Journal of Symbolic Computation 40 (2005) 1208–1224

Simplification techniques for maps in simplicial topology

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Abstract

This paper offers an algorithmic solution to the problem of obtaining "economical" formulae for some maps in Simplicial Topology, having, in principle, a high computational cost in their evaluation. In particular, maps of this kind are used for defining cohomology operations at the cochain level. As an example, we obtain explicit combinatorial descriptions of Steenrod kth powers exclusively in terms of face operators.

Keywords: Cohomology operations; Simplicial sets; Face and degeneracy operators

1. Introduction

In this paper we deal with problems in the field of Combinatorial Topology. We work with simplicial sets, which provide combinatorial descriptions of topological spaces. A simplicial set (see May, 1967) is a graded set $K = \{K_q\}_{q\geq 0}$ whose q-dimensional "building blocks" are q-simplices and whose "mortar" is face ($\partial_i : K_{q+1} \to K_q$) and degeneracy

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0747-7171/$ - see front matter
doi:10.1016/j.jsc.2004.04.008

($s_i : K_q \to K_{q+1}$) operators. It is an elementary fact that any composition of face and degeneracy operators of a simplicial set K can be expressed in the "normalized" form:

$$s_{j_t} \cdots s_{j_1} \partial_{i_1} \cdots \partial_{i_s},$$

where $j_t > \cdots > j_1 \geq 0$ and $i_s > \cdots > i_1 \geq 0$, due to certain commutativity properties. Roughly speaking, we are interested here not only in "normalizing" some compositions of face and degeneracy operators, but also in determining which of them involve exclusively face operators. In particular, we simplify compositions that are used for defining important cohomology operations such as Steenrod squares (Steenrod, 1947), Steenrod kth powers (Steenrod, 1952) or Adem secondary cohomology operations (Adem, 1952, 1958). In fact, from a simplicial viewpoint and taking into account that we deal with homological information given in terms of explicit chain homotopy equivalences (Real, 2000; Gonzalez-Diaz, 2000), the description of invariants in Algebraic Topology can be reduced to the study of compositions of certain specific maps given essentially in terms of face and degeneracy operators. The fundamental maps involved are the *AW*, *EML* and *SHI* operators given in the Eilenberg–Zilber Theorem (Eilenberg and Zilber, 1959). This theorem states that there is a chain homotopy equivalence (AW, EML, SHI) from the normalized chain complex $C^N(K \times L)$ of the cartesian product of K and L to the tensor product $C^N(K) \otimes C^N(L)$ of the normalized chain complexes $C^N(K)$ and $C^N(L)$. Whereas the number of summands in the formula for *AW* grows linearly, the number of summands in the formulae for *EML* and *SHI* grow exponentially, then in order to define "computable" algebraic–combinatorial invariants, it seems that the right strategy is reduced to determine compositions of maps in which the morphism *AW* is involved. For example, the cup product on cohomology is essentially determined at the cochain level by the morphism *AW* and the diagonal map. All of this fits well with the results of Kristensen (Kristensen, 1963; Kristensen and Madsen, 1967), where a representation result for stable primary and secondary cohomology operations in terms of cochain maps is given; and that of Klaus (Klaus, 2001, 2003), extending Kristensen's results to prove that any cohomology operation module p can be described in terms of polynomials of coface operators at the cochain level. This approach is also corroborated in Real (1996), Gonzalez-Diaz and Real (1999) and Gonzalez-Diaz and Real (2002a) where Steenrod squares, Steenrod kth powers and Adem secondary cohomology operations are seen at the cochain level essentially as compositions of the type

$$H = AW_{(p)} t_r SHI_{(p)} t_{r-1} \cdots SHI_{(p)} t_1 SHI_{(p)} : C^N(K^{\times p}) \to C^N(K)^{\otimes p} \tag{1}$$

where t_i are permutations of p factors and $AW_{(p)}$ and $SHI_{(p)}$ are, respectively, the *AW* and *SHI* operators given by the Eilenberg–Zilber Theorem for p simplicial sets. It is evident that an algorithm for computing these cohomology operations based on the previous formulation shows extremely high computational costs. Because of this, a normalization of compositions of face and degeneracy operators and a following step of the elimination of those summands of the normalized formula for H with a factor having a degeneracy operator in its expression are done in order. This "simplification" process allows us to reach a combinatorial description for H having the minimum number of face operators involved.

In this paper, we work with a general simplicial expression of type (1), where the t_i can be any permutation. We have developed a software using *Mathematica* that deduces its "minimal" simplicial formulation. In particular, the solution to this combinatorial problem

provides a way to design an efficient algorithm for computing any Steenrod cohomology operation on any cohomology class of any degree. This work has been presented in Gonzalez-Diaz and Real (2002b).

The paper is organized as follows: In Section 2 we review the necessary theoretical background. In Section 3 we develop simplification techniques for obtaining an "economical" formulation for operations of the type (1). Finally, Section 4 is devoted to showing an application of our method: an algorithm for computing the Steenrod kth power P^k_p on the cohomology of any locally finite simplicial set is developed.

2. Preliminaries

In this section we introduce the notation and terminology used throughout this paper. References for this material appear in May (1967) and Mac Lane (1995).

A *simplicial set* K is a graded set indexed by the non-negative integers together with *face* and *degeneracy operators* $\partial_i : K_q \to K_{q-1}$ and $s_i : K_q \to K_{q+1}$, $0 \le i \le q$, satisfying the following identities:

$$\begin{array}{lll} \textbf{(i)} & \partial_i \partial_j = \partial_{j-1}\partial_i, & i < j; \\ \textbf{(ii)} & s_i s_j = s_{j+1} s_i, & i \le j; \\ \textbf{(iii)} & \partial_i s_j = s_{j-1}\partial_i, & i < j; \\ & \partial_i s_j = s_j \partial_{i-1}, & i > j+1; \\ & \partial_j s_j = 1_{K_q} = \partial_{j+1} s_j. & \end{array}$$

The elements of K_q are called *q-simplices*. A simplex x is *degenerate* if $x = s_i(y)$ for some simplex y and degeneracy operator s_i; otherwise, x is *non-degenerate*. Let K and L be two simplicial sets. A map $f = \sum f_q : K_q \to L_q$ of degree zero is a *simplicial map* if it commutes with face and degeneracy operators, i.e., $f_q \partial_i = \partial_i f_{q+1}$ and $f_q s_i = s_i f_{q-1}$.

The *cartesian product* $K \times L$ is a simplicial set whose simplices and face and degeneracy operators are given by

$$(K \times L)_q = K_q \times L_q, \qquad \partial_i(x, y) = (\partial_i x, \partial_i y), \qquad s_i(x, y) = (s_i x, s_i y).$$

Let R be a commutative ring with identity $1 \neq 0$. The *chain complex* of a simplicial set K with coefficients in R, denoted by $C_*(K)$, is constructed as follows. Let $C_n(K)$ denote the free R-module on the set K_n. The face operators ∂_i linearly extend to module maps $\partial_i : C_n(K) \to C_{n-1}(K)$. The alternating sum

$$d_n = \sum_{i=0}^{n} (-1)^i \partial_i : C_n(K) \to C_{n-1}(K)$$

is an R-module endomorphism of degree -1 such that $d_n d_{n+1}$ is null for every $n \ge 0$; it is called the *differential* on $C_*(K)$. The normalized chain complex $C^N_*(K)$ is defined by the quotient

$$C^N_n(K) = C_n(K)/s(C_{n-1}(K)),$$

where $s(C_{n-1}(K))$ denotes the free R-module on the set of all the degenerate n-simplices of K. Since we always work with normalized chain complexes, we simplify notation

and write $C_*(K)$ instead of $C_*^N(K)$. $Z_n(K) = \ker d_n$ is the module of *n-cycles* in $C_*(K)$; $B_n(K) = \operatorname{Im} d_{n+1}$ is the module of *n-boundaries* in $C_*(K)$; the quotient $H_n(K) = Z_n(K)/B_n(K)$ is the nth *homology module* of K. The homology class of a cycle $a \in Z_n(K)$ is denoted by $[a]$.

Given an abelian group G, form the abelian group

$$C^n(K; G) = \operatorname{Hom}_R(C_n(K), G)$$

for each n; the elements of $C^n(K)$ are called the *n-cochains* of $C^*(K; G)$. The differential d on $C_*(K)$ induces a codifferential $\delta : C^*(K; G) \to C^{*+1}(K; G)$ of degree $+1$ via $\delta c = cd$; the *cohomology* of K is the family of abelian groups

$$H^n(K; G) = \ker \delta^n / \operatorname{Im} \delta^{n-1}.$$

$B^n(K; G) = \operatorname{Im} \delta^{n-1}$ is the module of *n-coboundaries*; $Z^n(K; G) = \ker \delta^n$ is the module of n-cocycles. Furthermore, if G is a ring, $H^*(K; G)$ is an algebra with respect to the *cup product*

$$\smile : H^i(K; G) \otimes H^j(K; G) \to H^{i+j}(K; G)$$

defined for $[c^i] \in H^i(K; G)$ and $[c^j] \in H^j(K; G)$ by $[c^i] \smile [c^j] = [c^i \smile c^j]$, where

$$\left(c^i \smile c^j\right)(x) = \mu(c^i(\partial_{i+1} \cdots \partial_{i+j} x) \otimes c^j(\partial_0 \cdots \partial_{i-1} x))$$

for $x \in C_{i+j}(K)$; here μ is the multiplication on G.

Whenever two graded objects x and y of degree p and q are interchanged we apply the *Koszul's convention* and introduce the sign $(-1)^{pq}$. The *tensor product* of chain complexes $C_*(K)$ and $C_*(L)$ is the chain complex $C_*(K) \otimes C_*(L)$ with differential $d_{C_*(K)\otimes C_*(L)} = d_{C_*(K)} \otimes 1_{C_*(L)} + 1_{C_*(K)} \otimes d_{C_*(L)}$. Thus if $x_p \in C_p(K)$ and $y_q \in C_q(L)$, an application of the Koszul convention gives

$$\begin{aligned} d_{C_*(K)\otimes C_*(L)}\left(x_p \otimes y_q\right) &= (d_{C_*(K)} \otimes 1_{C_*(L)} + 1_{C_*(K)} \otimes d_{C_*(L)})(x_p \otimes y_q) \\ &= d_{C_*(K)}(x_p) \otimes y_q + (-1)^q x_p \otimes d_{C_*(L)}(y_q). \end{aligned}$$

A module homomorphism $f : C_*(K) \to C_*(L)$ of degree zero such that $df = fd$ is a *chain map*. If $f : C_*(K) \to C_*(L)$ and $g : C_*(K') \to C_*(L')$ are chain maps, so is $f \otimes g : C_*(K) \otimes C_*(K') \to C_*(L) \otimes C_*(L')$. Examples of chain maps are:

- The *diagonal map* $\Delta : C_*(K) \to C_*(K^{\times n})$ defined by $\Delta(x) = (x, \overset{n \text{ times}}{\ldots}, x)$.
- The *cyclic permutations*

  $$t : C_*(K^{\times n}) \to C_*(K^{\times n}) \qquad \text{and} \qquad T : C_*(K)^{\otimes n} \to C_*(K)^{\otimes n}$$

  such that

  $$t(x_1, x_2, \ldots, x_n) = (x_2, \ldots, x_n, x_1)$$

  and

  $$T(x_1 \otimes x_2 \otimes \cdots \otimes x_n) = (-1)^{|x_1|(|x_2|+\cdots+|x_n|)}(x_2 \otimes \cdots \otimes x_n \otimes x_1).$$

A *contraction from* $C_*(K)$ *to* $C_*(L)$ is a triple of homomorphisms $r = (f, g, \phi)$, respectively referred to as the *projection*, *inclusion* and *homotopy operator*, with the following properties:

- $f : C_*(K) \to C_*(L)$ is a surjective chain map,
- $g : C_*(L) \to C_*(K)$ is an injective chain map,
- $\phi : C_*(K) \to C_{*+1}(K)$ is an endomorphism of degree $+1$,
- $d_{C_*(K)}\phi + \phi d_{C_*(K)} = 1_{C_*(K)} - gf$.

Furthermore, f, g and ϕ satisfy the following identities:

$$\phi g = 0, \qquad f\phi = 0 \qquad \text{and} \qquad \phi\phi = 0.$$

A contraction will be denoted by $r = (f, g, \phi) : C_*(K) \Rightarrow C_*(L)$. Two contractions $r = (f, g, \phi) : C_*(K) \Rightarrow C_*(L)$ and $r' = (f', g', \phi') : C_*(K') \Rightarrow C_*(L')$ can be canonically combined to form new contractions in the following ways:

- The *tensor product contraction* given by

$$r \otimes r' = (f \otimes f',\ g \otimes g',\ \phi \otimes g' f' + 1 \otimes \phi') : C_*(K) \otimes C_*(K') \Rightarrow C_*(L) \otimes C_*(L').$$

- If $L = K'$, the *composition contraction* given by

$$r'r = (f' f,\ gg',\ \phi + g\phi' f) : C_*(K) \Rightarrow C_*(L').$$

Let p and q be non-negative integers. A (p, q)-*shuffle* (α, β) is a partition

$$\{\alpha_1 < \cdots < \alpha_p\} \cup \{\beta_1 < \cdots < \beta_q\}$$

of the set $\{0, 1, \ldots, p + q - 1\}$. The signature of (α, β) is given by

$$\mathrm{sig}(\alpha, \beta) = \sum_{1 \le i \le p} \alpha_i - (i - 1).$$

Let $\gamma = \{\gamma_1, \ldots, \gamma_r\}$ be a set of integers. Then s_γ denotes the composition of the degeneracy operators $s_{\gamma_r} \cdots s_{\gamma_1}$.

An *Eilenberg–Zilber contraction* (Eilenberg and Zilber, 1959) from the chain complex $C_*(K \times L)$ to the tensor product of chain complexes $C_*(K)$ and $C_*(L)$ is a triple $r_{EZ} = (AW, EML, SHI)$ such as:

- The Alexander–Whitney operator $AW : C_*(K \times L) \longrightarrow C_*(K) \otimes C_*(L)$ is defined by

$$AW(x_m, y_m) = \sum_{0 \le i \le m} \partial_{i+1} \cdots \partial_m x_m \otimes \partial_0 \cdots \partial_{i-1} y_m,$$

where $(x_m, y_m) \in C_m(K \times L)$.

- The Eilenberg–Mac Lane operator $EML : C_*(K) \otimes C_*(L) \longrightarrow C_*(K \times L)$ is defined by

$$EML(x_p \otimes y_q) = \sum_{(\alpha,\beta) \in \{(p,q)\text{-shuffles}\}} (-1)^{\mathrm{sig}(\alpha,\beta)} (s_\beta x_p,\ s_\alpha y_q),$$

where $x_p \otimes y_q \in C_p(K) \otimes C_q(L)$.

- And the Shih operator $SHI : C_*(K \times L) \longrightarrow C_{*+1}(K \times L)$ is defined by

$$\begin{aligned} &SHI(x_0,\ y_0) = 0,\\ &SHI(x_m,\ y_m)\\ &= \sum_{T(m)} (-1)^{\epsilon(\alpha,\beta)}\ (s_{\bar{\beta}+\bar{m}}\partial_{m-q+1}\cdots\partial_m x_m,\ s_{\alpha+\bar{m}}\partial_{\bar{m}}\cdots\partial_{m-q-1}y_m), \end{aligned}$$

where

$$\begin{aligned} &T(m) = \{0 \le p \le m-q-1 \le m-1,\ (\alpha,\beta) \in \{(p+1,q)\text{-shuffles}\}\},\\ &\bar{m} = m-p-q,\\ &\alpha+\bar{m} = \{\alpha_1+\bar{m},\ldots,\alpha_{p+1}+\bar{m}\},\\ &\bar{\beta}+\bar{m} = \{\bar{m}-1,\beta_1+\bar{m},\ldots,\beta_q+\bar{m},\},\\ &\epsilon(\alpha,\beta) = \bar{m}-1+\mathrm{sig}(\alpha,\beta). \end{aligned}$$

A recursive formula for the *SHI* operator appears in Eilenberg and Mac Lane (1954). The explicit formula given here was stated by Rubio in Rubio (1991) and proved by Morace in the appendix of Real (1996). It is evident that the *AW* operator has a polynomial nature (concretely, the number of face operators involved in its formula is $O(m^2)$). However, the *EML* and *SHI* operator have an essential "exponential" character, because shuffles of degeneracy operators are involved in their respective formulations. In Prouté (1983), Prouté determines that *EML* is unique and there are only two possibilities for *AW*, both of its formulae being of the same complexity. Concerning *SHI*, all the possible formulae have in common their exponential nature.

There is a contraction from $C_*(K^{\times n})$ to $C_*(K)^{\otimes n}$ obtained by appropriately composing Eilenberg–Zilber contractions. For any positive integers $s < n$, let us denote by $r_{EZ(n,s)} = (AW_{(n,s)}, EML_{(n,s)}, SHI_{(n,s)})$ the contraction

$$r_{EZ(n,s)} \otimes 1^{\otimes s-1} = (AW \otimes 1^{\otimes s-1}, EML \otimes 1^{\otimes s-1}, SHI \otimes 1^{\otimes s-1})$$

from $C_*(K^{\times n-s} \times K) \otimes C_*(K)^{\otimes s-1}$ to $C_*(K^{\times n-s}) \otimes C_*(K) \otimes C_*(K)^{\otimes s-1}$. Then, the composition $r_{EZ(n,n-1)} \cdots r_{EZ(n,2)} r_{EZ(n,1)}$ is a contraction from $C_*(K^{\times n})$ to $C_*(K)^{\otimes n}$. We denote it by

$$r_{EZ(n)} = (AW_{(n)}, EML_{(n)}, SHI_{(n)}) : C_*(K^{\times n}) \Rightarrow C_*(K)^{\otimes n}.$$

Observe that the expression of $AW_{(n)}$ is:

$$\begin{aligned} AW_{(n)}(\mathbf{x}) &= AW_{(n,n-1)}\quad AW_{(n,n-2)}\cdots AW_{(n,2)}AW_{(n,1)}(\mathbf{x})\\ &= \sum_{0\le i_1\cdots\le i_{n-1}\le m} \partial_{i_1+1}\cdots\partial_m x_1\\ &\qquad \otimes \partial_0\cdots\partial_{i_1-1}\partial_{i_2+1}\cdots\partial_m x_2\\ &\qquad \vdots\\ &\qquad \otimes \partial_0\cdots\partial_{i_{n-2}-1}\partial_{i_{n-1}+1}\cdots\partial_m x_{n-1}\\ &\qquad \otimes \partial_0\cdots\partial_{i_{n-1}-1} x_n \end{aligned} \tag{2}$$

where $\mathbf{x} = (x_1,\ldots,x_n) \in C_m(K^{\times n})$. The number of face operators taking part in this formula is $O(n \cdot m^n)$.

On the other hand, the expression of $SHI_{(n)}$ in terms of the component morphisms of the previous Eilenberg–Zilber contractions is:

$$\begin{aligned}\sum_{1\le\ell+1<n} & EML_{(n,1)}\cdots EML_{(n,\ell)}SHI_{(n,\ell+1)}AW_{(n,\ell)}\cdots AW_{(n,1)}\\ &= SHI_{(n,1)}\\ &\quad + EML_{(n,1)}SHI_{(n,2)}AW_{(n,1)}\\ &\quad \vdots\\ &\quad + EML_{(n,1)}\cdots EML_{(n,n-2)}SHI_{(n,n-1)}AW_{(n,n-2)}\cdots AW_{(n,1)}.\end{aligned}$$

Observe that whereas the number of summands in the formula for $AW_{(n)}$ grows in polynomial time (fixed n), the number of summands in the formulae for $EML_{(n)}$ and $SHI_{(n)}$ grow exponentially.

3. Simplification techniques

Let us recall that our motivation here is to simplify any composition of the type

$$AW_{(p)}t_rSHI_{(p)}t_{r-1}\cdots SHI_{(p)}t_1SHI_{(p)} = \sum AW_{(p)}t_rESA_{(p,\ell_r)}\cdots t_1ESA_{(p,\ell_1)}$$

where every t_i is any kind of permutation of p factors,

$$ESA_{(p,\ell)} = EML_{(p,1)}\cdots EML_{(p,\ell)}SHI_{(p,\ell+1)}AW_{(p,\ell)}\cdots AW_{(p,1)}$$

and the sum is taken over the set $\{1\le i\le r, 0\le\ell_i\le p-2, 1\le k_i\le p-1\}$.

We will use the following basic properties:

- Any composition of face and degeneracy operators of K can be put in a unique "normalized" form:

  $$s_{j_t}\cdots s_{j_1}\partial_{i_1}\cdots\partial_{i_r},$$

  where $j_t>\cdots>j_1\ge 0$ and $i_s>\cdots>i_1\ge 0$.
- Any summand on the tensor product of n copies of $C_*(K)$ having a factor (in the normalized form) with degeneracy operators in its expression, is degenerate.

Let i, j, m be integers such that $0\le i\le j\le m$. The interval $[i, j)$ denotes the set of consecutive integers from i to $j-1$.

- The *face-interval* $\partial_{[i,j)}$, denotes the composition $\partial_0\cdots\partial_{i-1}\partial_{j+1}\cdots\partial_m$.
- If $i=0$ then $\partial_{[0,j)}=\partial_{j+1}\cdots\partial_m$.
- If $j=m$ then $\partial_{[i,m)}=\partial_0\cdots\partial_{i-1}$.
- In the case $i=j$ then $\partial_{[i,i)}=\partial_0\cdots\partial_{i-1}\partial_{i+1}\cdots\partial_m$.

The notation $\partial_{[i,j)}$ must be interpreted as the interval $[i, j)$ representing the indexes ℓ, $0\le\ell\le m-1$, such that $\partial_0\cdots\partial_{i_{j-1}-1}\partial_{i_j+1}\cdots\partial_m s_\ell$ is degenerate, whereas $j_1\le i_2$ define the following "composition":

$$\partial_{[i_1,j_1)}\partial_{[i_2,j_2)} = \partial_0\cdots\partial_{i_1-1}\partial_{j_1+1}\cdots\partial_{i_2-1}\partial_{j_2+1}\cdots\partial_m.$$

This composition can be extended without problems to the composition of n face-intervals.

With the new notation, we can rewrite the expression of $AW_{(n)}$ given in page 1213 as:

$$AW_{(n)}(\mathbf{x}) = \sum_{P(m,n)} \partial_{[1]}x_1 \otimes \partial_{[2]}x_2 \cdots \otimes \partial_{[n]}x_n,$$

where $[\ell]$ represents the interval $[i_{\ell-1}, i_\ell)$ and $P(m,n)$ is the set of all the possible partitions of $[0, m+1)$ in n intervals.

First, in order to gradually show our technique, let us simplify the composition $AW_{(n)}t^kESA_{(n,0)}(\mathbf{x}) = AW_{(n)}t^kSHI_{(n,1)}(\mathbf{x})$, where $1 \le k \le n-1$:

$$\begin{aligned} &AW_{(n)}t^kESA_{(n,0)}(\mathbf{x}) \\ &= \sum_{P(m+1,n)} \sum_{T(m)} (-1)^{\epsilon(\alpha,\beta)} \ \partial_{[1]}s_{\beta+\bar{m}}\partial_{m-q+1}\cdots\partial_m x_{k+1} \\ &\qquad\qquad \vdots \\ &\qquad\qquad \otimes \partial_{[n-k-1]}s_{\beta+\bar{m}}\partial_{m-q+1}\cdots\partial_m x_{n-1} \\ &\qquad\qquad \otimes \partial_{[n-k]}s_{\alpha+\bar{m}}\partial_{\bar{m}}\cdots\partial_{m-q-1}x_n \\ &\qquad\qquad \otimes \partial_{[n-k+1]}s_{\beta+\bar{m}}\partial_{m-q+1}\cdots\partial_m x_1 \\ &\qquad\qquad \vdots \\ &\qquad\qquad \otimes \partial_{[n]}s_{\beta+\bar{m}}\partial_{m-q+1}\cdots\partial_m x_k. \end{aligned} \tag{3}$$

On one hand,

$$(\alpha+\bar{m}) \cup (\beta+\bar{m}) = [\bar{m}-1, m+1) \quad \text{and} \quad \bar{m}-1 \in \beta+\bar{m}.$$

On the other hand, the non-degenerate summands of (3) satisfy that

$$(\alpha+\bar{m}) \cap [i_{n-k-1}, i_{n-k}) = \emptyset \ \text{ and } \ (\beta+\bar{m}) \cap ([0, i_{n-k-1}) \cup [i_{n-k}, m+1)) = \emptyset.$$

We immediately obtain that

$$\beta+\bar{m} \subset [i_{n-k-1}, i_{n-k}) \qquad \text{and} \qquad \alpha+\bar{m} \subset [0, i_{n-k-1}) \cup [i_{n-k}, m+1),$$

therefore $i_{n-k-1} \le \bar{m}-1$, $i_{n-k} = m-p$,

$$\beta+\bar{m} = [\bar{m}-1, i_{n-k}) \qquad \text{and} \qquad \alpha+\bar{m} = [i_{n-k}, m+1).$$

Now, we denote

$$i'_j = \begin{cases} i_j & 0 \le j < n-k, \\ i_j - q - 1 & n-k \le j \le n, \\ m & j = n+1. \end{cases}$$

and we can rewrite (3) as:

$$\sum_{P(m,n+1)} (-1)^{\tau_0} \ \partial_{[1]}x_{k+1} \otimes \cdots \otimes \partial_{[n-k-1]}x_{n-1} \otimes \partial_{[n-k]}\partial_{[n+1]}x_n \otimes \partial_{[n-k+1]}x_1 \otimes \cdots \otimes \partial_{[n]}x_k$$

where

$$\begin{aligned}\tau_0 &= \bar{m} - 1 + (p+1)q = i'_{n-k} + (i'_n - i'_{n-k})(i'_{n+1} - i'_n)\\ &= |1| + \cdots + |n-k| + (|n-k+1| + \cdots + |n|)|n+1|,\end{aligned}$$

$|\ell|$ being $i'_\ell - i'_{\ell-1}$.

In the same way, the expression of $AW_{(n)}t^k ESA_{(n,1)}(\mathbf{x})$ is:

$$\begin{aligned}\sum_{\substack{0\le\iota\le m\\ P(m+1,n),\ T(\iota)\\ (a,b)\in\{(\iota+1,m-\iota)\text{–sh.}\}}} & (-1)^{\mathrm{sig}(a,b)+\epsilon(\alpha,\beta)}\\ & \partial_{[1]}s_b s_{\beta+\bar{\iota}}\partial_{\iota-q+1}\cdots\partial_m x_{k+1}\\ & \vdots\\ & \otimes \partial_{[n-k-2]}s_b s_{\beta+\bar{\iota}}\partial_{\iota-q+1}\cdots\partial_m x_{n-2}\\ & \otimes \partial_{[n-k-1]}s_b s_{\alpha+\bar{\iota}}\partial_{\bar{\iota}}\cdots\partial_{\iota-q-1}\partial_{\iota+1}\cdots\partial_m x_{n-1}\\ & \otimes \partial_{[n-k]}s_a\partial_0\cdots\partial_{\iota-1}x_n\\ & \otimes \partial_{[n-k+1]}s_b s_{\beta+\bar{\iota}}\partial_{\iota-q+1}\cdots\partial_m x_1\\ & \vdots\\ & \otimes \partial_{[n]}s_b s_{\beta+\bar{\iota}}\partial_{\iota-q+1}\cdots\partial_m x_k.\end{aligned} \tag{4}$$

On one hand, $a \cup b = [0, m+1)$ and on the other hand, the non-degenerate summands satisfy that

$$a \cap [i_{n-k-1}, i_{n-k}) = \emptyset \qquad \text{and} \qquad b \cap ([0, i_{n-k-1}) \cup [i_{n-k}, m+1)) = \emptyset,$$

then $b = [i_{n-k-1}, i_{n-k})$ and $a = [0, i_{n-k-1}) \cup [i_{n-k}, m+1)$. We denote

$$i'_j = \begin{cases} i_j & 0 \le j < n-k,\\ i_{j+1} - m + \iota & n-k \le j \le n-2.\end{cases}$$

Therefore (4) becomes

$$\begin{aligned}\sum_{\substack{0\le\iota\le m\\ P(\iota+1,n-1),\ T(\iota)}} & (-1)^{\mathrm{sig}(a,b)+\epsilon(\alpha,\beta)} \quad \partial_{[1]}s_{\beta+\bar{\iota}}\partial_{\iota-q+1}\cdots\partial_m x_{k+1}\\ & \vdots\\ & \otimes \partial_{[n-k-2]}s_{\beta+\bar{\iota}}\partial_{\iota-q+1}\cdots\partial_m x_{n-2}\\ & \otimes \partial_{[n-k-1]}s_{\alpha+\bar{\iota}}\partial_{\bar{\iota}}\cdots\partial_{\iota-q-1}\partial_{\iota+1}\cdots\partial_m x_{n-1}\\ & \otimes \partial_0\cdots\partial_{\iota-1}x_n\\ & \otimes \partial_{[n-k]}s_{\beta+\bar{\iota}}\partial_{\iota-q+1}\cdots\partial_m x_1\\ & \vdots\\ & \otimes \partial_{[n-1]}s_{\beta+\bar{\iota}}\partial_{\iota-q+1}\cdots\partial_m x_k.\end{aligned} \tag{5}$$

and $\mathrm{sig}(a,b)$ is $(m-\iota)(\iota+1-i'_{n-k-1})$. Now, we can observe that if $k+1=n$ then the composition above is degenerate, else

$$i'_{n-k-2} \le \bar{\iota} - 1, \quad \beta + \bar{\iota} = [\bar{\iota} - 1, i'_{n-k-2}) \quad \text{and} \quad \alpha + \bar{\iota} = [i'_{n-k-1}, \iota + 1).$$

We denote

$$i''_j = \begin{cases} i'_j & 0 \le j < n-k-1, \\ i'_j - q - 1 & n-k-1 \le j \le n-2, \\ \iota - q & j = n-1, \\ \iota & j = n, \\ m & j = n+1 \end{cases}$$

therefore (5) is

$$\sum_{P(m,n+1)} (-1)^{\tau_1} \ \partial_{[1]}x_{k+1} \otimes \cdots \otimes \partial_{[n-k-2]}x_{n-2} \otimes \partial_{[n-k-1]}\partial_{[n]}x_{n-1} \otimes \partial_{[n+1]}x_n \otimes \partial_{[n-k]}x_1 \otimes \cdots \otimes \partial_{[n-1]}x_k$$

and the sign:

$$\begin{aligned} \tau_1 &= (m-\iota)(\iota + 1 - i'_{n-k-1}) + \bar{\iota} - 1 + (p+1)q \\ &= (i''_{n+1} - i''_n)(i''_n + 1 - i''_{n-k-1} + i''_{n-1} - i''_n - 1) \\ &\quad + i''_{n-k-1} + (i''_{n-1} - i''_{n-k-1})(i''_n - i''_{n-1}) \\ &= i''_{n-k-1} + (i''_{n-1} - i''_{n-k-1})(i''_{n+1} - i''_{n-1}) \\ &= |1| + \cdots + |n-k-1| + (|n-k| + \cdots + |n-1|)(|n| + |n+1|). \end{aligned}$$

Now, let us study the general case. As we said before, we are interested in simplifying any composition of the form

$$AW_{(n)}t_r ESA_{(n,\ell_r)} \cdots t_1 ESA_{(n,\ell_1)}. \tag{6}$$

We will do it inductively. Let $h : C_*(K^{\times n}) \to C_*(K)^{\otimes n}$ be a morphism of degree r whose normalized expression is:

$$h(\mathbf{x}) = \sum_{P(m,n+r)} (-1)^{\mathrm{sign}\{[1],\ldots,[n+r]\}} \partial_{[\ ]}x_{k_1} \otimes \partial_{[\ ]}x_{k_2} \cdots \otimes \partial_{[\ ]}x_{k_n}$$

such that $(x_{k_1}, \ldots, x_{k_n}) = t_\lambda(x_1, \ldots, x_n)$ where $t_\lambda : C_*(K^{\times n}) \to C_*(K^{\times n})$ is any permutation and each $\partial_{[\ ]}$ denotes a composition of non-consecutive elements of the set $\{\partial_{[1]}, \partial_{[2]}, \ldots, \partial_{[n+r]}\}$ where $\{[1], [2], \ldots, [n+r]\} \in P(m, n+r)$; moreover, each $\partial_{[j]}$, $1 \le j \le n+r$, appears exactly once in the expression of $h(\mathbf{x})$. Our goal is to simplify the composition $H = h\,ESA_{(n,\ell)}$, where $0 \le \ell \le n-2$.

Proposition 1. *If one of the following conditions holds on h:*

- *There is no face-interval preceding x_j for $1 \le j \le n$;*
- *There exists a factor in $h(\mathbf{x})$ with more than one face-interval preceding x_{n+1-u} for some $1 \le u \le \ell$;*
- *The face-interval $\partial_{[j]}$ immediately before $x_{n-\ell}$ in $h(\mathbf{x})$ satisfies that $j = \max\{v \text{ such that } \partial_{[v]} \text{ appears preceding some } x_u \text{ for } 1 \le u \le n-\ell\}$;*

then all the summands of H are degenerate.

From now on, let us suppose that $h(\mathbf{x})$ does not satisfy any of the conditions of the proposition above. Let us denote by $\partial_{[j_u]}$ the unique face-interval preceding x_{n+1-u} for $1 \leq u \leq \ell$.

Lemma 2. *If the composition $\partial_{[j_u-1]}\partial_{[j_u+1]}$ appears in the expression of h for some u, $1 \leq u \leq \ell$, then all the summands of H are degenerate.*

Theorem 3. SIMPLIFICATION ALGORITHM.

INPUT: `The morphism` $h : C_*(K^{\times n}) \to C_*(K)^{\otimes n}$ `of degree` r `described above such that it does not satisfy either Proposition 1 or Lemma 2.`

OUTPUT: `The simplified expression of` $H(\mathbf{x}) = h\,ESA_{(n,\ell)}(\mathbf{x})$.

`For` $u = 1$ `to` $u = \ell$ `do`
  `replace` $\partial_{[j_u]}$ `preceding` x_{n+1-u} `by` $\partial_{[n+r+2-u]}$.
`End for.`

Let $\{\partial_{[v_1]}, \ldots, \partial_{[v_{n+r-\ell}]}\}$, $v_1 < \cdots < v_{n+r-\ell}$, denote the set of the face-intervals preceding x_u for $1 \leq u \leq n-\ell$.

`For` $s = 1$ `to` $s = n + r - \ell$ `do`
  `replace` $\partial_{[v_s]}$ `by` $\partial_{[s]}$.
`End for.`
`Replace` $x_{n-\ell}$ `by` $\partial_{[n+r-\ell+1]}x_{n-\ell}$.

Starting from the sign of h of degree $m+1$, we obtain the sign of H of degree m as follows.

Step 1:

`For` $u = 1$ `to` $u = \ell$ `do`
  `replace` $|j_u|$ `by` $|n + r - u + 1| + 1$.
  `For` $j = j_u + 1$ `to` $j = n + r - u + 1$ `do`
    `replace` $|j|$ `by` $|j - 1|$.
  `End for;`
  `add` $(|n + r - u + 1| + 1)(|j_u| + \cdots + |n + r - u|)$
`End for.`

Let $\partial_{[v]}$ be the face-interval immediately before $x_{n-\ell}$. Starting from the modified sign of H do

Step 2:

`For` $j = n + r - \ell + 2$ `to` $j = n + r$ `do`
  `replace` $|j|$ `by` $|j + 1|$.
`End for;`
`replace` $|n + r - \ell + 1|$ `by` $|n + r - \ell + 2| - 1$;
`replace` $|v|$ `by` $|n + r - \ell + 1| + 1$;
`add` $|1| + \cdots + |v| + (|v + 1| + \cdots + |n + r - \ell|)|n + r - \ell + 1|$.

Proof. For the sake of simplicity but without loss of generality, we consider that the expression of $h(\mathbf{x})$ is

$$\sum_{P(m,n+r)} (-1)^{\mathrm{sign}\{[1],\ldots,[n+r]\}} \partial_{[\,]} x_1 \otimes \cdots \partial_{[\,]} x_{n-\ell} \otimes \otimes \partial_{[j_\ell]} x_{n-\ell+1} \otimes \cdots \partial_{[j_1]} x_n;$$

consequently, the expression of $H(\mathbf{x})$ is:

$$\begin{aligned}
\sum_{\substack{P(m+1,n+r),\ T(\iota_\ell) \\ 0\le \iota_\ell \le \iota_{\ell-1}-1 \le \cdots \le \iota_1 \le m \\ \{(a_j,b_j)\in\{(\iota_j+1,m-\iota_j)-\mathrm{sh.}\}:\ 1\le j\le \ell\}}} & (-1)^{\mathrm{sign}\{[1],\ldots,[n+r]\}+\mathrm{sig}(a_1,b_1)+\cdots+\mathrm{sig}(a_\ell,b_\ell)+\epsilon(\alpha,\beta)} \\
& \partial_{[\,]} s_{b_1} \cdots s_{b_\ell} s_{\beta+\overline{\iota_\ell}} \partial_{\iota_\ell - q+1} \cdots \partial_m x_1 \\
& \vdots \\
& \otimes \partial_{[\,]} s_{b_1} \cdots s_{b_\ell} s_{\beta+\overline{\iota_\ell}} \partial_{\iota_\ell - q+1} \cdots \partial_m x_{n-\ell-1} \\
& \otimes \partial_{[\,]} s_{b_1} \cdots s_{b_\ell} s_{\alpha+\overline{\iota_\ell}} \partial_{\overline{\iota_\ell}} \cdots \partial_{\iota_\ell - q-1} \partial_{\iota_\ell+1} \cdots \partial_m x_{n-\ell} \\
& \otimes \partial_{[j_\ell]} s_{b_1} \cdots s_{b_{\ell-1}} s_{a_\ell} \partial_0 \cdots \partial_{\iota_\ell-1} \partial_{\iota_{\ell-1}+1} \cdots \partial_m x_{n-\ell+1} \\
& \vdots \\
& \otimes \partial_{[j_2]} s_{b_1} s_{a_2} \partial_0 \cdots \partial_{\iota_2-1} \partial_{\iota_1+1} \cdots \partial_m x_{n-1} \\
& \otimes \partial_{[j_1]} s_{a_1} \partial_0 \cdots \partial_{\iota_1-1} x_n.
\end{aligned} \tag{7}$$

The non-degenerate summands of $H(\mathbf{x})$ satisfy that

$$a_1 = [0, i_{j_1-1}) \cup [i_{j_1}, m+1) \qquad \text{and} \qquad b_1 = [i_{j_1-1}, i_{j_1}).$$

Then,

$$\begin{aligned}
& i^1_j = i_j \ \text{ for } 0 \le j < j_1, \\
& i^1_j = i_{j+1} - m + \iota_1 \ \text{ for } j_1 \le j < n+r-1, \\
& i^1_{n+r-1} = \iota_1 + 1, \\
& i^1_{n+r} = m.
\end{aligned}$$

Therefore, we have that

$$\begin{aligned}
& i_j = i^1_j \ \text{ for } 0 \le j < j_1, \\
& i_j = i^1_{j-1} + i^1_{n+r} - i^1_{n+r-1} + 1 \ \text{ for } j_1 \le j \le n+r-1, \\
& i_{n+r} = i^1_{n+r} + 1.
\end{aligned}$$

So, in $\mathrm{sign}\{[1],\ldots,[n+r]\}$, $|j_1|$ is replaced by $|n+r|+1$, $|j|$ is replaced by $|j-1|$ for $j_1 < j \le n+r$ and

$$\begin{aligned}
\mathrm{sig}(a_1,b_1) &= (m-\iota_1)(\iota_1+1-i_{j_1-1}) = (i^1_{n+r} - i^1_{n+r-1}+1)(i^1_{n+r-1} - i'_{j_1-1}) \\
&= (|n+r|+1)(|j_1|+\cdots+|n+r-1|),
\end{aligned}$$

is added.

In general, fixed u, $1 \le u \le \ell$, we have that

$$a_u = \left[0, i^{u-1}_{j_u-1}\right) \cup \left[i^{u-1}_{j_u}, \iota_{u-1}+1\right) \qquad \text{and} \qquad b_u = \left[i^{u-1}_{j_u-1}, i^{u-1}_{j_u}\right).$$

Then,

$$
\begin{aligned}
&i^u_j = i^{u-1}_j \text{ for } 0 \le j < j_u,\\
&i^u_j = i^{u-1}_{j+1} - \iota_{u-1} + \iota_u \text{ for } j_u \le j < n+r-u,\\
&i^u_{n+r-u} = \iota_u + 1,\\
&i^u_{n+r-u+1} = \iota_{u-1}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&i^{u-1}_j = i^u_j \text{ for } 0 \le j < j_u \text{ and } n+r-u+2 \le j \le n+r,\\
&i^{u-1}_j = i^u_{j-1} + i^u_{n+r-u+1} - i^u_{n+r-u} + 1 \text{ for } j_u \le j \le n+r-u,\\
&i^{u-1}_{n+r-u+1} = i^u_{n+r-u+1} + 1.
\end{aligned}
$$

So, in sign$\{[1],\dots,[n+r]\}$, $|j_u|$ is replaced by $|n+r-u+1|+1$ and $|j|$ is replaced by $|j-1|$ for $j_u < j \le n+r-u+1$. Also,

$$
\begin{aligned}
\operatorname{sig}(a_u, b_u) &= (\iota_{u-1} - \iota_u)(\iota_u + 1 - i^{u-1}_{j_1-1})\\
&= (|n+r-u+1|+1)(|j_u| + \cdots + |n+r-u|),
\end{aligned}
$$

is added. Therefore, the expression of (7) is:

$$
\begin{aligned}
\sum_{\substack{P(\iota_\ell+1, n+r-\ell),\ T(\iota_\ell)\\ 0\le \iota_\ell \le \iota_{\ell-1} \le \cdots \le \iota_1 \le m}} \quad & (-1)^{\operatorname{sign}\{[1],\dots,[n+r]\}+\epsilon(\alpha,\beta)}\\
& \partial_{[\ ]} s_{\beta+\overline{\iota_\ell}} \partial_{\iota_\ell - q+1} \cdots \partial_m x_1\\
& \vdots\\
& \otimes \partial_{[\ ]} s_{\beta+\overline{\iota_\ell}} \partial_{\iota_\ell - q+1} \cdots \partial_m x_{n-\ell-1}\\
& \otimes \cdots \partial_{[v]} s_{\alpha+\overline{\iota_\ell}} \partial_{\overline{\iota_\ell}} \cdots \partial_{\iota_\ell - q-1} \partial_{\iota_\ell+1} \cdots \partial_m x_{n-\ell}\\
& \otimes \partial_0 \cdots \partial_{\iota_\ell - 1} \partial_{\iota_{\ell-1}+1} \cdots \partial_m x_{n-\ell+1}\\
& \vdots\\
& \otimes \partial_0 \cdots \partial_{\iota_2 - 1} \partial_{\iota_1+1} \cdots \partial_m x_{n-1}\\
& \otimes \partial_0 \cdots \partial_{\iota_1 - 1} x_n.
\end{aligned}
$$

Now, $\alpha + \overline{\iota_\ell} = [i^\ell_v, \iota_\ell + 1)$ and $\beta + \overline{\iota_\ell} = [\overline{\iota_\ell} - 1, i^\ell_v)$, then

$$
i^{\ell+1}_j = i^\ell_j \text{ for } 0 \le j \le v-1, \qquad i^{\ell+1}_j = i^\ell_j - q - 1 \text{ for } v \le j \le n+r-\ell-1,
$$

$$
i^{\ell+1}_{n+r-\ell} = \iota_\ell - q, \qquad i^{\ell+1}_{n+r-\ell+1} = \iota_\ell, \qquad i^{\ell+1}_{j+1} = i^\ell_j \text{ for } n+r-\ell+1 \le j \le n+r.
$$

That is,

$$
i^\ell_j = i^{\ell+1}_j \text{ for } 0 \le j \le v-1, \qquad i^\ell_j = i^{\ell+1}_j + q + 1 \text{ for } v \le j \le n+r-\ell-1,
$$

$$
i^\ell_{n+r-\ell} = i^{\ell+1}_{n+r-\ell+1} + 1, \qquad i^\ell_j = i^{\ell+1}_{j+1} \text{ for } n+r-\ell+1 \le j \le n+r.
$$

So, in sign$\{[1], \ldots, [n+r]\}$, $|j|$ is replaced by $|j+1|$ for $n+r-\ell+2 \leq j \leq n+r$, $|v|$ is replaced by $|n+r-\ell+1|+1$ and $|n+r-\ell+1|$ is replaced by $|n+r-\ell+2|-1$. Finally,

$$\begin{aligned}\epsilon(\alpha,\beta) &= \overline{\iota_\ell} - 1 + (p+1)q \\ &= i_v^{\ell+1} + (i_{n+r-\ell}^{\ell+1} - i_v^{\ell+1})(i_{n+r-\ell+1}^{\ell+1} - i_{n+r-\ell}^{\ell+1}) \\ &= |1| + \cdots + |v| + (|v+1| + \cdots + |n+r-\ell|)|n+r-\ell+1|\end{aligned}$$

is added. □

Theorem 4. *The number of face operators taking part in the normalized formula for* $AW_{(p)}t_r SHI_{(p)} \cdots t_1 SHI_{(p)}$ *is, in the worst case,* $O(p^{r+1}m^{p+r+1})$.

Proof. On one hand, the number of summands of the form (6) is $(p-1)^r$. On the other hand, the number of summands in the simplified formula for each morphism (6) is $O(m^{p+r})$ and the number of face operators in each summand is $O(pm)$. Therefore the number of face operators taking part in the normalized formula for $AW_{(p)}t_r SHI_{(p)}t_{r-1} \cdots SHI_{(p)}t_1 SHI_{(p)}$ is $O((p-1)^r m^{p+r} pm)$ that is $O(p^{r+1}m^{p+r+1})$. □

4. An example: Algorithm for computing P_p^k

In this section we study the computation of the cohomology operations Steenrod kth powers P_p^k (Steenrod, 1952) as an application of the technique given in the section above. First, we give the definition of these operations at the cochain level due to Steenrod (1952). We next show explicit formulae developed in Gonzalez-Diaz and Real (1999) for these operations in terms of Eilenberg–Zilber contractions at the cochain level. Finally, we develop an algorithm for computing P_p^k at the cohomology level on any locally finite simplicial set.

An infinite sequence of morphisms $\{D_r^n : C_*(K) \to C_*(K)^{\otimes n}\}_{r\geq 0}$ of degree r such that:

$$D_0^n = AW_{(n)}\Delta; \qquad d_{C_*(K)^{\otimes n}} D_r^n + (-1)^{r-1} D_r^n d_{C_*(K)} = \alpha_r D_{r-1}^n,\ r > 0; \tag{8}$$

where $\alpha_r : C_*(K)^{\otimes n} \to C_*(K)^{\otimes n}$ is defined by

$$\alpha_r = \begin{cases} T-1 & \text{if } r \text{ odd}, \\ 1+T+\cdots+T^{n-1} & \text{if } r \text{ even}, \end{cases}$$

called a *higher diagonal approximation* (Steenrod, 1952) "measures" the lack of commutativity of $AW_{(n)}$.

In the particular case of $p=2$, it is possible to define cochain mappings called *cup-i product*,

$$\smile_i : C^q(K;G) \otimes C^p(K;G) \to C^{q+p-i}(K;G)$$

by $c \smile_i c' = \mu(c \otimes c')D_i^2$. Observe that the expression of $c \smile_0 c'$ coincides with that of the cup product given in page 1211. Taking $[c] \in H^j(K;\mathbf{Z}_2)$, the cohomology operations *Steenrod squares* (Steenrod, 1947) are defined by $Sq^i[c] = [c \smile_{j-i} c] \in H^{j+i}(K;\mathbf{Z}_2)$.

Now, let $p > 2$ be a prime number. Starting from the sequence (8), the *Steenrod kth power* $P_p^k : H^q(K;\mathbf{Z}_p) \to H^{q+2k(p-1)}(K;\mathbf{Z}_p)$, $q \geq 2k$, is defined at the cochain level as follows. If $c \in Z^q(K;\mathbf{Z}_p)$, then

$$P_p^k(c) = R\,\mu c^{\otimes p} D^p_{(q-2k)(p-1)} \in Z^{q+2k(p-1)}(K;\mathbf{Z}_p), \tag{9}$$

where μ is the natural product on $\mathbf{Z}_p$ and $R = (-1)^{(p-1)(k+\frac{1}{2}q(q-1))}\left(\left(\frac{p-1}{2}\right)!\right)^{2k-q}$.

The acyclic model method (Eilenberg and Mac Lane, 1953) is used for guaranteeing the existence of the morphisms D_r^n (n and r being non-negative integers). An alternative to the previous method is to obtain the morphisms D_r^n using algebraic fibrations with a cartesian product of n copies of a given simplicial set K as the base space and a subgroup of the symmetric group S_n as the fiber space. This last point of view has been established in Real (1996) and Gonzalez-Diaz and Real (1999) for Steenrod operations, in Gonzalez-Diaz and Real (2002a) for secondary cohomology operations and generalized in Gonzalez-Diaz (2000) for any cohomology operation. In Gonzalez-Diaz and Real (1999) we obtain explicit formulae for a higher diagonal approximation in terms of the component morphisms of a given Eilenberg–Zilber contraction. Let $\gamma_j : C_*(K^{\times n}) \to C_*(K^{\times n})$ be defined by

$$\gamma_j = \begin{cases} t & \text{if } j \text{ odd} \\ t + \cdots + t^{n-1} & \text{if } j \text{ even.} \end{cases}$$

then

$$D_r^n = AW_{(n)}\gamma_r SHI_{(n)} \cdots \gamma_1 SHI_{(n)}\Delta = \sum AW_{(n)}t^{k_r} ESA_{(n,\ell_r)} \cdots t^{k_1} ESA_{(n\ell_1)}\Delta$$

where the sum is taken over all the possible $1 \leq \ell_i + 1, k_i < n$, where $k_i = 1$ if $i + r$ odd; for all $1 \leq i \leq r$.

Observe that an algorithm based on these formulae for D_r^n is not useful in practice, due to the exponential nature of the morphisms involved. Nevertheless, we can apply the Simplification Algorithm explained before in order to obtain a pure combinatorial definition of D_r^n only in terms of face operators. Notice that for obtaining a normalized expression of D_r^n, we have to apply Theorem 3 $(n-1)^{\lfloor r/2 \rfloor}(n-1)^r$ times in the worst case. However, taking into account Proposition 1, the non-degenerate summands of D_r^n can only appear when $k_i + \ell_i < n$ for $1 \leq i \leq r$. Moreover, if $k_i + \ell_i < n$ and $k_i < \ell_{i+1}$ then the non-degenerate summands of D_r^n can only appear when $k_i + \ell_i < \ell_{i+1}$ for $1 \leq i < r$. Examples of the simplification process are:

$$D_1^n(x) = \sum_{P(m,n+1)} (-1)^{\tau_1}\ \partial_{[1]}x \otimes \cdots \otimes \partial_{[n-\ell-2]}x \otimes \partial_{[n-\ell-1]}\partial_{[n-\ell+1]}x \otimes \partial_{[n-\ell+2]}x \otimes \cdots \otimes \partial_{[n+1]}x \otimes \partial_{[n-\ell]}x,$$

where $\tau_1 = |1| + \cdots + |n-\ell-1| + |n-\ell|(|n-\ell+1| + \cdots + |n+1|)$ and

$$D_2^n(x) = \sum_{\substack{0<\ell_2+1\leq\ell_1<n-1 \\ 0<k<n \\ P(m,n+2)}} (-1)^{\tau_2}\ \partial_{[1]}x \otimes \cdots \otimes \partial_{[n-k-\ell_1-1]}x \otimes \partial_{[n-k-\ell_1]}\partial_{[n-\ell_1+1]}x \otimes \partial_{[n-\ell_1+2]}x \otimes \cdots \otimes \partial_{[n-\ell_2-1]}x \otimes \partial_{[n-\ell_2]}\partial_{[n-\ell_2+2]}x$$

$$\otimes\, \partial_{[n-\ell_2+3]}x \otimes \cdots \otimes \partial_{[n+2]}x \otimes \partial_{[n-\ell_2+1]}x$$
$$\otimes\, \partial_{[n-k_1-\ell_1+1]}x \otimes \cdots \otimes \partial_{[n-\ell_1]}x$$

$$-\sum_{\substack{0<\ell+1,k<n\\ P(m,n+2)}} (-1)^{\tau_3}\ \partial_{[1]}x \otimes \cdots \otimes \partial_{[n-k-\ell-2]}x \otimes \partial_{[n-k-\ell-1]}\partial_{[n-\ell+2]}x$$
$$\otimes\, \partial_{[n-\ell+3]}x \otimes \cdots \otimes \partial_{[n+2]}x_n \otimes \partial_{[n-k-\ell]}\partial_{[n-\ell+1]}x$$
$$\otimes\, \partial_{[n-k-\ell+1]}x \otimes \cdots \otimes \partial_{[n-\ell]}x$$

where $\tau_2 = (|n-k_1-\ell_1+1|+\cdots+|n-\ell_1|)(|n-\ell_1+1|+\cdots+|n-\ell_2-1|$
$+|n-\ell_2+1|+1+\cdots+|n+1|+1)+|n-k_1-\ell_1+1|+\cdots+|n-\ell_2|$
$+|n-\ell_2+1|(|n-\ell_2+2|+1+\cdots+|n+2|+1)$
and $\tau_3 = |n-k-\ell-1|+(|n-k-\ell+1|+|n-\ell+1|)(|n-\ell+2|+1)+|n-\ell+3|+1+$
$\cdots+|n+2|+1+(|n-k-\ell+1|+\cdots+|n-\ell|)(|n-\ell+1|+|n-\ell+4|+\cdots+|n+1|)$.

Taking into account the sign and organization of the intervals in a general summand of the normalized expression of D_1^n and D_2^n, it should be possible to obtain a general expression of any D_r^n but this study exceeds the scope of this paper.

On the other hand, bearing in mind the expression at the cochain level of the Steenrod power operation $P_p^k(c)$ where $c \in Z^q(K, \mathbf{Z}_p)$, since c is a q-cochain, we only consider those summands in the normalized formula for $D^p_{(q-2k)(p-1)}$ with exactly $2k(p-1)$ face operators in each factor.

Since the explicit formulae for the Steenrod power operations P_p^k are given at the cochain level, in order to design an algorithm for computing them at the cohomology level, we first compute an explicit contraction (f, g, ϕ) from $C_*(K)$ to $H_*(K)$, K being a simplicial set finite in each degree and $\mathbf{Z}_p$ being the ground ring. This contraction can be constructed using the classical matrix algorithm (Munkres, 1984) based on reducing certain matrices (corresponding to the differential at each degree) to their Smith normal form (Gonzalez-Diaz and Real, 2003). The complexity of this method is $O(M^3)$ where M is the number of simplices of K.

Since the ground ring is a field, then the homology and cohomology are isomorphic. Moreover, if α is a generator of homology of degree q, then $\alpha^* : H_q(K) \to \mathbf{Z_p}$ such that

$$\alpha^*(\beta) = \begin{cases} 0 & \text{if } \alpha \neq \beta \in H_q(K) \\ 1 & \text{if } \beta = \alpha, \end{cases}$$

is a generator of cohomology of degree q. For fixed k, suppose that the normalized description of the morphism $D^p_{(q-2k)(p-1)}$ obtained using Theorem 3, and a contraction (f, g, ϕ) from $C_*(K)$ to $H_*(K)$ using the algorithm described above, are given. Then, (9) becomes at the cohomology level

$$P_p^k(\alpha^*) = \sum_{j=1}^{u} R\left(\mu(\alpha^* f)^{\otimes p} D^p_{(q-2k)(p-1)} g(\gamma_j)\right) \cdot \gamma_j^*$$

where $\{\gamma_1, \ldots, \gamma_u\}$ is a basis of $H_{q+2k(p-1)}$.

Summing up, we have designed an algorithm for computing any Steenrod reduced kth powers on any class of cohomology for any locally finite simplicial set.

Acknowledgment

This work was partially supported by the PAICYT research project FQM–296 from Junta de Andalucia.

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