Periodicity for a nematic liquid crystal model

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1. Introduction

Solid state

-

Temperature:
random order
molecules

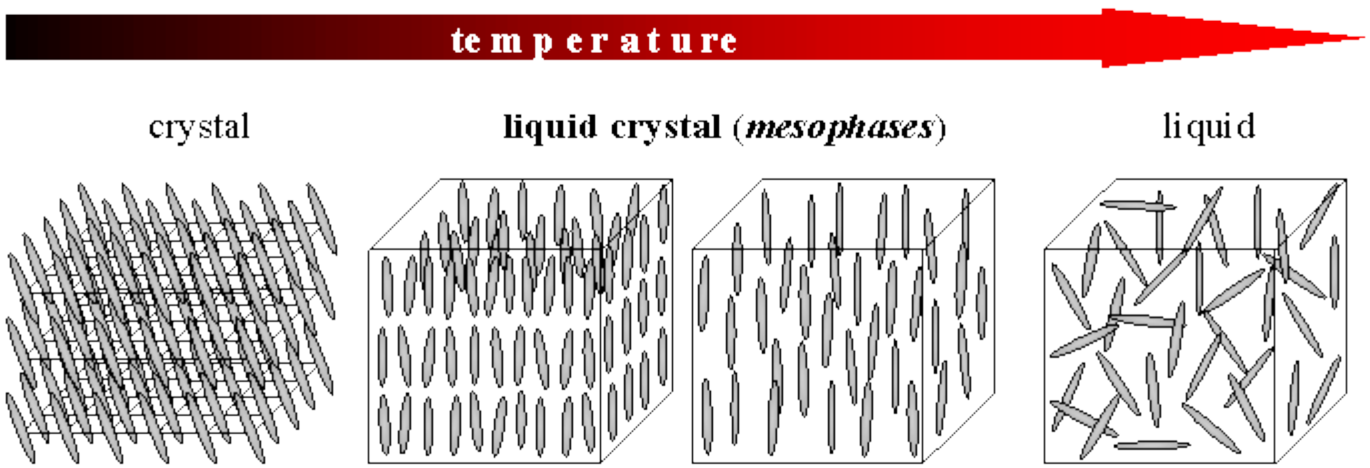
Cristal liquid:
optical characteristics of a liquid(anisotropic)
electro-magnetics characteristics of solid

+

Liquid state

1. Introduction

thermotropic liquid crystals



crystal
- 3-D lattice
- orientation
- solid

↳ ***anisotropic***

liquid crystal (mesophases)
- 1- (2-)D lattice
- orientation
- fluid

↳ ***anisotropic***

- no lattice
- orientation
- fluid

↳ ***anisotropic***

liquid
- no lattice
- no orientation
- fluid

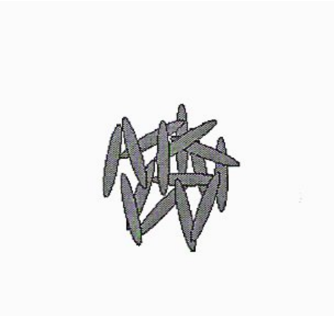
↳ ***isotropic***

Samples:

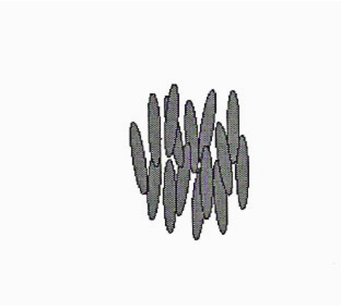
- Soap, soup
- Biological membranes
- The protein solution to generate silk of a spider
- DNA and polypeptides can form LC phases

Applications:

- Liquid Crystal Displays : wrist watches, pocket calculators, flat screens…
- Liquid Crystal Thermometers: to show a "map" of temperatures to find tumors, bad connections on a circuit board…
- Windows that can be changed from clear and opaque with the flip of a switch
- To make a stable hydrocarbon foam
- Optical Imaging and recording
- Nondestructive mechanical testing of materials under stress.

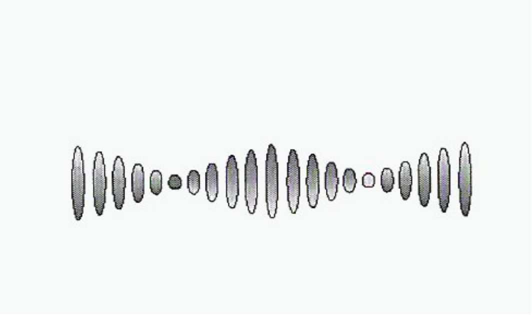


Isotropic phase

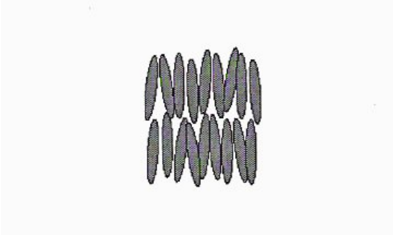


Nematic phase

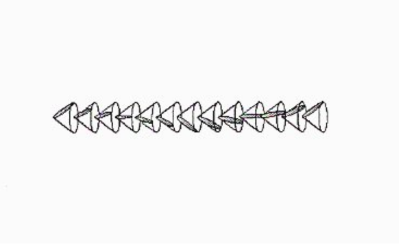
Average direction: **d**



Chiral nematic phase



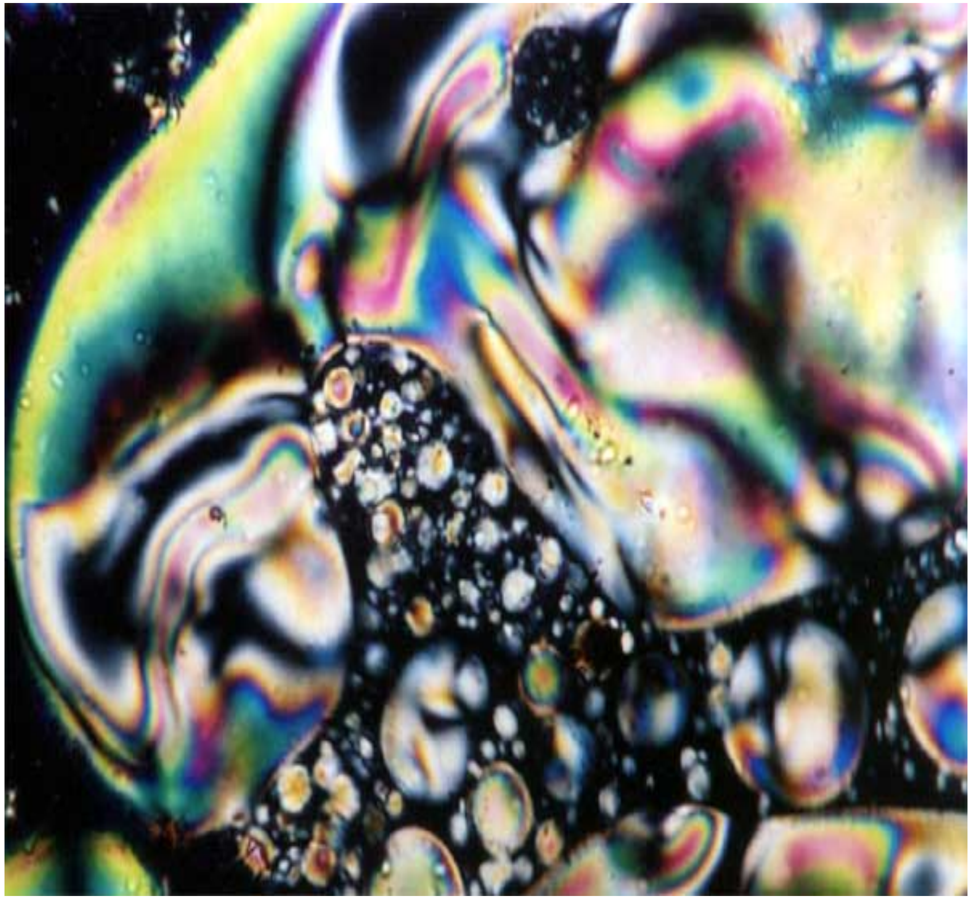
Smetic phase



Chiral Smetic phase

Nematic Crystal:

The molecular orientation (the material's optical properties) can be controlled with applied electric fields orientation



Schlieren texture of Liquid Crystal nematic phase

2. Statement of the problem.

Ericksen-Leslie version:

$\Omega \subset \mathbb{R}^N$ $(N = 2 \text{ or } 3)$, $\partial\Omega$ regular

Ginzburg-Landau penalization function: $f(\mathbf{d}) = \frac{1}{\varepsilon^2}(|\mathbf{d}|^2 - 1)\mathbf{d}, \quad \varepsilon > 0$

$\Rightarrow$ $|\mathbf{d}| = 1$ is partially conserved to $|\mathbf{d}| \leq 1$

$$\begin{cases} \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} - \nu \Delta \mathbf{u} + \nabla p = -\nabla \mathbf{d}^t \Delta \mathbf{d}, \quad \nabla \cdot \mathbf{u} = 0, \\ \partial_t \mathbf{d} + (\mathbf{u} \cdot \nabla)\mathbf{d} = (\Delta \mathbf{d} - f(\mathbf{d})), \quad |\mathbf{d}| \leq 1, \end{cases}$$

in $(0,T) \times \Omega$

$$\mathbf{u}(x,t) = 0, \qquad \mathbf{d}(x,t) = \mathbf{h}(x,t) \qquad \text{on } \partial\Omega \times (0,T)$$

$$\mathbf{u}(x,0) = \mathbf{u}(x,T), \qquad \mathbf{d}(x,0) = \mathbf{d}(x,T) \qquad \text{in } \Omega$$

2. Statement of the problem.

$$\boldsymbol{d}(x,t)_{|\partial\Omega\times(0,T)} = \boldsymbol{d}_0(x)$$

Trivial stationary (static) solution:

$$\boldsymbol{u} \equiv 0,$$

$$\boldsymbol{d} \quad \text{solution of the elliptic problem:} \quad -\Delta\boldsymbol{d} + f(\boldsymbol{d}) = 0 \quad \text{in } \Omega, \quad \boldsymbol{d}_{|\partial\Omega} = \boldsymbol{d}_0,$$

$$p \quad \text{such that} \quad \nabla p = -\nabla \boldsymbol{d}^t \Delta \boldsymbol{d}.$$

2. Statement of the problem.

Previous results: Existence weak solution, initial value problem with Dirichlet's boundary (d_0 indepent of time).
Existence and uniqueness classic solution, viscosity larger

F.H. Lin, C. Liu, *Non-parabolic dissipative systems modelling the flow of liquid crystals*, Comm.Pure Appl. Math, (1995)

Goal: Existence of reproductive weak solution (global in time) for a nematic liquid crystal model.

N=2 periodic solution

B. Climent Ezquerra, F. Guillén González, M. Rojas Medar; *Reproductivity for a nematic liquid crystal model*, Z. Angew Math. Phys.(2005)

Difficulties: The strongly nonlinear coupling

Time reproductive condition

Constraint $|\mathbf{d}| \leq 1$

Time dependent boundary conditions

Compatibility condition:

$$|\boldsymbol{h}| \leq 1 \quad \text{on } \partial\Omega \times (0,T) \quad \text{and} \quad \boldsymbol{h}(0) = \boldsymbol{h}(T) \quad \text{on } \partial\Omega.$$

Definition 1: *(u,d)* Reproductive solution

$$\boldsymbol{u} \in L^2(\boldsymbol{V}) \cap L^\infty(\boldsymbol{H})$$
$$\boldsymbol{d} \in L^\infty(H^1), \quad \Delta\boldsymbol{d} \in L^2(L^2), \quad \boldsymbol{d}|_{\partial\Omega\times(0,T)} = \mathbf{h}$$

verifying

$$\langle \partial_t \boldsymbol{u}, \mathbf{v}\rangle + ((\boldsymbol{u}\cdot\nabla)\boldsymbol{u}, \mathbf{v}) + (\nabla\boldsymbol{u}, \nabla\mathbf{v}) + (\nabla\boldsymbol{d}^t\Delta\boldsymbol{d}, \mathbf{v}) = 0 \quad \forall\, \mathbf{v} \in \boldsymbol{V} \cap \boldsymbol{L}^\infty,$$
$$\langle \partial_t \boldsymbol{d}, \mathbf{e}\rangle + ((\boldsymbol{u}\cdot\nabla)\boldsymbol{d}, \mathbf{e}) + (f(\boldsymbol{d}), \mathbf{e}) - (\Delta\boldsymbol{d}, \mathbf{e}) = 0 \quad \forall\, \mathbf{e} \in \boldsymbol{L}^3,$$
$$\boldsymbol{u}(0) = \boldsymbol{u}(T), \quad \boldsymbol{d}(0) = \boldsymbol{d}(T) \quad \text{in } \Omega.$$

Weak Maximun Principle for d

$e = d\,(|\boldsymbol{d}(x,t)|^2 - 1)_+$ test function $\Longrightarrow$

$$|\boldsymbol{d}(x,t)| \leq 1 \text{ a.e. in } \Omega \times (0,T)$$

$$f(\boldsymbol{d}) \longleftrightarrow \widetilde{f}(\boldsymbol{d}) = \begin{cases} f(\boldsymbol{d}) & \text{if} \quad |\boldsymbol{d}| \leq 1 \\ 0 & \text{if} \quad |\boldsymbol{d}| > 1 \end{cases}$$

$$|\widetilde{f}(\boldsymbol{d})| \leq 1/\varepsilon^2 \quad \forall \boldsymbol{d} \in \mathbb{R}^N$$

Lifting of boundary data: $\boldsymbol{h} \in H^1(0,T;\boldsymbol{H}^{1/2}(\partial\Omega)^N)$

We define $\widetilde{\boldsymbol{d}}(t)$ as

$$\begin{cases} -\Delta \widetilde{\boldsymbol{d}} = 0 & \text{in } \Omega, \\ \widetilde{\boldsymbol{d}}|_{\partial\Omega} = \boldsymbol{h}(t) & \text{on } \partial\Omega, \end{cases} \quad \Longrightarrow$$

$$\widetilde{\boldsymbol{d}} \in H^1(0,T;\boldsymbol{H}^1(\Omega)^N)$$

We define $\hat{\boldsymbol{d}}(t)$ as

$$\widehat{\boldsymbol{d}}(t) = \boldsymbol{d}(t) - \widetilde{\boldsymbol{d}}(t) \quad \Longrightarrow$$

$$\widehat{\boldsymbol{d}}(t) \in \boldsymbol{H}_0^1(\Omega)^N,$$

$$\Delta\widehat{\boldsymbol{d}} = \Delta\boldsymbol{d} \text{ in } \Omega \times (0,T)$$

$$\boldsymbol{d}(0) = \boldsymbol{d}(T) \text{ if and only if } \widehat{\boldsymbol{d}}(0) = \widehat{\boldsymbol{d}}(T)$$

Definition 2: *(u,d)* Reproductive solution

$$\boldsymbol{u} \in L^2(\boldsymbol{V}) \cap L^\infty(\boldsymbol{H}),$$
$$\widehat{\boldsymbol{d}} \in L^\infty(\boldsymbol{H}_0^1), \quad \Delta\widehat{\boldsymbol{d}} \in L^2(\boldsymbol{L}^2),$$

$$\langle \partial_t \boldsymbol{u}, \mathbf{v}\rangle + ((\boldsymbol{u}\cdot\nabla)\boldsymbol{u}, \mathbf{v}) + (\nabla\boldsymbol{u}, \nabla\mathbf{v}) + (\nabla\boldsymbol{d}^t \Delta\widehat{\boldsymbol{d}}, \mathbf{v}) = 0 \quad \forall\, \mathbf{v} \in \boldsymbol{V} \cap \boldsymbol{L}^\infty,$$
$$\langle \partial_t \widehat{\boldsymbol{d}}, \mathbf{e}\rangle + ((\boldsymbol{u}\cdot\nabla)\boldsymbol{d}, \mathbf{e}) + (f(\boldsymbol{d}), \mathbf{e}) - (\Delta\widehat{\boldsymbol{d}}, \mathbf{e}) = -(\partial_t \widetilde{\boldsymbol{d}}, \mathbf{e}) \quad \forall\, \mathbf{e} \in \boldsymbol{L}^3,$$
$$\boldsymbol{u}(0) = \boldsymbol{u}(T), \quad \widehat{\boldsymbol{d}}(0) = \widehat{\boldsymbol{d}}(T) \quad \text{in } \Omega.$$

Theorem

Let $T > 0$ and $\Omega \subset \mathbb{R}^N$ $(N = 2$ or $3)$ an open bounded domain with Lipschitz boundary. Assume $\mathbf{h} \in H^1(0,T;\boldsymbol{H}^{1/2}(\partial\Omega))$ such that

$$|\boldsymbol{h}| \leq 1 \quad \text{on } \partial\Omega \times (0,T) \quad \text{and} \quad \boldsymbol{h}(0) = \boldsymbol{h}(T) \quad \text{on } \partial\Omega.$$

Then there exists a weak reproductive solution $(\boldsymbol{u}, \boldsymbol{d})$

Let $\{\phi_i\}_{i\geq 1}$ and $\{\varphi_i\}_{i\geq 1}$ "special" basis of $\boldsymbol{V}$ and $\boldsymbol{H}_0^1(\Omega)$, respectively, formed by eigenfunctions of the Stokes and the Poisson problems following:

$$\left\{ \begin{array}{rcll} -\Delta\phi_i & = & \lambda_i\phi_i & \text{in } \Omega \\ \phi_i & = & 0 & \text{on } \partial\Omega \end{array} \right. \qquad \left\{ \begin{array}{rcll} -\Delta\varphi_i & = & \mu_i\varphi_i & \text{in } \Omega \\ \varphi_i & = & 0 & \text{on } \partial\Omega, \end{array} \right.$$

$\|\phi_i\|_1 = 1$, $\|\varphi_i\|_1 = 1$ for all i.

$$\boldsymbol{u}_m(t) = \sum_{j=1}^{m} \xi_{i,m}(t)\phi_i \qquad \hat{\boldsymbol{d}}_m(t) = \sum_{j=1}^{m} \zeta_{i,m}(t)\varphi_i,$$

3. The Galerkin initial-boundary problem

For each $m \geq 1$, given $\boldsymbol{u}_0 \in \boldsymbol{H}$ and $\boldsymbol{d}_0 \in \boldsymbol{H}^1$, there exits a unique solution $(\boldsymbol{u}_m, \boldsymbol{d}_m)$, with $\boldsymbol{u}_m : [0,T] \mapsto \boldsymbol{V}^m$ and $\boldsymbol{d}_m = \widehat{\boldsymbol{d}}_m + \widetilde{\boldsymbol{d}}$, with $\widehat{\boldsymbol{d}}_m : [0,T] \mapsto \boldsymbol{W}^m$, verifying the following variational formulation a.e. in $t \in (0,T)$:

$$
(\mathrm{Pm}) \left\{ \begin{array}{l}
(\partial_t \boldsymbol{u}_m(t), \mathbf{v}_m) + ((\boldsymbol{u}_m(t)\cdot\nabla)\boldsymbol{u}_m(t), \mathbf{v}_m) + (\nabla \boldsymbol{u}_m(t), \nabla \mathbf{v}_m) \\
\qquad +(\nabla \boldsymbol{d}_m^t(t)\Delta \widehat{\boldsymbol{d}}_m(t), \mathbf{v}_m) = 0 \quad \forall\, \mathbf{v}_m \in \boldsymbol{V}^m \\
(\partial_t \widehat{\boldsymbol{d}}_m(t), \mathbf{e}_m) + ((\boldsymbol{u}_m(t)\cdot\nabla)\boldsymbol{d}_m(t), \mathbf{e}_m) + (f(\boldsymbol{d}_m(t)), \mathbf{e}_m) \\
\qquad +(\nabla \widehat{\boldsymbol{d}}_m(t), \nabla \mathbf{e}_m) = -(\partial_t \widetilde{\boldsymbol{d}}(t), \mathbf{e}_m) \quad \forall\, \mathbf{e}_m \in \boldsymbol{W}^m \\
\boldsymbol{u}_m(0) = \boldsymbol{u}_{0m} = P_m(\boldsymbol{u}_0), \quad \boldsymbol{d}_m(0) = \boldsymbol{d}_{0m} = Q_m(\boldsymbol{d}_0),
\end{array} \right.
$$

4. “A priori” estimates

$$(\boldsymbol{u}_m \text{ system}, \boldsymbol{u}_m) + (\boldsymbol{d}_m \text{ system}, -\Delta \hat{\boldsymbol{d}}_m)$$

$$\frac{d}{dt}\left(\|\boldsymbol{u}_m\|_{L^2}^2 + \|\nabla \widehat{\boldsymbol{d}}_m\|_{L^2}^2\right) + 2\|\nabla \boldsymbol{u}_m\|_{L^2}^2 + \|\Delta \widehat{\boldsymbol{d}}_m\|_{L^2}^2 \leq 2\Big(\|f(\boldsymbol{d}_m)\|_{L^2}^2 + \|\partial_t \widetilde{\boldsymbol{d}}\|_{L^2}^2\Big)$$

$$\Downarrow$$

$(\boldsymbol{u}_m)$ is uniformly bounded in $L^\infty(\boldsymbol{H}) \cap L^2(\boldsymbol{V})$

$(\widehat{\boldsymbol{d}}_m)$ is uniformly bounded in $L^\infty(\boldsymbol{H}_0^1) \cap L^2(\boldsymbol{H}^2)$

$(\boldsymbol{d}_m)$ is uniformly bounded in $L^\infty(\boldsymbol{H}^1)$

$(\Delta \boldsymbol{d}_m)$ is uniformly bounded in $L^2(\boldsymbol{L}^2)$.

$(\partial_t \boldsymbol{u}_m)$ is uniformly bounded in $L^2((\boldsymbol{V} \cap \boldsymbol{L}^\infty)')$

$(\partial_t \widehat{\boldsymbol{d}}_m)$ is uniformly bounded in $L^2(\boldsymbol{L}^{3/2})$

4. Compactness

$(\boldsymbol{u}_m)$ is relatively compact in $L^2(\boldsymbol{H})$

$(\widehat{\boldsymbol{d}}_m)$ is relatively compact in $L^2(\boldsymbol{H}_0^1)$.

$(\boldsymbol{d}_m)$ is relatively compact in $L^2(\boldsymbol{H}^1)$.

Sufficient to pass to the limit in equations.

5. Uniqueness of approximate solution

Let $(\boldsymbol{u}_m^1, \boldsymbol{d}_m^1)$ and $(\boldsymbol{u}_m^2, \boldsymbol{d}_m^2)$ be two solutions;

$$\boldsymbol{u}_m = \boldsymbol{u}_m^1 - \boldsymbol{u}_m^2$$

$$\boldsymbol{d}_m = \boldsymbol{d}_m^1 - \boldsymbol{d}_m^2$$

$$(\boldsymbol{u}_m^1 \text{ system} - \boldsymbol{u}_m^2 \text{ system}, \boldsymbol{u}_m) + (\boldsymbol{d}_m^1 \text{ system} - \boldsymbol{d}_m^2 \text{ system}, -\Delta \hat{\boldsymbol{d}}_m)$$

⇓

$$\begin{cases} \dfrac{d}{dt}(\|\boldsymbol{u}_m\|_{L^2}^2 + \|\nabla \boldsymbol{d}_m\|_{L^2}^2) \leq a_m(t)(\|\boldsymbol{u}_m\|_{L^2}^2 + \|\nabla \boldsymbol{d}_m\|_{L^2}^2) \\ \|\boldsymbol{u}_m(0)\|_{L^2}^2 + \|\nabla \boldsymbol{d}_m(0)\|_{L^2}^2 = 0. \end{cases}$$

with $a_m(t)$ bounded in $\in L^1(0,T)$

$\boldsymbol{u}_m = 0$ and $\nabla \boldsymbol{d}_m = 0$

Since $\boldsymbol{d}_m = 0$ on $\partial\Omega$, then $\boldsymbol{d}_m = 0$

Energy inequality:

$$\frac{d}{dt}\left(\|\boldsymbol{u}_m\|_{L^2}^2+\|\nabla\widehat{\boldsymbol{d}}_m\|_{L^2}^2\right)+C\left(\|\boldsymbol{u}_m\|_{L^2}^2+\|\nabla\widehat{\boldsymbol{d}}_m\|_{L^2}^2\right)\leq 2\left(\|f(\boldsymbol{d}_m)\|_{L^2}^2+\|\partial_t\widetilde{\boldsymbol{d}}\|_{L^2}^2\right)$$

$$\begin{aligned} e^{CT}\left(\|\boldsymbol{u}_m(T)\|_{L^2}^2+\|\nabla\widehat{\boldsymbol{d}}_m(T)\|_{L^2}^2\right) &\leq \|\boldsymbol{u}_m(0)\|_{L^2}^2+\|\nabla\widehat{\boldsymbol{d}}_m(0)\|_{L^2}^2 \\ &+ 2\int_0^T e^{Ct}\left(\|f(\boldsymbol{d}_m)\|_{L^2}^2+\|\partial_t\widetilde{\boldsymbol{d}}\|_{L^2}^2\right). \end{aligned}$$

6. Reproductivity of approximate solution

Given $(\boldsymbol{u}_0^m, \boldsymbol{d}_0^m) \in V^m \times W^m$, we define the map

$$\begin{aligned} L^m : [0,T] &\mapsto \mathbb{R}^m \times \mathbb{R}^m \\ t &\mapsto (\xi_{1m}(t), ..., \xi_{mm}(t), \zeta_{1m}(t), ..., \zeta_{mm}(t)) \end{aligned}$$

$(\boldsymbol{u}_m(t), \widehat{\boldsymbol{d}}_m(t))$ solution of (Pm) for data $(\boldsymbol{u}_0^m, \boldsymbol{d}_0^m)$

$$\begin{aligned} \Phi^m : \mathbb{R}^m \times \mathbb{R}^m &\mapsto \mathbb{R}^m \times \mathbb{R}^m \\ L_0^m &\mapsto \Phi^m(L_0^m) = L^m(T) \end{aligned}$$

where $L^m(t)$ is related to the solution of problem (Pm) with initial data $L_0^m (= L^m(0))$

6. Reproductivity of approximate solution

Leray-Schauder's Theorem: $\forall \lambda \in [0,1]$, solutions $L_0^m(\lambda)$ of

$$L_0^m(\lambda) = \lambda \Phi^m(L_0^m(\lambda))$$

uniformly bounded (independent of λ)?

$$\|L^m(t)\|_{\mathbf{R}^m \times \mathbf{R}^m} = \left(\|\boldsymbol{u}_m(t)\|_{L^2}^2 + \|\nabla \widehat{\boldsymbol{d}}_m(t)\|_{L^2}^2 \right)^{1/2}$$

▶

$$e^{CT} \| \frac{1}{\lambda} L_0^m(\lambda) \|_{\mathbf{R}^m \times \mathbf{R}^m}^2 \leq \|L_0^m(\lambda)\|_{\mathbf{R}^m \times \mathbf{R}^m}^2 + K(T),$$

$$\|L_0^m(\lambda)\|_{\mathbf{R}^m \times \mathbf{R}^m}^2 \leq \frac{K(T)}{e^{CT} - 1}$$

7. The pass to the limit

d_m is relatively compact in $C([0,T];L^2)$

$$d_m(T) \longrightarrow d(T) \text{ in } L^2$$

$$\parallel$$

$$d_m(0) \longrightarrow d(0) \text{ in } L^2$$

$d \in C_w([0,T];H^1)$

$$\Rightarrow \quad d(T) = d(0) \text{ in } H^1(\Omega)$$

2D

Unique weak solution inital-boundary problem.

$$u(0) = u(T) := u_0 \qquad d(0) = d(T) := d_0$$

$$\Rightarrow \text{solution in } (0, \infty)$$

Solution periodic and regular

3D Regularity?

$$\frac{d}{dt}\left(\|\boldsymbol{u}\|_{L^2}^2 + \|\nabla\widehat{\boldsymbol{d}}\|_{L^2}^2\right) + 2\|\nabla\boldsymbol{u}\|_{L^2}^2 + \|\Delta\widehat{\boldsymbol{d}}\|_{L^2}^2 \le 2\left(\|f(\boldsymbol{d})\|_{L^2}^2 + \|\partial_t\widetilde{\boldsymbol{d}}\|_{L^2}^2\right),$$

⇓

$$\int_0^T \|\nabla\widehat{\boldsymbol{u}}\|_{L^2}^2 + \int_0^T \|\Delta\widehat{\boldsymbol{d}}\|_{L^2}^2 \le C\int_0^T \|f(\boldsymbol{d})\|_{L^2}^2 + \bar{\varepsilon}$$

$$f(\mathbf{d}) = \frac{1}{\varepsilon^2}(|\mathbf{d}|^2 - 1)\mathbf{d}$$

3D

$$(\boldsymbol{u}\,\text{system}, \boldsymbol{u}) + (\boldsymbol{d}\,\text{system}, -\Delta\hat{\boldsymbol{d}} + f(\boldsymbol{d}))$$

$$F(\boldsymbol{d}) = (|\boldsymbol{d}|^2 - 1)^2/4\varepsilon^2 \text{ and } f(\boldsymbol{d}) = \nabla_{\boldsymbol{d}} F(\boldsymbol{d})$$

$$\Downarrow$$

$$\frac{d}{dt}\left(\|\boldsymbol{u}\|_{L^2}^2 + \|\nabla\widehat{\boldsymbol{d}}\|_{L^2}^2 + 2\int_\Omega F(d)\right) + 2\,\|\nabla\boldsymbol{u}\|_{L^2}^2$$
$$+\|\Delta\widehat{\boldsymbol{d}} - f(\boldsymbol{d})\|_{L^2}^2 \le \|\partial_t\tilde{\boldsymbol{d}}\|_{L^2}^2 + \frac{2}{\varepsilon^2}\|\partial_t\tilde{\boldsymbol{d}}\|_{L^1}$$

$$\int_0^T \|\Delta\widehat{\boldsymbol{d}} - f(\boldsymbol{d})\|_{L^2}^2 << $$