REPRODUCTIVE AND TIME PERIODIC SOLUTIONS FOR INCOMPRESSIBLE FLUIDS

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The title

Time-conditions

- Initial condition: $\boldsymbol{u}(0) = \boldsymbol{u}_0$
- Condition of reproductivity: $\boldsymbol{u}(0) = \boldsymbol{u}(T)$

- Condition of reproductivity (or time-periodic condition) $\Rightarrow$**Reproductive solutions**
- Moreover if $\boldsymbol{u}(t+T) = \boldsymbol{u}(t) \qquad \forall t \in (0, +\infty)$ $\Rightarrow$ **Periodic solutions**

Incompressibility condition

$$\nabla \cdot u = 0$$

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$\Omega \subset \mathbb{R}^d$ ($d = 2$ or 3) bounded, regular enough domain.
$Q = (0, T) \times \Omega \qquad \Sigma = (0, T) \times \partial\Omega$.

Navier-Stokes equations

$$\begin{cases} \dfrac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} - \nu \Delta \boldsymbol{u} + \nabla p = \boldsymbol{f} \quad \text{in } Q, \\ \nabla \cdot \boldsymbol{u} = 0 \quad \text{in } Q, \\ \boldsymbol{u}(\boldsymbol{x}, t) = 0, \qquad \text{on } \Sigma, \\ + \text{ Condition in time} \end{cases}$$

Notation

$$\boldsymbol{H} = \{\boldsymbol{u} \in \boldsymbol{L}^2;\ \nabla \cdot \boldsymbol{u} = 0,\ \boldsymbol{u} \cdot \boldsymbol{n} = 0 \text{ on } \partial\Omega\},$$

$$\boldsymbol{V} = \{\boldsymbol{u} \in \boldsymbol{H}^1;\ \nabla \cdot \boldsymbol{u} = 0,\ \boldsymbol{u} = 0 \text{ on } \partial\Omega\}$$

Definition

If $\boldsymbol{u}_0 \in H$, $\boldsymbol{f} \in L^2(\boldsymbol{H}^{-1})$, $\boldsymbol{u}$ weak solution of the initial-boundary problem in $(0, T)$ if

$$\boldsymbol{u} \in L^2(\boldsymbol{V}) \cap L^\infty(\boldsymbol{H}), \int_Q \left\{ -\boldsymbol{u}\boldsymbol{v}_t + \nabla \boldsymbol{u} : \nabla \boldsymbol{u} - (\boldsymbol{u} \cdot \nabla)\boldsymbol{v}\boldsymbol{u} - \boldsymbol{f}\boldsymbol{v} \right\} = 0,$$

for all $\boldsymbol{v} \in C^1(\boldsymbol{H}) \cap C(\boldsymbol{V})$, with compact support and $u(0) = u_0$.

If $u_0 \in \mathbf{V}$ and $f \in L^2(\mathbf{L}^2)$ any weak solution will be a strong solution if $\boldsymbol{u} \in L^2(\boldsymbol{H}^2 \cap \boldsymbol{V}) \cap L^\infty(\boldsymbol{V})$,

then $\boldsymbol{u}_t \in L^2(\boldsymbol{H})$ and verifies the system pointwise a.e. in Q.

If $T = \infty$ OK.

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Main classical results for the initial-boundary problem

Weak solution

For any $u_0 \in \boldsymbol{H}$ and $\boldsymbol{f} \in L^2(0, T; \boldsymbol{H}^{-1}(\Omega))$, initial-boundary problem has (at least) a weak solution.

Regularity

If $u_0 \in \boldsymbol{V}$ and $\boldsymbol{f} \in L^\infty(0, \infty; \boldsymbol{L}^2(\Omega))$:

- Unique strong solution $(\boldsymbol{u}, p)$ local
- If $(\boldsymbol{u}_0, \boldsymbol{f})$ are small enough the strong solution is global.

Weak/strong uniqueness property

If a solution has the strong regularity, it coincides with any weak solution associated with the same data.

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Main classical results for the initial-boundary problem

$N = 2$

- If $u_0 \in \boldsymbol{H}$ and $\boldsymbol{f} \in L^2(0, T; \boldsymbol{H}^{-1}(\Omega))$ the weak solutions is unique.
- If $u_0 \in \boldsymbol{V}$ and $\boldsymbol{f} \in L^\infty(0, \infty; \boldsymbol{L}^2(\Omega))$ the strong solution is global.

Theorem

For any $\boldsymbol{f} \in L^2(0, T; \boldsymbol{H}^{-1}(\Omega))$, *there exists a weak solution of reproductive problem.*

Time periodic extension, $\widetilde{\boldsymbol{u}}$, of any weak reproductive solution $\boldsymbol{u}$ to the whole time interval $(0, +\infty)$ is a periodic weak solution corresponding to the data, $\widetilde{\boldsymbol{f}}$, defined as the time periodic extension of $\boldsymbol{f}$.

On the time-periodic weak solutions

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Main ideas proof existence reproductive solutions

Let $\boldsymbol{u}^k$ the unique approximate solution of the Galerkin initial boundary problem of Navier-Stokes in the finite-dimensional subspace $\boldsymbol{V}^k$, spanned by the first k elements of the "spectral" basis of $\boldsymbol{V}$ (orthogonal in $\boldsymbol{V}$ and orthonormal in $\boldsymbol{H}$), associated to a initial discrete data $\boldsymbol{u}_0^k \in \boldsymbol{V}^k$.

Main ideas proof existence reproductive solutions

Energy inequality + Poincaré inequality + integrating $[0, T] \Rightarrow$

$$e^{c_1 T} \|\boldsymbol{u}^k(T)\|_{L^2}^2 \leq \|\boldsymbol{u}^k(0)\|_{L^2}^2 + C \int_0^T e^{c_1 t} \|\boldsymbol{f}(t)\|_{H^{-1}}^2 dt. \quad (1)$$

We define the operator $L^k : [0, T] \to \mathbb{R}^k$,
$L^k(t) = (c_1^k(t), ..., c_k^k(t))$ where $c_i^k(t)$, $i = 1, \ldots, k$, are the coefficients of the expansion of $\boldsymbol{u}^k(t)$ in V^k.
Note that

$$\|L^k(t)\|_{\mathbb{R}^k} = \|\boldsymbol{u}^k(t)\|_{L^2}, \quad (2)$$

We define the operator $\Phi^k : \mathbb{R}^k \to \mathbb{R}^k$ as follows: Given $L_0^k \in \mathbb{R}^k$, $\Phi^k(L_0^k) = L^k(T)$, where $L^k(t)$ are the coefficients of the Galerkin solution with initial value with coefficients L_0^k.

Leray-Schauder Theorem

For all $\lambda \in [0, 1]$, the possible solutions of the equation $L_0^k(\lambda) = \lambda \Phi^k(L_0^k(\lambda))$, are bounded independently of λ?

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Since $L_0^k(0) = 0$, it suffices to consider $\lambda \in (0, 1]$ and

$$\Phi^k(L_0^k(\lambda)) = \frac{1}{\lambda} L_0^k(\lambda).$$

Definition of Φ^k, (1) and (2) $\Rightarrow$

$$e^{c_1 T} \|\frac{1}{\lambda} L_0^k(\lambda)\|^2_{\mathbb{R}^k} \leq \|L_0^k(\lambda)\|^2_{\mathbb{R}^k} + C \int_0^T e^{c_1 T} \|\boldsymbol{f}(t)\|^2_{H^{-1}} dt,$$

$$\|L_0^k(\lambda)\|^2_{\mathbb{R}^k} \leq \frac{C \int_0^T e^{c_1 T} \|\boldsymbol{f}(t)\|^2_{H^{-1}} dt}{e^{c_1 T} - 1} = M,$$

for each $\lambda \in (0, 1]$. $\square$

Weak periodic solutions–global solutions

$N = 2$
Assume $\boldsymbol{f} : [0, +\infty) \to \boldsymbol{H}^{-1}(\Omega)$ and T-time periodic.

The reproductive solution $\boldsymbol{u}$ associated to $\boldsymbol{u}(0) = \boldsymbol{u}(T) := \boldsymbol{u}_0$, is unique in $[0, T]$.

The solution $\overline{\boldsymbol{u}}(t) = \boldsymbol{u}(t - T)$ verifies $\boldsymbol{u}(T) = \boldsymbol{u}(2T) := \boldsymbol{u}_0$ and is unique in $[T, 2T]$.

$\cdots\cdots\cdots\cdot.$

Navier-Stokes 2D

The solution associated to $\boldsymbol{u}_0$ is periodic.

Weak periodic solutions–global solutions

$N = 3$

Uniqueness of weak solution is not known.

It is possible that the reproductive solution $\boldsymbol{u}$ and the weak solution $\widetilde{\boldsymbol{u}}$ associated to the initial data $\boldsymbol{u}_0 := \boldsymbol{u}(0) = \boldsymbol{u}(T)$ are different in $(0, T)$, although they coincide locally in time, near of the initial time $t = 0$.

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Boussinesq equations

$$
\begin{aligned}
\frac{\partial \boldsymbol{u}}{\partial t} - \nu \Delta \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nabla p &= \alpha \theta \boldsymbol{g} + \boldsymbol{f} \quad \text{in } Q, \\
\nabla \cdot \boldsymbol{u} &= 0 \quad \text{in } Q, \\
\frac{\partial \theta}{\partial t} - \chi \Delta \theta + (\boldsymbol{u} \cdot \nabla)\theta &= 0 \quad \text{in } Q, \\
\boldsymbol{u}_{|\Sigma} = 0, \qquad \theta_{|\Sigma} &= 0, \\
\boldsymbol{u}(0) = \boldsymbol{u}(T) \qquad \theta(0) &= \theta(T) \quad \text{in } \Omega.
\end{aligned}
$$

Micropolar equations

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - (\nu+\nu_r)\Delta\boldsymbol{u} + \nabla p = 2\nu_r \text{ rot } \mathbf{w} + \boldsymbol{f},$$

$$\text{div } \boldsymbol{u} = 0 \qquad \text{in } Q,$$

$$\frac{\partial \mathbf{w}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\mathbf{w} - (c_a+c_d)\Delta\mathbf{w} - (c_0+c_d-c_a)\nabla\text{div }\mathbf{w}$$

$$+4\nu_r\mathbf{w} = 2\nu_r \text{ rot } \boldsymbol{u} + \mathbf{g},$$

$$\boldsymbol{u}_{|\Sigma} = 0, \qquad \boldsymbol{w}_{|\Sigma} = 0,$$

$$\boldsymbol{u}(0) = \boldsymbol{u}(T), \qquad \boldsymbol{w}(0) = \boldsymbol{w}(T) \qquad \text{in } \Omega.$$

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Reproductivity and maximum principle

Given $\boldsymbol{u} : Q \to \mathbb{R}^3$ such that $\nabla \cdot \boldsymbol{u} = 0$ in Q and $\boldsymbol{u} \cdot \mathbf{n} = 0$ on $\partial\Omega$, we consider the (reproductive) convection-diffusion problem for the unknown $c : Q \to \mathbb{R}$ (a concentration):

$$\partial_t c - \Delta c + \boldsymbol{u} \cdot \nabla c = 0, \qquad c_{|\Sigma} = c_\Sigma, \qquad c(0) = c(T),$$

where $0 < \underline{c} \leq c_\Sigma \leq \overline{c}$ on Σ, for some constants $\underline{c}$ and $\overline{c}$.

Any reproductive solution satisfies the maximum principle.

In particular,

$$\partial_t (c - \overline{c}) - \Delta(c - \overline{c}) + (\boldsymbol{u} \cdot \nabla)(c - \overline{c}) = 0 \qquad \text{in } Q.$$

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Reproductivity and maximum principle

Multiplying by $(c - \overline{c})_+$ and integrating in Ω:

$$\frac{d}{dt}\int_\Omega |(c - \overline{c})_+|^2 + \int_\Omega |\nabla(c - \overline{c})_+|^2 \leq 0.$$

Integrating in $t \in (0, T)$ and using the periodic condition $c(0) = c(T)$:

$$\int_0^T \|\nabla(c - \overline{c})_+\|_{L^2}^2 = 0.$$

Hence $c \leq \overline{c}$ in Q hold. Similarly $c \geq \underline{c}$ in Q hold.

Reproductivity and maximum principle

Generalized Boussinesq system

$$\begin{cases} \partial_t \boldsymbol{u} - \nabla \cdot (\nu(\theta)\nabla \boldsymbol{u}) + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p = \alpha\theta\mathbf{g} + \boldsymbol{f}, \\ \nabla\cdot\boldsymbol{u} = 0, \\ \partial_t\theta - \nabla\cdot(k(\theta)\nabla\theta) + (\boldsymbol{u}\cdot\nabla)\theta = 0, \end{cases}$$

$$\boldsymbol{u} = 0, \qquad \theta = \theta_{\partial\Omega} \qquad \text{on } \partial\Omega\times[0,T),$$

$$\boldsymbol{u}(0) = \boldsymbol{u}(T), \qquad \theta(0) = \theta(T) \qquad \text{in } \Omega.$$

$\nu : \mathbb{R} \to \mathbb{R}^+$ and $k : \mathbb{R} \to \mathbb{R}^+$ are continuous functions.

Reproductivity and maximum principle

$$\theta_{\text{mín}} = \text{mín}\, \theta_{\partial\Omega} \qquad \theta_{\text{máx}} = \text{máx}\, \theta_{\partial\Omega}$$

Maximum principle $\implies \quad \theta_{\text{mín}} \leq \theta \leq \theta_{\text{máx}}.$

Then

$$\exists\, \nu_{\text{mín}} > 0, \quad k_{\text{mín}} > 0, \quad \nu_{\text{máx}} > 0, \quad k_{\text{máx}} > 0$$

such that

$$\nu_{\text{mín}} \leq \nu(s) \leq \nu_{\text{máx}}, \quad k_{\text{mín}} \leq k(s) \leq k_{\text{máx}}, \quad \forall s \in [\theta_{\text{mín}}, \theta_{\text{máx}}].$$

Changing ν by $\widetilde{\nu}$ and k by $\widetilde{k}$, where $\widetilde{\nu}$ and $\widetilde{k}$ are bounded functions, the same way that in the Navier-Stokes case.

Reproductivity and maximum principle

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Changing ν by $\widetilde{\nu}$ and k by $\widetilde{k}$, where $\widetilde{\nu}$ and $\widetilde{k}$ are bounded functions, the same way that in the Navier-Stokes case.

Penalized Nematic liquid crystal model

$$\begin{cases} \partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \mu\Delta\boldsymbol{u} + \nabla p = -\lambda\nabla\cdot(\nabla\boldsymbol{d}^t\nabla\boldsymbol{d}), \\ \nabla\cdot\boldsymbol{u} = 0, \\ \partial_t\boldsymbol{d} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{d} = \gamma(\Delta\boldsymbol{d} - \boldsymbol{f}_\varepsilon(\boldsymbol{d})). \end{cases}$$

$$\boldsymbol{u} = 0, \qquad \boldsymbol{d} = \boldsymbol{h} \qquad \text{on } \partial\Omega\times(0,T)$$

$$\boldsymbol{u}(0) = \boldsymbol{u}(T), \qquad \boldsymbol{d}(0) = \boldsymbol{d}(T) \qquad \text{in } \Omega.$$

$$\boldsymbol{f}_\varepsilon(\boldsymbol{d}) = \varepsilon^{-2}(|\boldsymbol{d}|^2 - 1)\boldsymbol{d}$$

Assuming $|\boldsymbol{h}| \leq 1$, we can apply the maximum principle argument obtaining $|\boldsymbol{d}| \leq 1$.

Reproductivity and maximum principle

We consider a equivalent problem changing $\boldsymbol{f}_\varepsilon$ by $\widetilde{\boldsymbol{f}}_\varepsilon$, the auxiliary function

$$\widetilde{\boldsymbol{f}}_\varepsilon(\boldsymbol{d}) = \begin{cases} \boldsymbol{f}_\varepsilon(\boldsymbol{d}) & \text{if} \quad |\boldsymbol{d}| \leq 1, \\ 0 & \text{if} \quad |\boldsymbol{d}| > 1. \end{cases}$$

The key is that $|\widetilde{\boldsymbol{f}}_\varepsilon(\boldsymbol{d})| \leq \dfrac{1}{\varepsilon^2}\ \forall \boldsymbol{d} \in \mathbb{R}^3$. Then, existence of weak reproductive solution of this model can be proved.

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Time-periodic boundary problem associated to 3D Navier Stokes model

The periodic extension of a reproductive solution $\boldsymbol{u}$ is a regular solution in $[0, +\infty)$ assuming small enough external forces $\boldsymbol{f}$.

This conclusion is also valid for Boussinesq equations or micropolar equations.

Proof regularity time-periodic solutions 3D N-S

Energy inequality + integrating in $(0,T)$ + condition of reproductivity $\Rightarrow$

$$\nu \int_0^T \|\nabla \boldsymbol{u}(t)\|_{L^2}^2 \leq \frac{1}{\nu} \int_0^T ||\boldsymbol{f}(t)||_{H^{-1}}^2.$$

$\int_0^T ||\boldsymbol{f}(t)||_{H^{-1}}^2$ small enough + Mean Value Theorem $\Rightarrow$

$$\exists t_\star \in [0,T] \text{ such that } \|\nabla \boldsymbol{u}(t_\star)\|_{L^2}^2 \leq \varepsilon.$$

Proof regularity time-periodic solutions 3D N-S

Let $\overline{\boldsymbol{u}}$ be the unique regular strong solution with initial data $\boldsymbol{u}(t_\star)$ and the same force $\boldsymbol{f}$.

One proves $\|\nabla\overline{\boldsymbol{u}}(t)\|_{L^2}^2 \leq 2\,\varepsilon$ for each $t \geq t_\star$

Uniqueness of weak-strong solution $\Rightarrow \overline{\boldsymbol{u}} \equiv \boldsymbol{u}$ in $[t_\star, T]$ and therefore $\boldsymbol{u}$ is regular in $[t_\star, T]$

In particular, $2\,\varepsilon \geq \|\nabla\overline{\boldsymbol{u}}(T)\|_{L^2}^2 = \|\nabla\boldsymbol{u}(T)\|_{L^2}^2 = \|\nabla\boldsymbol{u}(0)\|_{L^2}^2$, hence $\boldsymbol{u}$ is a strong solution in $[0, T]$.

In $[T, 2T]$, $\boldsymbol{u}(t - T) \equiv \overline{\boldsymbol{u}}(t)$ and so on.

Regularity of periodic solutions

3*D* penalized nematic liquid crystal

The regularity of the reproductive solutions for the 3*D* penalized nematic liquid crystal model is an open problem.

$3D$ penalized nematic liquid crystal. Regularity ?

$\widetilde{\boldsymbol{d}}$: adequate lifting function
$\widehat{\boldsymbol{d}} = \boldsymbol{d} - \widetilde{\boldsymbol{d}}$

Energy inequality:

- $$\frac{d}{dt}\left(\|\boldsymbol{u}\|_{L^2}^2 + \lambda\|\nabla\widehat{\boldsymbol{d}}\|_{L^2}^2\right) + 2\mu\|\nabla\boldsymbol{u}\|_{L^2}^2 + \lambda\gamma\|\Delta\widehat{\boldsymbol{d}}\|_{L^2}^2 \le C\left(\lambda\gamma\|\boldsymbol{f}_\varepsilon(\boldsymbol{d})\|_{L^2}^2 + \|\partial_t\widetilde{\boldsymbol{d}}\|_{L^2}^2\right).$$

- $$\frac{d}{dt}\left(\|\boldsymbol{u}\|_{L^2}^2 + \lambda\|\nabla\widehat{\boldsymbol{d}}\|_{L^2}^2 + 2\,\lambda\int_\Omega F_\varepsilon(\boldsymbol{d})\right) + 2\,\mu\|\nabla\boldsymbol{u}\|_{L^2}^2 + \lambda\gamma\|\Delta\widehat{\boldsymbol{d}} - \boldsymbol{f}_\varepsilon(\boldsymbol{d})\|_{L^2}^2 \le \frac{\lambda}{\gamma}\int_0^T \|\partial_t\tilde{\boldsymbol{d}}\|_{L^2}^2 + \frac{2\lambda}{\varepsilon^2}\int_0^T \|\partial_t\tilde{\boldsymbol{d}}\|_{L^1}.$$

Regularity of periodic solutions

Generalized Boussinesq model

The regularity of a time-periodic solution for the generalized Boussinesq model is an open problem.

Generalized Boussinesq model with Neumann boundary conditions for temperature

Assuming $\boldsymbol{f}$ small enough, reproductive solution has $H^2(\Omega)$-velocity and $H^3(\Omega)$-temperature regularity.

When Dirichlet boundary conditions for $\boldsymbol{u}$ and θ are assumed, it is not clear how to obtain appropriate differential inequalities in H^2 for velocity and H^3 for temperature.

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Periodic solutions

- Uniqueness of regular time periodic solutions remains open: H^3 regularity for the velocity $\leftrightarrow$ Dirichlet condition for velocity !!
- Argument of regular time periodic solution small data $\Rightarrow$ $\|\boldsymbol{u}(t_\star)\|_{H^1}^2 + \|\theta(t_\star)\|_{H^1}^2$ is small but $\|\theta(t_\star)\|_{H^2}^2$?