \left(L^2(\Omega)\right)^N$.

Indeed, (I) and (II) follow immediately. On the other hand, if $\xi,\eta \in C^0([0,T];H)$, for each $t \in [0,T]$ we obtain

$$\begin{aligned}
|g_1(t,\xi) - g_1(t,\eta)|^2 &\leq \int_\Omega \left(\int_{-h}^0 |G(t,s,\xi(s)(x)) - G(t,s,\eta(s)(x))|_{\mathbb{R}^N}\,ds\right)^2 dx \\
&\leq \int_\Omega \left(\int_{-h}^0 \gamma(s)|\xi(s)(x) - \eta(s)(x)|_{\mathbb{R}^N}\,ds\right)^2 dx \\
&\leq \int_\Omega \|\alpha\|^2_{L^2(-h,0)} \left(\int_{-h}^0 |\xi(s)(x) - \eta(s)(x)|^2_{\mathbb{R}^N}\,ds\right) dx \\
&\leq h\|\alpha\|^2_{L^2(-h,0)} \|\xi - \eta\|^2_{C^0([0,T];H)}.
\end{aligned}$$

Finally, it is also straightforward to verify (IV) and (V). Indeed, if $u, v \in L^2(-h,T;H)$ then, for each $t \in [0,T]$ it follows

$$\int_0^t |g_1(\tau,u_\tau) - g_1(\tau,v_\tau)|^2\, d\tau \le h\|\alpha\|^2_{L^2(-h,0)} \int_0^t \left(\int_{-h}^0 |u(s+\tau) - v(s+\tau)|^2\, ds \right) d\tau,$$

and, with the change of variable $r = s+\tau$, we have

$$\begin{aligned} \textstyle\int_0^t |g_1(\tau,u_\tau) - g_1(\tau,v_\tau)|^2\, d\tau &\le h\|\alpha\|^2_{L^2(-h,0)} \textstyle\int_0^t \left(\int_{\tau-h}^{\tau} |u(r) - v(r)|^2\, dr\right)\, d\tau \\ &\le hT\|\alpha\|^2_{L^2(-h,0)} \textstyle\int_{-h}^t |u(r) - v(r)|^2\, dr. \end{aligned}$$

4. **Local asymptotic behavior: stability of steady state solutions and Lyapunov functionals.** In this section we will analyze the long time behavior of solutions in a neighborhood of a stationary solution in the 2-dimensional case. First we will prove a general result ensuring the existence and, eventually, the uniqueness of such stationary solution. Then we will show four different approaches that can be used to study the stability properties: the Lyapunov function, the Lyapunov-Razumikhin method, the construction of Lyapunov functionals method, and a Gronwall-like lemma approach. All the cases will be related to the model considered in Example 1, i.e. for variable delays, since all the methods in this paper can be applied to this case. See [5] for some results on more general delay terms.

We would like to mention that the first technique requires a strong assumption on the delay function, which can be weakened by using a Razumikhin type argument (see Razumikhin [38], and Hale [26] for a modern and nice presentation of the method). But, on the other side, we will need to impose stronger assumptions on the coefficients of the model since it will be necessary to deal with strong solutions rather that weak one. The third approach will be based on the construction of Lyapunov functionals which will allow us to improve some of the previous sufficient conditions when one is able to construct such kind of functionals. Finally, a method based on a Gronwall-like lemma will enable us to impose only measurability on the variable delay function.

In the sequel, λ_1 will denote the first eigenvalue of operator A.

4.1. **Existence and uniqueness of stationary solutions.** Let us consider the following equation

$$\frac{du}{dt} + \nu Au + B(u) = f + g(t,u_t), \tag{4}$$

with $f \in V'$ independent of t. A stationary solution to (4), $u^\star$, satisfies

$$\nu Au^\star + B(u^\star) = f + g(t,u^\star), \quad \forall t \ge 0.$$

In order to carry out our analysis, we assume as in Example 1 that the forcing term g is given by

$$g(t,u_t) = G(u(t-\rho(t))),$$

where $G : \mathbb{R}^2 \to \mathbb{R}^2$ is a function satisfying

$$G(0) = 0 \tag{5}$$

and that there exists $M > 0$ for which

$$|G(u) - G(v)|_{\mathbb{R}^2} \le M|u-v|_{\mathbb{R}^2}, \forall u, v \in \mathbb{R}^2. \tag{6}$$

Assumptions for $\rho(t)$ include: $\rho \in C^1([0,+\infty))$, $\rho(t) \ge 0$ for all $t \ge 0$, $h = \sup_{t\ge 0}\rho(t) > 0$ and $\rho_* = \sup_{t\ge 0}\rho'(t) < 1$.

As it was proved in the last section, conditions (I)-(IV) and (V) hold and, consequently, we can ensure the existence and uniqueness of solutions (see Caraballo and Real [11] and also the existence of strong solutions according to Theorem 3.2).

Now we can establish a result on the existence and uniqueness of stationary solutions to our equation (4), i.e., there exists $u^\star \in V$ such that

$$\nu A u^\star + B(u^\star) = f + G(u^\star).$$

Theorem 4.1. *Suppose that G satisfies conditions* (5)-(6) *and $\nu > \lambda_1^{-1} M$. Then,*

(a) *for all $f \in V'$ there exists a stationary solution to (4);*

(b) *if $f \in (L^2(\Omega))^2$, the stationary solutions belong to $D(A)$;*

(c) *there exists a constant $C_3(\Omega) > 0$ such that if $(\nu - \lambda_1^{-1} M)^2 > C_3(\Omega)\|f\|_{V'}$, then the stationary solution to (4) is unique.*

Proof. (a) By the Lax-Milgram Theorem, for each $z \in V$, there exists a unique $u \in V$ such that

$$\nu((u,v)) + b(z,u,v) = \langle f, v\rangle + (G(z), v), \quad \forall v \in V. \tag{7}$$

Taking $v = u$ in (7), it follows that

$$\nu\|u\| \leq \|f\|_{V'} + \lambda_1^{-1} M \|z\|. \tag{8}$$

Let us pick $k > 0$ such that $k(\nu - \lambda_1^{-1} M) \geq \|f\|_{V'}$, and denote

$$\mathscr{C} = \{z \in V;\ \|z\| \leq k\}.$$

Then $\mathscr{C}$ is a convex and compact subset of $(L^4(\Omega))^2$. By (8), the mapping $z \mapsto u$ defined by (7), maps $\mathscr{C}$ into $\mathscr{C}$. If we can proof that this mapping is continuous in $\mathscr{C}$ with the topology induced by $(L^4(\Omega))^2$, then the Schauder Theorem implies the existence of a fixed point in $\mathscr{C}$, which clearly is a stationary solution to (4). To obtain he continuity of $z \mapsto u$, let $z_i \in \mathscr{C}$ and $u_i \in \mathscr{C}$ be such that

$$\nu((u_i, v)) + b(z_i, u_i, v) = \langle f, v\rangle + (G(z_i), v), \quad \forall v \in V, \quad i = 1, 2.$$

Then by (I2) we have,

$$\begin{aligned} \nu\|u_1 - u_2\|^2 &= b(z_2 - z_1, u_1, u_1 - u_2) + (G(z_1) - G(z_2), u_1 - u_2) \\ &\leq \kappa_2(\Omega)|z_1 - z_2|_{L^4(\Omega))^2}\|u_1 - u_2\| \\ &\quad + M\lambda_1^{-1/2}|z_1 - z_2|\|u_1 - u_2\|. \end{aligned} \tag{9}$$

As $V \subset (L^4(\Omega))^2$ and $(L^4(\Omega))^2 \subset (L^2(\Omega))^2$ with continuous injections, the continuity of the mapping $z \mapsto u$ in $\mathscr{C}$ follows from (9).

(b) If $f \in (L^2(\Omega))^2$, then every stationary solution u_* to (4) is also a solution to (1), but with initial data $u_0 = \phi(t) = u_*$ for $t \in [-h, 0)$, and forcing term $\tilde{f} = P(f + G(u_*)) \in H \subset L^2(0,T;H)$. Thus, the standard regularity results from the theory of the Navier-Stokes equation without delays can be applied.

(c) Let $f \in V$ and u_1 and u_2 stationary solutions to (4). Then, arguing as for the inequality (9) we have,

$$\begin{aligned} \nu\|u_1 - u_2\|^2 &\leq \kappa_2(\Omega)|u_1 - u_2|_{L^4(\Omega))^2}\|u_1\|\|u_1 - u_2\| \\ &\quad + M\lambda_1^{-1/2}|u_1 - u_2|\|u_1 - u_2\|. \end{aligned} \tag{10}$$

With

$$\nu\|u_1\|^2 = \langle f, u_1\rangle + (G(u_1), u_1) \leq \|f\|_{V'}\|u_1\| + \lambda_1^{-1} M \|u_1\|^2,$$

we obtain

$$(\nu - \lambda_1^{-1} L_1)\|u_1\| \leq \|f\|_{V'}. \tag{11}$$

Using inequality (11), and the continuous injection of V into $(L^4(\Omega))^2$, we obtain from (10) that there exists $C_3(\Omega) > 0$ such that

$$(\nu - \lambda_1^{-1}M)^2 \|u_1 - u_2\|^2 \leq C_3(\Omega)\|f\|_{V'}\|u_1 - u_2\|^2.$$

This completes the proof. □ □

4.2. **Exponential convergence of solutions: a direct approach for the model with variable delays.** Now we will prove that under appropriate assumptions, our model has a unique stationary solution, u_∞, and every weak solution approaches u_∞ exponentially fast as t goes to $+\infty$.

Theorem 4.2. *Assume that the forcing term $g(t,u_t)$ is given by $g(t,u_t) = G(u(t-\rho(t))$ with $\rho \in C^1(\mathbb{R}^+;[0,h])$ such that $\rho'(t) \leq \rho_* < 1$ for all $t \geq 0$. Then, there exist two constants $k_i > 0$, $i = 1,2$, depending only on Ω, such that if $f \in (L^2(\Omega))^2$ and $\nu > \lambda_1^{-1}M$ satisfy in addition*

$$2\nu\lambda_1 > \frac{(2-\rho_*)L_1}{1-\rho_*} + \frac{k_1|f|}{\nu - \lambda_1^{-1}M} + \frac{k_2|f|^3}{\nu^2(\nu - \lambda_1^{-1}M)^3}, \tag{12}$$

then there is a unique stationary solution $u_\infty$of (4) and every solution of (1) converges to u_∞ exponentially fast as $t \to +\infty$. More precisely, there exist two positive constants C and λ, such that for all $u_0 \in H$ and $\phi \in L^2(-h,0;V)$, the solution u of (1) with $f(t) \equiv f$ satisfies

$$|u(t) - u_\infty|^2 \leq Ce^{-\lambda t}\left(|u_0 - u_\infty|^2 + \|\phi - u_\infty\|^2_{L^2(-h,0;V)}\right), \tag{13}$$

for all $t \geq 0$.

Proof. Assume that $f \in (L^2(\Omega))^2$, and consider u, the solution of (3) for $f(t) \equiv f$, $\tau = 0$, and let $u_\infty \in D(A)$ be a stationary solution to (4). Let us write $w(t) = u(t) - u_\infty$, and observe that

$$\frac{\mathrm{d}}{\mathrm{d}t}w(t) + \nu Aw(t) + B(u(t)) - B(u_\infty) = G(u(t-\rho(t)) - G(u_\infty).$$

Now fix a positive λ to be determined later. By standard computations,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^{\lambda t}|w(t)|^2) &= \lambda \mathrm{e}^{\lambda t}|w(t)|^2 + \mathrm{e}^{\lambda t}\frac{\mathrm{d}}{\mathrm{d}t}|w(t)|^2 \\
&\leq \mathrm{e}^{\lambda t}(\lambda|w(t)|^2 - 2\nu\|w(t)\|^2 + 2b(w(t),w(t),u_\infty) \\
&\quad + 2M|w(t-\rho(t))||w(t)|) \\
&\leq \lambda_1^{-1}\mathrm{e}^{\lambda t}(\lambda + M - 2\nu\lambda_1)\|w(t)\|^2 \\
&\quad + 2\mathrm{e}^{\lambda t}|b(w(t),w(t),u_\infty)| + M\mathrm{e}^{\lambda t}|w(t-\rho(t))|^2.
\end{aligned} \tag{14}$$

Obviously

$$|b(w(t),w(t),u_\infty)| \leq c|w(t)|\|w(t)\||u_\infty|_\infty, \tag{15}$$

where we denote by $|u_\infty|_\infty$ the norm of u_∞ in $(L^\infty(\Omega))^2$. Observe that $H^2(\Omega) \subset L^\infty(\Omega)$ with continuous injection, and that there exists a constant $C(\Omega) > 0$ such that

$$|u|_{(H^2(\Omega))^2} \leq C(\Omega)|Au|, \ \forall u \in D(A) = (H^2(\Omega))^2 \cap V. \tag{16}$$

Thus, we obtain the existence of a constant $c_1 > 0$ depending only on Ω such that

$$|b(w(t),w(t),u_\infty)| \leq c_1\lambda_1^{-1/2}\|w(t)\|^2|Au_\infty|. \tag{17}$$

On the other hand since

$$\nu|Au_\infty| \leq |f| + |G(u_\infty)| + |B(u_\infty)| \leq |f| + M|u_\infty| + c'\|u_\infty\||u_\infty|_\infty,$$

from the continuous injection of $H^2(\Omega)$ into $L^\infty(\Omega)$, the inequality (16), and the Gagliardo-Nirenberg interpolation inequality, we obtain

$$\nu|Au_\infty| \le |f| + M|u_\infty| + c''\|u_\infty\||u_\infty|^{1/2}|Au_\infty|^{1/2}. \tag{18}$$

Notice that

$$c''\|u_\infty\||u_\infty|^{1/2}|Au_\infty|^{1/2} \le \frac{(c'')^2\lambda_1^{-1/2}}{2\nu}\|u_\infty\|^3 + \frac{\nu}{2}|Au_\infty|,$$

and thus from (18) we deduce

$$|Au_\infty| \le \frac{2}{\nu}|f| + \frac{2M\lambda_1^{-1/2}}{\nu}\|u_\infty\| + \frac{(c'')^2\lambda_1^{-1/2}}{\nu^2}\|u_\infty\|^3. \tag{19}$$

Using

$$\nu\|u_\infty\|^2 = (f,u_\infty) + (G(u_\infty),u_\infty) \le |f|\lambda_1^{-1/2}\|u_\infty\| + M\lambda_1^{-1}\|u_\infty\|^2,$$

we obtain from (19) that

$$\begin{aligned} |Au_\infty| &\le \frac{2}{\nu}|f| + \frac{2M\lambda_1^{-1}}{\nu(\nu - M\lambda_1^{-1})}|f| + \frac{(c'')^2\lambda_1^{-2}}{\nu^2(\nu - M\lambda_1^{-1})^3}|f|^3 \\ &= \frac{2}{(\nu - M\lambda_1^{-1})}|f| + \frac{(c'')^2\lambda_1^{-2}}{\nu^2(\nu - M\lambda_1^{-1})^3}|f|^3. \end{aligned} \tag{20}$$

From (14), (17), (20), and denoting

$$k_1 = 4c_1\lambda_1^{1/2}, \qquad k_2 = 2c_1\lambda_1^{-3/2}(c'')^2,$$

it follows that

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^{\lambda t}|w(t)|^2) \\ &\le \lambda_1^{-1}\mathrm{e}^{\lambda t}\left(\lambda + M - 2\nu\lambda_1 + \frac{k_1|f|}{(\nu - M\lambda_1^{-1})} + \frac{k_2|f|^3}{\nu^2(\nu - M\lambda_1^{-1})^3}\right)\|w(t)\|^2 \\ &\quad + M\mathrm{e}^{\lambda t}|w(t-\rho(t))|^2. \end{aligned} \tag{21}$$

Now, taking into account the properties of the function ρ, we deduce that if we denote $\tau(t) = t - \rho(t)$, the function τ is strictly increasing in $[0,+\infty)$, and that there exists a $\mu > 0$ such that $\tau^{-1}(t) \le t + \mu$ for all $t \ge -\rho(0)$. Thus, by the change of variable $\eta = s - \rho(s) = \tau(s)$, we have

$$\begin{aligned} \int_0^t \mathrm{e}^{\lambda s}|w(s-\rho(s))|^2\,\mathrm{d}s &= \int_{-\rho(0)}^{t-\rho(t)} \mathrm{e}^{\lambda\tau^{-1}(\eta)}|w(\eta)|\frac{1}{\tau'(\tau^{-1}(\eta))}\,\mathrm{d}\eta \\ &\le \frac{\mathrm{e}^{\lambda\mu}}{1-\rho_*}\int_{-h}^{t}\mathrm{e}^{\lambda\eta}|w(\eta)|^2\,\mathrm{d}\eta. \end{aligned} \tag{22}$$

If (12) is satisfied, then there exists $\lambda > 0$ small enough such that

$$\lambda + M - 2\nu\lambda_1 + \frac{k_1|f|}{(\nu - M\lambda_1^{-1})} + \frac{k_2|f|^3}{\nu^2(\nu - M\lambda_1^{-1})^3} + \frac{M\mathrm{e}^{\lambda\mu}}{1-\rho_*} \ge 0.$$

Integrating (21) over the interval $[0,t]$, and taking into account (22), we deduce that for this $\lambda > 0$

$$\mathrm{e}^{\lambda t}|w(t)|^2 \le |w(0)|^2 + \frac{M\mathrm{e}^{\lambda\mu}}{1-\rho_*}\int_{-h}^{0}\mathrm{e}^{\lambda\eta}|w(\eta)|^2\mathrm{d}\eta,$$

and thus (13) is satisfied. The uniqueness of u_∞ follows from the fact that if $\hat{u}_\infty$ is another stationary solution of (4), then $u(t) \equiv \hat{u}_\infty$ is a solution of (1) with $u_0 = \hat{u}_\infty$ and $\phi = \hat{u}_\infty$, and consequently, applying (13) and making $t \to +\infty$, one has $|\hat{u}_\infty - u_\infty|^2 \le 0$. □ □

4.3. **Exponential convergence of solutions: a Razumikhin approach.** In the previous subsection we proved a result on the exponential convergence of weak solutions to the unique stationary solution when the delay term g contains a variable delay which is continuously differentiable. However, it is possible to relax this restriction (assuming only continuity of the delay function) and prove a result for more general forcing terms by using a different method which is also widely used in dealing with the stability properties of delay differential equations. This method was firstly developed by Razumikhin [38] in the context of ordinary differential functional equations, and has already been applied to some stochastic ODEs and PDEs (e.g. Caraballo et al. [7]). However, one interesting point to be noted is that this method requires also some kind of continuity concerning the operators in the model and the solutions. This allows us to prove a result that weaken the assumptions on the delay function, but concerns only the strong solutions to (1).

First, we establish a result for a general delay term g, and afterwards we will discuss particularly the case of variable delay.

Theorem 4.3. *Assume that g satisfies conditions (I)-(III) for any $T > 0$, with X and Y as in Theorem 3.2, and moreover that for all $\xi \in C^0([-h,0];V)$ the mapping $t \in [0,+\infty) \mapsto g(t,\xi) \in (L^2(\Omega))^2$ is continuous. Suppose that for a given $\nu > 0$ and $f \in (L^2(\Omega))^2$ there exists a stationary solution u_∞ of* (4) *such that for some $\lambda > 0$ it holds*

$$\begin{aligned} &-\nu\langle A(\phi(0)-u_\infty),\phi(0)-u_\infty\rangle - \langle B(\phi(0))-B(u_\infty),\phi(0)-u_\infty\rangle \\ &+(g(t,\phi)-g(t,u_\infty),\phi(0)-u_\infty) \le -\lambda|\phi(0)-u_\infty|^2, \ \ t\ge 0, \end{aligned} \tag{23}$$

whenever $\phi \in C^0([-h,0];V)$ satisfies

$$\|\phi - u_\infty\|^2_{C(-h,0;H)} \le \mathrm{e}^{\lambda h}|\phi(0)-u_\infty|^2. \tag{24}$$

Then, the stationary solution u_∞ of (4) is unique, and for all $\psi \in C^0([-h,0];V)$, the strong solution $u(t;\psi)$ to (1) corresponding to this initial datum satisfies

$$|u(t;\psi)-u_\infty|^2 \le \mathrm{e}^{-\lambda t}\|\psi - u_\infty\|^2_{C^0([-h,0];H)}, \ \forall t \ge 0. \tag{25}$$

Proof. Suppose there exists an initial datum $\psi \in C^0([-h,0];V)$ such that (25) does not hold. Then, denoting

$$\sigma = \inf\{t > 0;\ |u(t;\psi)-u_\infty|^2 > \mathrm{e}^{-\lambda t}\|\psi - u_\infty\|^2\},$$

we obtain that for all $0 \le t \le \sigma$

$$\mathrm{e}^{\lambda t}|u(t;\psi)-u_\infty|^2 \le \mathrm{e}^{\lambda\sigma}|u(\sigma;\psi)-u_\infty|^2 = \|\psi-u_\infty\|^2_{C^0([-h,0];H)}, \tag{26}$$

and there is a sequence $\{t_k\}_{k\ge 1}$ in $\mathbb{R}^+$ such that $t_k \downarrow \sigma$, as $k \to \infty$, and

$$\mathrm{e}^{\lambda t_k}|u(t_k;\psi)-u_\infty|^2 > \mathrm{e}^{\lambda\sigma}|u(\sigma;\psi)-u_\infty|^2. \tag{27}$$

On the other hand, by virtue of (26) it is easy to deduce that

$$|u(\sigma+\theta;\psi)-u_\infty|^2 \le \mathrm{e}^{\lambda h}|u(\sigma;\psi)-u_\infty|^2 \quad \forall -h \le \theta \le 0,$$

which, in view of assumption (23), immediately implies that

$$\begin{aligned} &-\nu\langle A(u(\sigma;\psi)-u_\infty),u(\sigma;\psi)-u_\infty\rangle - \langle B(u(\sigma;\psi))-B(u_\infty),u(\sigma;\psi)-u_\infty\rangle \\ &+(g(\sigma,u_\sigma(\cdot;\psi))-g(\sigma,u_\infty),u(\sigma;\psi)-u_\infty) \le -\lambda|u(\sigma;\psi)-u_\infty|^2. \end{aligned} \tag{28}$$

As $u(\cdot;\psi) \in C^0([-h,+\infty);V)$, by the continuity of the operators in the problem, there exists

$\varepsilon_* >$ such that for all $\varepsilon \in (0, \varepsilon_*]$ and $t \in [\sigma, \sigma + \varepsilon]$,

$$-\nu\langle A(u(t;\psi) - u_\infty), u(t;\psi) - u_\infty\rangle - \langle B(u(t;\psi)) - B(u_\infty), u(t;\psi) - u_\infty\rangle \\ + (g(t, u_t(\cdot;\psi)) - g(t, u_\infty), u(t;\psi) - u_\infty) \leq -\lambda |u(t;\psi) - u_\infty|^2. \tag{29}$$

Thus, if we denote by $w(t) = u(t;\psi) - u_\infty$, we have

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|w(t)|^2 &= -\nu\langle Aw(t), w(t)\rangle - \langle B(u(t;\psi)) - B(u_\infty), w(t)\rangle \\ &\quad + (g(t, u_t(\cdot;\psi)) - g(t, u_\infty), w(t)) \end{aligned}$$

for all $t \in [\sigma, \sigma + \varepsilon]$, and after integrating we obtain

$$\begin{aligned} &\mathrm{e}^{\lambda(\sigma+\varepsilon)}|w(\sigma+\varepsilon;\psi)|^2 - \mathrm{e}^{\lambda\sigma}|u(\sigma;\psi) - u_\infty|^2 \\ &\quad = \int_\sigma^{\sigma+\varepsilon} \lambda \mathrm{e}^{\lambda t}|w(t;\psi)|^2 \mathrm{d}t \\ &\quad\quad + \int_\sigma^{\sigma+\varepsilon} \mathrm{e}^{\lambda t}\left(-2\nu\langle Aw(t), w(t)\rangle - 2\langle B(u(t;\psi)) - B(u_\infty), w(t)\rangle\right) \mathrm{d}t \\ &\quad\quad + \int_\sigma^{\sigma+\varepsilon} \mathrm{e}^{\lambda t}\left(g(t, u_t(\cdot;\psi)) - g(t, u_\infty), w(t)\right) \mathrm{d}t \leq 0. \end{aligned}$$

However, this contradicts (27), and hence (25) must be true. The uniqueness of the stationary solution is deduced in the same way as in Theorem 4.2. □ □

Remark 2. We wish now to provide a sufficient condition which implies (23) but easier to check in applications.

Corollary 4.4. *Assume that g satisfies conditions (I)-(V) for any $T > 0$, with X and Y as in Theorem 3.1, and that for all $\xi \in C^0([-h,0];V)$ the mapping $t \in [0,+\infty) \mapsto g(t,\xi) \in (L^2(\Omega))^2$ is continuous. Suppose $\nu > 0$ and $f \in (L^2(\Omega))^2$ are given so that there exists a stationary solution u_∞ of (4). There exist two constants, $k_i > 0$, $i = 1,2$, depending only on Ω, such that if*

$$2\nu\lambda_1 > 2M + \frac{k_1|f|}{\nu - \lambda_1^{-1}M} + \frac{k_2|f|^3}{\nu^2(\nu - \lambda_1^{-1}M)^3}, \tag{30}$$

then the stationary solution u_∞ of (4) is unique, and for all $\psi \in C^0([-h,0];V)$, the strong solution to (1) corresponding to this initial datum, $u(t;\psi)$, satisfies (25), i.e.,

$$|u(t;\psi) - u_\infty|^2 \leq \mathrm{e}^{-\lambda t}\|\psi - u_\infty\|^2_{C^0([-h,0];H)}, \ \forall t \geq 0.$$

Proof. Let $\phi \in C^0([-h,0];V)$ be such that

$$\|\phi - u_\infty\|^2_{C^0([-h,0];H)} \leq \mathrm{e}^{\lambda h}|\phi(0) - u_\infty|^2, \tag{31}$$

where $\lambda > 0$ is a constant to be chosen later on. Then,

$$\begin{aligned} & -\nu\langle A(\phi(0) - u_\infty), \phi(0) - u_\infty\rangle - \langle B(\phi(0)) - B(u_\infty), \phi(0) - u_\infty\rangle \\ & + (g(t,\phi) - g(t,u_\infty), \phi(0) - u_\infty) \\ \leq\ & -\nu\|\phi(0) - u_\infty\|^2 - b(\phi(0) - u_\infty, u_\infty, \phi(0) - u_\infty) \\ & + M\|\phi - u_\infty\|_{C(-h,0;H)}|\phi(0) - u_\infty| \\ \leq\ & -\nu\|\phi(0) - u_\infty\|^2 + M\lambda_1^{-1}\mathrm{e}^{\lambda h}\|\phi(0) - u_\infty\|^2 \\ & + |b(\phi(0) - u_\infty, \phi(0) - u_\infty, u_\infty)|. \end{aligned}$$

Now, using (17) and (20), and the notation used in the proof of Theorem 4.2, it follows immediately that

$$\begin{aligned} &-\nu\langle A(\phi(0)-u_\infty),\phi(0)-u_\infty\rangle-\langle B(\phi(0))-B(u_\infty),\phi(0)-u_\infty\rangle \\ &\quad+(g(t,\phi)-g(t,u_\infty),\phi(0)-u_\infty) \\ &\le\left(-\nu+M\lambda_1^{-1}e^{\lambda h}+\frac{k_1\lambda_1^{-1}|f|}{2(\nu-\lambda_1^{-1}M)}+\frac{k_2\lambda_1^{-1}|f|^3}{2\nu^2(\nu-\lambda_1^{-1}M)^3}\right)\|\phi(0)-u_\infty\|^2. \end{aligned} \tag{32}$$

Then, if (30) is fulfilled, there exists $\lambda>0$ such that

$$\lambda\lambda_1^{-1}-\nu+M\lambda_1^{-1}e^{\lambda h}+\frac{k_1\lambda_1^{-1}|f|}{2(\nu-\lambda_1^{-1}M)}+\frac{k_2\lambda_1^{-1}|f|^3}{2\nu^2(\nu-\lambda_1^{-1}M)^3}\ge 0,$$

and, for this fixed λ, we can obtain from (32) that

$$\begin{aligned} &-\nu\langle A(\phi(0)-u_\infty),\phi(0)-u_\infty\rangle-\langle B(\phi(0))-B(u_\infty),\phi(0)-u_\infty\rangle \\ &\quad+(g(t,\phi)-g(t,u_\infty),\phi(0)-u_\infty) \\ &\le-\lambda\lambda_1^{-1}\|\phi(0)-u_\infty\|^2\le-\lambda|\phi(0)-u_\infty|^2.\Box \end{aligned}$$

□

Remark 3. Notice that Theorem 4.2 and Corollary 4.4 ensure exponential convergence of solutions under very similar sufficient conditions. In fact, when the function g is the one considered in Example 1, i.e., $g(t,u_t)=G(u(t-\rho(t))$, then assumption (12) coincides with (30) when $\rho_*=0$ (i.e. when the delay function ρ is nonincreasing), but if $0<\rho_*<1$ then (12) implies (30).

4.4. **Exponential stability via the construction of Lyapunov functionals.** In the previous subsections we have analyzed the stability of the stationary solutions to our 2D Navier-Stokes model by using different Lyapunov functions. However, sometimes one is able to construct Lyapunov functionals rather than functions. This is not an easy task, in general, but when one succeeds, the results can be better as one is taking into account more information of the model.

There exists a well-known and established method to construct Lyapunov functionals for several kind of equations (difference equations, ordinary differential equations, partial differential equations, stochastic difference/differential equations, etc) which has been developed by Kolmanovskii and Shaikhet [33] and some collaborators of both of them. As a consequence of some general results proved in [14], we will establish here some stability results for our model in the case of variable delay. We would like to mention that to apply this method we strongly need the differentiability of the variable delay function, but the results that we will obtain are better than in the two previous cases.

Let us start by reviewing this procedure of constructing Lyapunov functionals.

Let $\tilde{A}(t,\cdot):V\to V'$; $f_1(t,\cdot):C^0([-h,0];H)\to V'$; $f_2(t,\cdot):C^0([-h,0];V)\to V'$ be three families of nonlinear operators defined for $t>0$ satisfying $\tilde{A}(t,0)=0$, $f_1(t,0)=0$, $f_2(t,0)=0$.

Consider the equation

$$\begin{cases} \dfrac{du(t)}{dt}=\tilde{A}(t,u(t))+f_1(t,u_t)+f_2(t,u_t), & t>0, \\ u(s)=\psi(s), & s\in[-h,0]. \end{cases} \tag{33}$$

Denote by $u(\cdot;\psi)$ the solution of (33) corresponding to the initial condition $\psi\in C^0([-h,0];V)$. We next recall a first result which is the key to prove the major result concerning the construction of Lyapunov functionals.

Theorem 4.5. *(*[14]*) Assume that there exists a functional $V(t,u_t)$ defined from $\mathbb{R}\times C^0([-h,0];H)$ into $[0,+\infty)$ such that the following conditions hold for some positive numbers c_1, c_2 and λ:*

$$\begin{aligned} V\quad(t,u_t) &\geq c_1 e^{\lambda t}|u(t)|^2, \quad t\geq 0\\ V\quad(0,u_0) &\leq c_2\|\psi\|^2_{C^0([-h,0];H)}\\ \frac{d}{dt}\quad V\quad(t,u_t) &\leq 0, \quad t\geq 0 \end{aligned}$$

for any $\psi\in C^0([-h,0];H)$ such that $u(\cdot;\psi)\in C^0([-h,+\infty);H)$. Then the trivial solution of Eq. (33) *is exponentially stable.*

Now we can describe a formal procedure to construct Lyapunov functionals. It is usually carried out in four steps, as follows.

Step 1. To transform (33) into an equation of the form

$$\frac{dz(t,u_t)}{dt} = A_1(t,u(t)) + A_2(t,u_t) \tag{34}$$

where $z(t,\cdot)$ and $A_2(t,\cdot)$ are families of nonlinear operators, $z(t,0)=0$, $A_2(t,0)=0$, operator $A_1(t,\cdot)$ depends only on t and $u(t)$, but does not depend on the previous values $u(t+s)$, $s<0$.

Step 2. The way in which Step 1 has to be done must imply that the trivial solution of the auxiliary equation without memory

$$\frac{dy(t)}{dt} = A_1(t,y(t))dt. \tag{35}$$

is exponentially stable and therefore there exists a Lyapunov function $v(t,y(t))$, which satisfies the conditions of Theorem 4.5.

Step 3. A Lyapunov functional $V(t,u_t)$ for Eq. (34) is constructed in the form $V=V_1+V_2$, where $V_1(t,u_t)=v(t,z(t,u_t))$. Here the argument y of the function $v(t,y)$ is replaced by the functional $z(t,u_t)$ from the left-hand part of (34).

Step 4. Usually, the functional $V_1(t,u_t)$ almost satisfies the conditions of Theorem 4.5. In order to fully satisfy these conditions, it is necessary to calculate and estimate $\frac{d}{dt}V_1(t,u_t)$. Then, the additional functional $V_2(t,u_t)$ can be chosen in a somehow standard way.

Notice that representation (34) is not unique. This fact allows, by using different type of representations of (34), or different ways of estimating $\frac{d}{dt}V_1(t,u_t)$, to construct different Lyapunov functionals and, as a result, to obtain different sufficient conditions for exponential stability.

Let us now consider the following evolution equation

$$\frac{du(t)}{dt} = \tilde{A}(t,u(t)) + F(u(t-\rho(t))), \tag{36}$$

where $\tilde{A}(t,\cdot), F: V\to V'$ are appropriate partial differential operators (see conditions below), which is a particular case of (33).

We will apply the method described above to construct Lyapunov functionals for (36), and, as a consequence, to obtain a sufficient condition ensuring the stability of the trivial solution. Although it is possible to perform several constructions of those Lyapunov functionals (see [14] for more details) we will only illustrate this method with one of them here.

Theorem 4.6. *(*[14]*) Suppose that operators in* (36) *satisfy the conditions*

$\langle\tilde{A}(t,u),u\rangle \leq -\gamma\|u\|^2, \quad \gamma>0$

$F : V \to V'$, $\|F(u)\|_* \leq \alpha \|u\|$, $u \in V$
$\rho(t) \in [0,h]$, $\dot{\rho}(t) \leq \rho_* < 1$

Then the trivial solution of Eq. (36) *is exponentially stable provided*

$$\gamma > \frac{\alpha}{\sqrt{1-\rho_*}}. \tag{37}$$

Proof. We only include a sketchy proof. According to the procedure, let us consider the auxiliary equation

$$\frac{d}{dt}y(t) = A(t,y(t)). \tag{38}$$

It is easy to see that the function $v(t,y) = e^{\lambda t}|y|^2$, $\lambda > 0$, is Lyapunov function for (38), i.e., it satisfies the conditions of Theorem 4.5. Besides, since $\gamma > 0$, for any fixed β there exists $\lambda > 0$ such that $2\gamma > \lambda\beta^2$, which allows us to obtain

$$\begin{aligned} \frac{d}{dt}v(t,y(t)) &= \lambda e^{\lambda t}|y(t)|^2 + 2e^{\lambda t}\langle A(t,y(t)), y(t)\rangle \\ &\leq -e^{\lambda t}(2\gamma - \lambda\beta^2)\|y(t)\|^2 \leq 0. \end{aligned}$$

We now construct a Lyapunov functional V for Eq. (36) in the form $V = V_1 + V_2$, where $V_1(t,u_t) = e^{\lambda t}|u(t)|^2$. For Eq. (36) we obtain

$$\begin{aligned} \frac{d}{dt}V_1(t,u_t) &= \lambda V_1(t,u_t) + 2e^{\lambda t}\langle A(t,u(t)) + F(u(t-\rho(t))), u(t)\rangle \\ &\leq e^{\lambda t}\left[\lambda|u(t)|^2 + 2\left(-\gamma\|u(t)\|^2 + \alpha\|u(t-\rho(t))\|\|u(t)\|\right)\right] \\ &\leq e^{\lambda t}\left[\lambda\beta^2\|u(t)\|^2 - 2\gamma\|u(t)\|^2 + \alpha\left(\varepsilon\|u(t-\rho(t))\|^2 + \frac{1}{\varepsilon}\|u(t)\|^2\right)\right] \\ &= e^{\lambda t}\left[\left(\lambda\beta^2 - 2\gamma + \frac{\alpha}{\varepsilon}\right)\|u(t)\|^2 + \varepsilon\alpha\|u(t-\rho(t))\|^2\right]. \end{aligned}$$

Set now

$$V_2(t,u_t) = \frac{\varepsilon\alpha}{1-\rho_*}\int_{t-\rho(t)}^t e^{\lambda(s+\rho_0)}\|u(s)\|^2 ds.$$

Then

$$\begin{aligned} \frac{d}{dt}V_2(t,u_t) &= \frac{\varepsilon\alpha}{1-\rho_*}\left(e^{\lambda(t+\rho_0)}\|u(t)\|^2 - (1-\dot{\rho}(t))e^{\lambda(t-\rho(t)+\rho_0)}\|u(t-\rho(t))\|^2\right) \\ &\leq \frac{\varepsilon\alpha e^{\lambda t}}{1-\rho_*}\left(e^{\lambda\rho_0}\|u(t)\|^2 - (1-\rho_*)e^{\lambda(\rho_0-\rho(t))}\|u(t-\rho(t))\|^2\right) \\ &\leq \varepsilon\alpha e^{\lambda t}\left(\frac{e^{\lambda\rho_0}}{1-\rho_*}\|u(t)\|^2 - \|u(t-\rho(t))\|^2\right). \end{aligned}$$

Thus, for $V = V_1 + V_2$ we have

$$\frac{d}{dt}V(t,u_t) \leq \left[\lambda\beta^2 - 2\gamma + \alpha\left(\frac{1}{\varepsilon} + \frac{\varepsilon e^{\lambda\rho_0}}{1-\rho_*}\right)\right]e^{\lambda t}\|u(t)\|^2.$$

Rewriting the expression in square brackets as

$$-2\gamma + \alpha\left(\frac{1}{\varepsilon} + \frac{\varepsilon}{1-\rho_*}\right) + \lambda\beta^2 + \varepsilon\alpha\frac{e^{\lambda\rho_0}-1}{1-\rho_*},$$

and choosing $\varepsilon = \sqrt{1-\rho_*}$, we obtain

$$\frac{d}{dt}V(t,u_t) \leq -\left[2\left(\gamma - \frac{\alpha}{\sqrt{1-\rho_*}}\right) - h(\lambda)\right]e^{\lambda t}\|u(t)\|^2 \tag{39}$$

with
$$h(\lambda)=\lambda\beta^2+\alpha\frac{e^{\lambda\rho_0}-1}{\sqrt{1-\rho_*}}.$$
Since $h(0)=0$, thanks to (37) there exists small enough $\lambda>0$ such that
$$2\left(\gamma-\frac{\alpha}{\sqrt{1-\rho_*}}\right)\geq h(\lambda).$$
It then follows directly from (39) that $\frac{d}{dt}V(t,u_t)\leq 0$, and the functional $V(t,u_t)$ constructed above satisfies the conditions in Theorem 4.5. □

□

Note that in particular, if $\rho(t)\equiv\rho_0$ then $\rho_*=0$ and condition (37) takes the form $\gamma>\alpha$.

Assume that the delay term is given by
$$g(t,u_t)=G(u(t-\rho(t))),$$
where $G:V\to V'$ is a Lipschitz continuous operator with Lipschitz constant $L_G>0$ and such that $G(0)=0$, and the delay function $\rho(t)$ satisfies the assumptions in Theorem 4.6. We will analyze the stability of the trivial solution for our Navier-Stokes model (1), which means that $f(t)\equiv 0$. Then, (1) can be set in this abstract formulation by simply letting $\tilde{A}(t,\cdot),F:V\to V^*$ be the operators defined as
$$\tilde{A}(t,u)=-\nu a(u,\cdot)-b(u,u,\cdot),\quad F(u)=G(u),\qquad u\in V.$$
It is not difficult to check that conditions in Theorem 4.6 hold true provided $\nu>L_G$. Then there exists a unique solution to this problem, which in addition satisfies $u\in C^0(0,T;H)$ for any $T>0$. When G maps V or H into H (i.e., G may contain partial derivatives up to first or zero order), the results in the two previous sections (see also Theorems 3.3 and 3.5 in [12]) guarantee the exponential stability of the trivial solution provided the viscosity parameter ν is large enough. For instance, in the case that G maps H into H, the null solution (which is now the unique stationary solution of our problem) is exponentially stable if
$$2\nu\lambda_1>\frac{(2-\rho_*)L_G}{1-\rho_*},\tag{40}$$
where λ_1 denotes the first eigenvalue of the Stokes operator (see also Corollary 3.7 in [7] for another sufficient condition when G maps U into H). However, the results obtained in [7] do not cover the more general situation in which G may contain second order partial derivatives. This motives our further consideration of this problem.

In the case when the operator $G:V\to V'$ and the function $g(t,u_t)=G(u(t-\rho(t)))$ are defined as above, we have that $\gamma=\nu,\alpha=L_G,\beta=\lambda_1^{-1/2}$ and assumptions in Theorem 4.6 hold assuming that
$$\nu>\frac{L_G}{\sqrt{1-\rho_*}}.$$
Remark 4. Observe that if $G:H\to H$ with Lipschitz constant L_{GH} then G is also Lipschitz continuous from V into V' and the Lipschitz constants are related by $L_G\leq\lambda_1^{-1}L_{GH}$. Thus if we assume that
$$\nu\lambda_1>\frac{L_{GH}}{\sqrt{1-\rho_*}}\tag{41}$$
it also follows that
$$\nu>\frac{L_G}{\sqrt{1-\rho_*}},$$
and consequently we have the exponential stability of the trivial solution. It is worth mentioning that condition (41) improves the condition established in [12], which is (40).

4.5. **Exponential stability for measurable delays via a Gronwall-like lemma.** In this subsection we will analyze the exponential stability of stationary solutions to the Navier-Stokes model with variable delays, by using a technique based on the application of a Gronwall-like lemma. As we will state below, in this case the only assumption that has to be imposed on the delay function is measurability, which is the weakest comparing with the ones imposed in the three previous methods. The results that we will state below can be considered as a particular case of a stochastic version of the Navier-Stokes model analyzed in [16].

To illustrate this technique in a more accessible way, we will consider our model (1) with $f(t) = 0$ and $g(t, u_t) = G(u(t - \rho(t)))$, where $G : H \to H$ is Lipschitz continuous as in the previous section with Lipschitz constant $L_G H > 0$ and $G(0) = 0$. For the delay function ρ we only assume that it is measurable and bounded, i.e., $\rho : [0, +\infty) \to [0, h]$.

Next we first introduce the Gronwall-like lemma which is crucial to prove the stability result.

Lemma 4.7. *(*[16, Lemma 3.2]*) Let $y(\cdot) : [-h, +\infty) \to [0, +\infty)$ be a function. Assume that there exist positive numbers γ, α_1 and α_2 such that the following inequality holds:*

$$y(t) \le \begin{cases} \alpha_1 e^{-\gamma t} + \alpha_2 \displaystyle\int_0^t e^{-\gamma(t-s)} \sup_{\theta \in [-h,0]} y(s+\theta)\, ds, \ t \ge 0, \\ \alpha_1 e^{-\gamma t}, \ t \in [-h, 0]. \end{cases} \tag{42}$$

Then,

$$y(t) \le \alpha_1 e^{-\mu t}, \text{ for } t \ge -h,$$

where $\mu \in (0, \gamma)$ is giving by the unique root of the equation $\frac{\alpha_2}{\gamma - \mu} e^{\mu h} = 1$ in this interval.

Now we can state the stability result as in next theorem.

Theorem 4.8. *Assume that $f(t) = 0$ and $g(t, u_t) = G(u(t - \rho(t)))$, where $G : H \to H$ is Lipschitz continuous with Lipschitz constant $L_{GH} > 0$ and satisfies $G(0) = 0$. Assume also that $\rho : [0, +\infty) \to [0, h]$ is a measurable function. Then the null solution of model* (1) *is exponentially stable provided*

$$2\nu\lambda_1 > L_{GH}.$$

Proof. Let us first choose a positive constant $\lambda > 0$ such that

$$\lambda - 2\nu\lambda_1 + L_{GH} > 0. \tag{43}$$

For $t \geq 0$, taking into account (43), the weak solution $u(\cdot)$ to model (1) corresponding to the initial datum ψ satisfies

$$\begin{aligned} e^{\lambda t}|u(t)|^2 &= |\psi(0)|^2 + \lambda \int_0^t e^{\lambda s}|u(s)|^2\,ds - 2\int_0^t e^{\lambda s}\langle \nu A u(s), u(s)\rangle\,ds \\ &\quad +2\int_0^t e^{\lambda s}(G(u(s-\rho(s))), u(s))\,ds \\ &\leq |\psi(0)|^2 + \lambda \int_0^t e^{\lambda s}|u(s)|^2\,ds - 2\nu\int_0^t e^{\lambda s}\|u(s)\|^2\,ds \\ &\quad +2L_{GH}\int_0^t e^{\lambda s}|u(s-\rho(s))||u(s)|\,ds \\ &\leq |\psi(0)|^2 + \lambda \int_0^t e^{\lambda s}|u(s)|^2\,ds - 2\nu\lambda_1\int_0^t e^{\lambda s}|u(s)|^2\,ds \\ &\quad +L_{GH}\int_0^t e^{\lambda s}|u(s)|^2\,ds + L_{GH}\int_0^t e^{\lambda s}|u(s-\rho(s))|^2\,ds \\ &\leq |\psi(0)|^2 + (\lambda - 2\nu\lambda_1 + L_{GH})\int_0^t e^{\lambda s}|u(s)|^2\,ds \\ &\quad +L_{GH}\int_0^t e^{\lambda s}|u(s-\rho(s))|^2\,ds \\ &\leq \|\psi\|^2_{C^0([-h,0];H)} + (\lambda - 2\nu\lambda_1 + 2L_{GH})\int_0^t e^{\lambda s}\sup_{\theta\in[-h,0]}|u(s+\theta)|^2\,ds. \end{aligned}$$

The result then follows from a direct application of Lemma 4.7. □ □

Remark 5. (a) The result can also be extended to cover the case in which the operator G is defined from H into V but assuming $\nu\lambda_1 > L_{GH}$ instead of $2\nu\lambda_1 > L_{GH}$. However, we still do not know if a general term $G : V \to V'$ can be handled with this method.

(b) Comparing with the result obtained by the method of Lyapunov functionals, the result obtained by the Gronwall approach is better, as the condition imposed on G is independent of any delay parameter and allow for larger range of other parameters ν and L_{GH}. This can be easily checked by comparing with condition (41). In the case $\rho_* = 0$, the condition required by this technique is $2\nu\lambda_1 > L_{GH}$ while the one required by (41) is $\nu\lambda_1 > L_{GH}$. In the case in which $G : H \to V'$ both conditions required are exactly the same.

5. **Conclusions, comments and future directions.** In this survey paper we have described several methods which can be used to analyze the local stability property of two dimensional Navier-Stokes models when some hereditary properties are taken into account in the model. Our analysis has been focused in the the case when a delay term appears in the forcing term. However, a more global analysis has been carried out in a series of research papers over the last decades for the cases with or without delays.

In the case of constant delays, the autonomous theory of global attractor may provide an appropriate framework to study the problem. But when one need to consider a more general delay term, such as variable or distributed delays, the problem becomes non-autonomous and it is necessary to develop a non-autonomous scheme for the long time behavior of the model. Several options are available, but we would like to emphasize the theory of pullback attractors, which has been widely developed over the last two decades (see [1], [29], [30], [32] and references therein).

For the two-dimensional Navier-Stokes model with delay, the analysis of the existence of pullback attractor was initiated in the paper [13], with general delays considered in the model. Many other works have been done later on, providing more information on the regularity of such an attractor (e.g., [24], [23]). Also there are other works dealing with the cases in which the delay appears not only in the forcing terms but also in the convective and diffusion terms (see [3], [25], [21]) and at the same time, there exist also several papers in which the delay is allowed to be unbounded (see [37], [22]).

Nevertheless, there are still many interesting problems related to Navier-Stokes models to be analyzed in future. One is related to the spatial discretization of the model which yields a lattice dynamical system, that can be used in the approximation of the solutions to our model. Recent results on the theory of lattice dynamical systems with or without delays can be applied to the analysis of the lattice dynamical systems generated by the Navier-Stokes models ([27, 28], [46], [9, 10]). In particular, for lattice Navier-Stokes system we can also prove the existence and uniqueness of solutions, construct Lyapunov functionals to obtain the stability of steady state solutions, etc.

When environmental noise is taken into account, the Navier-Stokes model become stochastic. Several variants of the stochastic Navier-Stokes model with delay and memory have already been studied (see, e.g. [16], [45]) from a local stability point of view. Yet there is still much research to be discovered in this field, such as the existence of random attractors as well as the properties and structures of random attractors. Similar studies can also be carried out for stochastic lattice Navier-Stokes models with delays.

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