A Graph-with-Loop Structure for a Topological Representation of 3D Objects*

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Abstract. Given a cell complex K whose geometric realization $|K|$ is embedded in $\mathbf{R}^3$ and a continuous function $h : |K| \to \mathbf{R}$ (called the *height function*), we construct a graph $G_h(K)$ which is an extension of the Reeb graph $R_h(|K|)$. More concretely, the graph $G_h(K)$ without loops is a subdivision of $R_h(|K|)$. The most important difference between the graphs $G_h(K)$ and $R_h(|K|)$ is that $G_h(K)$ preserves not only the number of connected components but also the number of "tunnels" (the homology generators of dimension 1) of K. The latter is not true in general for $R_h(|K|)$. Moreover, we construct a map $\psi : G_h(K) \to K$ identifying representative cycles of the tunnels in K with the ones in $G_h(K)$ in the way that if e is a loop in $G_h(K)$, then $\psi(e)$ is a cycle in K such that all the points in $|\psi(e)|$ belong to the same level set in $|K|$.

1 Reeb Graphs and Tunnels

We are interested in analyzing and visualizing intrinsic properties of geometric models and scientific data. Specifically, Reeb graphs [13], which express the connectivity of level sets, have been used in the past to construct data structures and user-interfaces for modeling and visualization applications [5].

Let X be a topological space and $h : X \to \mathbf{R}$ a continuos map. A *level set* is the primage of a constant value, $h^{-1}(t)$. Call a connected component of a level set a *contour*. Two points $x, y \in X$ are equivalent, $x \sim y$, if they belong to the same contour, that is, if $h(x) = h(y)$ and x and y are connected by a path on X. The *Reeb graph* of h, $R_h(X)$, is the quotient space defined by this equivalence relation. Observe that, by construction, the Reeb graph has a point for each contour and the connection is provided by $\psi : X \to R_h(X)$ that maps each point x to its equivalence class. Even though the Reeb graph loses a lot of the original topological structure, some things can be said: a tunnel in X that maps (by ψ) to a tunnel in $R_h(X)$ cannot be continuously deformed to a single point, and two tunnels in X that map to different tunnels in $R_h(X)$. The number of connected components of X, $\beta_0(X)$, is preserved and the number of tunnels of X, $\beta_1(X)$, cannot increase, i.e. $\beta_0(R_h(X)) = \beta_0(X)$ and $\beta_1(R_h(X)) \leq \beta_1(X)$.

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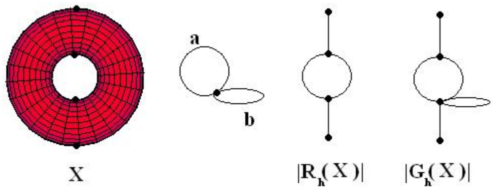


Fig. 1. From left to right, the torus X; representative cycles of the two tunnels of X (the tunnel a is obvious and the other one, b, is not); a geometric realization of the Reeb Graph $R_h(X)$ that h associates to each point on X its elevation; and a geometric realization of the graph with loops $G_h(X)$

In [2] the authors adapt concepts developed for smooth manifolds to discrete surface models, introducing an extended Reeb graph representation for a generic polyhedral surface. Their approach is based on the computation of a sufficiently dense number of contour lines and the definition of the Reeb graph from the contour set. However, such a construction is actually not an extension of the Reeb graph itself, but rather an application of its definition in the discrete domain. In [4], tight upper and lower bounds of the number of tunnels in the Reeb graph that depend on the genus, the number of boundary components and whether or not the 2-manifold is orientable, is given.

In this paper, we focus on objects embedded in $\mathbf{R}^3$. Several combinatorial structures may represent a cellular subdivision which models an object such as simplicial, cubical and simploidal complexes. Roughly speaking, the *cells* of a given simplicial complex are simplices (vertices, edges, triangles and tetrahedra); vertices, edges, squares and cubes constitute the collection of cells of a cubical complex; in the case of simploidal complexes, which generalize both simplicial and cubical complexes (see [3]), cells are cartesian products of simplices. In all the cases they fit together in a natural way to form the object (see [11,3]).

From now on, a graph $G = (V, E)$, where V is a set of vertices and E a set of edges, is considered as a particular complex only with vertices and edges. Given

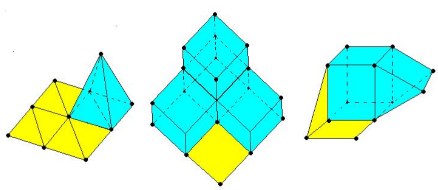


Fig. 2. A simplicial, cubical and simploidal complex

a complex K (simplicial, cubical or simploidal), a geometric realization of it (e.g. a 3D triangulated surface) $|K|$, and a continuous function $h : |K| \to \mathbf{R}$, our aim is the computation of a graph $G_h(K)$ and a function $\Psi : G_h(K) \to K$ with the following properties:

– $G_h(K)$ has the same number of connected components and tunnels than K.
– Each loop (an edge such that its endvertices are the same) of $G_h(K)$ maps to a non-contractible cycle c in K such that $|c|$ (the geometric realization of all the cells in c) lies in a contour of $|K|$, two loops in $G_h(K)$ map to non-homologous cycles in K, and each edge in $G_h(K)$ map to a path in K.
– If we do not consider the loops in $G_h(K)$, then $G_h(K)$ is a subdivision of the Reeb graph $R_h(K)$.

Therefore, $G_h(K)$ can be seen as an extension of $R_h(K)$ such that not only the number of connected components and tunnels of $G_h(K)$ and K coincide (this is not true, in general, for $R_h(K)$), but also there exists a one-by-one identification of tunnels in $G_h(K)$ with tunnels in K in the way that if e is a loop in $G_h(K)$, then $\psi(e)$ is a cycle in K such that all the points in $|\psi(e)|$ have the same height.

2 Algebraic-Topological Models for 3D Objects

This section introduces the algebraic topology background needed to understand the rest of the paper, which is essentially extracted from Munkres' book [12]. The concept of AT-model established in [8,9] for cohomology computation of 3D digital images is adapted here to solve the problem of computing the graph $G_h(K)$ and the function $\Psi : G_h(K) \to K$. Without lost of generality, we will consider that the ground ring is $\mathbf{Z}/2$.

A *chain complex* $\mathcal{C}$ is a sequence $\{C_q, d_q\}$ of abelian groups C_q and homomorphisms $d_q : C_{q+1} \to C_q$, such that, for all q, $d_q d_{q+1} = 0$. The set $\{d_q\}_{q \geq 0}$ is called the *differential* of $\mathcal{C}$. The chain complex $\mathcal{C}$ is *free* if C_q is a free abelian group for each q; it is *finite* if there exists an integer $n > 0$ such that $C_q = 0$ for $q > n$ and each abelian group C_q is finitely generated. All chain complexes considered here are finite and free. A chain c in $\mathcal{C}$ is a *q-cycle* if $c \in \text{Ker } d_q$. If $c \in \text{Im } d_{q+1}$ then a is called a *q-boundary.* Denote the groups of q–cycles and q–boundaries by Z_q and B_q respectively. Define the integer *qth homology group* to be the quotient group Z_q/B_q, denoted by $H_q(\mathcal{C})$. We say that c is a *representative q–cycle* of the homology generator $c + B_q$ (denoted by $[c]$). For each q, the qth homology group $H_q(\mathcal{C})$ is a finitely generated free abelian group. The rank of H_q, denoted by β_q, is called the qth *Betti number* of $\mathcal{C}$. Homology is a powerful topological invariant, which characterizes an object by its q-dimensional "holes" (connected components, tunnels and cavities).

Let K be a complex (simplicial, cubical or simploidal). A q-chain c is a formal sum of q-cells (where q is the dimension of the cell) in K. Let $\{\sigma_1^q, \dots, \sigma_{m_i}^q\}$ be the set of q-cells in K, then $c = \sum_{i=1}^{m_i} \lambda_i \sigma_i^q$, where $\lambda_i \in \{0, 1\}$. Alternatively, we can think of c as the set $\{\sigma_i^q$, such that $\lambda_i = 1\}$, and the sum of two q-chains as their symmetric difference. The q-chains together with the addition operation form the group of q-chains denoted as $C_q(K)$. The differential of a q-cell σ in K, $d_q(\sigma)$, is the sum of the $(q-1)$-cells in K that belong to the boundary of σ. By linearity, the differential can be extended to q–chains. The *chain complex* $C(K)$ is the sequence of chain groups $C_q(K)$ connected by the homomorphisms d_q. The homology of K is defined as the homology of $C(K)$. Since we work with objects embedded in $\mathbf{R}^3$, the homology groups are torsion–free (see [1, ch.10]). Moreover, Theorem of Universal Coefficient [12] ensures that all the homology information can be computed working with coefficients in $\mathbf{Z}/2$.

An *AT-model* for K is established in [8,9] and used to obtain the homology and representative cycles of homology generators of K. An AT-model can be computed starting from an ordering of the cells in K [8,9]. We deal here with a particular ordering based on a *cover forest* T of K (any two vertices are connected by exactly one path in T if and only if they are connected in K). Let $T = (V, E)$ be a cover forest of K where V is the set of all the vertices of K and E a subset of edges of K. $S = (\sigma_0, \ldots, \sigma_m)$ is a *T-filter* if it is an ordering of all the cells in K such that:

- for each j (where $0 \leq j \leq m$), $\{\sigma_0, \ldots, \sigma_j\}$ is a subcomplex of K;
- if $i < j$, the dimension of σ_i is less or equal than the dimension of σ_j;
- if $i < j$, σ_i and σ_j are two edges and $\sigma_j \in T$, then $\sigma_i \in T$ That is, the edges of the cover forest are in first positions in S).

Observe that σ_0 is always a vertex of K. An AT-model for K is then defined as the output of the following algorithm, having as the input a complex K and a T-filter S of K.

Algorithm 1. *[8,9] AT-model Algorithm.*

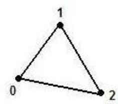
INPUT: `a` T`-filter` $S = (\sigma_0, \ldots, \sigma_m)$ `of` K,
$H := \{\sigma_0\}$, $f(\sigma_0) := \sigma_0$, $g(\sigma_0) := \sigma_0$, $\phi(\sigma_0) := 0$.
`For` $i = 1$ `to` m `do`
    `If` $fd(\sigma_i) = 0$, `then`
        $H := H \cup \{\sigma_i\}$, $f(\sigma_i) := \sigma_i$, $\phi(\sigma_i) := 0$, $g(\sigma_i) := \sigma_i + \phi d(\sigma_i)$.
    `If` $fd(\sigma_i) \neq 0$, `then:`
        $k :=$ `max` $\{j$ `such that` $\sigma_j \in fd(\sigma_i)$, $j = 1, ..., i-1\}$,
        $H := H \backslash \{\sigma_k\}$, $f(\sigma_i) := 0$, $\phi(\sigma_i) := 0$.
        `For` $j = 1$ `to` $i - 1$ `do if` $\sigma_k \in f(\sigma_j)$,
            $f(\sigma_j) := f(\sigma_j) + fd(\sigma_i)$, $\phi(\sigma_j) := \phi(\sigma_j) + \sigma_i + \phi d(\sigma_i)$.
OUTPUT: `the set` (S, H, f, g, ϕ).

Notice that in the ith step of the algorithm ($i = 1, \ldots, m$), exactly one homology generator is created or destroyed. The algorithm runs in time at most $\mathcal{O}(m^3)$.

Proposition 1. *Let K be a complex (simplicial, cubical or simploidal), let $T = (V, E)$ be a cover forest of K and S a T-filter of K. The output of Algorithm 1, (S, H, f, g, ϕ), satisfies that:*

- *H is a subset of S such that no edge in T is an edge in H. H generates a chain complex denoted by $\mathcal{H}$ with null differential;*
- *The number of vertices, edges and triangles in H equals the number of connected components, tunnels and cavities in $|K|$, respectively. In other words, the homology of K is isomorphic to $\mathcal{H}$.*
- *$f : C(K) \to \mathcal{H}$ satisfies that if c ad c' are two cycles in K such that $f(c) = f(c')$ then c and c' are homologous.*

– *$g : \mathcal{H} \to C(K)$ satisfies that $\{[g(h)] : h \in H\}$ is a set of homology generators of K. If $h, h' \in H$, $h \neq h'$, then $g(h)$ and $g(h')$ are not homologous. Moreover, if a is an edge in H, then $g(a)$ is a simple cycle in K and all the edges in $g(a) \setminus \{a\}$ are edges in T. In fact, $g(a) \setminus \{a\}$ is the simple path in T connecting the endvertices of a.*
– *$\phi : C(K) \to C(K)$ satisfies that if $x \in H$ then $\phi(x) = 0$ and there is no $y \in S$ such that $\phi(y) = x$. Moreover, if v is a vertex in K, then $\phi(v)$ is the simple path in T connecting v with the vertex in H that belongs to the same connected component in K than v.*



S	H	f	g	ϕ
$\langle 0\rangle$	$\langle 0\rangle$	$\langle 0\rangle$	$\langle 0\rangle$	0
$\langle 1\rangle$		$\langle 0\rangle$		$\langle 0,1\rangle$
$\langle 2\rangle$		$\langle 0\rangle$		$\langle 0,2\rangle$
$\langle 0,1\rangle$		0		0
$\langle 0,2\rangle$		0		0
$\langle 1,2\rangle$	$\langle 1,2\rangle$	$\langle 1,2\rangle$	$\langle 1,2\rangle + \langle 0,2\rangle + \langle 0,1\rangle$	0

Fig. 3. A a filter S of a simplicial complex K and the result of applying Algorithm 1 to S (an AT-model for K)

3 Computing a Graph-with-Loop Representation of a 3D Object

Let K be a complex (simplicial, cubical or simploidal); $|K|$ its geometric realization in $\mathbf{R}^3$; and $h : |K| \to \mathbf{R}$ a continuous function. Let e_{xy} denote an edge with endvertices x and y. We say that the *height of a point* $p \in |K|$ is t if $h(p) = t$ and the *height of a cell* $\sigma \in K$ is the minimum of the heights of all the points on $|\sigma|$. We say that K is an *h-complex* if:

– the set of the vertices of K can be partitioned into a finite number of subsets in terms of their height, $V = \bigcup_{i=1}^{r} V_i$, where $V_i = \{v \in V : h(v) = t_i$ and $t_1 < \cdots < t_r$.
– if e_{vw} is an edge in K then v and w belong to V_i for some $i = 1, \ldots, r$ or $v \in V_{i-1}$ and $w \in V_i$ for some $i = 2, \ldots, r$.

h-Complexes appear in a natural way when they are defined by the neighborhood relations of voxels of a 3D digital image and h is the real function that associates to each point on $|K|$ its elevation.

Let K be an h-complex and σ a cell in K. We say that σ is *horizontal* if the heights of all the points on $|\sigma|$ coincide; otherwise, it is *vertical*. For $i = 1, \ldots, r$, let K_i be the collection of all the horizontal cells in K with the same height t_i, $i = 0, 1, \ldots, r$. K_i is a subcomplex of K and if a cell σ is not in K_i, then σ is vertical. Let $T_i = (V_i, E_i)$ be a cover forest of K_i and S_i a T_i-filter of K_i. Denote by $(S_i, H_i, f_i, g_i, \phi_i)$, $i = 1, \ldots, r$, the AT-models obtained using Algorithm 1. Let V be the set of vertices in K, $T = (V, E)$ a cover forest of K (obtained

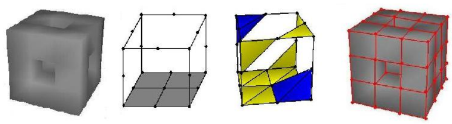


Fig. 4. From left to right: a digital image and 3 h-complexes associated to it considering the 6, 14 and 26-adjacency, respectively

after adding vertical edges in K in increasing ordering in height to the graph $(V, \bigcup_{i=1}^{r} E_i))$, and S a T-filter of K. Denote by (S, H, f, g, ϕ) the AT-model obtained using Algorithm 1.

Proposition 2. *The AT-model (S, H, f, g, ϕ) satisfies that:*

- *If a is a horizontal edge in H, then $a \in H_i$ for some i and $g(a) = g_i(a)$ is a simple cycle such that its edges are in K_i.*
- *If a is a vertical edge in H, then for each level i, $i = 1, \ldots, r$, $g(a)$ has an even number of vertical edges of height t_i.*

Now, let us explain how to construct the graph $G_h(K)$ and the function $\Psi : G_h(K) \to K$ using the AT-models $(S_i, H_i, f_i, g_i, \phi_i)$, $i = 1, \ldots, r$ and (S, H, f, g, ϕ) computed before. First, the vertices in $G_h(K)$ in each level i are the vertices in H_i, $i = 1, \ldots, r$. If v is a vertex in $G_h(K)$, then $\Psi(v) = v$. Second, for each level i and (horizontal) edge a in $H_i \cap H$, we add a loop α in $G_h(K)$ such that its endvertex is the vertex in the level i of $G_h(K)$ which belongs to the same connected component than $|a|$ in $|K_i|$. Define $\Psi(\alpha) = g_i(a) = g(a)$. Third, we add an edge e_{xy} between two vertices x and y in $G_h(K)$ if $x \in H_i$ and $y \in H_{i+1}$ for some i and $f(x) = f(y) = z \in H$ (i.e. x and y belong to the same connected component in K). Define $\Psi(e_{xy}) = \phi(x) + \phi(y)$ (the simple path in T connecting the vertices x and y). Finally, for each vertical edge e_{vw} in H, an edge b is added to $G_h(K)$. Since e_{vw} is vertical, then $v \in H_i$ and $w \in H_{i+1}$ for some i. The endvertices of b are the vertices in $G_h(K)$ which belong to the same connected component than v in K_i and w in K_{i+1}, respectively. Define $\Psi(b) = e_{vw} + \phi_i(v) + \phi_{i+1}(w)$.

Theorem 2. *Given a complex K and a continuous function $h : |K| \to \mathbf{R}$. If K is an h-complex, then:*

1. *The graph $G_h(K)$ and the complex K has the same number of tunnels and connected components.*
2. *For each loop $\alpha \in G_h(K)$, $\Psi(\alpha)$ is a simple cycle representative of a homology generator of K. If α_1 and α_2 are two different loops in $G_h(K)$, then $\Psi(\alpha_1)$ and $\Psi(\alpha_2)$ are two representative cycles of two non-equivalent generators of homology.*
3. *For each edge e_{xy} in $G_h(K)$ that comes from a vertical edge $e_{vw} \in H$, then $\Psi(e_{xy}) + \phi(x) + \phi(y) = g(e_{vw})$ is a representative cycle of a homology generator of K.*
4. *The graph $G_h(K)$ without loops is a subdivision of the Reeb graph $R_h(K)$.*
5. *The graph $G_h(K)$ and the function Ψ can be computed in $\mathcal{O}(m^3)$, where m is the number of cells in K.*

Proof. The number of tunnels of K is the number of edges in H. By construction, each horizontal edge in H produces a loop in $G_h(K)$ (i.e. a tunnel in $G_h(K)$). Each vertical edge e_{vw} in H produces a vertical edge β in $G_h(K)$. Let v in K_i and w in K_{i+1}. Let V and W be the two vertices in $G_h(K)$ that belong to the same connected component than v and w, respectively. Since $e_{vw} \in H$, then e_{vw} created a cycle when it was added. Therefore, v and w belong to the same connected component in K and so, there exists a path p between V and W in $G_h(K)$ apart from the edge β that produces e_{vw}, by construction. Then, $p + \beta$ is a cycle in $G_h(K)$. Moreover, $\Psi(p+\beta) = g(b)$ is a representative cycle of the homology generators of dimension 1 of K. Since representative cycles of a homology generator of dimension 1 of K map by ψ to a cycle in $R_h(K)$, then $\psi(g(b))$ is a cycle in $R_h(K)$. Since K is an h-complex, then a vertex in $R_h(K)$ corresponds to a contour in a level t_i, $i = 1, \dots, r$. Therefore, a vertex in $R_h(K)$ is a vertex in $G_h(K)$. $\square$

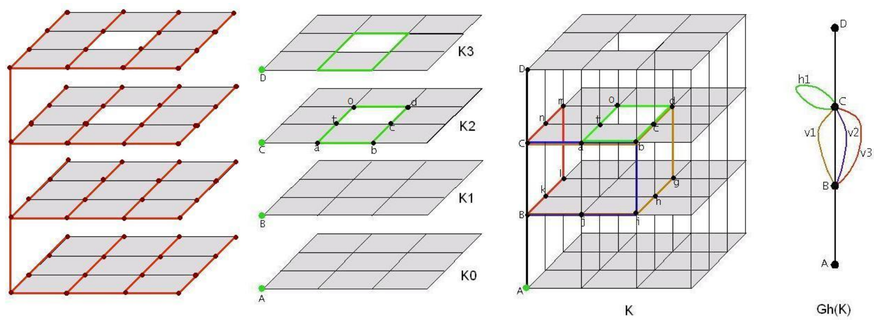


Fig. 5. From left to right: a cover forest T of the cubical complex K showed on the right of Figure 4; the complexes K_0, K_1, K_2 and K_3; a set of representative cycles of the generators $H_1(K)$; and the graph $G_h(K)$

Example 1. Let K be the cubical complex K on the right in Figure 4. A cover forest T of K; the complexes K_0, K_1, K_2 and K_3; a set of representative cycles of the generators $H_1(K)$; and the graph $G_h(K)$ The non-trivial identification of the edges and loops in $G_h(K)$ and K by Ψ are:

$G_h(K)$	Ψ
h_1	$e_{ab} + e_{bc} + e_{cd} + e_{do} + e_{of} + e_{af}$
v_1	$e_{Bj} + e_{ij} + e_{hi} + e_{gh} + e_{dg} + e_{cd} + e_{ac} + e_{ab} + e_{bC}$
v_2	$e_{Bj} + e_{ij} + e_{bi} + e_{ab} + e_{aC}$
v_3	$e_{Bk} + e_{\ell k} + e_{m\ell} + e_{mn} + e_{nC}$

The representative cycles of generators of $H_1(K)$ are:

$\alpha_0 = \Psi(h_1) = e_{ab} + e_{bc} + e_{cd} + e_{do} + e_{of} + e_{af}$
$\alpha_1 = \Psi(v_1) + e_{BC} = e_{Bj} + e_{ij} + e_{hi} + e_{gh} + e_{dg} + e_{cd} + e_{ac} + e_{ab} + e_{bC} + e_{BC}$
$\alpha_2 = \Psi(v_2) + e_{BC} = e_{Bj} + e_{ij} + e_{bi} + e_{ab} + e_{aC} + e_{BC}$
$\alpha_3 = \Psi(v_3) + e_{BC} = e_{Bk} + e_{\ell k} + e_{m\ell} + e_{mn} + e_{nC} + e_{BC}$

4 Conclusions and Future Work

It is possible to obtain representative cycles on the boundary of the given complex K if we compute a cover forest of K first adding the edges on the boundary. Another task is the generalization of the method to any dimension. The problem is that the homology of a complex of a dimension higher than 3 can have torsion groups. In order to capture the torsion part of the homology we could use the concept of λ-AT-model developed in [10].

A possible extension of this work is the construction of a discrete Morse complex $M_h(K)$ associated to a cell complex K, such that there is not only a one-by-one identification of all the homology generators of $M_h(K)$ with that of K, but also an isomorphism between cohomology rings. $M_h(K)$ can be constructed using a gradient vector field $\mathcal{V}_K$ associated to a discrete Morse function (see [6,7]) that can be obtained from an AT-model for K.

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