

RECTIFICATIONS OF A_∞ -ALGEBRAS

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In this article, in the setting of connected DG-modules, we prove that, for any A_∞ -algebra $(M, \{m_i\}_{i \geq 1})$, there is a chain contraction from a DG-algebra A_M onto the DG-module M such that the A_∞ -algebra structure induced by perturbation theory on M is precisely the original one. In fact, the mentioned DG-algebra can be considered a rectification of the A_∞ -algebra in the sense of Boardman and Vogt (1973). Appropriate dual results are given for A_∞ -coalgebras.

Key Words: A_∞ -Algebra; A_∞ -Coalgebra; Contraction; Tilde bar construction, Tilde cobar construction.

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1. INTRODUCTION

One classical result claims that, given a topological operad \mathcal{P} , each $W\mathcal{P}$ -space (strongly homotopy \mathcal{P} -space) can be “replaced” by a strict \mathcal{P} -space of the same homotopy type. More concretely, for each $W\mathcal{P}$ -space, X , there exists a strict \mathcal{P} -space, MX , together with a strong deformation retraction $(MX, X, r : MX \rightarrow X, i : X \rightarrow MX)$ in which i is a homomorphism of $W\mathcal{P}$ -spaces (Boardman and Vogt, 1973). Then, MX is called the *rectification* of X . Taking $\mathcal{P} = \text{Ass}$, the associative operad, the result means that any A_∞ -space is homotopy equivalent to an Ass -space, the latter being the rectification of the former. Basic ideas of the construction of MX on A_∞ -spaces are given in Markl et al. (2002).

The aim of this article is to construct a similar rectification in the setting of DG-operads and DG-algebras over them, rather than topological operads and algebras over them (which are topological spaces). This way, we assert that, working in the context of connected DG-modules, for any A_∞ -algebra $(M, \{m_i\}_{i \geq 1})$, there exists an associative DG-algebra, A_M , and a chain contraction from A_M to M , that allows A_M to be regarded as the rectification of M . In the dual case of A_∞ -coalgebras, our framework is the one of simply connected DG-modules.

However, we must emphasize some significant aspects of our work in order to make clear the differences with the mentioned topological result: (a) our theorem can be used for A_∞ -algebras which do not come from topology, that is, which are not

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chain complexes of topological A_∞ -spaces (for example, the A_∞ -algebra cohomology of a space $(H^*(X), \{m_i\}_{i \geq 1})$); (b) in our situation, the rectification A_M of an A_∞ -algebra $(M, \{m_i\}_{i \geq 1})$ is a free graded algebra, so it provides a free DG-algebra model $A_M \rightarrow (M, \{m_i\}_{i \geq 1})$ of an A_∞ -algebra (cf. Baues and Lemaire, 1977; Huebschmann and Kadeishvili, 1991, where free models for DG-algebras are considered); (c) it is known that a chain contraction from A_M to the chain complex (M, m_1) generates, via the tensor trick and the Basic Perturbation Lemma (Gugenheim, 1977; Gugenheim and Stasheff, 1986; Gugenheim et al., 1991), an A_∞ -algebra structure on M . In our case, such an A_∞ -structure is *precisely* the initial one.

Our method to construct the rectification A_M is specific to algebraic framework since the main tool used is the Basic Perturbation Lemma. Namely, we construct first, for a chain complex (M, m_1) , an explicit contraction from $\overline{\Omega}T^c(sM)$, the reduced cobar construction of the tensor coalgebra of the suspension of M , to M and then, using the Basic Perturbation Lemma for the suitable perturbation data defined by the A_∞ -operations $\{m_i\}_{i > 1}$, we obtain a contraction from $\overline{\Omega}\widetilde{B}(M)$ to M , being $A_M = \overline{\Omega}\widetilde{B}(M)$ the aimed DG-algebra. We remark that the Munkholm's (1974) contraction (called *trivialized extension*) from $\overline{\Omega}\widetilde{B}(A)$ to a DG-algebra A can be obtained exactly in the same way, but considering the perturbation data defined by the multiplication of A instead of the A_∞ -operations.

This article is organized as follows. In the next section we recall the preliminaries needed and state the notation used throughout the article. The third section is devoted to our description of a family of morphisms that provide an A_∞ -(co)algebra structure via the tensor trick and the Basic Perturbation Lemma. In the fourth section, we establish the main result of the article by which any A_∞ -(co)algebra can be seen as a chain contraction from a (simply) connected DG-(co)algebra onto a DG-module.

2. NOTATIONS AND PRELIMINARIES

Although relevant notions of homological algebra are recalled here, most of common concepts are not explicitly given. They might be consulted, for instance, in Cartan and Eilenberg (1956) and Mac Lane (1995).

Take a commutative ground ring, Λ , with unit. A *differential graded module* or DG-module, (M, d) , is a module M , graded on the non-negative integers, $M = \bigoplus_{n \geq 0} M_n$, endowed with a morphism of graded modules d of degree -1 such that, $d^2 = 0$.

A DG-module M is called *connected* if $M_0 = \Lambda$ and *simply connected* if it is connected and $M_1 = 0$. Given a connected DG-module, M , the *reduced module* \overline{M} is such that $\overline{M}_n = M_n$ for $n > 1$ and $\overline{M}_0 = 0$. In this article, we shall always refer to connected DG-modules in the context of A_∞ -algebras and simply connected DG-modules when dealing with A_∞ -coalgebras.

We will denote the module $M \otimes \cdots \otimes M$ by $M^{\otimes n}$, with $M^{\otimes 0} = \Lambda$; we will use the notation $f^{\otimes n}$ for the morphism

$$f \otimes \cdots \otimes f: M^{\otimes n} \rightarrow N^{\otimes n}.$$

We will respect the Koszul convention for signs.

On the other hand, if $f: M^{\otimes i} \rightarrow N$ is a DG-module morphism and n is a non-negative integer, we will denote by $f^{[n]}: M^{\otimes n} \rightarrow N^{\otimes n-i+1}$ the morphism

$$f^{[n]} = \sum_{j=0}^{n-i} 1^{\otimes j} \otimes f \otimes 1^{\otimes n-i-j}.$$

The morphism $f^{[1]}: \bigoplus_{j \geq i} M^{\otimes j} \rightarrow \bigoplus_{k \geq 1} N^{\otimes k}$ will be the one such that $f^{[1]}|_{M^{\otimes n}} = f^{[n]}$.

If (M, d_M) is a DG-module, the *suspension* of M is defined as the DG-module (sM, d_{sM}) , where $(sM)_n = M_{n-1}$ and $d_{sM} = -d_M$. The *desuspension* of M is given by $(s^{-1}M)_n = M_{n+1}$ and differential $-d_M$ too. We will denote by \uparrow and \downarrow the suspension and desuspension morphisms, which shift the degree by $+1$ and -1 , respectively. A given morphism of graded modules of degree k , $f: M \rightarrow N$, induces $sf: sM \rightarrow sN$, given by $sf = (-1)^k \uparrow f \downarrow$. In fact, $d_{sM} = s d_M = -\uparrow d_M \downarrow$.

A *DG-algebra*, (A, d_A, μ_A) , is a DG-module endowed with an associative product, μ_A , compatible with the differential d_A and which has a unit $\eta_A: \Lambda \rightarrow A$, that is, $\mu_A(\eta_A \otimes 1) = \mu_A(1 \otimes \eta_A) = 1$. Sometimes we will use the notation $*_A$ for the product on A . If there is no confusion, subscripts will be omitted. A *DG-coalgebra* (C, d_C, Δ_C) is a DG-module provided with a compatible coproduct and counit $\xi_C: C \rightarrow \Lambda$ (so, $(\xi_C \otimes 1)\Delta_C = (1 \otimes \xi_C)\Delta_C = 1$).

Given a DG-module (M, d) , the *tensor module* of M is denoted by $T(M)$ and is constructed in the following way:

$$T(M) = \bigoplus_{n \geq 0} M^{\otimes n}.$$

The *tensor graduation* of $T(M)$, $| \cdot |_t$, is given by:

$$|a_1 \otimes \cdots \otimes a_n|_t = \sum_{i=1}^n |a_i|.$$

The differential structure in $T(M)$ is provided by the *tensor differential*, d_t , which is the morphism $d_M^{[1]}$.

A product, μ , and a coproduct, Δ , can be naturally defined on $T(M)$, as follows:

- $\mu((a_1 \otimes \cdots \otimes a_n) \otimes (a_{n+1} \otimes \cdots \otimes a_{n+p})) = a_1 \otimes \cdots \otimes a_{n+p}$;
- $\Delta(a_1 \otimes \cdots \otimes a_n) = \sum_{i=0}^n (a_1 \otimes \cdots \otimes a_i) \otimes (a_{i+1} \otimes \cdots \otimes a_n)$.

Therefore, $T(M)$ acquires both structures of DG-algebra (denoted by $T^a(M)$) and DG-coalgebra ($T^c(M)$), though they are not compatible to each other (that is, $(T(M), \mu, \Delta)$ is not a Hopf algebra).

Every morphism of DG-modules $f: M \rightarrow N$ induces another one $T(f): T(M) \rightarrow T(N)$, such that $T(f)|_{M^{\otimes n}} = f^{\otimes n}$.

The *reduced bar construction* of a connected DG-algebra A , $\bar{B}(A)$, is a DG-coalgebra whose module structure is given by

$$T(s\bar{A}) = \bigoplus_{n \geq 0} (s\bar{A})^{\otimes n}.$$

A typical element of $\bar{B}(A)$ is denoted by $\bar{a} = [a_1 | \cdots | a_n]$ and $[] = 1 \in \Lambda$. The degree of a is given by the sum of the *tensor degree* $|a_1| + \cdots + |a_n|$ and the *simplicial degree* n ,

which appeals to the number of components or length of the element (also referred to as simplicial dimension).

The total differential $d_{\bar{B}}$ is given by the sum of the tensor differential, d_t (which is the natural one on the tensor product) and the *simplicial differential*, d_s (that depends on the product on A):

$$d_t = -(\uparrow d_A \downarrow)^{[1]}; \quad d_s = (\uparrow \mu_A \downarrow^{\otimes 2})^{[1]}.$$

The coproduct $\Delta_{\bar{B}} : \bar{B}(A) \rightarrow \bar{B}(A) \otimes \bar{B}(A)$ is the natural one on the tensor module.

Given a simply connected DG-coalgebra C , the *reduced cobar construction*, $\bar{\Omega}(C)$, is a DG-algebra whose underlying module is

$$T(s^{-1}\bar{C}) = \bigoplus_{n \geq 0} (s^{-1}\bar{C})^{\otimes n}.$$

A typical element of $\bar{\Omega}(C)$ will be written $\bar{c} = \langle c_1 | \dots | c_n \rangle$, being $\langle \rangle = 1 \in \Lambda$. The total degree of \bar{c} is $|\bar{c}|_{\bar{\Omega}} = |c_1| + \dots + |c_n| - n$. We will refer to the length of the element, n , as the *cosimplicial degree* (or dimension).

The total differential $d_{\bar{\Omega}}$ is given by the sum of the tensor differential and the *cosimplicial differential* $d_{cos} = (\downarrow^{\otimes 2} \Delta_C \uparrow)^{[1]}$. The product on $\bar{\Omega}(C)$ is the natural one on the underlying module.

In the context of homological perturbation theory, the main input data are *chain contractions* (or simply, contractions) (see Eilenberg and Mac Lane, 1953; Huebschmann and Kadeishvili, 1991): a contraction $c : \{N, M, f, g, \phi\}$ from a DG-module N to a DG-module M , consists of a particular homotopy equivalence determined by three morphisms f, g , and ϕ ; being $f : N_{\star} \rightarrow M_{\star}$ (projection) and $g : M_{\star} \rightarrow N_{\star}$ (inclusion) two DG-module morphisms and $\phi : N_{\star} \rightarrow N_{\star+1}$ a homotopy operator, that is, $fg = 1_M$, and $\phi d_N + d_N \phi + gf = 1_N$. Moreover, these data are also required to satisfy

$$f\phi = 0, \quad \phi g = 0, \quad \phi\phi = 0.$$

Notice that the homology of both DG-modules are isomorphic. We will also use the notation $(f, g, \phi) : N \Rightarrow M$ or simply $N \xrightarrow{c} M$ when confusion cannot arise.

Given a DG-module contraction,

$$c : \{N, M, f, g, \phi\}$$

we can establish the following ones (Gugenheim and Lambe, 1989; Gugenheim et al., 1991):

- (i) The *suspension contraction* of c , sc , which consists of the suspended DG-modules and the induced morphisms.

$$sc : \{sN, sM, sf, sg, s\phi\},$$

being $sf = \uparrow f \downarrow$, $sg = \uparrow g \downarrow$ and $s\phi = -\uparrow \phi \downarrow$, which are briefly expressed by f, g and $-\phi$.

(ii) The *tensor module contraction*, $T(c)$, between the tensor modules of N and M .

$$T(c) : \{T(N), T(M), T(f), T(g), T(\phi)\},$$

where

$$T(\phi)|_{N^{\otimes n}} = \phi^{[\otimes n]} = \sum_{i=0}^{n-1} 1^{\otimes i} \otimes \phi \otimes (gf)^{\otimes n-i-1}.$$

Now, we recall the concept of *perturbation datum*. Let N be a graded module and let $f : N \rightarrow N$ be a morphism of graded modules. The morphism f is defined to be *pointwise nilpotent* whenever for all $x \in N$, $x \neq 0$, there exists a positive integer n such that $f^n(x) = 0$. A *perturbation of a DG-module* N consists in a morphism of graded modules $\delta : N \rightarrow N$ of degree -1 , such that $(d_N + \delta)^2 = 0$. A *perturbation datum* of the contraction $c : \{N, M, f, g, \phi\}$ is a perturbation δ of the DG-module N satisfying that the composition $\phi\delta$ is pointwise nilpotent.

The main tool when dealing with contractions is the *Basic Perturbation Lemma* (Brown, 1967; Gugenheim, 1972; Gugenheim and Stasheff, 1986; Lambe and Stasheff, 1987; Gugenheim and Lambe, 1989; Huebschmann and Kadeishvili, 1991; Real, 2000; Shih, 1962), which is an algorithm whose input is a contraction of DG-modules $c : \{N, M, f, g, \phi\}$ and a perturbation datum δ of c and whose output is a new contraction $c_\delta : \{(N, d_N + \delta), (M, d_M + d_\delta), f_\delta, g_\delta, \phi_\delta\}$ defined by the formulas

$$d_\delta = f\delta\Sigma_c^\delta g, \quad f_\delta = f(1 - \delta\Sigma_c^\delta \phi), \quad g_\delta = \Sigma_c^\delta g, \quad \phi_\delta = \Sigma_c^\delta \phi,$$

where $\Sigma_c^\delta = \sum_{i \geq 0} (-1)^i (\phi\delta)^i$.

The pointwise nilpotency of the composition $\phi\delta$ guarantees that the sums are finite for each particular element.

3. FROM CONTRACTIONS TO A_∞ -STRUCTURES

We find the origin of A_∞ -(co)algebras in Stasheff (1963), where Stasheff set the concept of *strongly homotopy associativity* in the search of a homotopy invariant that plays the role of associativity.

We recall here the definition of A_∞ -algebra (respectively, A_∞ -coalgebra) (Kadeishvili, 1980; Prouté, 1984). An A_∞ -algebra (respectively, A_∞ -coalgebra), is a DG-module (M, m_1) (respectively, (M, Δ_1)) endowed with a family of morphisms of graded modules

$$m_i : M^{\otimes i} \rightarrow M \quad (\text{respectively, } \Delta_i : M \rightarrow M^{\otimes i})$$

of degree $i - 2$ such that, for $i \geq 1$,

$$\sum_{n=1}^i \sum_{k=0}^{i-n} (-1)^{n+k+nk} m_{i-n+1}(1^{\otimes k} \otimes m_n \otimes 1^{\otimes i-n-k}) = 0, \quad (1)$$

$$\left(\text{respectively, } \sum_{n=1}^i \sum_{k=0}^{i-n} (-1)^{n+k+nk} (1^{\otimes i-n-k} \otimes \Delta_n \otimes 1^{\otimes k}) \Delta_{i-n+1} = 0 \right). \quad (2)$$

Starting from a contraction between a connected DG-algebra A and a DG-module M , the application of the tensor trick (Gugenheim, 1977; Gugenheim and Stasheff, 1986; Gugenheim et al., 1991) and the Basic Perturbation Lemma provide a way of constructing a family of morphisms that makes the module inherit an A_∞ -algebra structure. In fact, the first transference of an A_∞ -algebra structure, in this sense, was made by Kadeishvili (1980) for the case $M = H(A)$. Using this technique, in the following theorem we will express these morphisms with regard to the component morphisms of the initial contraction. We draft a proof of the theorem with the only purpose of showing the tools used in the context of homological perturbation theory.

Theorem 3.1 (Gugenheim et al., 1991; Kadeishvili, 1980). *Let (A, d_A, μ) and (M, d_M) be a connected DG-algebra and a DG-module, respectively, and $c : \{A, M, f, g, \phi\}$ a contraction between them. Then the DG-module M is endowed with an A_∞ -algebra structure by the morphisms*

$$m_1 = -d_M$$

$$m_n = (-1)^{n+1} f \mu^{(1)} \phi^{[\otimes 2]} \mu^{(2)} \dots \phi^{[\otimes n-1]} \mu^{(n-1)} g^{\otimes n}, \quad n \geq 2$$

where

$$\mu^{(k)} = \sum_{i=0}^{k-1} (-1)^{i+1} 1^{\otimes i} \otimes \mu_A \otimes 1^{\otimes k-i-1}.$$

Proof. Starting from $c : \{A, M, f, g, \phi\}$, we can construct the contraction

$$T(sc) : \{T^c(s\bar{A}), T^c(s\bar{M}), Tf, Tg, T(-\phi)\}.$$

Now, in order to get the differential of the bar construction on the initial DG-module, we consider the simplicial differential as a perturbation datum of this contraction. We can easily check the pointwise nilpotency of $T(-\phi)d_s$ since $T(-\phi)$ does not affect the simplicial dimension of the element, while d_s decreases this amount by one and so will be zero after a finite number of steps. Then, by applying the Basic Perturbation Lemma, a new contraction is obtained,

$$(\tilde{f}, \tilde{g}, \tilde{\phi}) : \bar{B}(A) \Rightarrow (T^c(s\bar{M}), \tilde{d}),$$

where $(T^c(s\bar{M}), \tilde{d})$ is called the *tilde bar construction* of M (Stasheff, 1963), denoted by $\tilde{B}(M)$ and the formula obtained for the perturbed differential is

$$\tilde{d} = d_t + \sum_{i \geq 0} (-1)^i Tf d_s (T(-\phi)d_s)^i Tg.$$

We call

$$\tilde{d}_1 = d_t, \quad \tilde{d}_i = (-1)^i Tf d_s (T(-\phi)d_s)^{i-2} Tg \quad \text{for } i \geq 2.$$

Let us consider the induced morphisms $m_n : M^{\otimes n} \rightarrow M$ of degree $n - 2$, with the formula

$$m_n = (-1)^{[n/2]} \downarrow (\tilde{d}_n|_{(sM)^{\otimes n}}) \uparrow^{\otimes n},$$

where the brackets refer to the integer part. Then, it is easy to check that

$$m_n = (-1)^{n+1} f\mu^{(1)} \phi^{[\otimes 2]} \mu^{(2)} \dots \phi^{[\otimes n-1]} \mu^{(n-1)} g^{\otimes n}.$$

Finally, the fact that $\tilde{d}^2 = 0$ can be translated into relations between the morphisms m_n which are, actually, the ones recalled in Eq. (1). \square

An analogous theorem can be established for the case of an A_∞ -coalgebra structure, whose proof is completely dual to the one given above.

Theorem 3.2 (Gugenheim et al., 1991; Kadeishvili, 1980). *Let (C, d_C, Δ) and (M, d_M) be a simply connected DG-coalgebra and a DG-module, respectively, and $c : \{C, M, f, g, \phi\}$ a contraction between them. Then the DG-module M is endowed with an A_∞ -coalgebra structure by the morphisms*

$$\begin{aligned} \Delta_1 &= -d_M \\ \Delta_n &= (-1)^{[n/2]+n+1} f^{\otimes n} \Delta^{(n)} \phi^{[\otimes n-1]} \Delta^{(n-1)} \dots \phi^{[\otimes 2]} \Delta^{(2)} g, \quad n \geq 2 \end{aligned}$$

where

$$\Delta^{(k)} = \sum_{i=0}^{k-2} (-1)^i 1^{\otimes i} \otimes \Delta_C \otimes 1^{\otimes k-i-2}.$$

In this case, the simple connection of the DG-coalgebra guarantees that the formulas implied in the contraction $\overline{\Omega}(C) \Rightarrow \tilde{\Omega}(M)$, obtained by the Basic Perturbation Lemma, are finite.

Now we are concerned about the inverse process: given an A_∞ -(co)algebra, finding a suitable contraction that generates, in the sense given in this section, such a structure.

4. FROM A_∞ -STRUCTURES TO CONTRACTIONS

In this section, we explicitly construct a rectification of an A_∞ -algebra. As a result, we establish that such a structure can be structurally represented as a contraction from a DG-algebra onto a DG-module. Recall that by ‘‘DG-module’’ we shall mean a *connected* DG-module.

Munkholm (1974) obtained a contraction between the reduced cobar construction of the reduced bar construction of a DGA-algebra A , $\overline{\Omega} \overline{B}A$, and the DGA-algebra itself.

$$c_A : \{\overline{\Omega} \overline{B}A, A, \alpha_A, \rho_A, h_A\}.$$

The same contraction can be obtained by means of perturbation theory, for the connected case: starting from a “basic” contraction

$$\{\overline{\Omega T^c}(s\overline{A}), A, \alpha, \rho, h\},$$

and taking the perturbation datum $\delta = (\downarrow (-d_s) \uparrow)^{[1]}$, the contraction c_A is obtained by applying the Basic Perturbation Lemma. This technique will allow us to extend this result to the case of A being an A_∞ -algebra, obtaining a contraction between the reduced cobar construction of the tilde bar construction of A , $\overline{\Omega} \widetilde{B}A$, and A .

Now we expose the main theorem of the article. Afterwards, we will dualize the result for A_∞ -coalgebras.

Theorem 4.1. *Let M be a DG-module endowed with an A_∞ -algebra structure. Then there exists a contraction, $c_M : \{A_M, M, f, g, \phi\}$, between a connected DG-algebra A_M and the DG-module M , such that the application of the tensor trick (followed by the Basic Perturbation Lemma) to c_M provides the original A_∞ -algebra structure on M .*

Proof. Let $(M, \{m_i\}_{i \geq 1})$ be an A_∞ -algebra and consider the contraction

$$\{\overline{\Omega T^c}(s\overline{M}), M, \alpha, \rho, h\},$$

whose component morphisms are described below.

In order to make clearer the formulas, we denote by c_i the i th component an element of the cobar, $\langle c_1 | \cdots | c_n \rangle$, with length $k(i) \geq 1$, that is, $c_i = [a_{i,1} | \cdots | a_{i,k(i)}]$ and we will only specify the whole expression $[a_{i,1} | \cdots | a_{i,k(i)}]$ in the case we want to emphasize its composition. Particularly, when $k(i) = 1$, we write $[a_i]$.

- $\alpha : \overline{\Omega T^c}(s\overline{M}) \rightarrow M,$

$$\begin{aligned} \alpha \langle c_1 | \cdots | c_n \rangle &= 0 \quad \text{if } n \geq 1 \text{ or } k(1) \geq 1; \\ \alpha \langle [a] \rangle &= a; \end{aligned}$$

- $\rho : M \rightarrow \overline{\Omega T^c}(s\overline{M}),$

$$\rho(a) = \langle [a] \rangle$$

- $h : \overline{\Omega T^c}(s\overline{M}) \rightarrow \overline{\Omega T^c}(s\overline{M})$

$$\begin{aligned} h \langle c_1 | \cdots | c_n \rangle &= 0 \quad \text{if } n = 0 \text{ or } n = 1 \text{ or } k(1) \geq 1; \\ h \langle [a_1] | c_2 | \cdots | c_n \rangle &= (-1)^{|a_1|+1} \langle [a_1 | a_{2,1} | \cdots | a_{2,k(2)}] | c_3 | \cdots | c_n \rangle; \end{aligned}$$

Now the perturbation δ will consist in including (m_2, m_3, \dots) in the tensor differential of the cobar, so that $(\overline{\Omega T^c}(s\overline{M}), d_{\overline{\Omega}} + \delta)$, becomes $\overline{\Omega} \widetilde{B}(M)$:

$$\delta = \left(\downarrow \left(- \sum_{k \geq 2} (\uparrow m_k \downarrow^{\otimes k})^{[1]} \right) \uparrow \right)^{[1]}.$$

The composition $h\delta$ is pointwise nilpotent, since δ decreases the simplicial degree of the components on the bar construction and h decreases the number of components on the cobar.

Then, we apply the Basic Perturbation Lemma, obtaining the contraction

$$c_M : \{\widetilde{\Omega B}(M), M, f, g, \phi\},$$

where $\widetilde{\Omega B}(M)$ will be the algebra A_M satisfying the theorem.

Notice that for an element $\omega = \langle [a_1] | c_2 | \cdots | c_n \rangle$

$$\begin{aligned} \delta h(\omega) &= (-1)^{|a_1| + \zeta_{k(2)+1}} \left\{ [m_{k(2)+1}(a_1 \otimes a_{2,1} \otimes \cdots \otimes a_{2,k(2)})] | c_3 | \cdots | c_n \right\} \\ &\quad + \text{other summands with } k(1) \geq 1, \end{aligned}$$

where $\zeta_{k(2)+1} = k(2)|a_1| + (k(2) - 1)|a_{2,1}| + \cdots + |a_{2,k(2)-1}| + [(k(2) + 1)/2]$.

Then, we can describe recursively $(\delta h)^i$, up to sign:

$$(\delta h)^i(\omega) = \left\{ [m_{k(i+1)+1}(\pi_{1,1}((\delta h)^{i-1}\omega) \otimes a_{i+1,1} \otimes \cdots \otimes a_{i+1,k(i+1)})] | c_{i+2} | \cdots | c_n \right\},$$

where $\pi_{1,1} \langle [a_{1,1} | \cdots | a_{1,k(1)}] | c_2 | \cdots | c_n \rangle = a_{1,1}$.

Taking into account these notes, we can describe the morphisms of the contraction above:

- $f(\omega) = \pm m_{k(n)+1}(\pi_{1,1}((\delta h)^{n-2}\omega) \otimes a_{n,1} \otimes \cdots \otimes a_{n,k(n)})$.

In the particular case of the element $\langle [a_1] | c_2 \rangle$,

$$f \langle [a_1] | c_2 \rangle = (-1)^{1+|a_1| + \zeta_{k(2)+1}} m_{k(2)+1}(a_1 \otimes a_{2,1} \otimes \cdots \otimes a_{2,k(2)}). \quad (3)$$

- $g = \rho$, since $\delta \langle [a] \rangle = 0$.
- $\phi(\omega) = \sum_{i=1}^{n-1} (-1)^{i-1} h(\delta h)^{i-1} \langle [a_1] | c_2 | \cdots | c_n \rangle$

$$\begin{aligned} &= (-1)^{|a_1|+1} \left\{ [a_1 | a_{2,1} | \cdots | a_{2,k(2)}] | c_3 | \cdots | c_n \right\} \\ &\quad + \sum_{i=2}^{n-1} \pm \left\{ [m_{k(i)+1}(\pi_{1,1}((\delta h)^{i-2}\omega) \otimes a_{i,1} \otimes \cdots \otimes a_{i,k(i)}) | a_{i+1,1} | \cdots | a_{i+1,k(i+1)}] \right. \\ &\quad \left. | c_{i+2} | \cdots | c_n \right\}. \end{aligned}$$

Particularly, for an element $\langle [a_1] | c_2 \rangle$,

$$\phi \langle [a_1] | c_2 \rangle = (-1)^{|a_1|+1} \left\{ [a_1 | a_{2,1} | \cdots | a_{2,k(2)}] \right\}. \quad (4)$$

Besides, the perturbed differential on M is zero since $\delta\rho = 0$, what means that M still remains with the same differential structure.

We can check now that the structure of A_∞ -algebra generated on M via the tensor trick coincides with the original infinite structure on M .

Take the contraction $T(sc_M) : \{T(s\bar{A}_M), T(s\bar{M}), Tf, Tg, T(-\phi)\}$ and the simplicial differential, d_s , that depends on the juxtaposing product $\mu_{\bar{\Omega}}$, as a perturbation datum. The pointwise nilpotency of $T(-\phi)d_s$ is due to the fact that d_s

does not modify the sum of the lengths of the components of the cobar, meanwhile $T(-\phi)$ decreases, at least by one, this amount. Then, applying the Basic Perturbation Lemma, the following contraction is obtained

$$(f_s, g_s, \phi_s) : \overline{B}(A_M) \Rightarrow (T(s\overline{M}), \tilde{d}).$$

In order to shorten the formulas ahead, we will denote the iterated composition of morphisms $\phi^{[\otimes i]} \mu^{(i)} \dots \phi^{[\otimes j]} \mu^{(j)}$, with $i \leq j$, by $\Psi^{(i,j)}$ and $\phi^{[\otimes i]} \mu^{(i)}$, simply by $\Psi^{(i)}$. So, the family of morphisms given in Section 3 that provides an A_∞ -algebra structure for M can be expressed as follows:

$$\tilde{m}_n = (-1)^{n+1} f \mu^{(1)} \Psi^{(2,n-1)} g^{\otimes n}.$$

We will prove that each morphism \tilde{m}_n is exactly the original m_n of the A_∞ -algebra structure on M . Take $a = a_1 \otimes \dots \otimes a_n \in M^{\otimes n}$. Then,

$$\begin{aligned} \mu^{(n-1)} g^{\otimes n}(a) &= \sum_{i=0}^{n-2} (-1)^{i+1} \langle [a_1] \rangle \otimes \dots \otimes \langle [a_i] \rangle \otimes \langle [a_{i+1}] | [a_{i+2}] \rangle \\ &\quad \otimes \langle [a_{i+3}] \rangle \otimes \dots \otimes \langle [a_n] \rangle. \end{aligned}$$

The only non-null summand of $\phi^{[\otimes n-1]}$ that can be now applied on each summand on the right hand side is $1^{\otimes i} \otimes \phi \otimes (gf)^{\otimes n-i-2}$, where ϕ is applied (following (4)) to the only element of cosimplicial dimension 2.

$$\begin{aligned} \phi^{[\otimes n-1]} \mu^{(n-1)} g^{\otimes n}(a) &= \sum_{i=0}^{n-2} (-1)^{i+1+\gamma_i} \langle [a_1] \rangle \otimes \dots \otimes \langle [a_i] \rangle \otimes \langle [a_{i+1} | a_{i+2}] \rangle \\ &\quad \otimes \langle [a_{i+3}] \rangle \otimes \dots \otimes \langle [a_n] \rangle, \end{aligned}$$

where $\gamma_i = |a_1| + \dots + |a_i| + |a_{i+1}| + 1$.

Notice that the obtained summands have again, only factors of cosimplicial dimension 1 and this condition remains every time that $\Psi^{(k)}$, for any k , is applied: take an element from $(A_M)^{\otimes j}$ in the form $\bar{\omega} = \langle c_1 \rangle \otimes \dots \otimes \langle c_j \rangle$, then,

$$\mu^{(j-1)}(\bar{\omega}) = \sum_{i=0}^{j-2} (-1)^{i+1} \langle c_1 \rangle \otimes \dots \otimes \langle c_i \rangle \otimes \langle c_{i+1} | c_{i+2} \rangle \otimes \langle c_{i+3} \rangle \otimes \dots \otimes \langle c_j \rangle,$$

so $\phi^{[\otimes j-1]}$ (on each summand above) is reduced to $1^{\otimes i} \otimes \phi \otimes (gf)^{\otimes j-i-2}$ where ϕ is applied to the only factor of cosimplicial degree 2 and will be non-null only if $c_{i+1} = [a_{i+1}]$,

$$\begin{aligned} \Psi^{(j-1)}(\bar{\omega}) &= \sum_{i=0}^{j-2} (-1)^{i+1+\gamma_i} \langle c_1 \rangle \otimes \dots \otimes \langle c_i \rangle \otimes \langle [a_{i+1} | a_{i+2,1} | \dots | a_{i+2,k(i+2)}] \rangle \\ &\quad \otimes gf \langle c_{i+3} \rangle \otimes \dots \otimes gf \langle c_j \rangle, \end{aligned} \tag{5}$$

with $\gamma_i = |c_1| + \dots + |c_i| + i + |a_{i+1}| + 1$.

Besides, the elements $f\langle c_{i+3} \rangle, \dots, f\langle c_j \rangle$ will be non-null only if $k(i+3) = 1 = \dots = k(j)$. In this case, the i th summand of (5) will be

$$\langle c_1 \rangle \otimes \dots \otimes \langle c_i \rangle \otimes \left[[a_{i+1} | a_{i+2,1} | \dots | a_{i+2,k(i+2)}] \right] \otimes \langle [a_{i+3}] \rangle \otimes \dots \otimes \langle [a_j] \rangle,$$

which has only factors with 1 component.

This way, in order to obtain a non-null result for the i th summand, the element to which $\Psi^{(j-1)}$ is applied must be

$$\langle c_1 \rangle \otimes \dots \otimes \langle c_i \rangle \otimes \langle [a_{i+1}] \rangle \otimes \langle c_{i+2} \rangle \otimes \langle [a_{i+3}] \rangle \otimes \dots \otimes \langle [a_j] \rangle. \quad (6)$$

As for the application of the whole sequence $\mu^{(1)}\Psi^{(2,n-1)}$, starting from a tensor product of n elements from the cobar construction, the morphism $\mu^{(i)}$ always decreases by one the number of factors from the cobar construction and $\phi^{[\otimes i]}$ does not touch this amount. This way, after applying $\mu^{(1)}\Psi^{(2,n-1)}$, only one factor is obtained to which f will be applied. Taking this into account together with the fact that the only elements that survive to $\Psi^{(j-1)}$ are those in the form (6), we conclude that the only summand of $\Psi^{(j-1)}$ that will pass through $f\mu^{(1)}\Psi^{(2,j-2)}$ is that of $i = j - 2$ in the sum (5), whenever $c_{j-1} = [a_{j-1}]$. Then,

$$\Psi^{(j-1)}(\bar{\omega}) = (-1)^{j-1+\gamma_{j-2}} \langle c_1 \rangle \otimes \dots \otimes \langle c_{j-2} \rangle \otimes \left[[a_{j-1} | a_{j,1} | \dots | a_{j,k(j)}] \right] + \dots,$$

where dots represent the rest of summands that, from now on, we will omit.

Now, by induction on k , one can easily prove that

$$\Psi^{(n-k,n-1)} g^{\otimes n}(a) = (-1)^{\varrho(k)} \langle [a_1] \rangle \otimes \dots \otimes \langle [a_{n-k-1}] \rangle \otimes \langle [a_{n-k} | \dots | a_n] \rangle,$$

where $\varrho(k) = \sum_{i=0}^{k-1} (n-i) + \sum_{i=1}^k i |a_{n-i}| + k \sum_{i=k+1}^{n-1} |a_{n-i}|$.

So, for the case $k = n - 2$,

$$\Psi^{(2,n-1)} g^{\otimes n}(a) = (-1)^{\varrho(n-2)} \langle [a_1] \rangle \otimes \langle [a_2 | \dots | a_n] \rangle,$$

where

$$\begin{aligned} \varrho(n-2) &= \sum_{i=0}^{n-3} (n-i) + \sum_{i=1}^{n-2} i |a_{n-i}| + (n-2) |a_1| \\ &\equiv [(n-1)/2] + \sum_{i=1}^{n-2} i |a_{n-i}| + (n-2) |a_1| \pmod{2}. \end{aligned}$$

Taking up again the calculation of \tilde{m}_n ,

$$\begin{aligned} \tilde{m}_n(a_1 \otimes \dots \otimes a_n) &= (-1)^{n+1+\varrho(n-2)} f \mu^{(1)} \langle [a_1] \rangle \otimes \langle [a_2 | \dots | a_n] \rangle \\ &= (-1)^{n+\varrho(n-2)} f \langle [a_1] | [a_2 | \dots | a_n] \rangle \\ \text{(using 3)} &= (-1)^{n+\varrho(n-2)+1+|a_1|+\zeta_n} m_n(a_1 \otimes \dots \otimes a_n), \end{aligned}$$

where $\zeta_n = \sum_{i=1}^{n-1} i|a_{n-i}| + [n/2]$ and hence

$$n + \varrho(n-2) + 1 + |a_1| + \zeta_n \equiv 0 \pmod{2}.$$

That is, the A_∞ -algebra structure obtained coincides with the original one defined on M . \square

This theorem, together with the Theorem 3.1, provide a structural representation of an A_∞ -algebra as a contraction.

Corollary 4.2. *A DG-module M is endowed with an A_∞ -algebra structure if and only if there exists a contraction $c : \{A_M, M, f, g, \phi\}$, between a connected DG-algebra A_M and a DG-module M .*

Now, we show the dual result to the theorem given above, in the case of A_∞ -coalgebras, omitting its proof since it follows a similar scheme (though quite more tedious). From now on, all the DG-modules will be considered to be *simply connected*. This way, any A_∞ -coalgebra can be represented as a contraction from a simply connected DG-coalgebra.

Theorem 4.3. *Let M be a DG-module endowed with an A_∞ -coalgebra structure. Then there exists a contraction $c : \{C_M, M, f, g, \phi\}$, between a simply connected DG-coalgebra C_M and the DG-module M , such that the application of the tensor trick to c (and the Basic Perturbation Lemma) yields the initial A_∞ -coalgebra structure on M .*

In this case, the simply connection guarantees that the perturbation process is finite and a contraction $c_M : \{\bar{B}\tilde{\Omega}(M), M, f, g, \phi\}$ can be constructed.

Corollary 4.4. *A DG-module M is endowed with an A_∞ -coalgebra structure if and only if there exists a contraction $c : \{C_M, M, f, g, \phi\}$, between a simply connected DG-coalgebra C_M and a DG-module M .*

We are aware that, at this point, the development of a categorical framework for A_∞ -(co)algebras in terms of chain contractions would be a natural direction of our future efforts.

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