

A second order PVM flux limiter method.

Application to magnetohydrodynamics and shallow stratified flows

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Abstract

In this work we propose a second order flux limiter finite volume method, named PVM-2U-FL, that only uses information of the two external waves of the hyperbolic system. This method could be seen as a natural extension of the well known WAF method introduced by Prof. Toro in [21]. We prove that independently of the number of unknowns of the 1D system, it recovers the second order accuracy at regular zones, while in presence of discontinuities, the scheme degenerates to PVM-2U method, which can be seen as an improvement of the HLL method (see [4], [8]). Another interesting property of the method is that it does not need any spectral decomposition of the Jacobian or Roe matrix associated to the flux function. Therefore, it can be easily applied to systems with a large number of unknowns or in situations where no analytical expression of the eigenvalues or eigenvectors are known. In this work, we apply the proposed method to Magnetohydrodynamics and to stratified multilayer flows. Comparison with the two-waves WAF and HLL-MUSCL methods are also presented. The numerical results show that PVM-2U-FL is the most efficient and accurate among them.

Key words: Finite Volumes, flux limiters, riemann solver, second order, magnetohydrodynamic, multilayer, stratified flows

1 Introduction

The goal of this article is to design a robust, simple and fast second order flux limiter numerical scheme for solving one dimensional hyperbolic systems. An interesting technique to obtain second order accurate and robust schemes is to use a non-linear combination of first and second order methods in terms of flux limiters functions. An example of this type scheme can be defined

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by a suitable combination of Roe method (which is only first order) near discontinuities, and the Lax-Wendroff method (which is second order in space and time) in regular areas. Note that the previous scheme requires the explicit knowledge of the eigenstructure of the system, which is not straight forward for some hyperbolic systems, making this scheme computationally expensive in those cases.

It is also well known that the use of incomplete Riemann solvers as Rusanov, Lax-Friedrichs, HLL, among others (see [11], [25], [5], [8], [29]) allows one to reduce the computing time required by a Roe solver (see, for instance, [9]). Although Roe scheme gives, in general, a better resolution of the discontinuities than incomplete Riemann solvers, when combined with high order methods may be indistinguishable.

In [4] Castro and Fernández-Nieto introduce a family of incomplete simple Riemann solvers named as PVM (Polynomial Viscosity Matrix), for conservative and nonconservative hyperbolic systems, defined in terms of viscosity matrices computed by a suitable polynomial evaluation of a Roe linearization, that overcome the difficulty of the computation of the spectral decomposition of Roe matrices. PVM schemes can be seen as the natural extension of the one proposed in [8] for balance laws, and, more generally, for nonconservative systems.

An interesting numerical scheme that uses flux limiters functions is the WAF (Weighted Average Flux) method, introduced by Toro in [21]. It is a one-step Godunov-type method to solve hyperbolic conservation laws that achieves second order accuracy by averaging the solution of the Riemann problem with piecewise constant initial data. As it is well known, due to Godunov's theorem, linear schemes with high order accuracy generate spurious oscillations near discontinuities. To avoid this problem, WAF method uses flux limiter functions. The resulting scheme is a non-linear TVD (Total Variation Diminishing) scheme with second order accuracy. WAF scheme has been extensively used to approximate hyperbolic systems, see for example [22], [23], [10], [24], [28]. It has been also used as the base of higher order numerical solvers (see [27]).

Nevertheless, WAF method needs the explicit knowledge of the structure of the approximated Riemann problem to achieve second order accuracy. For example, if we only consider the information of the two external waves and we use the HLL intermediate flux we obtain a WAF method – that we will name in what follows HLL-WAF method – that has second order accuracy for 1D 2x2 hyperbolic conservative systems.

The main objective of this paper is to obtain a new flux limiter scheme that only uses the information of the two external waves, like the HLL-WAF scheme, and that achieves second order accuracy for 1D $N \times N$ hyperbolic systems with $N \geq 2$. The resulting scheme can be seen as a natural extension of the original HLL-WAF scheme and it is defined in terms of a non-linear combination of a suitable PVM scheme, that is first order, with the second order Lax-Wendroff scheme.

The paper is organized as follows: in Section 2, first we summarize how WAF and, in particular, HLL-WAF methods are derived. Next, HLL-WAF method is rewritten as a non-linear combination of two PVM schemes. In section 3, the new flux limiter scheme is defined and finally, some numerical tests for the 1D ideal magnetohydrodynamics and the multilayer shallow-water systems are presented. Comparison with HLL-WAF and the second order HLL methods with MUSCL (see [12], [13], [32]) state reconstruction are also provided.

2 Preliminaries

In this section we summarize the derivation of WAF method introduced by Prof. E.F. Toro in [21]. Let us consider the conservative hyperbolic system

$$w_t + F(w)_x = 0, \quad x \in [0, L], \quad t \in [0, T], \quad (2.1)$$

where $w(x, t)$ takes values on an open convex set $\mathcal{O} \subset \mathbb{R}^N$, F is a regular function from \mathcal{O} to \mathbb{R}^N .

Let us consider a partition of the domain $\{x_i\}_i = \{i\Delta x\}_i$ where, by simplicity, Δx is supposed to be constant, and we denote $t^n = n\Delta t$ where Δt is the time step. A finite volume method in conservative form to approximate (2.1) can be written as

$$w_i^{n+1} = w_i^n - \frac{\Delta t}{\Delta x} (\mathcal{F}_{i+1/2}^n - \mathcal{F}_{i-1/2}^n), \quad (2.2)$$

where w_i^n denotes an approximation of the mean value of the solution on the control volume $(x_{i-1/2}, x_{i+1/2})$ at time $t = t^n$:

$$w_i^n \approx \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} w(x, t^n) dx,$$

and $\mathcal{F}_{i+1/2}^n = \mathcal{F}(w_i^n, w_{i+1}^n)$ denotes the numerical flux function that characterizes each method.

Let us consider a Riemann problem associated to (2.1) with initial data w_i^n and w_{i+1}^n :

$$\begin{cases} w_t + F(w)_x = 0, \\ w(x, 0) = \begin{cases} w_i & x < 0; \\ w_{i+1} & x > 0, \end{cases} \end{cases} \quad (2.3)$$

where we have removed superindex n for sake of simplicity. In what follows, the dependency of the intercell $i + 1/2$ will be dropped for clarity if there is no ambiguity.

Let us denote S_l for $l = 1, \dots, N$ the approximation of the characteristic velocities and let us consider the computational cell $V = [-\Delta x/2, \Delta x/2] \times [0, \Delta t]$. Then, the WAF numerical flux is obtained by integrating the physical flux in V using the midpoint rule for the time integral:

$$\mathcal{F}_{i+1/2}^{\text{WAF}} = \frac{1}{\Delta x} \int_{-\Delta x/2}^{\Delta x/2} F(\tilde{w}(x, \frac{\Delta t}{2})) dx, \quad (2.4)$$

where \tilde{w} is an approximated solution of the Riemann problem (2.3), composed by $N + 1$ constant states.

If we define ω_k , $k = 0, \dots, N + 1$ (see Figure 1 for the case $N = 2$) as

$$\omega_k = \frac{1}{2}(c_k - c_{k-1}); \quad \text{with } c_0 = -1, \quad c_{N+1} = 1, \quad \text{and } c_l = \frac{\Delta t}{\Delta x} S_l, \quad \text{for } 1 \leq l \leq N, \quad (2.5)$$

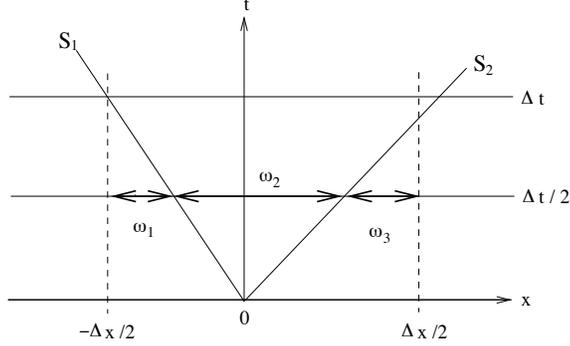


Figure 1: Computational grid and waves to compute the WAF method for problem (2.1) ($N = 2$)

then, $\mathcal{F}_{i+1/2}^{\text{WAF}}$ can be rewritten as

$$\mathcal{F}_{i+1/2}^{\text{WAF}} = \sum_{k=1}^{N+1} \omega_k F_{i+1/2}^{(k)}, \quad (2.6)$$

where $F_{i+1/2}^{(k)}$ is the value of the flux function in the interval k .

Finally, taking into account the definition of ω_k , we have that:

$$\mathcal{F}_{i+1/2}^{\text{WAF}} = \frac{1}{2}(F_i + F_{i+1}) - \frac{1}{2} \sum_{k=1}^N c_k \Delta F_{i+1/2}^{(k)}, \quad (2.7)$$

where $\Delta F_{i+1/2}^{(k)} = F_{i+1/2}^{(k+1)} - F_{i+1/2}^{(k)}$, and $F_i = F(w_i)$.

The WAF scheme is second order accurate in time and space, therefore according to Godunov's theorem, it produces spurious oscillations for non-smooth solutions. To overcome this fact, a TVD stabilization must be performed. If we denote by $\chi(v)$ a flux limiter function, then a limiter function can be defined by

$$\Psi(v, c) = 1 - (1 - |c|)\chi(v),$$

and the TVD-WAF flux function becomes as follows:

$$\mathcal{F}_{i+1/2}^{\text{WAF}} = \frac{1}{2}(F_i + F_{i+1}) - \frac{1}{2} \sum_{k=1}^N \text{sign}(c_k) \Psi_k \Delta F_{i+1/2}^{(k)}, \quad (2.8)$$

where

$$\Psi_k = \Psi(v^{(k)}, c_k) = 1 - (1 - |c_k|)\chi(v^{(k)}). \quad (2.9)$$

Some suitable choices for χ can be found in [26]. In this work we consider the Beam-Warming limiter:

$$\chi(v^{(k)}) = \min(\max(0, v^{(k)}), 1).$$

For $v^{(k)} = v^{(k)}(S_k)$ at the interface $x_{i+1/2}$ we consider the following definition:

$$v^{(k)}(S_k) = \begin{cases} \frac{\bar{m}((p_{i+1} - p_{i-1})/2, p_{i+1} - p_i, p_i - p_{i-1})}{p_{i+1} - p_i}, & \text{if } S_k > 0, \\ \frac{\bar{m}((p_{i+2} - p_i)/2, p_{i+1} - p_i, p_{i+2} - p_{i+1})}{p_{i+1} - p_i}, & \text{if } S_k < 0, \\ 1 & \text{if } |p_{i+1} - p_i| \leq \varepsilon \end{cases} \quad 1 \leq k \leq N. \quad (2.10)$$

In this definition \bar{m} is the minmod limiter:

$$\bar{m}(a, b, c) = \frac{\text{sgn}(a) + \text{sgn}(b)}{2} \frac{\text{sgn}(b) + \text{sgn}(c)}{2} \min(|a|, |b|, |c|);$$

$\{p_j\}_{j=i-1}^{j=i+2}$ is a set of scalar values that depend on the problem and ε is a small parameter (in the numerical tests we will consider $\varepsilon = \Delta x^3$).

Finally, using the former definition of Ψ_k and c_k , TVD-WAF flux can be written as follows:

$$\mathcal{F}_{i+1/2}^{\text{WAF}} = \frac{1}{2}(F_i + F_{i+1}) - \frac{1}{2} \sum_{k=1}^N \text{sgn}(S_k)(1 - \chi_k) \Delta F_{i+1/2}^{(k)} - \frac{1}{2} \frac{\Delta t}{\Delta x} \sum_{k=1}^N S_k \chi_k \Delta F_{i+1/2}^{(k)}, \quad (2.11)$$

where $\chi_k = \chi(v^{(k)})$.

2.1 Two-waves WAF method

In this section we consider the TVD-WAF method resulting when only the fastest (S_R) and the slowest (S_L) wave of the Riemann problem are used. Let us denote by χ_L and χ_R the corresponding flux limiter evaluations. Now, as in the original paper of Prof. Toro (see [21]), using the HLL flux to evaluate the intermediate flux $F_{i+1/2}^{(2)}$,

$$F_{i+1/2}^{(2)} = \frac{S_R F_i - S_L F_{i+1} + S_R S_L (w_{i+1} - w_i)}{S_R - S_L}$$

and taking into account that $F_{i+1/2}^{(1)} = F_i = F(w_i)$ and $F_{i+1/2}^{(3)} = F_{i+1} = F(w_{i+1})$, the two-waves WAF scheme (HLL-WAF in what follows) can be written:

$$\begin{aligned} \mathcal{F}_{i+1/2}^{\text{HLL-WAF}} = \frac{1}{2}(F_i + F_{i+1}) & - \frac{1}{2} (\nu_1(\chi_L, \chi_R)(w_{i+1} - w_i) + \nu_2(\chi_L, \chi_R)(F_{i+1} - F_i)) \\ & - \frac{1}{2} \frac{\Delta t}{\Delta x} (\mu_1(\chi_L, \chi_R)(w_{i+1} - w_i) + \mu_2(\chi_L, \chi_R)(F_{i+1} - F_i)) \end{aligned} \quad (2.12)$$

where

$$\begin{aligned}
\nu_1(\chi_L, \chi_R) &= \frac{S_L S_R ((1 - \chi_L) \operatorname{sgn}(S_L) - (1 - \chi_R) \operatorname{sgn}(S_R))}{S_R - S_L}, \\
\nu_2(\chi_L, \chi_R) &= \frac{(1 - \chi_R) |S_R| - (1 - \chi_L) |S_L|}{S_R - S_L}, \\
\mu_1(\chi_L, \chi_R) &= \frac{S_L S_R (S_L \chi_L - S_R \chi_R)}{S_R - S_L}, \\
\mu_2(\chi_L, \chi_R) &= \frac{S_R^2 \chi_R - S_L^2 \chi_L}{S_R - S_L}.
\end{aligned} \tag{2.13}$$

In [4] a family of first order finite volume methods named PVM- l method is proposed. For the case of conservative systems in the form of (2.3), they can be defined in terms of a numerical flux function written as follows

$$\mathcal{F}_{i+1/2} = \frac{1}{2}(F_{i+1} + F_i) - \frac{1}{2}Q_{i+1/2}(w_{i+1} - w_i). \tag{2.14}$$

The numerical viscosity matrix $Q_{i+1/2}$, is defined in terms of a Roe Matrix $A_{i+1/2}$ associated to $F(w)$, that is, $A_{i+1/2}$ verifies

$$F_{i+1} - F_i = A_{i+1/2}(w_{i+1} - w_i). \tag{2.15}$$

In particular $Q_{i+1/2}$ is given by a polynomial evaluation of this Roe Matrix as

$$Q_{i+1/2} = P_l^{i+1/2}(A_{i+1/2}), \tag{2.16}$$

where $P_l^{i+1/2}(x)$ is a polynomial of degree l verifying

$$P_l^{i+1/2}(x) = \sum_{j=0}^l \alpha_j^{i+1/2} x^j, \quad \text{such that} \quad P_l^{i+1/2}(x) \geq |x| \quad \forall x \in [S_L, S_R]. \tag{2.17}$$

Taking into account the Roe property (2.15), we can write (2.12) under the structure of (2.14) by simply defining

$$Q_{i+1/2}^{HLL-WAF}(\chi_L, \chi_R) = Q_{o1, i+1/2}^{HLL-WAF}(\chi_L, \chi_R) + \frac{\Delta t}{\Delta x} Q_{o2, i+1/2}^{HLL-WAF}(\chi_L, \chi_R), \tag{2.18}$$

with

$$\begin{aligned}
Q_{o1, i+1/2}^{HLL-WAF}(\chi_L, \chi_R) &= \nu_1(\chi_L, \chi_R)I + \nu_2(\chi_L, \chi_R)A_{i+1/2} \\
Q_{o2, i+1/2}^{HLL-WAF}(\chi_L, \chi_R) &= \mu_1(\chi_L, \chi_R)I + \mu_2(\chi_L, \chi_R)A_{i+1/2}.
\end{aligned} \tag{2.19}$$

That is, the usual two-waves HLL-WAF method (2.12) can be seen as a combination of two PVM schemes whose viscosity matrices are $Q_{o1, i+1/2}$ and $Q_{o2, i+1/2}$, respectively, associated to the first degree polynomials:

$$P_1^{o1}(x) = \nu_1(\chi_L, \chi_R) + \nu_2(\chi_L, \chi_R)x \quad \text{and} \quad P_1^{o2}(x) = \mu_1(\chi_L, \chi_R) + \mu_2(\chi_L, \chi_R)x.$$

The HLL scheme can also be interpreted as a PVM method (see [4] for details) for a viscosity matrix

$$Q_{i+1/2} = P_{1U}(A_{i+1/2})$$

and the polynomial

$$P_{1U}(x) = \frac{S_R|S_L| - S_L|S_R|}{S_R - S_L} + \frac{|S_R| - |S_L|}{S_R - S_L}x. \quad (2.20)$$

Indeed, since HLL-WAF method is based on the HLL method, we can find a relation between their definitions as PVM schemes through the polynomials that define respectively their viscosity matrices. Thus, we can check that

$$\begin{aligned} Q_{o1,i+1/2}^{HLL-WAF}(\chi_L, \chi_R) &= \frac{\text{sgn}(S_L)(1 - \chi_L) + \text{sgn}(S_R)(1 - \chi_R)}{2} A_{i+1/2} \\ &+ \frac{\text{sgn}(S_R)(1 - \chi_R)}{2} P_{1M}(A_{i+1/2}) - \frac{\text{sgn}(S_L)(1 - \chi_L)}{2} P_{1M}(A_{i+1/2}) \\ Q_{o2,i+1/2}^{HLL-WAF}(\chi_L, \chi_R) &= \frac{S_L\chi_L + S_R\chi_R}{2} A_{i+1/2} + \frac{S_R\chi_R}{2} P_{1M}(A_{i+1/2}) - \frac{S_L\chi_L}{2} P_{1M}(A_{i+1/2}). \end{aligned} \quad (2.21)$$

where

$$P_{1M}(x) = \frac{-2S_RS_L}{S_R - S_L} + \frac{S_R + S_L}{S_R - S_L}x. \quad (2.22)$$

Note that for the case $S_L < 0 < S_R$, we have that $P_{1M}(x) = P_{1U}(x)$, the polynomial associate to the HLL method. Moreover, it can be seen that

$$P_{1U}(x) = \frac{\text{sgn}(S_L) + \text{sgn}(S_R)}{2}x + \frac{\text{sgn}(S_R) - \text{sgn}(S_L)}{2}P_{1M}(x). \quad (2.23)$$

Remark 2.1. *Notice that:*

- If $\chi_L = \chi_R = 0$, then $Q_{i+1/2}^{HLL-WAF}(\chi_L, \chi_R) = Q_{o1,i+1/2}^{HLL-WAF}(\chi_L = 0, \chi_R = 0)$. Then, $\mathcal{F}_{i+1/2}^{HLL-WAF}$ reduces to the usual HLL flux.
- If $\chi_L = \chi_R = 1$, then $Q_{i+1/2}^{HLL-WAF}(\chi_L, \chi_R) = \frac{\Delta t}{\Delta x} Q_{o2,i+1/2}^{HLL-WAF}(\chi_L = 1, \chi_R = 1)$. Then, for the case $N = 2$, $\mathcal{F}_{i+1/2}^{HLL-WAF}$ reduces to the usual Lax-Wendroff method. Therefore, HLL-WAF method achieves second order accuracy in space and time for 1D, 2×2 systems, but it is not true for $N > 2$.

3 A two-waves PVM flux limiter method (PVM-2U-FL)

As we have seen in previous section HLL-WAF method can be interpreted as an improvement of the HLL method to achieve a second order accuracy scheme satisfying a TVD property through a flux limiter function for 2×2 1D systems. In this section we propose a new method based on the same idea, where the first order method is now replace by a suitable PVM scheme such

as the resulting method achieves second order accuracy at regular areas independently of the dimension of the system. In particular we present a new two-waves PVM flux limiter scheme with the following properties:

- if $\chi_L = \chi_R = 0$ the method reduces to the first order PVM-2U method introduced in [8] and extended in [4].
- if $\chi_L = \chi_R = 1$ the method reduces to the usual Lax-Wendroff method for $N \geq 2$.

Let us recall that the PVM-2U method is defined by the second degree polynomial, $P_{2U}(x)$ verifying that:

$$P_{2U}(S_L) = |S_L|, \quad P_{2U}(S_R) = |S_R| \quad \text{and} \quad P'_{2U}(S_M) = \text{sgn}(S_M), \quad (3.1)$$

where

$$S_M = \begin{cases} S_L & \text{if } |S_L| \geq |S_R|, \\ S_R & \text{if } |S_L| < |S_R|. \end{cases}$$

Moreover, the following relation can be derived:

$$P_{2U}(x) = \frac{\text{sgn}(S_L) + \text{sgn}(S_R)}{2}x + \frac{\text{sgn}(S_R) - \text{sgn}(S_L)}{2}P_{2,\bar{\alpha}}(x). \quad (3.2)$$

where

$$P_{2,\bar{\alpha}}(x) = \bar{\alpha}P_{2M}(x) + (1 - \bar{\alpha})P_{1M}(x), \quad (3.3)$$

with

$$\bar{\alpha} = \frac{(S_R - S_L)\text{sgn}(S_M) - (S_R + S_L)}{4S_M - 2(S_L + S_R)}, \quad (3.4)$$

the polynomial $P_{1M}(x)$ is defined by (2.22) and

$$P_{2M}(x) = -\frac{S_R + S_L}{S_R - S_L}x + \frac{2}{S_R - S_L}x^2.$$

Remark 3.1. *The PVM-2U method can be seen as a generalization of HLL method, in the sense that the PVM-2U method can be obtained from the HLL method by replacing the polynomial P_{1M} by the polynomial $P_{2,\bar{\alpha}}$ (see equations (2.23) and (3.2)). Moreover, if $\bar{\alpha} = 0$ then $P_{2,\bar{\alpha}} = P_{1M}$.*

Remember that HLL-WAF method has been defined by (2.21) in terms of the polynomial $P_{1M}(x)$. We propose to define a new flux-limiter type scheme in terms of the polynomial $P_{2,\bar{\alpha}}(x)$ as follows:

$$\mathcal{F}_{i+1/2}^{2U-FL} = \frac{F_i + F_{i+1}}{2} - \frac{1}{2}Q_{i+1/2}^{2U-FL}(\chi_L, \chi_R)(w_{i+1} - w_i), \quad (3.5)$$

where $Q_{i+1/2}^{2U-FL}(\chi_L, \chi_R)$ is defined by:

$$Q_{i+1/2}^{2U-FL}(\chi_L, \chi_R) = Q_{o1,i+1/2}^{2U-FL}(\chi_L, \chi_R) + \frac{\Delta t}{\Delta x}Q_{o2,i+1/2}^{2U-FL}(\chi_L, \chi_R), \quad (3.6)$$

with

$$\begin{aligned} Q_{o1,i+1/2}^{2U-FL}(\chi_L, \chi_R) &= \frac{\text{sgn}(S_L)(1 - \chi_L) + \text{sgn}(S_R)(1 - \chi_R)}{2} A_{i+1/2} \\ &+ \frac{\text{sgn}(S_R)(1 - \chi_R)}{2} P_{2,\alpha_R}(A_{i+1/2}) - \frac{\text{sgn}(S_L)(1 - \chi_L)}{2} P_{2,\alpha_L}(A_{i+1/2}), \end{aligned} \quad (3.7)$$

and

$$Q_{o2,i+1/2}^{2U-FL}(\chi_L, \chi_R) = \frac{S_L \chi_L + S_R \chi_R}{2} A_{i+1/2} + \frac{S_R \chi_R}{2} P_{2,\alpha_R}(A_{i+1/2}) - \frac{S_L \chi_L}{2} P_{2,\alpha_L}(A_{i+1/2}), \quad (3.8)$$

where

$$\alpha_K = 1 - (1 - \chi_K)(1 - \bar{\alpha}), \quad K = L, R,$$

with $\bar{\alpha}$ defined by (3.4). Note that $Q_{o1,i+1/2}^{2U-FL}(\chi_L, \chi_R)$ (respectively $Q_{o2,i+1/2}^{2U-FL}(\chi_L, \chi_R)$) can be obtained from $Q_{o1,i+1/2}^{HLL-WAF}(\chi_L, \chi_R)$ (respectively $Q_{o2,i+1/2}^{HLL-WAF}(\chi_L, \chi_R)$), by replacing the polynomial P_{1M} by the polynomials P_{2,α_R} or P_{2,α_L} if the right or left limiters are involved, respectively.

Proposition 3.1. *We have the following results:*

- a) *If $\chi_L = \chi_R = 0$ then, $Q_{i+1/2}^{2U-FL}(\chi_L = 0, \chi_R = 0) = Q_{o1,i+1/2}^{2U-FL}(\chi_L = 0, \chi_R = 0) = P_{2U}(A_{i+1/2})$. Therefore, the two-waves PVM Flux limiter method reduces to the first order PVM-2U method.*
- b) *If $\chi_L = \chi_R = 1$ then, $Q_{i+1/2}^{2U-FL}(\chi_L = 1, \chi_R = 1) = \frac{\Delta t}{\Delta x} Q_{o2,i+1/2}^{2U-FL}(\chi_L = 1, \chi_R = 1) = \frac{\Delta t}{\Delta x} A_{i+1/2}^2$. That is, the two-waves PVM Flux limiter method reduces to the Lax-Wendroff method.*

PROOF:

Let us suppose that $\chi_R = \chi_L = 0$, then $\alpha_L = \alpha_R = \bar{\alpha}$. Therefore, using equations (3.3) and (3.4) we have that $P_{2,\alpha_L}(x) = P_{2,\alpha_R}(x) = \bar{\alpha}P_{2M}(x) + (1 - \bar{\alpha})P_{1M}(x) = P_{2,\bar{\alpha}}(x)$. Then, by (3.2) we obtain that

$$Q_{o1,i+1/2}^{2U-FL}(\chi_L = 0, \chi_R = 0) = P_{2U}(A_{i+1/2}).$$

If $\chi_R = \chi_L = 1$, then $\alpha_R = \alpha_L = 1$, therefore $P_{2,\alpha_L} = P_{2,\alpha_R} = P_{2M}(x)$. And $P_{2M}(x)$ is the polynomial such that $Q_{o2,i+1/2}^{2U-FL}(\chi_L = 1, \chi_R = 1) = A_{i+1/2}^2$. □

Finally, $Q_{i+1/2}^{2U-FL}$ can be written in a more compact form as follows:

$$Q_{i+1/2}^{2U-FL}(\chi_L, \chi_R) = \gamma_{0,i+1/2} Id + \gamma_{1,i+1/2} A_{i+1/2} + \gamma_{2,i+1/2} A_{i+1/2}^2, \quad (3.9)$$

where $\gamma_{j,i+1/2} = \gamma_j(\chi_L, \chi_R) \in \mathbb{R}$, $j = 1, 2, 3$, are given by:

$$\gamma_0(\chi_L, \chi_R) = \frac{-2S_R S_L}{S_R - S_L} \left(\beta_R(1 - \alpha_R) - \beta_L(1 - \alpha_L) \right), \quad (3.10)$$

$$\gamma_1(\chi_L, \chi_R) = \beta_R + \beta_L + \frac{S_R + S_L}{S_R - S_L} \left(\beta_R(1 - 2\alpha_R) - \beta_L(1 - 2\alpha_L) \right), \quad (3.11)$$

$$\gamma_0(\chi_L, \chi_R) = \frac{2}{S_R - S_L} \left(\beta_R \alpha_R - \beta_L \alpha_L \right), \quad (3.12)$$

with

$$\beta_K = \frac{\text{sgn}(S_K)(1 - \chi_K)}{2} + \frac{\Delta t}{\Delta x} \frac{S_K \chi_K}{2}, \quad K = L, R.$$

Remark 3.2. Note that if we set $\alpha_L = \alpha_R = 0$, then $P_{2,\alpha_L}(x) = P_{2,\alpha_R}(x) = P_{1M}(x)$. Then, we recover the classical HLL-WAF method (see equations (3.3) and (2.21)).

3.1 Extension to nonconservative systems

Let us now consider the extension of the new two-waves PVM flux limiter scheme previously defined to nonconservative systems of the form:

$$w_t + F(w)_x + B(w) \cdot w_x = G(w)H_x, \quad (3.13)$$

where $w(x, t)$ takes values on an open convex set $\mathcal{O} \subset \mathbb{R}^N$, F is a regular function from \mathcal{O} to \mathbb{R}^N , B is a regular matrix function from \mathcal{O} to $\mathcal{M}_{N \times N}(\mathbb{R})$, G is a function from \mathcal{O} to \mathbb{R}^N , and H is a function from \mathbb{R} to \mathbb{R} .

By adding to (3.13) the equation $H_t = 0$, the system (3.13) can be rewritten under the form

$$W_t + \mathcal{A}(W) \cdot W_x = 0, \quad (3.14)$$

where W is the augmented vector

$$W = \begin{bmatrix} w \\ H \end{bmatrix} \in \Omega = \mathcal{O} \times \mathbb{R} \subset \mathbb{R}^{N+1}$$

and $\mathcal{A}(W)$ is the matrix whose block structure is given by:

$$\mathcal{A}(W) = \left[\begin{array}{c|c} A(w) & -G(w) \\ \hline 0 & 0 \end{array} \right],$$

where

$$A(w) = J(w) + B(w), \quad \text{being } J(w) = \frac{\partial F}{\partial w}(w).$$

Solutions of (3.14) may develop discontinuities and, due to the non-divergence form of the equations, the notion of weak solution in the sense of distributions cannot be used. The theory introduced by Dal Maso, LeFloch, and Murat [7] is followed here to define weak solutions of

(3.14). This theory allows one to define the nonconservative product $\mathcal{A}(W) \cdot W_x$ as a bounded measure provided a family of Lipschitz continuous paths $\Phi : [0, 1] \times \Omega \times \Omega \rightarrow \Omega$ is prescribed, which must satisfy certain natural regularity conditions, in particular

$$\Phi(0; W_L, W_R) = W_L, \quad \Phi(1; W_L, W_R) = W_R, \quad (3.15)$$

and

$$\Phi(s; W, W) = W. \quad (3.16)$$

For example, a family of straight segments can be considered:

$$\Phi(s; W_L, W_R) = W_L + s(W_R - W_L).$$

We consider here path-conservative numerical schemes in the sense defined in [15], that is, numerical schemes of the general form:

$$W_i^{n+1} = W_i^n - \frac{\Delta t}{\Delta x} (\mathcal{D}_{i-1/2}^+ + \mathcal{D}_{i+1/2}^-), \quad (3.17)$$

where Δx is, for simplicity, assumed to be constant; W_i^n is the approximation provided by the numerical scheme of the cell average of the exact solution at the i -th cell, $I_i = [x_{i-1/2}, x_{i+1/2}]$ at the n -th time level $t^n = n\Delta t$, and

$$\mathcal{D}_{i+1/2}^\pm = \mathcal{D}^\pm(W_i^n, W_{i+1}^n),$$

where \mathcal{D}^- and \mathcal{D}^+ are two Lipschitz continuous functions from $\Omega \times \Omega$ to Ω satisfying:

$$\mathcal{D}^\pm(W, W) = 0, \quad \forall W \in \Omega, \quad (3.18)$$

and for every $W_L, W_R \in \Omega$,

$$\mathcal{D}^-(W_L, W_R) + \mathcal{D}^+(W_L, W_R) = \int_0^1 \mathcal{A}(\Phi(s; W_L, W_R)) \frac{\partial \Phi}{\partial s}(s; W_L, W_R) ds.$$

These conditions provide a generalization of the concept of conservative scheme introduced by Lax for systems of conservation laws. In particular, if the system (3.14) admits a conservative subsystem, a path-conservative numerical scheme is conservative in the sense of Lax for that subsystem. The influence of the family of paths in the numerical approximation of shocks and the difficulties related to the convergence to the weak solutions of the system have been discussed in [3] and [16].

Let us introduce the following notation:

$$A_\Phi(W_L, W_R) = J(w_L, w_R) + B_\Phi(W_L, W_R). \quad (3.19)$$

Here, $J(w_L, w_R)$ is a Roe linearization of the Jacobian of the flux F in the usual sense:

$$J(w_L, w_R) \cdot (w_R - w_L) = F(w_R) - F(w_L); \quad (3.20)$$

and $B_\Phi(W_L, W_R)$ is a matrix satisfying:

$$B_\Phi(W_L, W_R) \cdot (w_R - w_L) = \int_0^1 B(\Phi(s; W_L, W_R)) \frac{\partial \Phi}{\partial s}(s; W_L, W_R) ds. \quad (3.21)$$

$G_\Phi(W_L, W_R)$ is a vector satisfying:

$$G_\Phi(W_L, W_R)(H_R - H_L) = \int_0^1 G(\Phi(s; W_L, W_R)) \frac{\partial \Phi_H}{\partial s}(s; W_L, W_R) ds. \quad (3.22)$$

Following [4] and [5], a natural extension of the two-waves PVM flux limiter method for nonconservative systems is the following:

$$w_i^{n+1} = w_i^n - \frac{\Delta t}{\Delta x} ((D_{i+1/2}^{2U-FL})^+ + (D_{i+1/2}^{2U-FL})^-), \quad (3.23)$$

where $(D_{i+1/2}^{2U-FL})^\pm = (D(w_i, w_{i+1}, H_i, H_{i+1})^{2U-FL})^\pm$,

$$\begin{aligned} (D_{i+1/2}^{2U-FL})^\pm &= \frac{1}{2} \left(F(w_{i+1}) - F(w_i) + B_{i+1/2}(w_{i+1} - w_i) - G_{i+1/2}(H_{i+1} - H_i) \right. \\ &\quad \left. \pm Q_{i+1/2}^{2U-FL}(\chi_L, \chi_R)(w_{i+1} - w_i - (A_{i+1/2}^*)^{-1} G_{i+1/2}(H_{i+1} - H_i)) \right), \end{aligned} \quad (3.24)$$

with $B_{i+1/2} = B_\Phi(W_i, W_{i+1})$, $G_{i+1/2} = G_\Phi(W_i, W_{i+1})$ and $Q_{i+1/2}^{2U-FL}(\chi_L, \chi_R)$ defined by (3.9) with $A_{i+1/2} = A_\Phi(W_i, W_{i+1})$ and $A_{i+1/2}^*$ is some suitable evaluation of the Roe matrix $A_{i+1/2}$ on a stationary solution and it is related to the well-balancing properties of the scheme (see [5] and [4] for more details).

Nevertheless, as pointed in [5] the second order accuracy of the method is not reached when $Q_{i+1/2}^{2U-FL} = \frac{\Delta t}{\Delta x} A_{i+1/2}^2$, as in the conservative case. According to [5] we consider the following modification of the numerical scheme:

$$w_i^{n+1} = w_i^n - \frac{\Delta t}{\Delta x} ((D_{i-1/2}^{2U-FL})^+ + (D_{i+1/2}^{2U-FL})^-) + \frac{\Delta t^2}{4\Delta x^2} \left(\mathcal{R}(\chi_L, \chi_R)_{i-1/2} + \mathcal{R}(\chi_L, \chi_R)_{i+1/2} \right) \quad (3.25)$$

with

$$\begin{aligned} \mathcal{R}(\chi_L, \chi_R)_{i+1/2}^n &= \frac{1}{2} \left(\chi_L DA(W_i)[A_{i+1/2}(w_{i+1} - w_i) - G_{i+1/2}(H_{i+1} - H_i), w_{i+1} - w_i] \right. \\ &\quad + \chi_R DA(W_{i+1})[A_{i+1/2}(w_{i+1} - w_i) - G_{i+1/2}(H_{i+1} - H_i), w_{i+1} - w_i] \\ &\quad - \chi_L DA(W_i)[w_{i+1} - w_i, A_{i+1/2}(w_{i+1} - w_i) - G_{i+1/2}(H_{i+1} - H_i)] \\ &\quad - \chi_R DA(W_{i+1})[w_{i+1} - w_i, A_{i+1/2}(w_{i+1} - w_i) - G_{i+1/2}(H_{i+1} - H_i)] \\ &\quad - \chi_L G_w(w_i)(A_{i+1/2}(w_{i+1} - w_i) - G_{i+1/2}(H_{i+1} - H_i))(H_{i+1} - H_i) \\ &\quad \left. - \chi_R G_w(w_{i+1})(A_{i+1/2}(w_{i+1} - w_i) - G_{i+1/2}(H_{i+1} - H_i))(H_{i+1} - H_i) \right) \end{aligned} \quad (3.26)$$

with $DA(W)[U, V] = \left(\sum_{l=1}^N u_l \partial_{w_l} A(W) \right) V$ and $\partial_{w_l} A(W)$ is the $N \times N$ matrix whose (i, j) element is $\partial_{w_l} a_{ij}(W)$ (see [5]). $G_w(w)$ denotes the Jacobian matrix of $G(w)$.

Note that if $\chi_L = \chi_R = 1$ then $\mathcal{R}(\chi_L = 1, \chi_R = 1)_{i+1/2}/\Delta x^2$ is a second order approximation of the term

$$\begin{aligned}\mathcal{R}(W) &= DA(W)[A(W)w_x - G(w)H_x, w_x] - DA(W)[w_x, A(W)w_x - G(w)H_x] \\ &- G_w(w)(A(W)w_x - G(w)H_x)H_x\end{aligned}\quad (3.27)$$

Proposition 3.2. *We have the following results:*

- a) *If $\chi_L = \chi_R = 1$ the method defined by (3.24), (3.25), (3.26) coincides with a Lax-Wendroff method for nonconservative problems introduced in [5], and it is second order accurate.*
- b) *If $\chi_L = \chi_R = 0$ this method coincides with the first order PVM-2U method introduced in [4] for nonconservative problems.*

The proof is analogous to Proposition 3.1 taking into account that

$$w(t+\Delta t) = w(t) + \Delta t(A(w)w_x - G(w)H_x) + \frac{\Delta t^2}{2}\partial_x(A(w)(A(w)w_x - G(w)H_x)) + \frac{\Delta t^2}{2}\mathcal{R}(W) + \mathcal{O}(\Delta t^3),$$

where $\mathcal{R}(W)$ is defined by (3.27).

4 Application to Magnetohydrodynamics

Let us consider the one-dimensional ideal magnetohydrodynamics (MHD) equations:

$$\left\{\begin{array}{l}\partial_t \rho + \partial_x(\rho v_x) = 0, \\ \partial_t(\rho v_x) + \partial_x(\rho v_x^2 + P^* - B_x^2) = 0, \\ \partial_t(\rho v_y) + \partial_x(\rho v_x v_y - B_x B_y) = 0, \\ \partial_t(\rho v_z) + \partial_x(\rho v_x v_z - B_x B_z) = 0, \\ \partial_t B_x = 0, \\ \partial_t B_y + \partial_x(v_x B_y - v_y B_x) = 0, \\ \partial_t B_z + \partial_x(v_x B_z - v_z B_x) = 0, \\ \partial_t E + \partial_x(v_x(E + P^*) - B_x(v_x B_x + v_y B_y + v_z B_z)) = 0,\end{array}\right.\quad (4.1)$$

where ρ represents the mass density, (v_x, v_y, v_z) and (B_x, B_y, B_z) are the velocity and magnetic fields, and E is the total energy. If q and B denote the magnitudes of the velocity and magnetic fields, the total energy can be expressed as

$$E = \frac{1}{2}\rho q^2 + \frac{1}{2}B^2 + \rho\varepsilon,$$

where the specific internal energy ε is related to the hydrostatic pressure P through the equation of state $P = (\gamma - 1)\rho\varepsilon$, γ being the adiabatic constant. The total pressure P^* is then defined

as $P + P_M$, where $P_M = \frac{1}{2}B^2$ is the magnetic pressure. Notice that system (4.1) can be written in the form of conservative hyperbolic system (2.1) with

$$w = \begin{pmatrix} \rho \\ \rho v_x \\ \rho v_y \\ \rho v_z \\ B_x \\ B_y \\ B_z \\ E \end{pmatrix}, \quad F(w) = \begin{pmatrix} \rho v_x \\ \rho v_x^2 + P^* - B_x^2 \\ \rho v_x v_y - B_x B_y \\ \rho v_x v_z - B_x B_z \\ 0 \\ v_x B_y - v_y B_x \\ v_x B_z - v_z B_x \\ v_x(E + P^*) - B_x(v_x B_x + v_y B_y + v_z B_z) \end{pmatrix}.$$

Let us define $(b_x, b_y, b_z) = (B_x, B_y, B_z)/\sqrt{\rho}$, $b^2 = b_x^2 + b_y^2 + b_z^2$, and the acoustic sound speed $a = \sqrt{\gamma P/\rho}$. The Alfven speed is given by $c_a = |b_x|$ and the fast and slow waves, c_f and c_s , are defined as

$$c_{f,s}^2 = \frac{1}{2}(a^2 + b^2 \pm \sqrt{(a^2 + b^2)^2 - 4a^2 b_x^2}).$$

The eight characteristic velocities of system (4.1) are then

$$\lambda_1 = u - c_f, \quad \lambda_2 = u - c_a, \quad \lambda_3 = u - c_s, \quad \lambda_4 = \lambda_5 = u, \quad \lambda_6 = u + c_s, \quad \lambda_7 = u + c_a, \quad \lambda_8 = u + c_f,$$

where the characteristic fields associated to $\lambda_{1,8}$, $\lambda_{3,6}$, $\lambda_{2,7}$ and $\lambda_{4,5}$ are called, respectively, the fast, slow, Alfven and entropy waves. The spectral structure of system (4.1) is further analyzed in [1, 19]; in particular, the system admits a complete set of eigenvectors.

A Roe matrix for system (4.1) was originally presented in [1] for the case $\gamma = 2$. Instead, the extension introduced in [2] is considered here, as it is valid for arbitrary values of γ . This Roe matrix will be used in order to construct PVM-2U-FL method. Comparisons with the original HLL-WAF and the second order HLL method with MUSCL state reconstruction (HLL-MUSCL) are also presented.

For this problem, we consider the following definition for the two external waves (see Davis [6]):

$$S_L = \min(\lambda_{1,i+1/2}, \lambda_{1,i}), \quad S_R = \max(\lambda_{8,i+1/2}, \lambda_{8,i+1}),$$

where $\lambda_{1,i}$ and $\lambda_{8,i}$ are the minimum and the maximum eigenvalues of the Jacobian matrix of the flux evaluated at W_i , respectively. And $\lambda_{1,i+1/2}$, $\lambda_{8,i+1/2}$ are the ones corresponding to Roe matrix.

For the definition of the flux limiter function at each interface $x_{i+1/2}$ (see Section 2) we define a set of scalar values $\{p_j\}_{j=i-1}^{j=i+2}$. Here, we consider

$$p_j = E_j^n, \quad j = i - 1, \dots, i + 2, \quad (4.2)$$

with E_j^n being the approximation of the averaged value of the total energy E at the control volume $(x_{j-1/2}, x_{j+1/2})$ at the corresponding time $t = t^n$,

$$E_j^n \approx \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} E(x, t^n) dx.$$

4.1 Brio-Wu shock tube problem

The test developed in this section was proposed in [1] to show the formation of a compound wave consisting of an intermediate shock followed by a slow rarefaction wave. For each variable, the solution consists of five constant states separated by a left-moving fast rarefaction wave, a slow compound wave, a contact discontinuity, a right-moving slow shock and a right-moving fast rarefaction wave.

To solve this problem we consider the Riemann problem for the MHD system (4.1) with initial data

$$(\rho, v_x, v_y, v_z, B_x, B_y, B_z, P) = \begin{cases} (1, 0, 0, 0, 0.75, 1, 0, 1) & \text{for } x \leq 0, \\ (0.125, 0, 0, 0, 0.75, -1, 0, 0.1) & \text{for } x > 0, \end{cases}$$

and $\gamma = 2$.

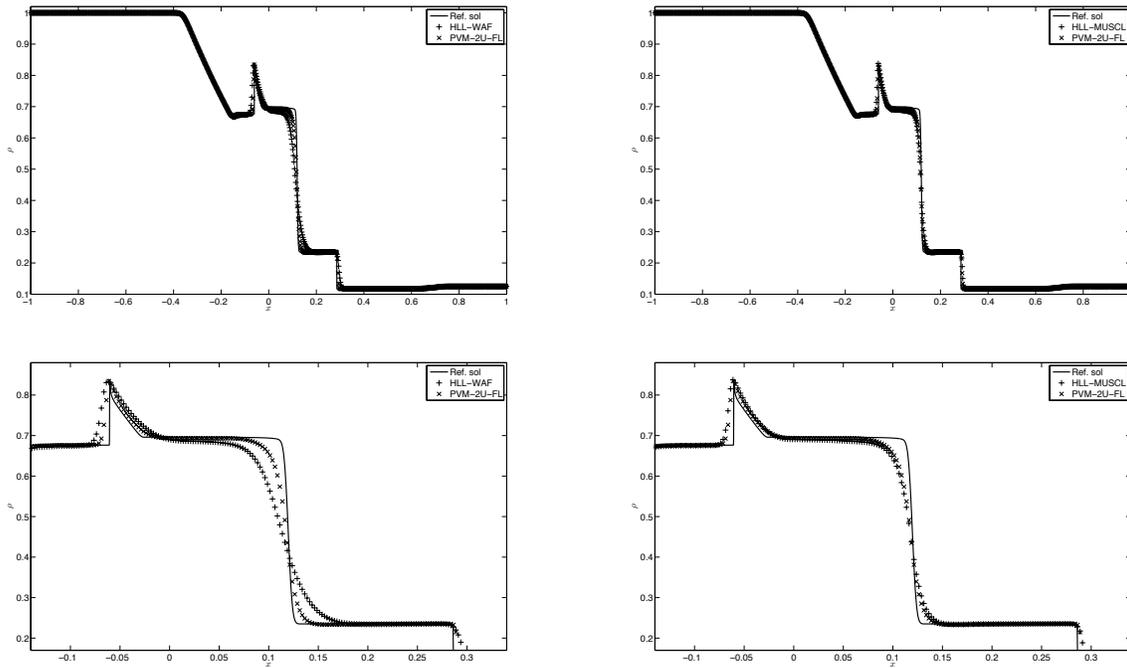


Figure 2: Mass density ρ for the Brio-Wu shock tube problem 4.1: HLL-WAF and PVM-2U-FL schemes (left) and HLL-MUSCL and PVM-2U-FL schemes (right). General view (top) and a zoom (down).

The problem has been solved from a final time $t = 0.2$ in the interval $[-1, 1]$ with 800 grid points and CFL number 0.8. It is found that PVM-2U-FL provides the best results. The results are shown in Figures 2-5, together with a reference solution that has been computed using the HLL method with 25600 points. Finally, we represent in Figure 6 an efficiency curve, in particular CPU time vs error is shown in log scale for different mesh sizes from 100 up to 1600 grid points. Note that, PVM-2U-FL is the most efficient among them and HLL-WAF and

HLL-MUSCL provide similar results. Observe that for a fixed mesh, PVM-2U-FL is also the most accurate, being HLL-MUSCL more accurate than HLL-WAF in this case.

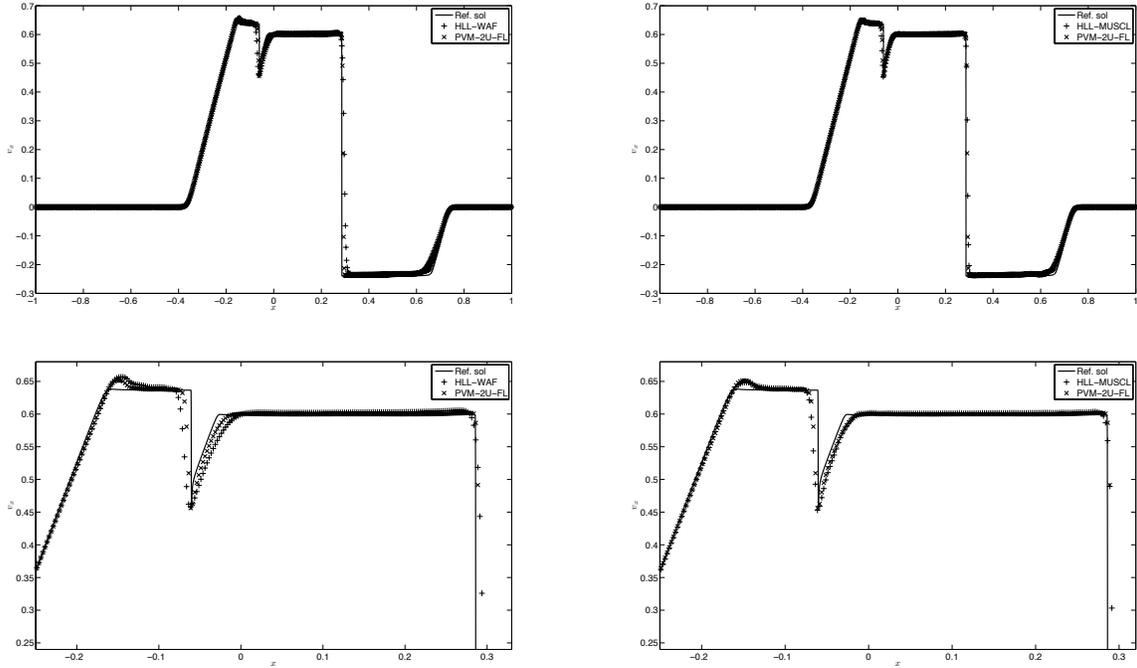


Figure 3: Velocity v_x for the Brio-Wu shock tube problem 4.1: HLL-WAF and PVM-2U-FL schemes (left) and HLL-MUSCL and PVM-2U-FL schemes (right). General view (top) and a zoom (down).

4.2 High Mach shock tube problem

This problem was presented also in [1] with the aim of testing the robustness of the numerical schemes for high Mach number flows. The initial conditions are

$$(\rho, v_x, v_y, v_z, B_x, B_y, B_z, P) = \begin{cases} (1, 0, 0, 0, 0, 1, 0, 1000) & \text{for } x \leq 0, \\ (0.125, 0, 0, 0, 0, -1, 0, 0.1) & \text{for } x > 0, \end{cases}$$

and we take $\gamma = 2$. The Mach number of the right-moving shock is 15.5. The problem has been solved in $[-1, 1]$ using 200 grid points, CFL coefficient 0.8 and final time $t = 0.012$. For this test we also found that PVM-2U-FL provide the best results. They are plotted in Figures 7-10. A reference solution computed using HLL method with 25600 points has been also considered.

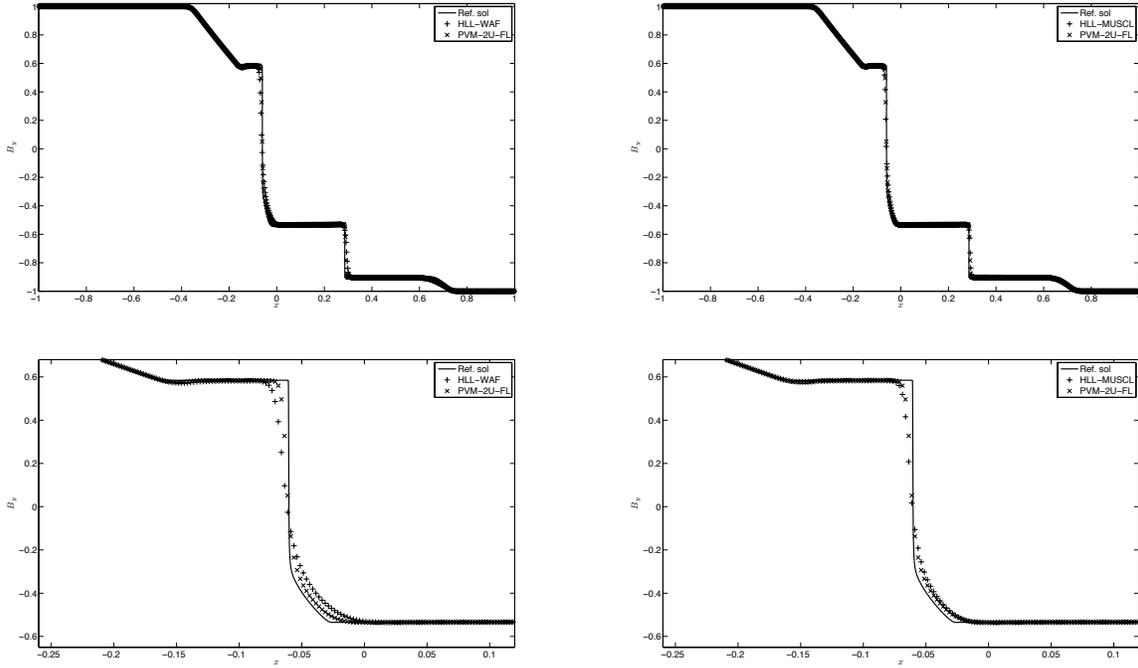


Figure 4: Magnetic field B_y for the Brio-Wu shock tube problem 4.1: HLL-WAF and PVM-2U-FL schemes (left) and HLL-MUSCL and PVM-2U-FL schemes (right). General view (top) and a zoom (down).

4.3 Non-planar Riemann problem

A non-planar Riemann problem with solution containing two strong rotational waves was proposed in [30]. The initial conditions are given by

$$(\rho, v_x, v_y, v_z, B_x, B_y, B_z, P) = \begin{cases} (1.7, 0, 0, 0, 1.1, 1, 0, 1.7) & \text{for } x \leq 0, \\ (0.2, 0, 0, 1.4968909, 1.1, \cos \beta, \sin \beta, 0.2) & \text{for } x > 0, \end{cases}$$

where $\beta = 2.3$. Notice that although the problem has an unique solution, the initial conditions are close to initial conditions for which the problem admits non-unique solutions (see [30]). Figures 11-15 show the solution computed in the interval $[-1, 1.5]$ with 800 grid points, CFL number 0.8, $\gamma = 5/3$ and final time $t = 0.4$. Finally, an efficiency curve is shown in Figure 16, where CPU time vs error is shown in log scale for different mesh sizes from 100 up to 1600 grid points. Similar results to those in Test 1 are obtained: PVM-2U-FL is the most efficient among them and HLL-WAF and HLL-MUSCL provide similar results. Observe that PVM-2U-FL is also the most accurate among them for a fixed mesh, being HLL-MUSCL more accurate than HLL-WAF for a given mesh.

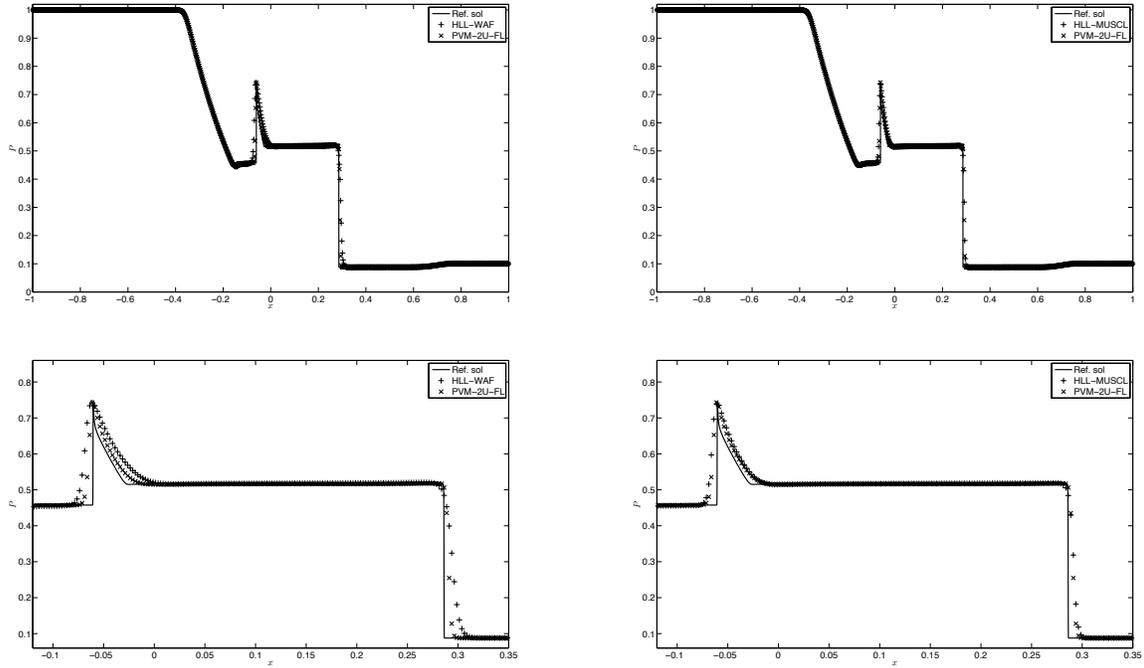


Figure 5: Hydrostatic pressure P for the Brio-Wu shock tube problem 4.1: HLL-WAF and PVM-2U-FL schemes (left) and HLL-MUSCL and PVM-2U-FL schemes (right). General view (top) and a zoom (down).

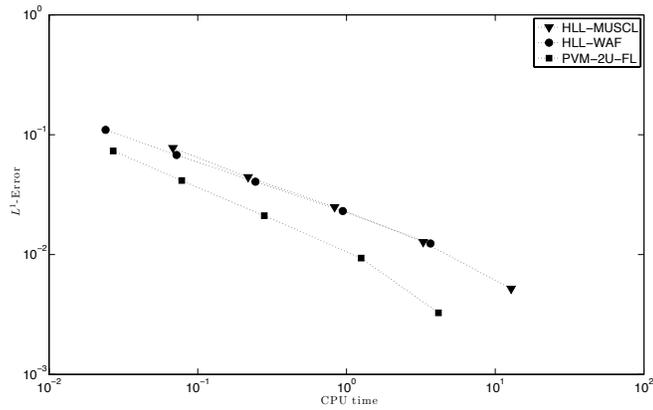


Figure 6: Brio-Wu shock tube problem 4.1: efficiency curve for HLL-MUSCL, HLL-WAF and PVM-2U-FL schemes.

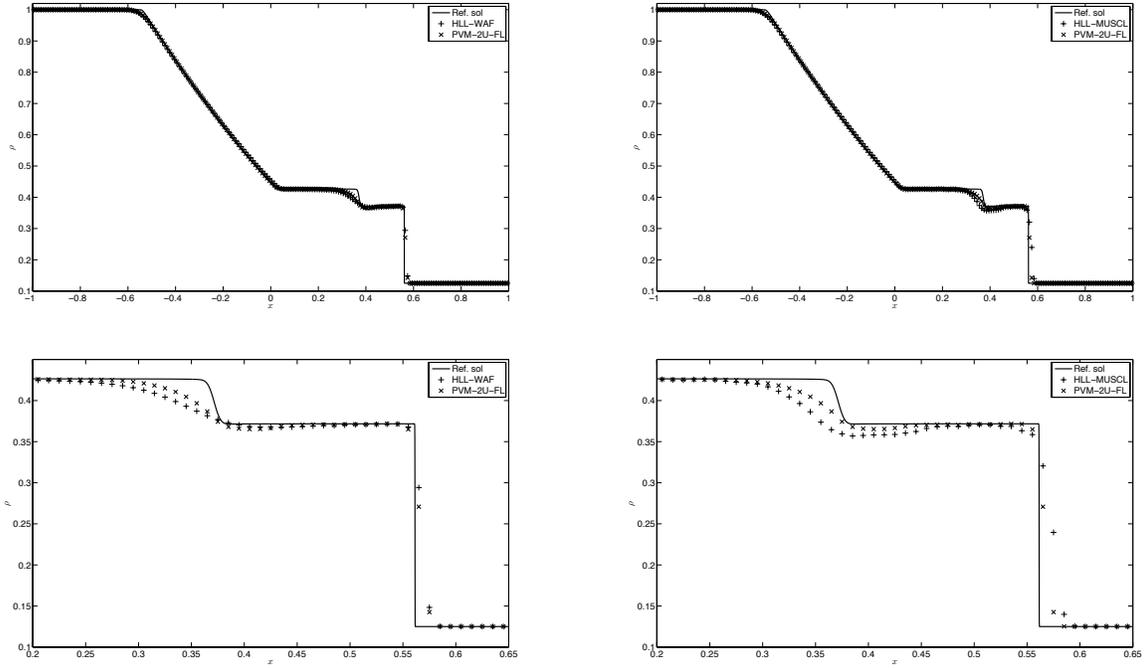


Figure 7: Mass density ρ for the high Mach shock tube problem 4.2: HLL-WAF and PVM-2U-FL schemes (left) and HLL-MUSCL and PVM-2U-FL schemes (right). General view (top) and a zoom (down).

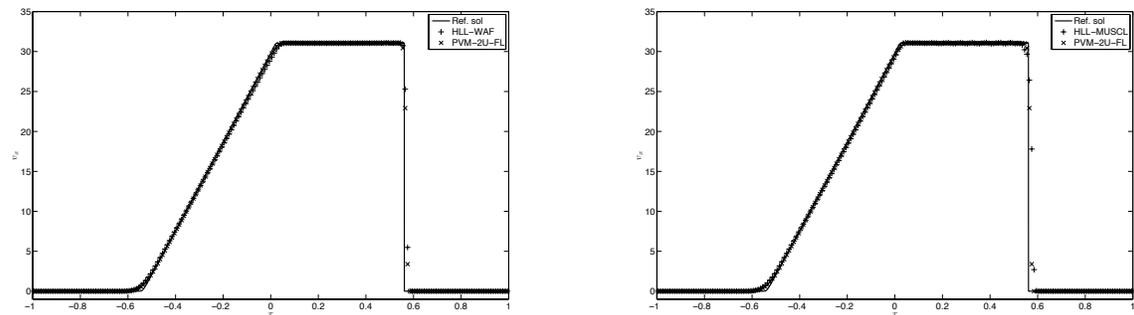


Figure 8: Velocity v_x for the high Mach shock tube problem 4.2: HLL-WAF and PVM-2U-FL schemes (left) and HLL-MUSCL and PVM-2U-FL schemes (right).

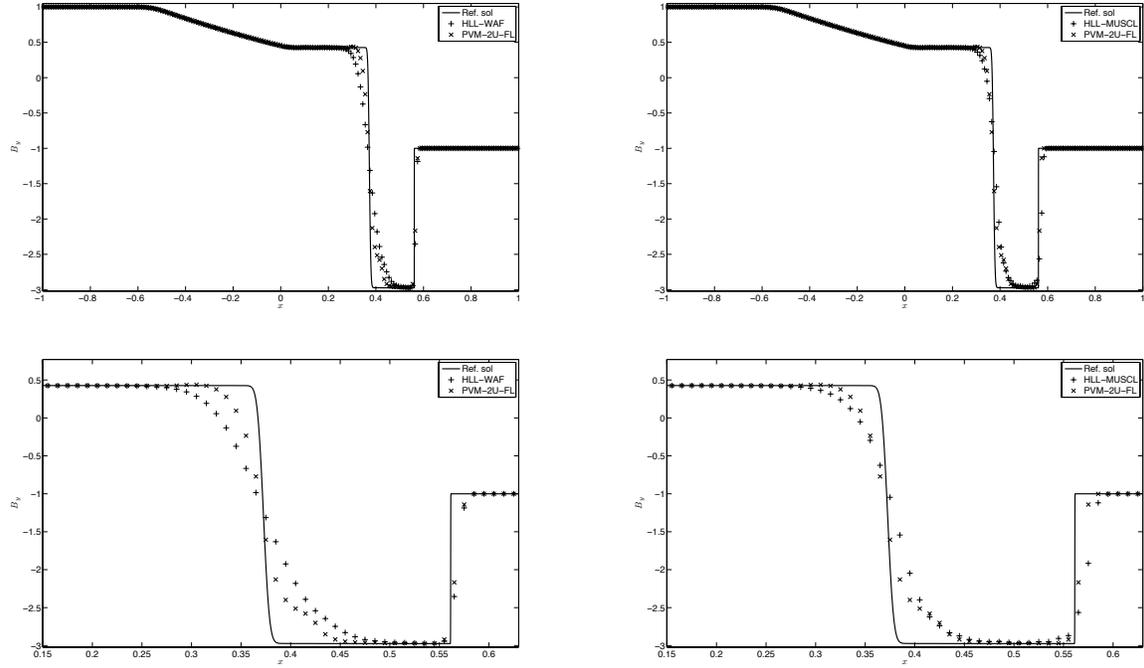


Figure 9: Magnetic field B_y for the high Mach shock tube problem 4.2: HLL-WAF and PVM-2U-FL schemes (left) and HLL-MUSCL and PVM-2U-FL schemes (right). General view (top) and a zoom (down).

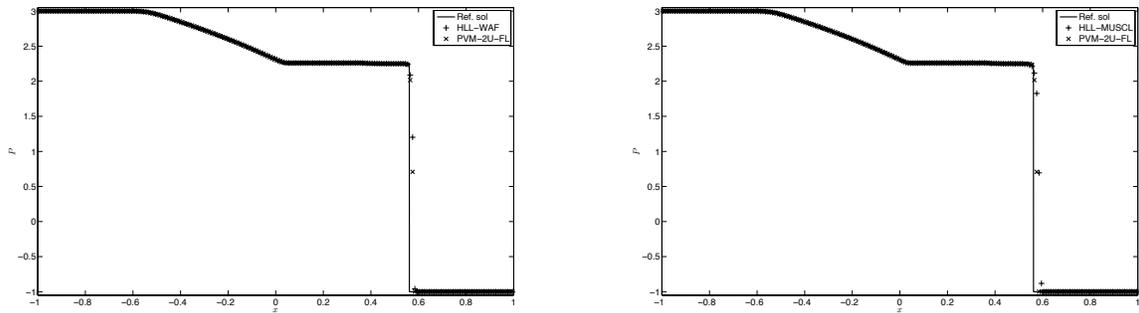


Figure 10: Hydrostatic pressure P for the high Mach shock tube problem 4.2: HLL-WAF and PVM-2U-FL schemes (left) and HLL-MUSCL and PVM-2U-FL schemes (right).

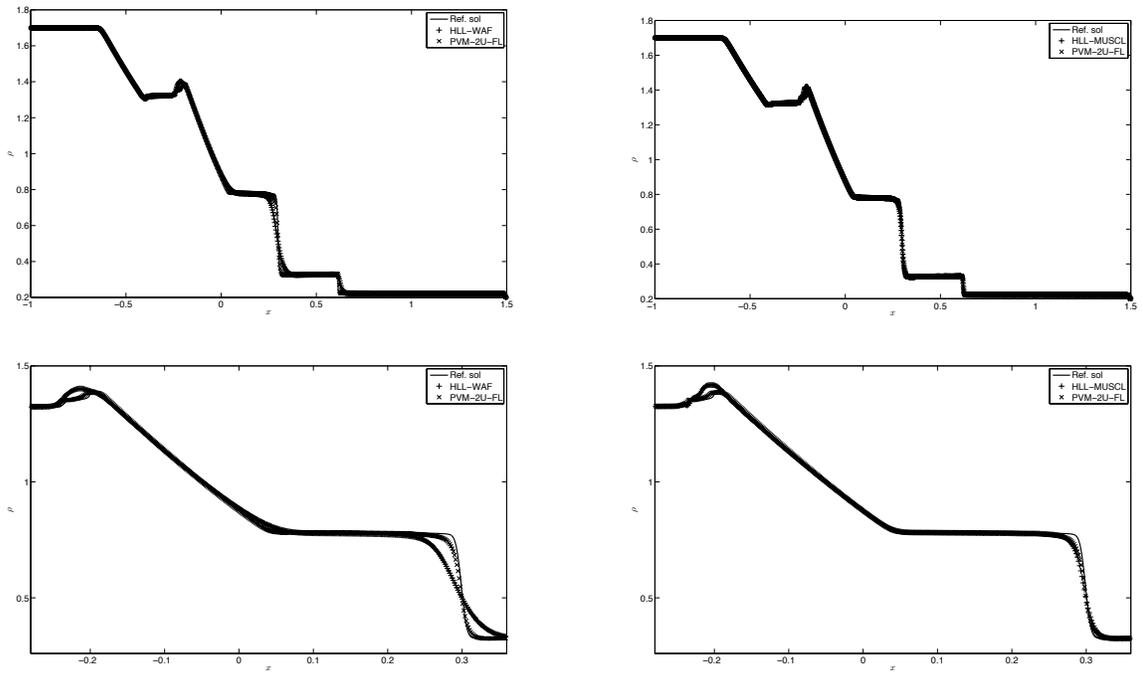


Figure 11: Mass density ρ for the non-planar Riemann problem 4.3: HLL-WAF and PVM-2U-FL schemes (left) and HLL-MUSCL and PVM-2U-FL schemes (right). General view (top) and a zoom (down).

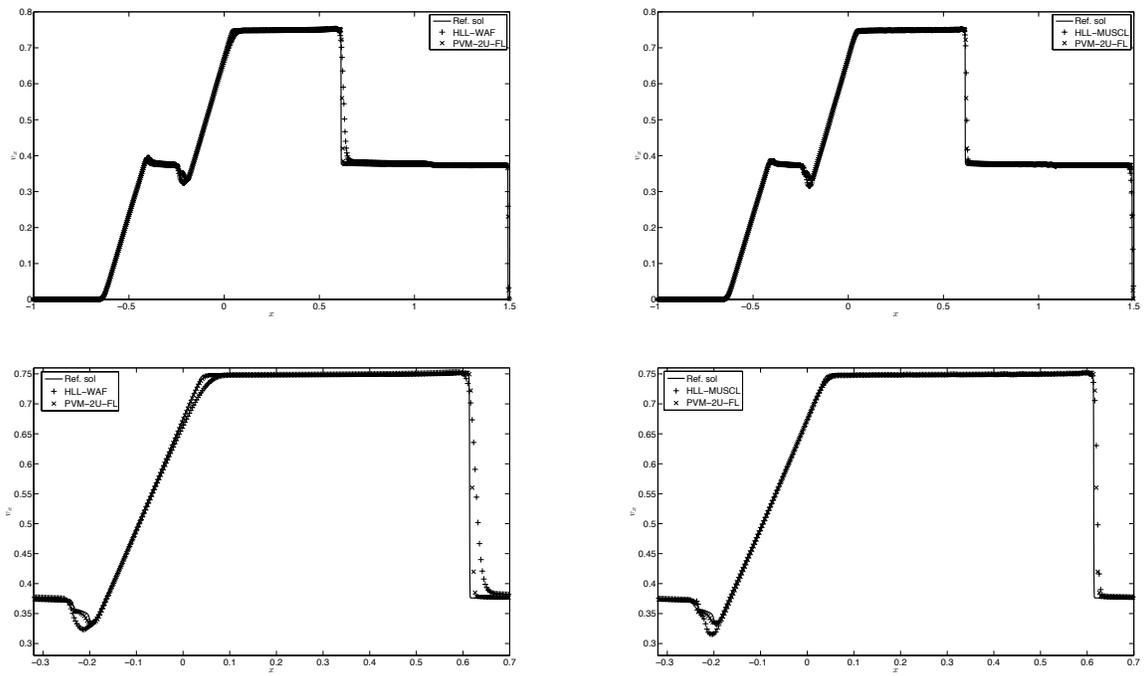


Figure 12: Velocity v_x for the non-planar Riemann problem 4.3: HLL-WAF and PVM-2U-FL schemes (left) and HLL-MUSCL and PVM-2U-FL schemes (right). General view (top) and a zoom (down).

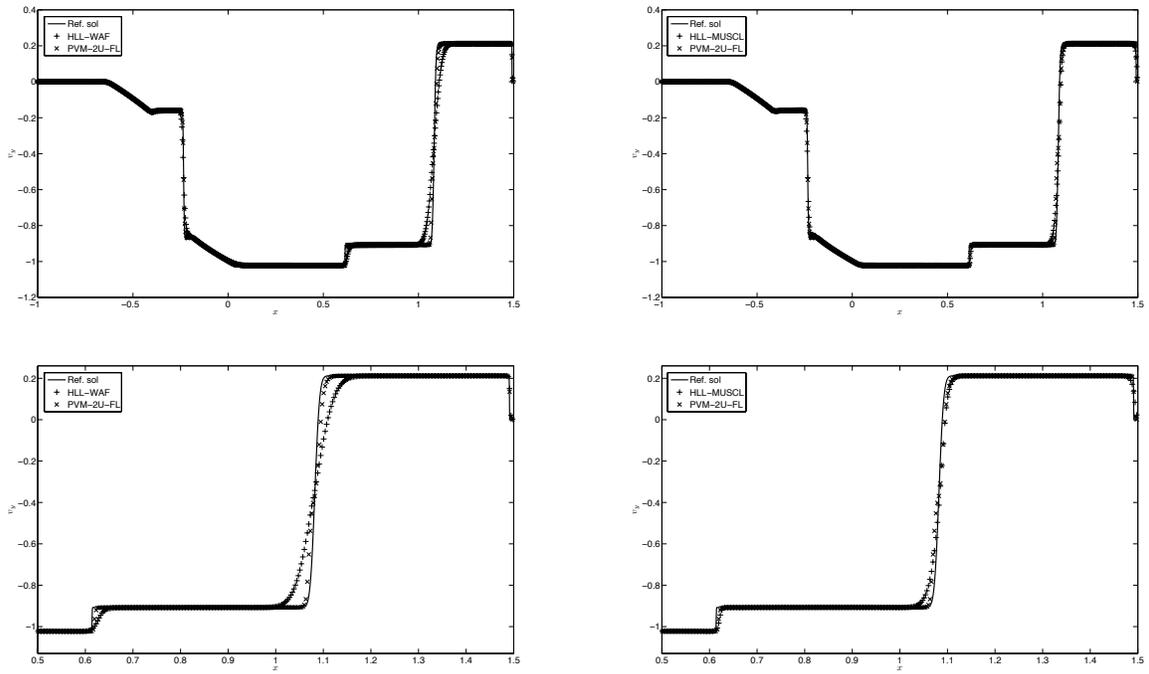


Figure 13: Velocity v_y for the non-planar Riemann problem 4.3: HLL-WAF and PVM-2U-FL schemes (left) and HLL-MUSCL and PVM-2U-FL schemes (right). General view (top) and a zoom (down).

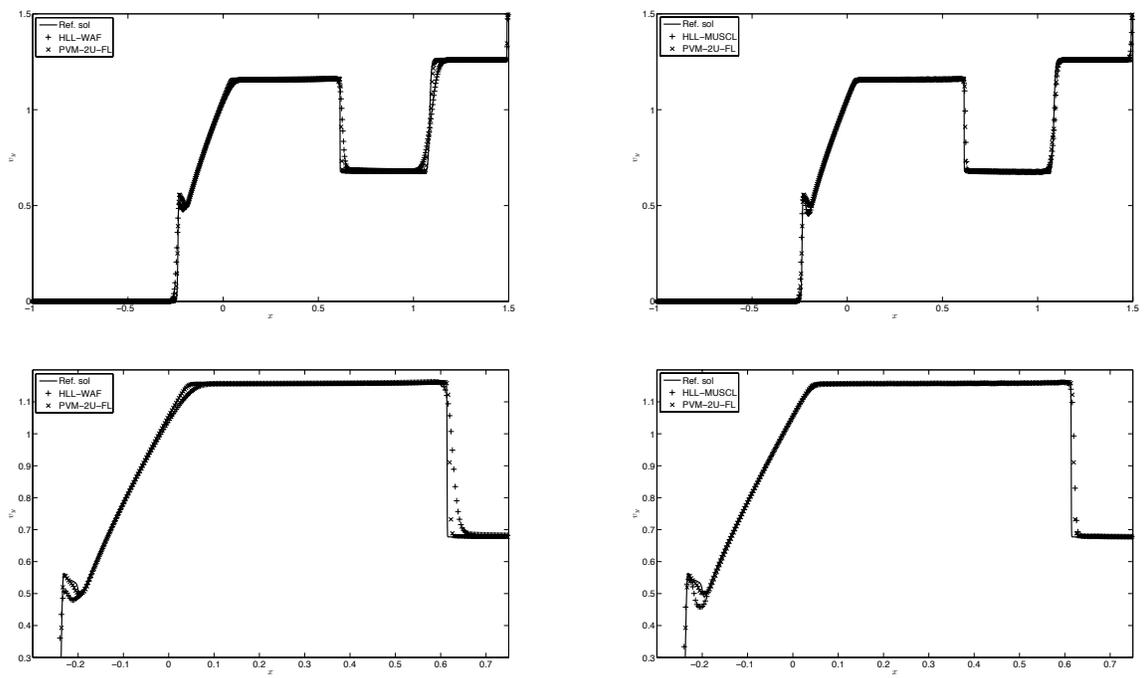


Figure 14: Velocity v_z for the non-planar Riemann problem 4.3: HLL-WAF and PVM-2U-FL schemes (left) and HLL-MUSCL and PVM-2U-FL schemes (right). General view (top) and a zoom (down).

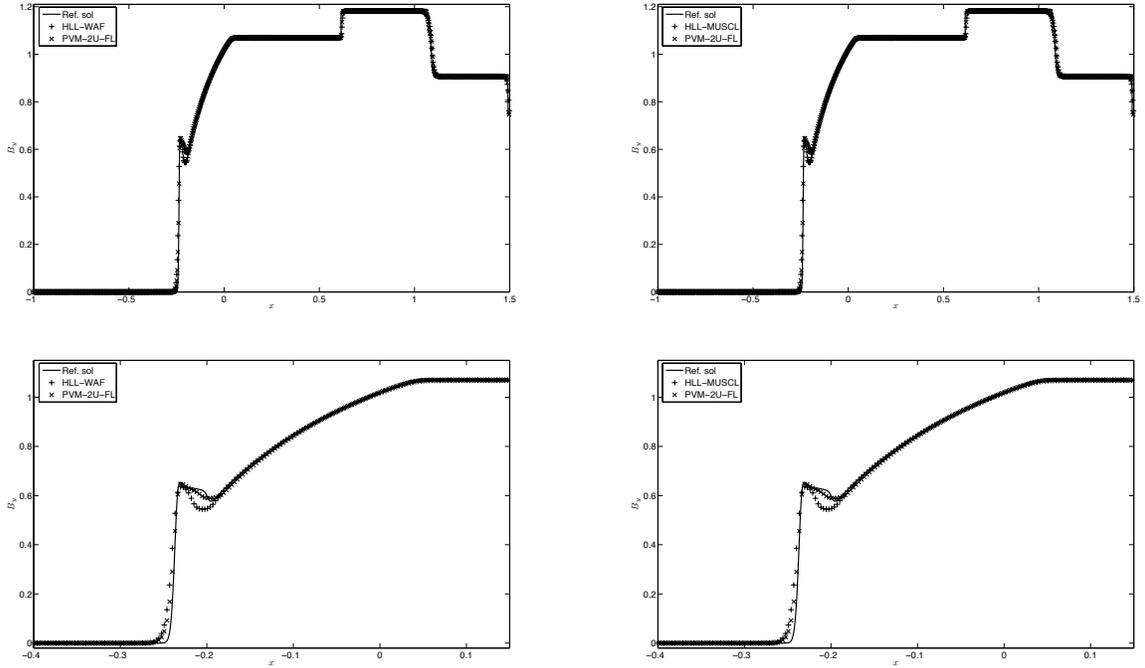


Figure 15: Magnetic field B_z for the non-planar Riemann problem 4.3: HLL-WAF and PVM-2U-FL schemes (left) and HLL-MUSCL and PVM-2U-FL schemes (right). General view (top) and a zoom (down).

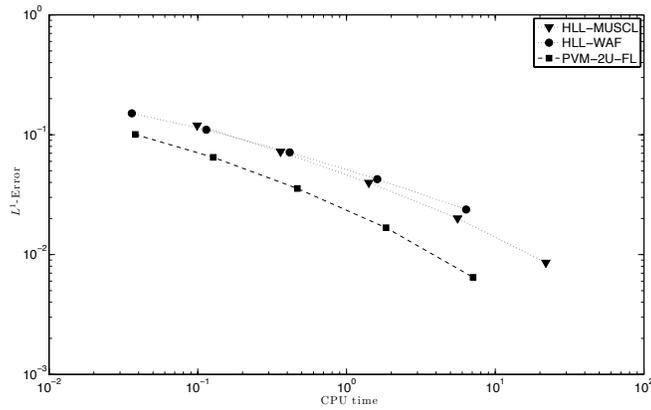


Figure 16: Non-planar Riemann problem 4.3: efficiency curve for HLL-MUSCL, HLL-WAF and PVM-2U-FL schemes.

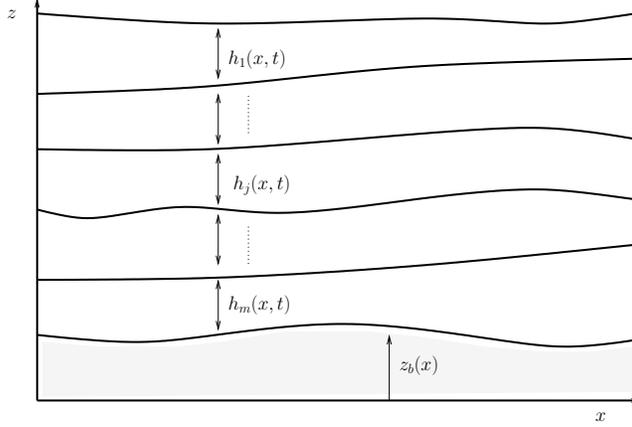


Figure 17: Multilayer stratified flow

5 Application to multilayer stratified shallow flows

Let us consider the multilayer shallow water system where layers are supposed to be immiscible given by:

$$\begin{cases} \partial_t h_j + \partial_x q_j = 0, \\ \partial_t q_j + \partial_x \left(\frac{q_j^2}{h_j} + \frac{1}{2} g h_j^2 \right) + g h_j \partial_x (z_b + \sum_{k>j} h_k + \sum_{k<j} \frac{\rho_k}{\rho_j} h_k) = 0. \end{cases} \quad j = 1, \dots, m, \quad (5.1)$$

where m is the number of layers, h_j , $j = 1, \dots, m$ are the fluid depths, $q_j = h_j u_j$ are the discharges, u_j are the velocities and $z_b(x)$ is the topography. g is the gravity constant and ρ_j the densities of each layer verifying

$$0 < \rho_1 < \dots < \rho_m.$$

Notice that, h_1 is the heigh of the layer of fluid on the top and h_m is the heigh of the layer of fluid over the bottom (See Figure 17).

This system can be written under the structure of system (3.13) with

$$w = \begin{bmatrix} w_1 \\ \vdots \\ w_m \end{bmatrix}, \quad w_j = \begin{bmatrix} h_j \\ q_j \end{bmatrix}, \quad F(w) = \begin{bmatrix} F_1 \\ \vdots \\ F_m \end{bmatrix}, \quad F_j = \begin{bmatrix} q_j \\ \frac{q_j^2}{h_j} + \frac{1}{2} g h_j^2 \end{bmatrix},$$

$$G(w) = \begin{bmatrix} G_1 \\ \vdots \\ G_m \end{bmatrix}, \quad G_j = \begin{bmatrix} 0 \\ g h_j \end{bmatrix},$$

and $H = H_{ref} - z_b(x)$ being H_{ref} a constant reference height. Finally, $B(w)$ is the $2m \times 2m$ matrix defined by the elements $B_{lj}(w)$ $l, j = 1, \dots, 2m$ with

$$B_{lj}(w) = \begin{cases} \frac{\rho_i}{\rho_l} gh_l & \text{for } j = 1 + 2k, \quad k = 0, \dots, l-2, \\ gh_l & \text{for } j = 1 + 2k, \quad k = l, \dots, m-1, \\ 0 & \text{otherwise.} \end{cases}$$

Finally, the term $\mathcal{R}(w)$ (see equation (3.27)), is given by:

$$\mathcal{R}(w) = \begin{bmatrix} R_1(w) \\ \vdots \\ R_m(w) \end{bmatrix},$$

where

$$R_j(w) = \begin{bmatrix} 0 \\ g\partial_x q_j(\partial_x z_b + \sum_{k>j} \partial_x h_k + \sum_{k<j} \frac{\rho_k}{\rho_j} \partial_x h_k) - g\partial_x h_j(\sum_{k>j} \partial_x q_k + \sum_{k<j} \frac{\rho_k}{\rho_j} \partial_x q_k) \end{bmatrix},$$

for $j = 1, \dots, m$.

In order to define the numerical scheme (3.24), the matrices $A_{i+1/2}$, $B_{i+1/2}$ and $A_{i+1/2}^*$ and the vectors $G_{i+1/2}$ and $\mathcal{R}(\chi_L, \chi_R)_{i+1/2}$ should be defined. Here $G_{i+1/2}$ is defined by

$$G_{i+1/2} = \begin{bmatrix} G_{1,i+1/2} \\ \vdots \\ G_{m,i+1/2} \end{bmatrix}, \quad G_{j,i+1/2} = \begin{bmatrix} 0 \\ gh_{j,i+1/2} \end{bmatrix},$$

$B_{i+1/2}$ is the $2m \times 2m$ matrix defined by the elements $B_{i+1/2}^{lj}$ $l, j = 1, \dots, 2m$ with

$$B_{i+1/2}^{lj} = \begin{cases} \frac{\rho_i}{\rho_l} gh_{l,i+1/2} & \text{for } j = 1 + 2k, \quad k = 0, \dots, l-2, \\ gh_{l,i+1/2} & \text{for } j = 1 + 2k, \quad k = l, \dots, m-1, \\ 0 & \text{otherwise.} \end{cases}$$

with

$$h_{l,i+1/2} = \frac{h_{l,i} + h_{l,i+1}}{2}, \quad l = 1, \dots, m.$$

The matrix $A_{i+1/2} = J_{i+1/2} + B_{i+1/2} J_{i+1/2}$ is a matrix with diagonal blocks $J_{l,i+1/2}$, $l = 1 \dots m$ given by

$$J_{l,i+1/2} = \begin{bmatrix} 0 & 1 \\ gh_{l,i+1/2} - u_{l,i+1/2}^2 & 2u_{l,i+1/2} \end{bmatrix}$$

with

$$u_{l,i+1/2} = \frac{u_{l,i}\sqrt{h_{l,i}} + u_{l,i+1}\sqrt{h_{l,i+1}}}{\sqrt{h_{l,i}} + \sqrt{h_{l,i+1}}}, \quad l = 1 \dots m.$$

The matrix $A_{i+1/2}^*$ is obtained by setting $u_{l,i+1/2} = 0$, $l = 1 \dots m$ in matrix $A_{i+1/2}$. Finally, $\mathcal{R}(\chi_L, \chi_R)_{i+1/2}$, is defined by

$$\mathcal{R}(\chi_L, \chi_R)_{i+1/2} = \begin{bmatrix} R_{1,i+1/2}(\chi_L, \chi_R) \\ \vdots \\ R_{m,i+1/2}(\chi_L, \chi_R) \end{bmatrix},$$

with

$$R_{j,i+1/2}(\chi_L, \chi_R) = \begin{bmatrix} 0 \\ g \frac{\chi_L + \chi_R}{2} \left((q_{j,i+1} - q_{j,i})(z_{b,i+1} - z_{b,i} + \right. \\ \left. + \sum_{k>j} (h_{k,i+1} - h_{k,i}) + \sum_{k<j} \frac{\rho_k}{\rho_j} (h_{k,i+1} - h_{k,i})) - \right. \\ \left. - (h_{j,i+1} - h_{j,i}) \left(\sum_{k>j} (q_{k,i+1} - q_{k,i}) + \sum_{k<j} \frac{\rho_k}{\rho_j} (q_{k,i+1} - q_{k,i}) \right) \right) \end{bmatrix},$$

where $h_{j,i}$, $q_{j,i}$ and $z_{b,i}$ denote the approximation of $h_j(t = t^n)$, $q_j(t = t^n)$ and z_b at the control volume $[x_{i-1/2}, x_{i+1/2}]$, respectively, for $j = 1, \dots, m$.

The external eigenvalues for the case of stratified shallow flows are related to the propagation speed of barotropic perturbations (see [20]). Then, for the definition of the two external waves we consider the following definition,

$$S_L = U_{con,i+1/2} - c_{i+1/2}, \quad S_R = U_{con,i+1/2} + c_{i+1/2}.$$

where,

$$U_{con,i+1/2} = \frac{\sum_{l=1}^m u_{l,i+1/2} h_{l,i+1/2}}{\sum_{l=1}^m h_{l,i+1/2}} \quad \text{and} \quad c_{i+1/2} = \sqrt{g \sum_{l=1}^m h_{l,i+1/2}}.$$

Let us denote by $\eta_i^l = z_{b,i} + \sum_{j=l}^m h_{j,i}^n$, the l -interface, $l = 1, \dots, m$ at time $t = t^n$. Note that for $l = 1$ we obtain the free surface. For the definition of the flux limiter function at each interface $x_{i+1/2}$ (see Section 2) we must define a set of scalar values $\{p_j\}_{j=i-1}^{j=i+2}$. For the case of multilayer stratified flows studied here we define it as follows:

$$p_{i-1} = \eta_{i-1}^{I_o}, \quad p_i = \eta_i^{I_o}, \quad p_{i+1} = \eta_{i+1}^{I_o}, \quad p_{i+2} = \eta_{i+2}^{I_o}, \quad (5.2)$$

where I_o is the number of the interface at which the maximum of $|\eta_{i+1}^I - \eta_i^I|$ is reached.

As in the previous section, comparison with HLL-WAF and HLL-MUSCL are provided for different tests for two and ten layers.

5.1 Internal dam break for the two-layer shallow water equations

In this test we consider an internal dam break. The domain length is 10 meters. The initial condition is

$$h_1(x, 0) = \begin{cases} 0.9 & \text{if } x < 5, \\ 0.1 & \text{if } x \geq 5, \end{cases}$$

$$h_2(x, 0) = 1 - h_1(x, 0), \quad q_1(x, 0) = q_2(x, 0) = 0 \quad \forall x \in [0, 10].$$

Open boundary conditions are imposed. The problem has been solved until time $t = 20$ with 200 grid points and CFL number 0.9. A reference solution is computed using HLL method with 25600 points. In Figure 18 we compare the numerical results corresponding to the interface position for HLL-WAF, HLL-MUSCL and PVM-2U-FL methods. It is found that PVM-2U-FL provide the best results.

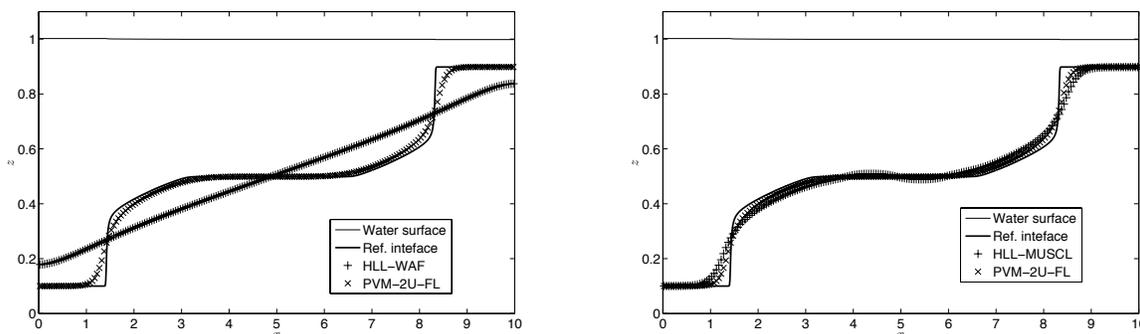


Figure 18: Free surface and comparison of the interface for the internal dam break problem 5.1: HLL-WAF and PVM-2U-FL schemes (left) and HLL-MUSCL and PVM-2U-FL schemes (right).

5.2 Stationary transcritical non-smooth flow for the two-layer shallow water equations

For this test we study the convergence towards a stationary solution with a shock for the two-layer shallow water equations. As initial condition a dam break problem over a non flat topography is considered. The initial conditions are $q_1(x, 0) = q_2(x, 0) = 0$,

$$h_1(x, 0) = \begin{cases} 0.48 & \text{if } x < 5, \\ 0.5 & \text{if } x \geq 5, \end{cases} \quad h_2(x, 0) = 1 - h_1(x, 0) - z_b(x),$$

and the bottom topography is defined by

$$z_b(x) = \frac{1}{2}e^{-(x-5)^2}. \quad (5.3)$$

The ratio of the densities is $\rho_1/\rho_2 = 0.99$. We look for a stationary solution by imposing free boundary conditions. The numerical results presented in Figure 19 correspond to $t = 100$ s. The problem has been solved with 200 grid points in $[0, 10]$ and CFL number 0.9. A reference solution has been computed using HLL method with 25600 points. Figure 19 shows the free surface and interface computed with HLL-WAF, HLL-MUSCL and PVM-2U-FL methods. As in the previous tests, PVM-2U-FL provides the best results.

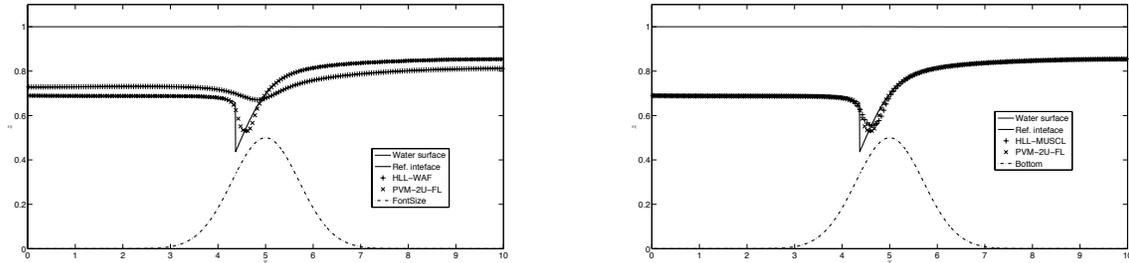


Figure 19: Free surface and comparison of the interface for problem 5.2: HLL-WAF and PVM-2U-FL schemes (left) and HLL-MUSCL and PVM-2U-FL schemes (right).

5.3 Stationary non-smooth flow for the ten-layer shallow water equations

In this test we consider the multilayer system with 10 layers, that is, a system with 20 unknowns: height and discharge for each layer. The test that we consider here is a generalization of the one considered in the previous section. We consider as initial condition a set of internal dam breaks and we study its convergence towards a stationary solution. As in the previous test, the last interface presents a strong discontinuity.

As initial condition we consider that the fluid is at rest, $q_j(x, 0) = 0$, $j = 1, \dots, 10$, and a set of internal dam breaks (see Figure 20):

$$h_{2k-1}(x, 0) = \begin{cases} 0.9 & \text{if } x < 5, \\ 0.1 & \text{if } x \geq 5, \end{cases} \quad h_{2k}(x, 0) = 1 - h_{2k-1}(x, 0), \quad k = 1, 2, 3, 4,$$

$$h_9(x, 0) = \begin{cases} 0.48 & \text{if } x < 5, \\ 0.02 & \text{if } x \geq 5, \end{cases} \quad h_{10}(x, 0) = \begin{cases} 0.52 - z_b(x) & \text{if } x < 5, \\ 0.98 - z_b(x) & \text{if } x \geq 5, \end{cases}$$

where the bottom function z_b is the same one that in previous test (equation (5.3)). The following ratio of densities is considered:

$$\frac{\rho_1}{\rho_{10}} = 0.974, \quad \frac{\rho_2}{\rho_{10}} = 0.976, \quad \frac{\rho_3}{\rho_{10}} = 0.978, \quad \frac{\rho_4}{\rho_{10}} = 0.980, \quad \frac{\rho_5}{\rho_{10}} = 0.982,$$

$$\frac{\rho_6}{\rho_{10}} = 0.984, \quad \frac{\rho_7}{\rho_{10}} = 0.986, \quad \frac{\rho_8}{\rho_{10}} = 0.988, \quad \frac{\rho_9}{\rho_{10}} = 0.99.$$

As in the previous test, open boundary conditions are imposed. Figure 21 shows the bottom topography and the different interfaces computed with the PVM-2U-FL, HLL-WAF and HLL-MUSCL methods at $t = 400$ s. A zoom of the stationary solution near the bump is presented in Figure 22

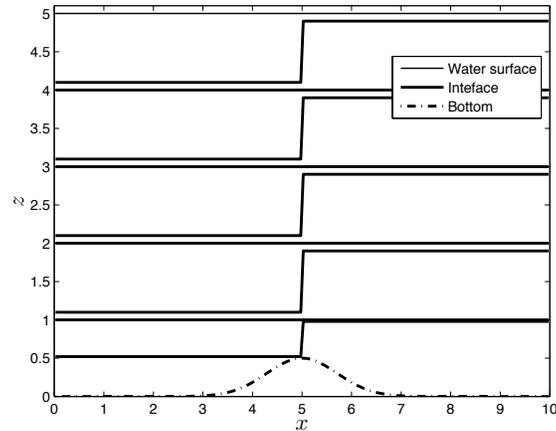


Figure 20: Initial free surface and interface for problem 5.3

An efficiency curve is shown in Figure 23, where CPU time vs error is shown in log scale for different mesh sizes from 50 up to 800 grid points. PVM-2U-FL is the most efficient method among them and HLL-WAF is the less efficient method in this case. We observe again that PVM-2U-FL is also the most accurate among them for a fixed mesh, being HLL-MUSCL more accurate than HLL-WAF for a given mesh.

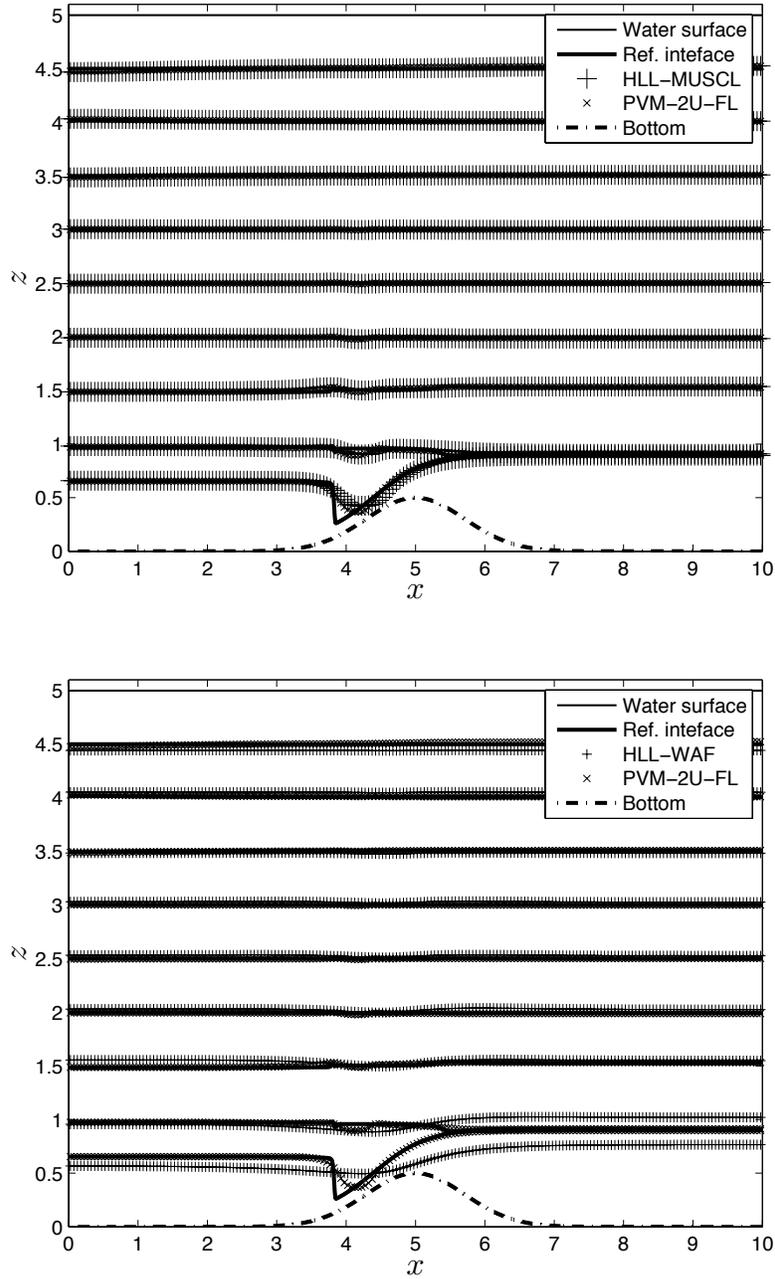


Figure 21: Free surface and comparison of the interface for problem 5.3: HLL-WAF and PVM-2U-FL schemes (top) and HLL-MUSCL and PVM-2U-FL schemes (down).

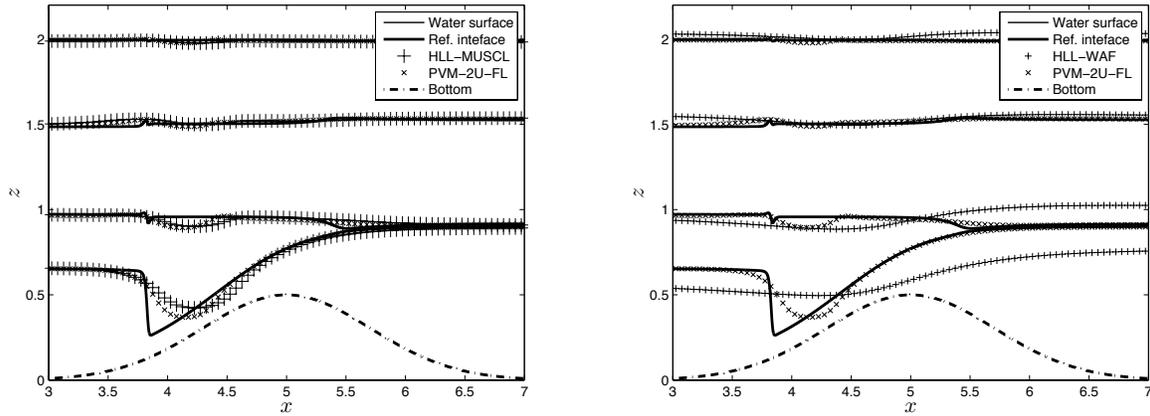


Figure 22: Zoom of Figure 21: HLL-WAF and PVM-2U-FL schemes (left) and HLL-MUSCL and PVM-2U-FL schemes (right).

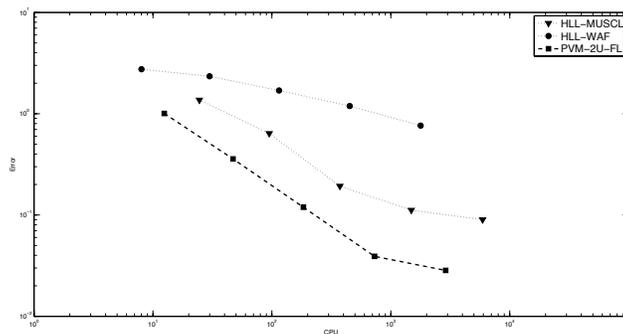


Figure 23: Internal dam break problem 5.3: efficiency curve for HLL-MUSCL, HLL-WAF and PVM-2U-FL schemes.

6 Conclusions

In this paper we present a computationally fast and efficient second order flux limiter finite volume method. It can be seen as a generalization of the HLL-WAF method, in the sense that it uses flux limiter functions to combine an incomplete Riemann solver with another one that allows to recover the second order accuracy in smooth regions. Moreover, both methods: HLL-WAF and PVM-2U-FL, only uses information of the two external waves. But: (i) the HLL-WAF method only recovers second order accuracy for 1D systems with two unknowns, while PVM-2U-FL recovers second order for arbitrary 1D systems; (ii) HLL-WAF degenerates to HLL method near discontinuities, while the PVM-2U-FL method degenerates to the PVM-2U method. Let us remark that the PVM-2U can be seen as a generalization of the HLL method.

Application to conservative and nonconservative systems are provided. In both cases, the numerical tests show that PVM-2U-FL is the most efficient among the compared methods and HLL-WAF and HLL-MUSCL provide similar results for MHD, but HLL-MUSCL provides better results than HLL-WAF for the multilayer shallow water system. Moreover, PVM-2U-FL is also the most accurate among them for a fixed mesh, being HLL-MUSCL more accurate than HLL-WAF for a given mesh. The extension to higher dimensions is not straightforward and it will be considered in future works.

References

- [1] M. Brio, C. C. Wu. *An upwind differencing scheme for the equations of ideal magnetohydrodynamics*. J. Comput. Phys. 75: 400–422, (1988).
- [2] P. Cargo, G. Gallice. *Roe matrices for ideal MHD and systematic construction of Roe matrices for systems of conservation laws*. J. Comput. Phys. 136: 446–466, (1997).
- [3] M.J. Castro, P.G. LeFloch, M.L. Muñoz, and C. Parés. *Why many theories of shock waves are necessary: Convergence error in formally path-consistent schemes*. Jour. Comp. Phys. 3227: 8107–8129, (2008).
- [4] M.J. Castro, E.D. Fernández-Nieto. *A class of computationally fast first order finite volume solvers: PVM methods*. SIAM J. Sci. Comput. 34(4): 2173–2196, (2012).
- [5] M.J. Castro, A. Pardo, C. Parés, and E.F. Toro. *On some fast well-balanced first order solvers for nonconservative systems*. Math. Comp, 79 (271): 1427–1472, (2010).
- [6] S. F. Davis. *Simplified Second-Order Godunov-Type Methods*. SIAM J. Sci. Stat. Comput., 9: 445–473, (1988).
- [7] G. Dal Maso, P.G. LeFloch, F. Murat. *Definition and weak stability of nonconservative products*. J. Math. Pures Appl. 74: 483–548, (1995).

- [8] P. Degond, P.F. Peyrard, G. Russo, Ph. Villedieu. *Polynomial upwind schemes for hyperbolic systems*. C. R. Acad. Sci. Paris 328: 479–483, (1999).
- [9] B. Einfeldt. *On Godunov-type methods for gas dynamics*. SIAM J. Numer. Anal. 25: 294–318, (1988).
- [10] E.D. Fernández-Nieto, G. Narbona-Reina. *Extension of waf type methods to non-homogeneous shallow water equations with pollutant*, J. Sci. Comput. 36(2): 193–217, (2008).
- [11] A. Harten, P.D. Lax, B. van Leer B. *On Upstream Differencing and Godunov Type Schemes for Hyperbolic Conservation Laws*. SIAM Review, 25(1): 35–61, (1983).
- [12] N.E. Kolgan. *Application of the minimum-derivative principle in the construction of finite-difference schemes for numerical analysis of discontinuous solutions in gas dynamics*. Uchenye Zapiski TsAGI [Sci. Notes Central Inst. Aerodyn], 3(6): 68–77. 137, 138, (1972).
- [13] N.E. Kolgan. *Finite-difference schemes for computation of three dimensional solutions of gas dynamics and calculation of a flow over a body under an angle of attack*. Uchenye Zapiski TsAGI [Sci. Notes Central Inst. Aerodyn], 6(2):1-6. 137, 138, (1975).
- [14] C. Parés, M.J. Castro. *On the well-balance property of Roe’s method for nonconservative hyperbolic systems. Applications to Shallow-Water Systems*. M2AN, Vol. 38(5): 821–852, (2004).
- [15] C. Parés. *Numerical methods for nonconservative hyperbolic systems: a theoretical framework*. SIAM J. Num. Anal. 44(1): 300–321, (2006).
- [16] C. Parés and M.L. Muñoz Ruíz. *On some difficulties of the numerical approximation of nonconservative hyperbolic systems*. Boletín SEMA, 47:23–52, (2009).
- [17] Y. Loukili, A. Soulaymani, *Numerical Tracking of Shallow Water Waves by the Unstructured Finite Volume WAF Approximation*, Int. J. Comput. Meth. Eng. Sci. Mech. 8(2):75–88, (2007).
- [18] P.L. Roe. *Approximate Riemann solvers, parameter vectors and difference schemes*. J. Comp. Phys., 43:357–371, (1981).
- [19] S. Serna. *A characteristic-based nonconvex entropy-fix upwind scheme for the ideal magnetohydrodynamics equations*. J. Comput. Phys. 228, 4232–4247, (2009).
- [20] J.B. Schijf and J.C. Schonfeld. *Theoretical considerations on the motion of salt and fresh water*. In *Proc. of the Minn. Int. Hydraulics Conv.*, 321–333. Joint meeting IAHR and Hyd. Div. ASCE. (1953).
- [21] E.F. Toro. *A Weighted Average Flux Method for Hyperbolic Conservation Laws*. Proceedings of the Royal Society of London, Series A, Mathematical and Physical Sciences, 423:401–418, (1989).

- [22] E.F. Toro, *Riemann problems and the the WAF method for solving two-dimensional shallow water equations*, Phil. Trans. Roy. Soc. London, A338, pp.43-68, (1992).
- [23] E.F. Toro, *The weighted average flux method applied to the time dependent euler equations*, Phil. Trans. Roy. Soc. London, A341: 499–530, (1992).
- [24] E. F. Toro, R. C. Millington, L.A.M. Nejad, *Primitive upwind numerical methods for hyperbolic partial differential equations*, in Bruneau, C. H. (ed.), Sixteenth International Conference on Numerical Methods for Fluids Dynamics. Lecture notes in physics, Springer-Verlag, Berlin, 421–426, (1998).
- [25] E.F. Toro, S.J. Billett. *Centred TVD schemes for hyperbolic conservation laws*. IMA Journal of Numerical Analysis (20): 47–79, (2000).
- [26] E.F. Toro: *Shock-Capturing Methods for Free-Surface Shallow Flows*, Wiley, England, (2001).
- [27] E.F. Toro, V.A. Titarev, *TVD Fluxes for the High-Order ADER Schemes*, J. Sci. Computing, (24).3: 285–309, (2005).
- [28] E.F. Toro, V.A. Titarev, *ADER schemes for scalar hyperbolic conservation laws with source terms in three space dimensions*, J. Com. Phys (202): 196–215, (2005).
- [29] E.F. Toro, V.A. Titarev. *MUSTA fluxes for systems of conservation laws*. J. Comput. Phys. 216(2): 403–429, (2006).
- [30] M. Torrilhon, D. S. Balsara. *High order WENO schemes: investigations on non-uniform convergence for MHD Riemann problems*. J. Comput. Phys. (201): 586–600, (2004).
- [31] I. Toumi. *A weak formulation of Roe approximate Riemann solver*. J. Comp. Phys. 102(2): 360–373, (1992).
- [32] Van Leer B. *Toward the ultimate conservative difference scheme. III. Upstream-centered finite-difference schemes for ideal compressible flows*. Journal of Computational Physics (7).23: 263–275, (1977).