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Cohomology of some Nilpotent Lie Algebras *

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1. Introduction

In the theory of Lie algebras, algebra of derivations constitutes an essential tool in the resolution of lots of problems. These problems are the classification of Lie algebras [3], the description of the first space of cohomology, or the study of problems linked to physics (e.g the counterexamples to the Gel'fand-Kirillov Conjecture [1], [2]).

But, to determine the algebra of derivations associated to a Lie algebra involved many calculations that can be simplified for using of the software *Mathematica*. This software have been used earlier by some authors in order to solve problems into Lie algebras, (see, e.g., [4], [6], [8]).

For the first time, M. Vergne did a cohomological study of the variety of nilpotent Lie algebras, to see [10]. In the above paper, the classification of the filiform Lie algebras "naturally" graded plays a decisive role. In some sense, these algebras are the basic structure of the filiform Lie algebras (those of maximal nilindex, $n-1$). Thus, knowledge of the naturally graded algebras of a some class of Lie algebras give an important information about the structure of all the class of algebras [7]. This allows easily to determine the algebra of derivations, and so some different geometric elements as, for example, the description of the first space of cohomology or as the determination of the dimensions of the orbits.

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Gómez and Jiménez-Merchán [5], [9] give the classification of naturally graded algebras of quasi-filiform Lie algebras (those of nilindex $n-2$). This authors extend the results obtained by Vergne. And thus, our goal is to show a cohomological study of the naturally graded quasi-filiform Lie algebras by determining the corresponding algebra of derivations.

2. Preliminary

Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{K}$. A linear endomorphism d of $\mathfrak{g}$ is called a *derivation* of $\mathfrak{g}$ if it satisfies

$$d([X,Y]) = [d(X),Y] + [X,d(Y)]\,, \qquad \forall\, X,Y \in \mathfrak{g}\,. \tag{1}$$

It is easy to see that the set $\mathcal{D}er(\mathfrak{g})$ of all derivations of $\mathfrak{g}$ is a Lie algebra over $\mathbb{K}$ for the bracket $[d^1,d^2] = d^1 \circ d^2 - d^2 \circ d^1$.

If $X \in \mathfrak{g}$, the endomorphism of $\mathfrak{g}$ defined by $Y \longrightarrow [X,Y]$ is denoted $ad(X)$. The Jacobi identity implies that $ad(X)$ is a derivation of $\mathfrak{g}$ for all $X \in \mathfrak{g}$. This derivations are called inner derivation. The set $\mathcal{A}d(\mathfrak{g})$ of inner derivations of $\mathfrak{g}$ is an ideal of $\mathcal{D}er(\mathfrak{g})$. Next, we show a theorem that permits to express easily the space of derivations of any Lie algebra as a direct sum of others Lie algebras.

Theorem 2.1. ([8]) *If $\mathfrak{g}$ is a direct sum $\mathfrak{g} = \bigoplus_{i=1}^{k} \mathfrak{g}_i$ of Lie algebras over $\mathbb{K}$, then*

$$\mathcal{D}er(\mathfrak{g}) = \left(\bigoplus_{i=1}^{k} \mathcal{D}er(\mathfrak{g}_i)\right) \oplus \left(\bigoplus_{i\neq j} \mathcal{D}(\mathfrak{g}_i,\mathfrak{g}_j)\right),$$

and $\mathcal{D}(\mathfrak{g}_i,\mathfrak{g}_j)$ denote the set of all derivations d_{ij} of $\mathfrak{g}$ satisfying $d_{ij}(\mathfrak{g}_p) = 0$ if $p \neq i$, $d_{ij}(\mathfrak{g}_i) \subset \mathcal{Z}(\mathfrak{g}_j)$ and $d_{ij}([\mathfrak{g}_i,\mathfrak{g}_i]) = 0$.

Let $\mathfrak{g}$ be a complex Lie algebra. Then, $\mathfrak{g}$ is naturally filtered by the descending central sequence

$$\mathcal{C}^i(\mathfrak{g}) = \mathfrak{g}\,, \quad i \leq 0\,, \qquad \mathcal{C}^i(\mathfrak{g}) = [\mathfrak{g}, \mathcal{C}^{i-1}(\mathfrak{g})]\,, \quad i \geq 1\,.$$

This allows us to naturally associate with any nilpotent Lie algebra with nilindex $k = \inf\{i \in \mathbb{N} : \ \mathcal{C}^i(\mathfrak{g}) = \{0\}\}$ a graded Lie algebra, denoted by $\operatorname{gr}\mathfrak{g}$, with the same nilindex, defined by

$$\operatorname{gr}\mathfrak{g} = \bigoplus_{i\in Z} \mathfrak{g}_i\,, \qquad \mathfrak{g}_i = \mathcal{C}^{i-1}(\mathfrak{g})/\mathcal{C}^i(\mathfrak{g})\,.$$

Because of the nilpotence, the graduation is finite, i.e.,

$$\operatorname{gr}\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_k$$

with $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$, for $i+j \leq k$, verifying that $\dim(\mathfrak{g}_1) \geq 2$ and $\dim(\mathfrak{g}_i) \geq 1$, for $2 \leq i \leq k$. Lie algebra $\mathfrak{g}$ is said to be *naturally graded* if $\operatorname{gr}\mathfrak{g}$ is isomorphic to $\mathfrak{g}$, which from now on will be noted by $\operatorname{gr}\mathfrak{g} = \mathfrak{g}$. Examples of these algebras are as follows.

Let be $\mathcal{L}_n$ the Lie algebra defined in the basis $\{X_0, X_1, \ldots, X_{n-1}\}$ by

$$\mathcal{L}_n : \{[X_0, X_i] = X_{i+1} \qquad 1 \leq i \leq n-2\,.$$

Let be $\mathcal{Q}_n$ the Lie algebra defined in the basis $\{X_0, X_1, \ldots, X_{n-1}\}$, with $n = 2q$, by

$$\begin{cases} [X_0, X_i] = X_{i+1} & 1 \leq i \leq n-2 \\ [X_i, X_{n-1-i}] = (-1)^{i-1} X_{n-1} & 1 \leq i \leq q-1\ . \end{cases}$$

In [10], M. Vergne proves that if $\mathfrak{g}$ is naturally graded filiform Lie algebra with dimension n is either isomorphic to $\mathcal{L}_n$, if n is odd, or isomorphic to $\mathcal{L}_n$ or $\mathcal{Q}_n$, if n is even.

In a recent work [5], Gómez and Jiménez-Merchán show that any naturally graded quasi-filiform Lie algebra is isomorphic to one of the following

$$\mathcal{L}_{n-1} \bigoplus \mathbb{C}\,, \qquad \mathcal{Q}_{n-1} \bigoplus \mathbb{C}\,,$$

$$\mathcal{L}_{(n,r)} = \begin{cases} [X_0, X_i] = X_{i+1} & 1 \leq i \leq n-3 \\ [X_i, X_{r-i}] = (-1)^{i-1} Y & 1 \leq i \leq \frac{r-1}{2} \end{cases}$$

with $n \geq 5$, $3 \leq r \leq n-2$ and r odd,

$$\mathcal{Q}_{(n,r)} = \begin{cases} [X_0, X_i] = X_{i+1} & 1 \leq i \leq n-3 \\ [X_i, X_{r-i}] = (-1)^{i-1} Y & 1 \leq i \leq \frac{r-1}{2} \\ [X_i, X_{n-2-i}] = (-1)^{i-1} X_{n-2} & 1 \leq i \leq \frac{n-3}{2} \end{cases}$$

with $n \geq 7$, n odd, $3 \leq r \leq n-4$ and r odd,

$$\mathcal{T}_{(n,n-3)} = \begin{cases} [X_0, X_i] = X_{i+1} & 1 \leq i \leq n-3 \\ [Y, X_1] = \frac{n-4}{2} X_{n-2} & \\ [X_i, X_{n-3-i}] = (-1)^{i-1}(X_{n-3} + Y) & 1 \leq i \leq \frac{n-4}{2} \\ [X_i, X_{n-2-i}] = \frac{n-2-2i}{2}(-1)^{i-1} X_{n-2} & 1 \leq i \leq \frac{n-4}{2} \end{cases}$$

with $n \geq 6$, n even,

$$\mathcal{T}_{(n,n-4)} = \begin{cases} [X_0, X_i] = X_{i+1} & 1 \leq i \leq n-3 \\ [Y, X_i] = \frac{n-5}{2} X_{n-4+i} & 1 \leq i \leq 2 \\ [X_i, X_{n-4-i}] = (-1)^{i-1}(X_{n-4} + Y) & 1 \leq i \leq \frac{n-5}{2} \\ [X_i, X_{n-3-i}] = \frac{n-3-2i}{2}(-1)^{i-1} X_{n-3} & 1 \leq i \leq \frac{n-5}{2} \\ [X_i, X_{n-2-i}] = \frac{n-3-i}{2}(-1)^{i}(i-1) X_{n-2} & 2 \leq i \leq \frac{n-3}{2} \end{cases}$$

with $n \geq 7$, n odd.

For convenience, the adapted basis for 2-filiform Lie algebras will be denoted by $\{X_0, X_1, \ldots, X_{n-2}, Y\}$.

The space $H^1(\mathfrak{g}, \mathfrak{g})$ can be interpreted as the space of the "outer" derivations of the Lie algebra $\mathfrak{g}$. Thus, the quotient space $\mathcal{D}er(\mathfrak{g})/\mathcal{A}d(\mathfrak{g})$ is the space $H^1(\mathfrak{g}, \mathfrak{g})$. In this way, the dimension of the orbit $\mathcal{O}(\mathfrak{g})$ can be given by

$$\dim(\mathcal{O}(\mathfrak{g})) = n^2 - \dim(\mathcal{D}er(\mathfrak{g}))\,.$$

In the next section, we give the above cohomological elements for the naturally graded quasi-filiform Lie algebras by determining the algebra of derivations of those algebras.

In the last section, we show a program using the language of symbolic calculus *Mathematica*. This software allows us to compute the space of derivations for the mentioned algebras in concrete dimensions. These results lead to conjecture the structure of such space of derivations in generic dimension, which is our first goal in this paper.

3. Cohomology of Quasi-filiform Lie Algebras

If $\mathfrak{g}$ is a graded quasi-filiform algebra of dimension n, we have the following decomposition

$$t_r = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_r \oplus \cdots \oplus \mathfrak{g}_{n-2}$$

with $\mathfrak{g}_1 = \langle X_0, X_1 \rangle$, $\mathfrak{g}_i = \langle X_i \rangle$ for $2 \leq i \leq n-2$ and $i \neq r$, and $\mathfrak{g}_r = \langle X_r, Y \rangle$, verifying that $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$, for $i + j \leq n-2$.

Theorem 3.1. *Let $ad(X_i)$, $0 \leq i \leq n-3$, $h_3, h_5, \ldots, h_{r-4}, h_{r-2}, h_{r-1}, \ldots,$ $h_{n-4}, h_{n-3}, t_0, t_1, t_2, g_{r-1}, f_{r-1}$ (and t_3, only for $n = 5$) be the endomorphisms* *of* $\mathcal{L}_{(n,r)}$ *defined by*

$$\begin{aligned}
h_k(X_i) &= X_{k+i} \quad (1 \le i \le n-2-k) \quad \textit{if } (k > r-3) \\
&\qquad\qquad\qquad\qquad \textit{or } (k \le r-3 \textit{ and } k \ne \dot{2})\,; \\
t_0(X_0) &= X_0\,, \quad t_0(X_i) = (i-1)X_i \ \ (1 \le i \le n-2)\,, \quad t_0(Y) = (r-2)Y\,; \\
t_1(X_0) &= X_1\,, \quad t_1(X_r) = Y\,; \\
t_2(X_i) &= X_i \quad (1 \le i \le n-2)\,, \quad t_2(Y) = 2Y\,; \\
t_3(X_1) &= X_0\,, \quad t_3(Y) = X_3\,, \quad \textit{only for } n = 5\,; \\
g_{r-1}(X_0) &= X_r\,; \\
f_{r-1}(X_0) &= Y\,.
\end{aligned}$$

Then, the above endomorphisms form a basis of $\mathcal{D}er(\mathcal{L}_{(n,r)})$. *Thus,*

$$\dim(\mathcal{D}er(\mathcal{L}_{(n,r)})) = \begin{cases} \frac{4n+1-r}{2} & \textit{if } n > 5 \\ 10 & \textit{if } n = 5\,. \end{cases}$$

Proof. It is an easy calculation verifying that the above endomorphisms are independent and belong to $\mathcal{D}er(\mathcal{L}_{(n,r)})$. Remain to prove that these derivations span to the whole space.

Any derivation $d \in \mathcal{D}er(\mathcal{L}_{(n,r)})$ can be expressed by $d = \sum_{3-n}^{n-3} d_i$, with $d_i(\mathfrak{g}_j) \subset \mathfrak{g}_{i+j}$, $1 \le j \le n-2$.

Note that the ideals $\mathcal{C}^i(\mathcal{L}_{(n,r)})$ are conserved by the derivations; these ideals are characteristic. Thus, in the formal decomposition of d, $d = \bigoplus_{i \in \mathbb{Z}} d_i$, we have $d_k = 0$ for $k \le -1$. So $d = d_0 + d_1 + \cdots + d_{n-3}$.

We are going to determine, recursively, the subspaces of derivations d_0, d_1, $\ldots$, d_{n-3}.

- Consider $d_0 \in \mathcal{D}er(\mathcal{L}_{(n,r)})$. It obvious that

$$\begin{cases} d_0(X_k) = \begin{cases} \alpha_k X_0 + \beta_k X_1 & \text{if } k \in \{0,1\} \\ \alpha_k X_k + \delta_k^r \beta_r Y & \text{if } 2 \le k \le n-2 \end{cases} \\ \\ d_0(Y) = \alpha_{n-1} X_r + \beta_{n-1} Y \end{cases}$$

for some $\alpha_0, \ldots, \alpha_{n-1}, \beta_0, \beta_1, \beta_r, \beta_{n-1} \in \mathbb{K}$. It follows from the equality

$$d_0[a,b] - [d_0(a), b] - [a, d_0(b)] = 0$$

for $(a,b) = (X_0, X_k)$ with $1 \le k \le n-3$, that $\beta_r = \beta_0$ and $\alpha_{k+1} = k\alpha_0 + \beta_1$. This same equation, with $(a,b) = (X_1, X_{r-1})$, $(a,b) = (X_i, X_{r-i})$ for $2 \le i \le$

$(r-1)/2$, yields $\alpha_{n-1} = \alpha_1$, $\beta_{n-1} = (r-2)\alpha_0 + 2\beta_1$ and $\alpha_1 = 0$ if $n > 5$ and α_1 free if $n = 5$.

Considering the values $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$ for the triple $(\alpha_0, \beta_0, \beta_1)$, we arrive at t_0, t_1 and t_2, respectively, if $n > 5$ and $(\alpha_0, \alpha_1, \beta_0, \beta_1)$ if $n = 5$. Thus, we have

$$d_0 = t_0\alpha_0 + t_1\beta_0 + t_2\beta_1 + \delta_n^5 t_3\alpha_1$$

so $\dim(d_0) = 3$ if $n > 5$ and $\dim(d_0) = 4$ if $n = 5$.

• In the same way, we can obtain d_k, $1 \le k \le n-3$ and $k \ne r-1$. The generic expression for the values non null of d_k can be given by

$$\begin{cases} d_k(X_i) = \begin{cases} \alpha_i X_{k+1} & \text{if } i \in \{0,1\} \\ \alpha_i X_{i+k} + \delta_i^{r-k}\beta_{r-k} Y & \text{if } 2 \le i \le n-2-k \end{cases} \\ d_k(Y) = \alpha_{n-1} X_{r+k} \qquad \text{if } \ 1 \le k \le n-2-r. \end{cases}$$

Using the equality (1) for the pairs (X_0, X_i) with $1 \le i \le n-2-k$, and (X_1, X_{r-1}) we obtain that

$$\begin{aligned} \alpha_i &= \alpha_1 && 2 \le i \le n-2-k \\ \beta_{r-k} &= (-1)^k \alpha_0 && \\ \alpha_{n-1} &= 0 && \\ \alpha_1 &= 0 && \text{if } \ k = \dot{2} \ \text{ and } k \le r-3 \\ \alpha_1 & \quad \text{free} && \text{if } \ k \ne \dot{2} \ \text{ and } \ k \le r-3 \text{ or } k > r-3 \ . \end{aligned}$$

Considering the values $(1,0)$ and $(0,1)$ for the vector (α_0, α_1), we arrive at $ad(X_0)$ and $ad(X_1)$ for $k = 1$, and $ad(X_k)$, and h_k for $k \ne \dot{2}$ and $k \le r-3$ or $k > r-3$. If $k = \dot{2}$ and $k \le r-3$ we only obtain $ad(X_k)$. So,

$$\dim(d_k) = \begin{cases} 1 & \text{if } \ k = \dot{2} \ \text{ and } k \le r-3 \\ 2 & \text{if } \ k \ne \dot{2} \ \text{ and } \ k \le r-3 \text{ or } k > r-3 \end{cases}$$

with $1 \le k \le n-4$, $k \ne r-1$.

• Finally, we obtain d_{r-1}. In this case we have the following non null values for the subspace of derivations d_{r-1}

$$\begin{cases} d_{r-1}(X_i) = \begin{cases} \alpha_i X_r + \beta_i Y & \text{if } i \in \{0,1\} \\ \alpha_i X_{i+r-1} & \text{if } 2 \le i \le n-r-1 \end{cases} \\ d_{r-1}(Y) = \alpha_{n-1} X_{2r-1} \qquad \text{if } 3 \le r \le \frac{n-1}{2}. \end{cases}$$

As the pairs (X_0, X_i) with $1 \leq i \leq n-1-r$ and (X_1, X_{r-1}) have to verify (1), then we have that

$$\alpha_{n-1} = 0$$
$$\alpha_i = \alpha_1 \qquad 2 \leq i \leq n-1-r \ .$$

Now, if we consider the values $(1,0,0,0)$, $(0,1,0,0)$, $(0,0,1,0)$, $(0,0,0,1)$ for the vector $(\alpha_0, \alpha_1, \beta_0, \beta_1)$ we get $ad(X_{r-1})$, h_{r-1}, g_{r-1} and f_{r-1}, respectively. Thus, $\dim(d_{r-1}) = 4$.

• Computation of d_{n-3}. From $d_{n-3} \in \mathcal{D}er(\mathcal{L}_{(n,r)})$ and $d_{n-3}(\mathfrak{g}_i) \subset \mathfrak{g}_{i+n-3}$, we obtain that

$$\begin{cases} d_{n-3}(X_i) = \begin{cases} \alpha_0 X_{n-2} & \text{if } i = 0 \\ \alpha_1 X_{n-2} & \text{if } i = 1 \\ 0 & \text{if } 2 \leq i \leq n-2 \end{cases} \\ d_{n-3}(Y) = 0. \end{cases}$$

Substituting the vector (α_0, α_1) by the values $(1,0)$ and $(0,1)$ we lead to $ad(X_{n-3})$ and h_{n-3}. Thus $\dim(d_{n-3}) = 2$.

All precedents results lead to determine the dimension and a basis of $\mathcal{D}er(\mathcal{L}_{(n,r)})$, being

$$\dim(\mathcal{D}er(\mathcal{L}_{(n,r)})) = \sum_{i=0}^{n-3} \dim(d_i) = \begin{cases} \frac{4n+1-r}{2} & \text{if } n > 5 \\ 10 & \text{if } n = 5. \end{cases}$$ ∎

And thus, the knowing of the space $\mathcal{D}er(\mathcal{L}_{(n,r)})$ lead to determine the dimensions of the spaces $H^1(\mathcal{L}_{(n,r)}, \mathcal{L}_{(n,r)})$ and $\mathcal{O}(\mathcal{L}_{(n,r)})$.

Corollary 3.2. (i) *The linear maps* $h_3, \ldots, h_{r-4}, h_{r-2}, h_{r-1}, \ldots, h_{n-4}, h_{n-3}, t_0, t_1, t_2, g_{r-1}, f_{r-1}$ *(and* t_3 *if* $n = 5$*) module* $\mathcal{A}d(\mathcal{L}_{(n,r)})$*, form a basis of* $H^1(\mathcal{L}_{(n,r)}, \mathcal{L}_{(n,r)})$*. Thus,*

$$\dim(H^1(\mathcal{L}_{(n,r)}, \mathcal{L}_{(n,r)})) = \begin{cases} \frac{2n-r+5}{2} & \text{if } n > 5 \\ 7 & \text{if } n = 5. \end{cases}$$

(ii)

$$\dim(\mathcal{O}(\mathcal{L}_{(n,r)})) = \begin{cases} \frac{2n^2-4n+r-1}{2} & \text{if } n > 5 \\ 15 & \text{if } n = 5. \end{cases}$$

Proof. In fact

$$H^1(\mathfrak{g},\mathfrak{g}) = Z^1(\mathfrak{g},\mathfrak{g})/B^1(\mathfrak{g},\mathfrak{g}) = \mathcal{D}er(\mathfrak{g})/\mathcal{A}d(\mathfrak{g}),$$
$$\dim(H^1(\mathcal{L}_{(n,r)},\mathcal{L}_{(n,r)})) = \dim(\mathcal{D}er(\mathcal{L}_{(n,r)})) - \dim(\mathcal{A}d(\mathcal{L}_{(n,r)})),$$
$$\dim(\mathcal{O}(\mathcal{L}_{(n,r)})) = n^2 - \dim(\mathcal{D}er(\mathcal{L}_{(n,r)}).$$

∎

In the same manner, we obtain a basis for the space $\mathcal{D}er(\mathcal{Q}_{(n,r)}$.

THEOREM 3.3. *Let $ad(X_i)$, $0 \le i \le n-3$, h_3, h_5, $\dots$, h_{n-4}, h_{n-3}, t_0, t_1, g_{r-1}, f_{r-1} (and h_{r-1} if $r = \frac{n-1}{2}$ or h_{n-2-r} if $r \le \frac{n-3}{2}$) be the endomorphisms of $\mathcal{Q}_{(n,r)}$ defined by*

$$\begin{aligned}
h_k(X_i) &= X_{k+i} \quad (1 \le i \le n-2-k,\ 3 \le k \le n-4,\ k \ne \dot{2}) \\
&\qquad\qquad \text{and } (k = n-3); \\
t_0(X_0) &= X_0, \quad t_0(X_i) = iX_i \quad (1 \le i \le n-2), \quad t_0(Y) = rY; \\
t_1(X_0) &= X_1, \quad t_1(X_i) = X_i \quad (1 \le i \le n-3), \quad t_1(X_r) = X_r + Y, \\
&\qquad t_1(X_{n-2}) = 2X_{n-2}, \quad t_1(Y) = 2Y; \\
g_{r-1}(X_0) &= X_r, \quad g_{r-1}(X_{n-1-r}) = X_{n-2}; \\
f_{r-1}(X_0) &= Y; \\
h_{r-1}(X_i) &= X_{r-1+i} \quad (1 \le i \le n-1-r), \\
&\qquad h_{r-1}(Y) = 2X_{n-2} \quad (\text{only for } r = \frac{n-1}{2}); \\
h_{n-2-r}(X_i) &= X_{i+n-2-r} \quad (1 \le i \le r), \\
&\qquad h_{n-2-r}(Y) = 2X_{n-2} \quad (\text{only for } r \le \frac{n-3}{2}).
\end{aligned}$$

Then, the above endomorphisms form a basis of $\mathcal{D}er(\mathcal{Q}_{(n,r)})$. Thus,

$$\dim(\mathcal{D}er(\mathcal{Q}_{(n,r)})) = \begin{cases} \frac{3n+3}{2} & \text{if } r \le \frac{n-1}{2} \\ \frac{3n+1}{2} & \text{if } r > \frac{n-1}{2}. \end{cases}$$

Proof. Any derivation $d \in \mathcal{D}er(\mathcal{Q}_{(n,r)})$ can be expressed in the form $d = d_0 + d_1 + \cdots + d_{n-3}$ with $d_k \in \mathcal{D}er(\mathcal{Q}_{(n,r)})$, $0 \le k \le n-3$ and $d_k(\mathfrak{g}_i) \subset \mathfrak{g}_{i+k}$.

From the computation of the subspace of derivations d_0 and tacking into account the pairs (X_0, X_i) with $1 \le i \le n-3$, (X_1, X_3), (X_0, Y) and (X_i, X_{r-i})

with $2 \leq i \leq \frac{r-1}{2}$, (X_1, X_{n-3}) and (X_i, X_{n-2-i}) with $1 \leq i \leq \frac{n-3}{2}$ for (1), we obtain the following restrictions

$$\begin{aligned} \alpha_i &= i\alpha_0 + \beta_0 && 2 \leq i \leq n-3 \\ \alpha_{n-2} &= (n-2)\alpha_0 + 2\beta_0 \\ \alpha_{n-1} &= 0 \\ \beta_1 &= \alpha_0 + \beta_0 \\ \beta_r &= \beta_0 \\ \beta_{n-1} &= r\alpha_0 + 2\beta_0 \end{aligned}$$

and then, by considering the values $(1,0)$ and $(0,1)$ for the pair (α_0, β_0), we lead to t_0 and t_1 respectively, and so $\dim(d_0) = 2$.

By computation of d_k, for $1 \leq k \leq n-3$; $k \neq r-1$, and as precedent reasonings, we have

$$\begin{aligned} \alpha_i &= \alpha_1 && 2 \leq i \leq n-3-k \\ \alpha_{n-2-k} &= (-1)^k\alpha_0 + \alpha_1 \\ \beta_{r-k} &= (-1)^k\alpha_0 \\ (-1)^k\alpha_1 + \alpha_1 &= 0 && r-k \geq 3 \end{aligned}$$

$$k = n-2-r \begin{cases} r \geq \frac{n+1}{2} & \alpha_1 = 0 \\ r < \frac{n+1}{2} & \alpha_1 \text{ free.} \end{cases}$$

Now, if $k \neq \dot{2}$ or $k = n-2-r$ with $r \geq \frac{n+1}{2}$ we assign the values $(\alpha_0, \alpha_1) = (1,0)$ and $(\alpha_0, \alpha_1) = (0,1)$ for the parameters α_0 and α_1 obtaining a basis for the subspace of derivations d_k, that is, $ad(X_k)$ and h_k with $h_1 = ad(X_0)$. For $k = \dot{2}$ or $k = n-2-r$ with $r < \frac{n+1}{2}$ there is only one parameter (α_0) obtaining in this case the derivation $ad(X_k)$. So, we arrive at

$$\dim(d_k) = \begin{cases} 1 & \text{if } k = \dot{2} \text{ or } k = n-2-r, \ \text{ with } r \geq \frac{n+1}{2} \\ 2 & \text{if } k \neq \dot{2} \text{ or } k = n-2-r, \ \text{ with } r < \frac{n+1}{2} \end{cases}$$

with $1 \leq k \leq n-4$ and $k \neq r-1$.

In this way, we can consider at a basis for the subspace d_{r-1} as

$$\begin{cases} (ad(X_{r-1}), g_{r-1}, f_{r-1}) & \text{if } r \neq \frac{n-1}{2} \\ (ad(X_{r-1}), g_{r-1}, f_{r-1}, h_{r-1}) & \text{if } r = \frac{n-1}{2}. \end{cases}$$

Thus $\dim(d_{r-1}) = 3$ if $r \neq \frac{n-1}{2}$ and $\dim(d_{r-1}) = 4$ if $r = \frac{n-1}{2}$, and similarly, we obtain a basis of the space of derivations d_{n-3} forms by, $ad(X_{n-3})$ and h_{n-3}. Thus, $\dim(d_{n-3}) = 2$.

From all the precedent results, we conclude that

$$\dim(\mathcal{D}er(\mathcal{Q}_{(n,r)})) = \sum_{i=0}^{n-3} \dim(d_i) = \begin{cases} \frac{3n+3}{2} & \text{if } r \leq \frac{n-1}{2} \\ \frac{3n+1}{2} & \text{if } r > \frac{n-1}{2}. \end{cases} \qquad \blacksquare$$

And thus, the knowing of the space $\mathcal{D}er(\mathcal{Q}_{(n,r)})$ lead to determine the dimensions of the spaces $H^1(\mathcal{Q}_{(n,r)}, \mathcal{Q}_{(n,r)})$ and $\mathcal{O}(\mathcal{Q}_{(n,r)})$ respectively.

Corollary 3.4. (i) *The linear maps* $h_3, h_5, \ldots, h_{n-4}, h_{n-3}, t_0, t_1, g_{r-1}, f_{r-1}$ *(and* h_{r-1} *if* $r = \frac{n-1}{2}$ *or* h_{n-2-r} *if* $r \leq \frac{n-3}{2}$*), module* $\mathcal{A}d(\mathcal{Q}_{(n,r)})$ *form a basis of* $H^1(\mathcal{Q}_{(n,r)}, \mathcal{Q}_{(n,r)})$. *So,*

$$\dim(H^1(\mathcal{Q}_{(n,r)}, \mathcal{Q}_{(n,r)})) = \begin{cases} \frac{n+7}{2} & \text{if } r \leq \frac{n-1}{2} \\ \frac{n+5}{2} & \text{if } r > \frac{n-1}{2} \end{cases}.$$

(ii)

$$\dim(\mathcal{O}(\mathcal{Q}_{(n,r)})) = \begin{cases} \frac{2n^2-3n-3}{2} & \text{if } r \leq \frac{n-1}{2} \\ \frac{2n^2-3n-1}{2} & \text{if } r > \frac{n-1}{2} \end{cases}.$$

By continuing with the precedent reasoning, next we study the derivations of the Lie algebra $\mathcal{T}_{(n,n-3)}$ that let us to determine $\dim(H^1(\mathcal{T}_{(n,n-3)}, \mathcal{T}_{(n,n-3)}))$ and $\dim(\mathcal{O}(\mathcal{T}_{(n,n-3)}))$ respectively. From now on, we omit some proofs because of they are similar to Theorem 3.1.

Theorem 3.5. (i) *Let* $ad(X_i)$, $0 \leq i \leq n-3$, $ad(Y)$, h_{n-4}, t_0, t_1, g_{n-4} *and* f_{n-4} *be the endomorphisms of* $\mathcal{T}_{(n,n-3)}$ *defined by*

$$t_0(X_0) = X_0\,, \quad t_0(X_i) = iX_i \ (1 \leq i \leq n-2)\,, \quad t_0(Y) = (n-3)Y;$$

$$t_1(X_0) = X_1\,, \quad t_1(X_i) = \frac{n-2}{2}X_i \quad (1 \leq i \leq n-4)\,,$$

$$t_1(X_{n-3}) = \frac{n}{2}X_{n-3} + Y\,, \quad t_1(X_{n-2}) = (n-2)X_{n-2}\,,$$

$$t_1(Y) = \frac{n-4}{2}X_{n-3} + (n-3)Y\,;$$

$$g_{n-4}(X_0) = X_{n-3}\,, \quad g_{n-4}(X_2) = -\frac{n-4}{2}X_{n-2}\,;$$

$$f_{n-4}(X_0) = Y\,, \quad f_{n-4}(X_2) = \frac{n-4}{2}X_{n-2}\,.$$

Then, the above endomorphisms form a basis of $\mathcal{D}er(\mathcal{T}_{(n,n-3)})$. *Thus,*

$$\dim(\mathcal{D}er(\mathcal{T}_{(n,n-3)})) = n+4\,.$$

(ii) *The endomorphisms* t_0, t_1, h_{n-4}, g_{n-4} *and* f_{n-4}, *module* $\mathcal{A}d(\mathcal{T}_{(n,n-3)})$ *form a basis of the space* $H^1(\mathcal{T}_{(n,n-3)}, \mathcal{T}_{(n,n-3)})$ *and*

$$\dim(H^1(\mathcal{T}_{(n,n-3)}, \mathcal{T}_{(n,n-3)})) = 5\,.$$

(iii)

$$\dim(\mathcal{O}(\mathcal{T}_{(n,n-3)})) = n^2 - n - 4\,.$$

THEOREM 3.6. (i) *Let* $ad(X_i)$, $0 \le i \le n-3$, $ad(Y)$, t_0, t_1, h_{n-3}, l_{n-5}, *and* f_{n-5} *be the endomorphisms of* $\mathcal{T}_{(n,n-4)}$ *with* $n \ge 9$ *defined by*

$$h_{n-3}(X_1) = X_{n-2}\,;$$

$$t_0(X_0) = X_0\,, \quad t_0(X_i) = iX_i \ (1 \le i \le n-2)\,, \quad t_0(Y) = (n-4)Y;$$

$$t_1(X_0) = X_1\,, \quad t_1(X_i) = \frac{n-3}{2}X_i \quad (1 \le i \le n-5)\,,$$

$$t_1(X_{n-4}) = \frac{n-1}{2}X_{n-4} + Y\,, \quad t_1(X_{n-3}) = (n-3)X_{n-3}\,,$$

$$t_1(X_{n-2}) = (n-3)X_{n-2}\,, \quad t_1(Y) = \frac{n-5}{2}X_{n-4} + (n-4)Y;$$

$$l_{n-5}(X_1) = X_{n-4} + Y\,, \quad l_{n-5}(X_2) = X_{n-3}\,, \quad l_{n-5}(X_3) = X_{n-2}\,;$$

$$f_{n-5}(X_0) = Y\,, \quad f_{n-5}(X_2) = \frac{n-5}{2}X_{n-3}\,, \quad f_{n-5}(X_3) = (n-5)X_{n-3}\,.$$

Then, the above endomorphisms form a basis of the space $\mathcal{D}er(\mathcal{T}_{(n,n-4)})$. *Thus, we have*

$$\dim(\mathcal{D}er(\mathcal{T}_{(n,n-4)})) = n+4\,.$$

(ii) *A basis of* $H^1(\mathcal{T}_{(n,n-4)}, \mathcal{T}_{(n,n-4)})$ *can be defined by* t_0, t_1, h_{n-3}, l_{n-5} *and* f_{n-5}, *module* $\mathcal{A}d(\mathcal{T}_{(n,n-4)})$. *So, we have*

$$\dim(H^1(\mathcal{T}_{(n,n-4)}, \mathcal{T}_{(n,n-4)})) = 5\,.$$

(iii)

$$\dim(\mathcal{O}(\mathcal{T}_{(n,n-4)})) = n^2 - n - 4\,.$$

Remark 3.7. The case $n = 7$ needs a special treatment because of the restrictions obtained by impose that d_2 be a space of derivations are different at the other dimensions. Thus, we have the following theorem

THEOREM 3.8. (i) *Let* $ad(X_i)$, $0 \leq i \leq 4$, $ad(Y)$, t_0, t_1, h_4, l_2, g_2 *and* f_2 *be the endomorphisms of* $\mathcal{T}_{(7,3)}$, *with* t_0, t_1, h_4, f_2 *as the precedent theorem, and* l_2, g_2 *defined by*

$$l_2(X_1) = X_3\,, \qquad l_2(X_2) = X_4\,, \qquad l_2(X_3) = X_5\,, \qquad l_2(Y) = -2X_5\,;$$
$$g_2(X_0) = X_3\,, \qquad g_2(X_2) = -X_4\,, \qquad g_2(X_3) = -2X_5\,, \qquad g_2(Y) = 2X_5\,.$$

Then, the above endomorphisms form a basis of the space $\mathcal{D}er(\mathcal{T}_{(7,3)})$, *and*

$$\dim(\mathcal{D}er(\mathcal{T}_{(7,3)})) = 12\,.$$

(ii) *A basis of* $H^1(\mathcal{T}_{(7,3)}, \mathcal{T}_{(7,3)})$ *can be given by* h_4, t_0, t_1, l_2, g_2 *and* f_2, *module* $\mathcal{A}d(\mathcal{T}_{(7,3)})$. *So, we have*

$$\dim(H^1(\mathcal{T}_{(7,3)}, \mathcal{T}_{(7,3)})) = 6\,.$$

(iii)

$$\dim(\mathcal{O}(\mathcal{T}_{(7,3)})) = 37\,.$$

At this point, now we are going to study the algebra of derivations of the quasi-filiform naturally graded Lie algebras that are trivial extensions of the filiform naturally graded algebras obtained by Vergne, that is $\mathcal{L}_{n-1} \bigoplus \mathbb{C}$ and $\mathcal{Q}_{n-1} \bigoplus \mathbb{C}$. We first consider the algebra $\mathcal{L}_{n-1} \bigoplus \mathbb{C}$ in the following theorem.

THEOREM 3.9. (i) *Let* $ad(X_i)$, $0 \leq i \leq n-3$ t_0, t_1, t_2, $h_2, \ldots, h_{n-3}$, g_0, g_1, g_2, *and* $Id_{\mathbb{C}}$ *be the endomorphisms of* $\mathcal{L}_{n-1} \bigoplus \mathbb{C}$ *defined by*

$$t_0(X_0) = X_0\,, \qquad t_0(X_i) = (i-1)X_i \quad (2 \leq i \leq n-2)\,;$$
$$t_1(X_0) = X_1\,;$$
$$t_2(X_i) = X_i \quad (1 \leq i \leq n-2)\,;$$
$$h_k(X_i) = X_{k+i} \quad (1 \leq i \leq n-2-k,\ 2 \leq k \leq n-3)\,;$$
$$g_0(X_0) = Y\,;$$
$$g_1(X_1) = Y\,;$$
$$g_2(Y) = X_{n-2}\,.$$

Then, the above endomorphisms form a basis of $\mathcal{D}er(\mathcal{L}_{n-1} \bigoplus \mathbb{C})$. *And, thus, we have*

$$\dim(\mathcal{D}er(\mathcal{L}_{n-1} \bigoplus \mathbb{C})) = 2n + 1 .$$

(ii) *A basis of the space* $H^1(\mathcal{L}_{n-1} \bigoplus \mathbb{C}, \mathcal{L}_{n-1} \bigoplus \mathbb{C})$ *can be given by* $t_0,\ t_1,\ t_2,\ g_0, g_1, g_2, h_k, k \in \{2, \ldots, n-3\}, Id_{\mathbb{C}}$, *module* $\mathcal{A}d(\mathcal{L}_{n-1} \bigoplus \mathbb{C})$. *Therefore, we have that*

$$\dim(H^1(\mathcal{L}_{n-1} \bigoplus \mathbb{C}, \mathcal{L}_{n-1} \bigoplus \mathbb{C})) = n + 3 .$$

(iii)

$$\dim(\mathcal{O}(\mathcal{L}_{n-1} \bigoplus \mathbb{C})) = n^2 - 2n - 1 .$$

Proof. Using Theorem 2.1 we have

$$\mathcal{D}er(\mathcal{L}_{n-1} \bigoplus \mathbb{C}) = \mathcal{D}er(\mathcal{L}_{n-1}) \oplus \mathcal{D}er(\mathbb{C}) \oplus \mathcal{D}(\mathcal{L}_{n-1}, \mathbb{C}) \oplus \mathcal{D}(\mathbb{C}, \mathcal{L}_{n-1})\,;$$

if $d \in \mathcal{D}er(\mathcal{L}_{n-1} \bigoplus \mathbb{C})$, then there exist

$$\overline{d}_1 \in \mathcal{D}er(\mathcal{L}_{n-1}),\ \ \overline{d}_2 \in \mathcal{D}er(\mathbb{C}),\ \ \overline{d}_{12} \in \mathcal{D}(\mathcal{L}_{n-1}, \mathbb{C}),\ \ \overline{d}_{21} \in \mathcal{D}(\mathbb{C}, \mathcal{L}_{n-1})$$

such that $d = \overline{d}_1 + \overline{d}_2 + \overline{d}_{12} + \overline{d}_{21}$, verifying the conditions of Theorem 2.1.

(a) Computation of $\mathcal{D}er(\mathcal{L}_{n-1})$: This space has been already studied by Goze and Khakimdjanov [8], obtaining a basis of $\mathcal{D}er(\mathcal{L}_{n-1})$ compose by the linear mappings $t_0,\ t_1,\ t_2,\ h_2, \ldots, h_{n-3}$ together to the inner derivations.

(b) Computation of $\mathcal{D}er(\mathbb{C})$: Trivial.

(c) Computation of $\mathcal{D}(\mathcal{L}_{n-1}, \mathbb{C})$: The linear mappings g_0 and g_1, above described form a basis of the space $\mathcal{D}(\mathcal{L}_{n-1}, \mathbb{C})$. In fact, if $d \in \mathcal{D}(\mathcal{L}_{n-1}, \mathbb{C})$ then it verifies that

$$d(\mathbb{C}) = 0\,, \qquad d(\mathcal{L}_{n-1}) \subset \mathcal{Z}(\mathbb{C})\,, \qquad d([\mathcal{L}_{n-1}, \mathcal{L}_{n-1}]) = 0$$

leading to

$$d(X_0) = \alpha_0 Y$$
$$d(X_1) = \alpha_1 Y\ .$$

Now, without no more than substituting the values $(1,0)$ and $(0,1)$ for the vector (α_0, α_1) we arrive at g_0 and g_1, respectively.

(d) Computation of $\mathcal{D}(\mathbb{C},\mathcal{L}_{n-1})$: Using the precedent reasoning, we obtain that a basis of the space $\mathcal{D}(\mathbb{C},\mathcal{L}_{n-1})$ can be given by the endomorphism g_2, which is defined in the enunciate. Thus,

$$\dim(\mathcal{D}er(\mathcal{L}_{n-1}\bigoplus\mathbb{C}))=2n+1$$

then, we obtain (ii) and (iii) in the same way to the precedents. ∎

Secondly, we consider the Lie algebra $\mathcal{Q}_{n-1}\oplus\mathbb{C}$.

THEOREM 3.10. (i) *Let* $ad(X_i)$, $0\leq i\leq n-3$, t_0, t_1, h_3, h_5, $h_7,\dots,$ h_{n-4}, h_{n-3}, g_0, g_1, g_2 *and* $Id_{\mathbb{C}}$ *be the endomorphisms of* $\mathcal{Q}_{n-1}\oplus\mathbb{C}$ *defined by*

$$\begin{aligned}
&t_0(X_0)=X_0\,, \qquad t_0(X_i)=iX_i \quad (1\leq i\leq n-2);\\
&t_1(X_0)=X_1\,, \qquad t_1(X_{n-2})=2X_{n-2}\,, \qquad t_1(X_i)=X_i \ \ (1\leq i\leq n-3);\\
&h_k(X_i)=X_{k+i} \quad (1\leq i\leq n-2-k,\ 3\leq k\leq n-4,\ k \text{ odd})\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{and } (k=n-3)\,;\\
&g_0(X_0)=Y\,;\\
&g_1(X_1)=Y\,;\\
&g_2(Y)=X_{n-2}\,.
\end{aligned}$$

Then, the above endomorphisms form a basis of $\mathcal{D}er(\mathcal{Q}_{n-1}\oplus\mathbb{C})$. *Therefore,*

$$\dim(\mathcal{D}er(\mathcal{Q}_{n-1}\bigoplus\mathbb{C}))=\frac{3n+5}{2}\,.$$

(ii) *A basis of the space* $H^1(\mathcal{Q}_{n-1}\oplus\mathbb{C},\mathcal{Q}_{n-1}\oplus\mathbb{C})$ *can be given by* t_0, t_1, h_3, h_5, $h_7,\dots,h_{n-4}$, h_{n-3}, $Id_{\mathbb{C}}$, g_0, g_1 *and* g_2, *module* $\mathcal{A}d(\mathcal{Q}_{n-1}\oplus\mathbb{C})$. *Thus,*

$$\dim(H^1(\mathcal{Q}_{n-1}\bigoplus\mathbb{C},\mathcal{Q}_{n-1}\bigoplus\mathbb{C}))=\frac{n+9}{2}\,.$$

(iii)

$$\dim(\mathcal{O}(\mathcal{Q}_{n-1}\bigoplus\mathbb{C}))=\frac{2n^2-3n-5}{2}\,.$$

In [9], Gómez and Jiménez-Merchán prove that in low dimensions only appear those algebras of the family that have sense in each case, except for the dimensions 7 and 9, where also appear the algebras $\mathcal{E}_{(7,3)}$, $\mathcal{E}^1_{(9,5)}$ and $\mathcal{E}^2_{(9,5)}$, respectively. These new algebras are defined by

$$\mathcal{E}_{(7,3)} = \begin{cases} [X_0, X_i] = X_{i+1} & 1 \le i \le 4 \\ [Y, X_i] = X_{3+i} & 1 \le i \le 2 \\ [X_1, X_2] = X_3 + Y & \\ [X_1, X_i] = X_{i+1} & 3 \le i \le 4, \end{cases}$$

$$\mathcal{E}^1_{(9,5)} = \begin{cases} [X_0, X_i] = X_{i+1} & 1 \le i \le 6 \\ [Y, X_i] = 2X_{5+i} & 1 \le i \le 2 \\ [X_1, X_4] = X_5 + Y & \\ [X_1, X_5] = 2X_6 & \\ [X_1, X_6] = 3X_7 & \\ [X_2, X_3] = -X_5 - Y & \\ [X_2, X_4] = -X_6 & \\ [X_2, X_5] = -X_7. & \end{cases}$$

$$\mathcal{E}^2_{(9,5)} = \begin{cases} [X_0, X_i] = X_{i+1} & 1 \le i \le 6 \\ [Y, X_i] = 2X_{5+i} & 1 \le i \le 2 \\ [X_1, X_4] = X_5 + Y & \\ [X_1, X_5] = 2X_6 & \\ [X_1, X_6] = X_7 & \\ [X_2, X_3] = -X_5 - Y & \\ [X_2, X_4] = -X_6 & \\ [X_2, X_5] = X_7 & \\ [X_3, X_4] = -2X_7. & \end{cases}$$

The results obtained from calculate the space of derivations for these algebras are summarized as follows.

THEOREM 3.11. (i) *Let $ad(X_i)$, $0 \le i \le 4$, $ad(Y)$, t_0, h_2, h_4, g_2 and f_2 be the endomorphisms of $\mathcal{E}_{(7,3)}$ defined by*

$$t_0(X_0) = X_0\,, \qquad t_0(X_i) = iX_i \quad (1 \le i \le 5)\,, \qquad t_0(Y) = 3Y\,;$$
$$h_k(X_i) = X_{k+i} \quad (1 \le i \le 5-k,\ k \in \{2,4\})\,;$$
$$g_2(X_0) = X_3\,, \qquad g_2(X_2) = -X_4\,, \qquad g_2(X_3) = -X_5\,;$$
$$f_2(X_0) = Y\,, \qquad f_2(X_2) = X_4\,, \qquad f_2(X_3) = 2X_5\,, \qquad f_2(Y) = -X_5\,.$$

Then, the above endomorphism form a basis of the space of derivations $\mathcal{D}er(\mathcal{E}_{(7,3)})$. *Therefore, we have that*

$$\dim(\mathcal{D}er(\mathcal{E}_{(7,3)})) = 11\,.$$

(ii) *A basis of the space* $H^1(\mathcal{E}_{(7,3)}, \mathcal{E}_{(7,3)})$, *can be given by* t_0, h_2, h_4, g_2 *and* f_2, *module* $\mathcal{A}d(\mathcal{E}_{(7,3)})$. *So,*

$$\dim(H^1(\mathcal{E}_{(7,3)}, \mathcal{E}_{(7,3)})) = 5\,.$$

(iii)

$$\dim(\mathcal{O}(\mathcal{E}_{(7,3)})) = 43\,.$$

THEOREM 3.12. (i) *Let* $ad(X_i)$, $0 \le i \le 6$, $ad(Y)$, t_0, h_4, h_6, *and* f_4 *be the endomorphisms of* $\mathcal{E}^1_{(9,5)}$ *defined by*

$$\begin{aligned}
&t_0(X_0) = X_0\,, \qquad t_0(X_i) = iX_i \quad (1 \le i \le 7)\,, \qquad t_0(Y) = 5Y\,;\\
&h_4(X_1) = X_5 - 2Y\,, \qquad h_4(X_i) = X_{4+i} \quad (2 \le i \le 3)\,;\\
&h_6(X_1) = X_7\,;\\
&f_4(X_0) = Y\,, \quad f_4(X_1) = -3Y\,, \quad f_4(X_2) = 2X_6\,, \quad f_4(X_3) = 4X_7\,.
\end{aligned}$$

Then, the above endomorphisms form a basis of $\mathcal{D}er(\mathcal{E}^1_{(9,5)})$, *and*

$$\dim(\mathcal{D}er(\mathcal{E}^1_{(9,5)})) = 12\,.$$

(ii) *A basis for the first cohomological space* $H^1(\mathcal{E}^1_{(9,5)}, \mathcal{E}^1_{(9,5)})$ *is formed by* t_0, h_4, h_6 *and* f_4, *module* $\mathcal{A}d(\mathcal{E}^1_{(9,5)})$. *So, we have,*

$$\dim(H^1(\mathcal{E}^1_{(9,5)}, \mathcal{E}^1_{(9,5)})) = 4\,.$$

(iii)

$$\dim(\mathcal{O}(\mathcal{E}^1_{(9,5)})) = 69\,.$$

THEOREM 3.13. (i) *Let* $ad(X_i)$, $0 \le i \le 6$, $ad(Y)$, t_0, h_4, h_6 *and* f_4 *the endomorphisms of* $\mathcal{E}^2_{(9,5)}$ *defined by*

$$\begin{aligned}
&t_0(X_0) = X_0\,, \qquad t_0(X_i) = iX_i \quad (1 \le i \le 7)\,, \qquad t_0(Y) = 5Y\,;\\
&h_4(X_1) = X_5 - 2Y\,, \qquad h_4(X_i) = X_{4+i} \quad (1 \le i \le 3)\,;\\
&h_6(X_1) = X_7\,;\\
&f_4(X_0) = Y\,, \quad f_4(X_1) = -Y\,, \quad f_4(X_2) = 2X_6\,, \quad f_4(X_3) = 4X_7\,.
\end{aligned}$$

Then, the above endomorphisms form a basis of $\mathcal{D}er(\mathcal{E}^2_{(9,5)})$. *And thus, we have that*

$$\dim(\mathcal{D}er(\mathcal{E}^2_{(9,5)})) = 12\,.$$

(ii) *A basis of the space* $H^1(\mathcal{E}^2_{(9,5)}, \mathcal{E}^2_{(9,5)})$ *is formed by* t_0, h_4, h_6 *and* f_4, *module* $\mathcal{A}d(\mathcal{E}^2_{(9,5)})$. *Thus,*

$$\dim(H^1(\mathcal{E}^2_{(9,5)}, \mathcal{E}^2_{(9,5)})) = 4\,.$$

(iii)

$$\dim(\mathcal{O}(\mathcal{E}^2_{(9,5)})) = 69\,.$$

4. Cohomological Calculus with *Mathematica*

In this section, we present a program with *Mathematica*, that allows us to calculate the space of derivations in concrete dimensions for any algebra that admits a graduation of the type t_r. This fact, permits to conjecture the structure of a basis of the mentioned space and its dimension in the case of generic dimension, case that has been already solved in the previous section.

The program is structured in the following way.

1. Generation of the subspaces of derivations d_k $(0 \leq k \leq n-3)$ that permit to express easily the algebra of derivations.

2. Determination and resolution of the equation that result by impose that every one of the above subspaces d_k is constituted by derivations of the algebra.

3. Substitution of the solutions obtained in the above step, in the initial expression of the subspace d_k, obtaining in this way, the dimensions for each one of them.

4. Dimension and a basis of the whole algebra of derivations through the sum of the dimensions and the union of the basis of the preceding subspaces of derivations.

The three first steps are implemented by the function `dimder[k_,n_,r_]` that gives the dimension and a basis for the subspace d_k subject to the graduation t_r, providing n is the dimension of the algebra.

```
dimder[k_,n_,r_]:=Module[{i,j,l,m,p,q,t,s},dim=n;der=k;grad=r;
                 Clear[ec,sol,a1,b1,d1,parder,d];Lec={};
```

we introduce the linearity of the subspace d_k

```
d[0]:=0; d[a_  x_]:=a d[x]; d[x_+y_]:= d[x]+d[y]; d[x[i_]]:=0;
```

if $k = 0$ we generate the subspace d_0

```
If[der==0,d[x[0]]=a[0]x[0]+b[0] x[1];d[x[1]]=a[1]x[0]+b[1]x[1];
        For[p=2,p<=dim-2,p++,If[p!=grad,d[x[p]]=a[p]x[p] ,{}]];
                     d[x[grad]]=a[grad]x[grad]+b[grad]x[dim-1];
                 d[x[dim-1]]=a[dim-1]x[grad]+b[dim-1]x[dim-1];,
```

for $k > 0$ it is generated the corresponding subspace d_k

```
If[grad==(der+1),d[x[0]]=a[0]x[der+1]+b[0]x[dim-1];
   d[x[1]]=a[1]x[der+1]+b[1]x[dim-1],d[x[0]]=a[0]x[der+1];
   d[x[1]]=a[1]x[der+1]];

    For[p=2,p<=dim-2-der,p++,
     If[p!=grad-der,d[x[p]]=a[p]x[p+der] ,{}]];
      If[2<=(grad-der)&&(grad-der)<=dim-2-der,
   d[x[grad-der]]=a[grad-der]x[grad]+b[grad-der]x[dim-1],{}];
If[2<=grad && grad<=dim-2-der,d[x[dim-1]]=a[dim-1]x[grad+der],
 {}]];
```

Now, we make through the function `ec`, the equations that result by impose to d_k that be a space of derivations, and the union of these equations is given in the list `Lec`.

```
ec[i_,j_,l_]:=ec[i,j,l]=Coefficient[Expand[Plus[
  -d[mu[x[i],x[j]]],mu[d[x[i]],x[j]],mu[x[i],d[x[j]]]]],x[l]];
   For[i=0,i<dim-1,i++,For [j=i+1,j<dim,j++,For[l=0,l<dim ,l++,
Lec= Union[Lec,{ec[i,j,l]}]]]];
```

Thanks to an adequate use of the instruction `Solve` [11], we can solve the preceding equations without to indicate the variables to eliminate. Of all the possible solutions, that is, selection of free parameters, is enough to choose one in particular (the first) to obtain a basis of the space of derivations.

Note that all the solutions have the same number of free parameters because of this number is an invariant of the algebra.

```
sol[m_]:=Join[Table[a[i],{i,0,m-1}],Table[b[j],{j,0,m-1}]] /.
                                                  Solve[Lec==0];
a1[i_]:=sol[dim][[1,i+1]] ;
b1[j_]:=sol[dim][[1,j+1+dim]];
d1[x[i_]]:=d[x[i]] /. {a[i]-> a1[i], b[i]->b1[i]};
```

We calculate the dimension of the space by tacking into account the number of parameters (`parder`) that appear in the solution. This allows us to obtain a certain basis for each subspace.

```
parder[m_,der]:=Select[Variables[Table[d1[x[i]],{i,0,m-1}]],
                  FreeQ[#,x]&];
d2[i_,t_,s_]:=d1[x[i]]/. Dispatch[Join[{parder[dim,s]
                  [[t]]->1},Table[a[l]->0,{l,0,dim-1}],
                           Table[b[j]->0,{j,0,dim-1}]]];
Length[parder[dim,der]]];
```

The step 4, is implemented with the functions `DimDer[n_,r_]` and `BaseDer[n_,r_]` that give the dimension and a basis for the whole algebra of derivations.

```
DimDer[n_,r_]:=Module[{k}, dim=n;
        grad=r;  Sum[dimder[k,dim,grad],{k,0,dim-3}]]

BaseDer[n_,r_]:=Module[{k,q,i},dim=n;grad=r;For[k=0,k<=dim-3,
        k++,For[q=1,q<dimder[k,dim,grad]+1,q++,Do[
        If[d2[i,q,k]===0,{},Print["d[x[",i,"]]:=",d2[i,q,k]]]
                              ,{i,0,dim-1}]; Do[Print[" "]]]]]
```

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