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Effects of additive and multiplicative noise on the dynamics of a parabolic equation

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Abstract: We consider the effects of additive and multiplicative noise on the asymptotic behavior of a fourth order parabolic equation arising in the study of phase transitions. On account that the deterministic model presents three different time scales, in this paper we have established some conditions under which the third time scale, which encounter finite dimensional behavior of the system, is preserved under both additive and multiplicative linear noise. In particular we have proved the existence of a random attractor in both cases, and observed that the order of magnitude of the third time scale is also preserved.

Keywords: Random attractor, Phase Transitions, White Noise

1 Introduction

The study of phase transitions has been an important subject of research over the last decades (see [1]). Several models have been introduced, among the others, the Cahn-Hilliard (see for example [2]) equations have been intensively studied. In [3] the authors introduced a model related to that of Cahn and Hilliard (C-H) in the sense that if u is the solution of the equation

$$\begin{cases} u_t = -\varepsilon^2 u_{xxxx} + \frac{1}{2}[W'(u_x)]_x \\ u = u_{xx} = 0, \quad \text{on } \partial I, \\ u(0,t) = u_0, \end{cases} \tag{1}$$

then u_x solve the C-H equation. In (1), I is an open interval, the function W is the so called double well potential, that is, $W(p) = (p^2-1)^2$ and ε is a small parameter. The equation (1) represents the L^2-gradient dynamics associated to the energy functional (see [10] and [14]):

$$F_\varepsilon(u) = \frac{1}{2}\int_I u_{xx}^2 dx + \frac{1}{2}\int_I W(u_x)dx.$$

The global dynamics of (1) have been studied in [3] by numerical experiments and, in particular, the existence of three time scales with different dynamical behavior has been pointed out.

In a first time scale of order $O(\varepsilon^2)$, the energy of the initial datum is drastically reduced and a microstructure appears in the region in which the gradients of the initial data are in the set in which the potential W is non convex, that is $(-\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3})$ (see [10] for an analysis of the first time scale using variational techniques).
In the second time scale of order $O(1)$, the region without microstructure evolves in a heat equation like behavior, while the region with microstructure remains almost stationary.
In the last time scale of order $O(\varepsilon^{-2})$, equation (1) exhibits a finite dimensional behavior, the solution is approximately the union of consecutive segments with slops ± 1.
In the papers [11], [12], [13], and [4], the third time scale has been studied. In [11] the authors prove the existence of an exponential attractor $\mathscr{M}_\varepsilon$ whose dimension is of order $O(\varepsilon^{-10})$. In particular the time for which the solutions enter the absorbing set is of order (ε^{-2}). In [12] the authors found an estimate of the dimension, of order $O(\varepsilon^{-1})$ (in accordance with the numerical experiments presented in [3]), of the global attractor $\mathscr{A}_\varepsilon$ by the volume elements evolution method and they proved the existence of an inertial manifold whose dimension is of order $O(\varepsilon^{-19})$. Moreover, some estimates on the regularity estimates and on the embedding dimension are

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performed. Since natural systems are subjected to random perturbation we consider interesting to reconsider the problem (1) in the context of stochastic processes. Namely, when some multiplicative and additive noise may appear in the formulation.

Thus, in Section 2 below we state the problem in the multiplicative case, state some necessary preliminary definitions on the theory of random attractors, and prove that our problem generates a random dynamical system. In Section 3 we prove the existence of a random attractor. Finally, in Section 4 we consider the case in which the noise appears as an additive one. We do not intend to establish all the results in a formal and detailed way, but only wish to provide the main estimates in order to justify the existence of a random attractor, highlighting the main differences with the multiplicative case. This additive case requires of some additional technicalities which make the analysis more involved.

2 Preliminary definitions

We consider the following stochastic version of the system (1) with multiplicative noise

$$\begin{cases} dX = (-\varepsilon^2\triangle^2 X + \frac{1}{2}W''(\nabla X)\triangle X)dt + \sigma X \circ dw, \\ X = X_{xx} = 0, \quad \text{on } \partial I, \\ X(0,t) = X_0, \end{cases} \tag{2}$$

where $w(t)$ is a two-sided standard Wiener process defined on a probability basis $(\Omega, \mathscr{F}, \{\mathscr{F}\}_{t\in\mathbb{R}}, \mathbb{P})$, where $\{\mathscr{F}\}_{t\in\mathbb{R}}$ is a filtration that is an increasing collection of sigma-algebras on Ω and $\mathscr{F}_0$ contains all the null sets.
In order to study the dynamics of equation (2) we first recall some basic definitions of the theory of random dynamical systems.
Let $\{\theta_t : \Omega \to \Omega, t \in \mathbb{R}\}$ be a family of measure preserving transformations such that $(t,\omega) \to \theta_t\omega$ is measurable, $\theta_0 = id$ and $\theta_{t+s} = \theta_t\theta_s$ for all $s,t \in \mathbb{R}$. The flow θ_t together with the corresponding probability space $(\Omega, \mathscr{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$ is called a measurable dynamical system. Finally we suppose that θ_t is ergodic.
A continuous random dynamical system is a measurable map

$$\varphi : \mathbb{R}^+ \times \Omega \times X \to X,$$

such that $\mathscr{P}$-a.s.

(i)$\varphi(0,\omega) = id$ on X,
(ii)$\varphi(t+s,\omega) = \varphi(t,\theta_s\omega)\varphi(s,\omega), \ \forall t,s \in \mathbb{R}^+$,

where (X,d) is a Polish space with Borel σ-algebra $\mathscr{B}$ over θ on $(\Omega, \mathscr{F}, \mathbb{P})$.
In [16] the authors introduced, in the context of random dynamical systems, the concept of attractor for stochastic partial differential equations in order to study the qualitative dynamical behavior of the solutions.

Definition 1. *A random compact set $\{K(\omega)\}_{\omega\in\Omega}$ is a family of compact sets indexed by ω such that for every $x \in X$ the mapping $\omega \to d(x,K(\omega))$ is measurable with respect to $\mathscr{F}$.*

Definition 2. *A random set $K(\omega) \subset X$ is said to absorb the set B if for almost all $\omega \in \Omega$ there exists $t(B,\omega)$ such that for all $t \geq t(B,\omega)$*

$$\varphi(t,\theta_{-t}\omega)B \subset K(\omega).$$

Definition 3. *A random set $\mathscr{A}(\omega)$ is said to be a random attractor associated to the random dynamical system φ if $\mathbb{P}$-a.s.*

(i)$\mathscr{A}(\omega)$ is a random compact set.
(ii)$\varphi(t,\omega)\mathscr{A}(\omega) = \mathscr{A}(\theta_t\omega)$, $\forall t \geq 0$,
(iii)attracts all $B \subset X$ bounded and non-random, that is

$$\lim_{t\to\infty} \operatorname{dist}(\varphi(t,\theta_{-t}\omega)B, \mathscr{A}(\omega)) = 0,$$

where dist *denotes the usual Hausdorff semi-distance.*

In the sequel we will use the fact that $\varphi(t,\theta_{-t}\omega)$ can be interpreted as the position at $t = 0$ of the trajectory which was in x at time $-t$ and we consider the attraction property at t goes to $-\infty$. This is the so called pullback convergence which means to look at the position of the solutions at present time when the initial ones go to $-\infty$.
In the next section we will prove the existence of a random attractor for equation (2) using the following theorem (see [16]):

Theorem 1. *Assume that there exists a compact set $D(\omega)$ absorbing every bounded non-random set $B \subset X$. Then, the set*

$$\mathscr{A}(\omega) = \overline{\bigcup_{B\subset X} \Lambda_B(\omega)}$$

is a random attractor for φ, where the union is taken over all $B \subset X$ bounded, and $\Lambda_B(\omega)$ is the omega-limit set of B given by

$$\Lambda_B(\omega) = \bigcap_{n\geq 0} \overline{\bigcup_{t\geq n} \varphi(t,\theta_{-t}\omega)B}.$$

Moreover in [16] the author proved that random attractors are unique and, by the ergodicity of θ_t, there exists a compact set $K \subset X$ such that $\mathbb{P}$-a.s. the random attractor is the omega limit set of K, that is:

$$\mathscr{A}(\omega) = \bigcap_{n\geq 0} \overline{\bigcup_{t\geq n} \varphi(t,\theta_{-t}\omega)K}.$$

In order to prove the existence of the random attractor, the idea is to transform the stochastic evolution equation (2) containing a noise term into an evolution equation without noise but with a random coefficient, that is a real function which takes random values. In this case all the techniques and tools for the study of standard evolutions equations are available.
For this purpose we consider the following change of variable

$$\psi = T^{-1}(\omega)X, \tag{3}$$

where

$$T(\omega) = e^{\sigma z^*(\omega)},$$

and where z^* is the stationary solution of the equation

$$dz = -zdt + dw_t, \qquad \omega(t) = w_t(\omega), \tag{4}$$

which is the so called Ornstein-Uhlenbeck process.

In particular, if φ is a random dynamical system and T and T^{-1} are measurable then

$$(t,\omega,x) \to T^{-1}(\theta_t\omega, \varphi(t,\omega,T(\omega,x))) := \psi(t,\omega,x)$$

is a random dynamical systems (see [6]). Before proceeding with the analysis of the random system we recall some important properties of $z^*(\omega)$ (see [6] for more details):

Proposition 1. *There exists a $\{\theta_t\}_{t\in\mathbb{R}}$-invariant subset $\bar{\Omega} \in \mathscr{F}$ of $\Omega = C_0(\mathbb{R},\mathbb{R})$ of full measure such that*

$$\lim_{t\to\pm\infty} \frac{|\omega(t)|}{t} = 0, \qquad \text{for } \omega \in \bar{\Omega},$$

and for such ω the random variable is given by

$$z^*(\omega) := -\int_{-\infty}^{0} e^{\tau}\omega(\tau)d\tau$$

and is well defined. Moreover, for $\omega \in \bar{\Omega}$, the mapping

$$(t,\omega) \to z^*(\theta_t\omega) = -\int_{-\infty}^{0} e^{\tau}\theta_t(\omega(\tau))d\tau$$
$$= -\int_{-\infty}^{0} e^{\tau}(\omega(t+\tau) - \omega(\tau))d\tau$$

is a stationary solution of equation (4) with continuous trajectories. In addition, for $\omega \in \bar{\Omega}$:

$$\lim_{t\to\pm\infty} \frac{|z^*(\theta_t\omega)|}{|t|} = 0, \qquad \lim_{t\to\pm\infty} \frac{1}{t}\int_0^t z^*(\theta_\tau\omega)d\tau = 0,$$

and

$$E|z^*| = \lim_{t\to\pm\infty} \frac{1}{t}\int_0^t |z^*(\theta_\tau\omega)|d\tau < \infty.$$

By the change of variable (3) the equation (2) becomes:

$$\psi_t = -\varepsilon^2\psi_{xxxx} + \frac{T^{-1}(\theta_t\omega)}{2}[W'(T(\theta_t\omega)\psi_x)]_x + \sigma z^*(\theta_t\omega)\psi, \tag{5}$$

$$\psi(0) = \psi_0, \tag{6}$$

The existence and uniqueness of solution follows from classical methods such as Galerkin approximations (see for example [26] Theorem 3.1.1), and we will omit the details. In particular, we can prove the following result:

Theorem 2. *For $\mathbb{P}-$a.s. $\omega \in \Omega$ and all $T > 0$, the problem (5) admits a unique solution ψ such that*

if $\psi_0 \in H = L^2(I)$ then $\psi \in C^0([0,T],H) \cap L^2(0,T;V)$;

if $\psi_0 \in V = H^2(I)$ then $\psi \in C([0,T],V) \cap L^2(0,T;D(A))$;

where $D(A) = \{\psi \in H^4(I) : \quad \psi = \psi_{xx} = 0 \text{ on } \partial I\}$ is the domain of the differential operator $A = -\frac{\partial^4}{\partial x^4}$.

The continuity of the cocycle is guarranteed by the following stronger result:

Proposition 2. *The solution of (5) is Lipschitz continuous with respect to the initial data.*

Proof. Let ψ_i , $i = 1,2$ be two solutions associated respectively to the initial data $\psi_{0,i}$, $i = 1,2$ and set $u = \psi_1 - \psi_2$. Then the equation fulfilled by u is the following:

$$u_t = -\varepsilon^2 u_{xxxx} + \sigma z^*(\theta_t\omega)u + \frac{T^{-1}(\theta_t\omega)}{2}\left[W'(T(\theta_t\omega)\psi_{1x}) - W'(T(\theta_t\omega)\psi_{2x})\right]_x,$$

Multiplying the previous equation by u in H yields:

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 + \varepsilon^2\|u_{xx}\|^2 + \frac{T^{-1}(\theta_t\omega)}{2}\int_I \left[W'(T(\theta_t\omega)\psi_{1x}) - W'(T(\theta_t\omega)\psi_{2x})\right]u_x dx + \sigma z^*(\theta_t\omega)\|u\|^2 = 0,$$

from which

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 + \varepsilon^2\|u_{xx}\|^2 \le 2\|u_x\|^2 + 2T^2\int_I \bar{u}u_x^2 dx + \sigma|z^*|\|u_x\|^2,$$

where

$$\bar{u} = \psi_{1x}\psi_{2x}.$$

In the next section we will prove the absorbing properties in several spaces, in particular, we will find the existence of random variables $r_i(\omega)$ such that

$$\|\psi(0)\| \le r_0(\omega), \quad \|\psi_x(0)\| \le r_2(\omega), \quad \|\psi_{xx}(0)\| \le r_3(\omega),$$

where again we intend the above estimates in the following sense

$$S(t,\theta_{-t}\omega)\psi_0 = \psi(0).$$

Here we make use of that in order to conclude the proof (see section below for the complete details).

In particular we use:

$$\|\bar{u}\|_\infty \le \|\psi_{1x}\|_\infty\|\psi_{2x}\|_\infty$$
$$\le \|\psi_1\|\|\psi_2\|\|\psi_{1xx}\|\|\psi_{2xx}\|$$
$$\le r_0^2(\omega)r_3^2(\omega) := K(\omega)$$

Then, we can easily obtain

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 + \varepsilon^2\|u_{xx}\|^2 \le 2(1 + T^2K(\omega) + \frac{1}{2}\sigma|z^*|)\|u_x\|^2$$
$$:= 2h(\omega,t)\|u_x\|^2$$
$$\le \varepsilon^2\|u_{xx}\|^2 + \frac{h^2(\omega,t)}{\varepsilon^2}\|u\|^2,$$

and integrating we arrive at

$$\|S(t,\theta_{-t}\omega)\psi_{1,0} - S(t,\theta_{-t}\omega)\psi_{2,0}\|$$
$$= \|\psi_1(0,\omega,-t,\psi_{1,0}) - \psi_2(0,\omega,-t,\psi_{2,0})\|$$
$$= \|\psi_1(0) - \psi_2(0)\|$$
$$\le e^{\frac{1}{\varepsilon^2}\int_{-t}^0 h^2(\omega,t)dt}\|\psi_{0,1} - \psi_{0,2}\|.$$

3 Existence of the random attractor

In order to use Theorem 1 to prove the existence of the random attractor, we prove the existence of a compact random absorbing set.
We start by proving the existence of an absorbing ball in H.

Proposition 3. *There exists a random variable $r_0(\omega)$ such that for any $\rho > 0$ there exists $t(\omega,\rho) < -1$ satisfying*

$$\|\psi(0,\omega;t_0,\psi_0)\| \leq r_0(\omega),$$

for all $t_0 \leq t(\omega,\rho)$ and $\|\psi_0\| \leq \rho$.

Proof. If we multiply the equation (5) by ψ we obtain the following inequality:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\psi\|^2 + \varepsilon^2\|\psi_{xx}\|^2 + 2[T(\theta_t\omega)]^2\|\psi_x\|_4^4 \\ &= 2\|\psi_x\|^2 + \sigma z^*(\theta_t\omega)\|\psi\|^2 \\ &\leq (2+\sigma z^*(\theta_t\omega))\|\psi_x\|^2 \\ &\leq (2+\sigma z^*(\theta_t\omega))\|\psi\|\|\psi_{xx}\|,\end{aligned}$$

from which

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\psi\|^2 + \frac{\varepsilon^2}{2}\|\psi_{xx}\|^2 &\leq (2+\sigma z^*(\theta_t\omega))^2\|\psi\| \\ &\leq \frac{(2+\sigma z^*(\theta_t\omega))^4}{\varepsilon^2} + \frac{\varepsilon^2}{4}\|\psi_{xx}\|^2.\end{aligned}$$

Then

$$\begin{aligned}\frac{d}{dt}\|\psi\|^2 + \frac{\varepsilon^2}{2}\|\psi_{xx}\|^2 \\ \leq \frac{2[2+\sigma z^*(\theta_t\omega)]^4}{\varepsilon^2} := \mathscr{F}(\theta_t\omega).\end{aligned} \tag{7}$$

Integrating the previous inequality in $[t_0,-1]$ with $t_0 \leq -1$:

$$\begin{aligned}\|\psi(-1)\|^2 \leq \|\psi(t_0)\|^2 e^{-\frac{\varepsilon^2}{2}(-1-t_0)} \\ + \int_{t_0}^{-1} e^{-\frac{\varepsilon^2}{2}(-1-s)}\mathscr{F}(\theta_s\omega)ds,\end{aligned}$$

from which

$$\begin{aligned}\|\psi(-1)\|^2 &\leq e^{\frac{\varepsilon^2}{2}}\left\{\|\psi(t_0)\|^2 e^{\frac{\varepsilon^2}{2}t_0} + \int_{t_0}^{-1} e^{\frac{\varepsilon^2}{2}s}\mathscr{F}(\theta_s\omega)ds\right\} \\ &\leq e^{\frac{\varepsilon^2}{2}}\left\{T^{-2}(\theta_{t_0}\omega)\|X(t_0)\|^2 e^{\frac{\varepsilon^2}{2}t_0} \right. \\ &\qquad \left. + \int_{-\infty}^{-1} e^{\frac{\varepsilon^2}{2}s}\mathscr{F}(\theta_s\omega)ds\right\}.\end{aligned}$$

Now, if we fix the initial datum X_0 in $B(0,\rho) \subset H$, then there exists a time $t(\omega,\rho)$ such that for all $t_0 \leq t(\omega,\rho)$ and for all $X_0 \in B(0,\rho)$ we have:

$$\|\psi(-1)\|^2 \leq r_1^2(\omega),$$

where

$$r_1^2(\omega) = e^{\frac{\varepsilon^2}{2}}\left\{1 + \int_{-\infty}^{-1} e^{\frac{\varepsilon^2}{2}s}\mathscr{F}(\theta_s\omega)ds\right\}.$$

In detail, it is sufficient to chose $t(\omega,\rho)$ such that:

$$T^{-2}(\theta_{t_0}\omega)\rho^2 e^{\frac{\varepsilon^2}{2}t_0} \leq 1,$$

and this is possible since $\mathbb{P}$-a.s. $T^{-2}(\theta_{t_0}\omega)e^{\frac{\varepsilon^2}{2}t_0} \to 0$ as $t_0 \to -\infty$. Using again (7), for all $t \in [-1,0]$ we have that

$$\|\psi(t)\|^2 \leq \|\psi(-1)\|^2 e^{-\frac{\varepsilon^2}{2}(t+1)} + \int_{-1}^{t} e^{-\frac{\varepsilon^2}{2}(t-s)}\mathscr{F}(\theta_s\omega)ds,$$

and then

$$\|\psi(0)\|^2 \leq r_1^2(\omega)e^{-\frac{\varepsilon^2}{2}} + \int_{-1}^{0} e^{\frac{\varepsilon^2}{2}s}\mathscr{F}(\theta_s\omega)ds := r_0^2(\omega).$$

We remark that, from Proposition 1, it follows that

$$\int_{-\infty}^{-1} e^{2\varepsilon^2 s}\mathscr{F}(\theta_s\omega)ds < \infty.$$

Remark. For any fixed $\omega \in \Omega$ the time for which the absorbing property is fulfilled is of order $O(\varepsilon^{-2})$, and this is in perfect accordance with the numerical experiments in [3] and with the autonomous case (see [11] and [12]).

Integrating inequality (7), we deduce the following estimate that will be useful later

$$\begin{aligned}\int_{-1}^{0}\|\psi\|^2 dt &\leq \int_{-1}^{0}\|\psi_x\|^2 dt \\ &\leq \int_{-1}^{0}\|\psi_{xx}\|^2 dt \\ &\leq \frac{1}{2\varepsilon^2}\left\{r_1^2(\omega) + \int_{-1}^{0}\mathscr{F}(\theta_t\omega)dt\right\} := R(\omega),\end{aligned}$$

while integrating in $(t_0,0)$ and letting $t_0 \to -\infty$ we derive

$$\int_{-\infty}^{0}\|\psi_{xx}\|^2 dt \leq \frac{1}{\varepsilon^2}\left\{\frac{\rho^2}{2} + \int_{-\infty}^{0}\mathscr{F}(\theta_t\omega)dt\right\}.$$

From the previous proposition we obtain the absorbing property in H and we can pass to the proof of the absorbing property in

$$H_0^1(I) := \{v \in L^2(I) : \ v_x \in L^2(I) \text{ and } v = 0 \text{ on } \partial I\}.$$

Proposition 4. *There exists a random variable $r_2(\omega)$ such that for any $\rho > 0$ there exists $t(\omega,\rho) < -1$ satisfying*

$$\|\psi(0,\omega;t_0,\psi_0)\|_{H_0^1(I)} \leq r_2(\omega),$$

for all $t_0 \leq t(\omega,\rho)$ and $\|\psi_0\| \leq \rho$.

Proof. If we multiply equation (5) by ψ_{xx} we obtain

$$\frac{1}{2}\frac{d}{dt}\|\psi_x\|^2+\varepsilon^2\|\psi_{xxx}\|^2+6[T(\theta_t\omega)]^2\int_I\psi_x^2\psi_{xx}^2$$
$$=2\|\psi_{xx}\|^2+\sigma z^*(\theta_t\omega)\|\psi_x\|^2$$
$$\leq(2+\sigma z^*(\theta_t\omega))\|\psi_{xx}\|^2, \qquad (8)$$

from which

$$\frac{d}{dt}\|\psi_x\|^2\leq 2\left[\sigma z^*(\theta_t\omega)+\frac{1}{\varepsilon^2}\right]\|\psi_x\|^2, \qquad (9)$$

and by the uniform Gronwall lemma

$$\|\psi_x(0)\|^2 \leq R(\omega)\exp\left(2\int_{-1}^0\left[\sigma z^*(\theta_s\omega)+\frac{1}{\varepsilon^2}\right]ds\right)$$
$$:= r_2^2(\omega).$$

Before proving the absorbing property for the norm $\|\psi_{xx}\|$, we perform some important estimates. We consider the following inequality

$$\int_{t+t_0}^t\|\psi_x\|^2ds\leq\int_{t+t_0}^t\|\psi_{xx}\|^2ds$$
$$\leq\frac{1}{\varepsilon^2}\left\{\frac{\rho^2}{2}+\int_{t+t_0}^t\mathscr{F}(\theta_s\omega)ds\right\},$$

and apply the uniform Gronwall lemma in $(t+t_0,t)$ with $t,t_0\leq 0$:

$$\|\psi_x(t)\|^2\leq\left\{\frac{\frac{1}{\varepsilon^2}\left\{\frac{\rho^2}{2}+\int_{t+t_0}^t\mathscr{F}(\theta_s\omega)ds\right\}}{-t_0}\right\}\times$$
$$\times\exp\left(2\int_{t+t_0}^t\left[\sigma z^*(\theta_s\omega)+\frac{1}{\varepsilon^2}\right]ds\right).$$

From the Birkhof ergodic theorem (see Da Prato and Zabczyk [17, Chapter 1] and the stationarity of the Ornstein-Uhlenbeck process we have that

$$\lim_{t_0\to-\infty}-\frac{1}{t_0}\int_{t+t_0}^t|z^*(\theta_p\omega)|dp=E(|z^*|).$$

Then, there exists $T_0(\omega)$ such that for all $t_0\leq T_0(\omega)$ we have

$$-\frac{1}{t_0}\int_{t+t_0}^t|z^*(\theta_p\omega)|dp\leq 2E(|z^*|).$$

Moreover, by the same argument, there exists $T_1(\omega)\leq T_0(\omega)$ such that for all $t_0\leq T_1(\omega)$:

$$-\frac{1}{t_0}\int_{t+t_0}^t\mathscr{F}(\theta_s\omega)ds$$
$$=-\frac{1}{t_0}\int_{t+t_0}^t\frac{2}{\varepsilon^2}(2+\sigma z^*(\theta_s\omega))^4ds$$
$$\leq-\frac{1}{t_0}\int_{t+t_0}^t\frac{16}{\varepsilon^2}[2^4+(\sigma z^*(\theta_s\omega))^4]ds$$
$$\leq\frac{2^8}{\varepsilon^2}+\frac{16\sigma^4}{\varepsilon^2}\left(\frac{1}{-t_0}\int_{t+t_0}^t(z^*(\theta_s\omega))^4ds\right)$$
$$\leq\frac{2^8}{\varepsilon^2}+\frac{16\sigma^4}{\varepsilon^2}2E([z^*]^4):=f(\omega).$$

Then,

$$\|\psi_x(t)\|^2\leq\left\{\frac{\rho^2}{2\varepsilon^2(-T_1(\omega))}+f(\omega)\right\}e^{4\sigma E(|z|)+\frac{2}{\varepsilon^2}T_1(\omega)}$$
$$:=\tilde{R}^2(\omega).$$

Additionally, integrating equation (8) in $(-1,0)$, we have

$$\int_{-1}^0\|\psi_{xxx}\|^2dt\leq\frac{2}{\varepsilon^2}\left\{r_1^2(\omega)+2\left(\sigma M(\omega)+\frac{2}{\varepsilon^2}\right)R(\omega)\right\}$$
$$:=R'(\omega),$$

where

$$M(\omega):=\max_{t\in[-1,0]}[T(\theta_t\omega)]^4.$$

Proposition 5. *There exists a random variable $r_3(\omega)$ such that, for any $\rho>0$, there exists a constant $T(\omega,\rho):=\min\{t(\omega,\rho),T_1(\omega)\}<-1$ satisfying*

$$\|\psi_{xx}(0,\omega;t_0,\psi_0)\|\leq r_3(\omega),$$

for all $t_0\leq T(\omega,\rho)$ and $\|\psi_0\|\leq\rho$.

Proof. In order to prove the absorbing property in V we multiply equation (5) by ψ_{xxxx}:

$$\frac{1}{2}\frac{d}{dt}\|\psi_{xx}\|^2+\varepsilon^2\|\psi_{xxxx}\|^2$$
$$=6[T(\theta_t\omega)]^2\int_I\psi_x^2\psi_{xx}\psi_{xxxx}dx$$
$$+2\|\psi_{xxx}\|^2+\sigma z^*(\theta_t w)\|\psi_{xx}\|^2$$
$$\leq 6[T(\theta_t\omega)]^2\|\psi_x\|_\infty^2\|\psi_{xx}\|\|\psi_{xxxx}\|+\frac{\varepsilon^2}{2}\|\psi_{xxxx}\|^2$$
$$+\frac{2}{\varepsilon^2}\|\psi_{xx}\|^2+\sigma z^*(\theta_t w)\|\psi_{xx}\|^2$$
$$\leq 6[T(\theta_t\omega)]^2\tilde{R}\|\psi_{xx}\|^2\|\psi_{xxxx}\|+\frac{\varepsilon^2}{2}\|\psi_{xxxx}\|^2$$
$$+(\frac{2}{\varepsilon^2}+\sigma z^*(\theta_t\omega))\|\psi_{xx}\|^2,$$

from which:

$$\frac{d}{dt}\|\psi_{xx}\|^2\leq\frac{18}{\varepsilon^2}[T(\theta_t\omega)]^4\|\psi_{xx}\|^4 \qquad (10)$$
$$+\left(\frac{2}{\varepsilon^2}+2\sigma z^*(\theta_t(\omega))\right)\|\psi_{xx}\|^2.$$

If we integrate equation (10) in $(s,0)$ for $s\in(-1,0)$ we have

$$\|\psi_{xx}(0)\|^2\leq\|\psi_{xx}(s)\|^2+\frac{18}{\varepsilon^2}\int_s^0[T(\theta_t\omega)]^4\|\psi_{xx}\|^4dt$$
$$+\int_s^0\left(\frac{2}{\varepsilon^2}+2\sigma z^*(\theta_t(\omega))\right)\|\psi_{xx}\|^2dt.$$

Integrating again with respect to s in $(-1,0)$ we obtain

$$\begin{aligned}\|\psi_{xx}(0)\|^2 &\le \int_{-1}^{0}\|\psi_{xx}(s)\|^2 dsx \\ &+\frac{18}{\varepsilon^2}\int_{-1}^{0}[T(\theta_s\omega)]^4\|\psi_{xx}\|^4 ds \\ &+\int_{-1}^{0}\left(\frac{2}{\varepsilon^2}+2\sigma z^*(\theta_s(w))\right)\|\psi_{xx}\|^2 ds \\ &\le R(\omega)+\frac{18M(\omega)}{\varepsilon^2}\tilde{R}^2(\omega)R'(\omega) \\ &+\left(\frac{2}{\varepsilon^2}+2\sigma\max_{s\in(-1,0)}|z^*(\theta_s\omega)|\right)R(\omega) \\ &:= r_3^2(\omega),\end{aligned}$$

where we have used

$$\begin{aligned}\int_{-1}^{0}\|\psi_{xx}\|^4 ds &\le \int_{-1}^{0}\|\psi_x\|^2\|\psi_{xxx}\|^2 ds \\ &\le \tilde{R}^2(\omega)\int_{-1}^{0}\|\psi_{xxx}\|^2 ds \\ &\le R^2(\omega)R'(\omega).\end{aligned}$$

From the above propositions we get that there exists a random ball $\mathscr{B}(\omega)$ in $L^2(I)\cap H_0^1(I)\cap H^2(I)$ which absorbs any bounded non random subset of H at time 0 for any $t_0 \le \min\{T_1(\omega), T_0(\omega), t(\omega,\rho)\}$. The compactness of $\mathscr{B}(\omega)$ in $L^2(I)$ follows from the compact embedding of $H_0^1(I)$ in $L^2(I)$. Then, by Theorem 1, we conclude that

Theorem 3. *The equation (5) admits a random attractor $\mathscr{A}_\varepsilon(\omega)$ in H.*

4 Additive noise

In this final section we consider some remarks on the stochastic version of (1) with additive noise and compare the results with that of multiplicative noise. The system takes the form:

$$\begin{cases} dX=(-\varepsilon^2\triangle^2 X+\frac{1}{2}W''(\nabla X)\triangle X)dt+\sigma\phi(x)dw(t), \\ X=X_{xx}=0, \quad \text{on } \partial I, \\ X(t_0,x)=X_0(x), \end{cases} \tag{11}$$

where $\sigma>0$ and $\phi\in H$ satisfies $\phi_{xxxx}\in L^\infty(I)$. As in the previous section we consider a transformation involving the Orstein-Uhlenbeck process:

$$v(t)=X(t)-\sigma z^*(\theta_t\omega)\phi.$$

Then the equation can be rewritten as

$$\begin{aligned}v_t+\varepsilon^2 v_{xxxx} &= \frac{1}{2}\left[W'(v_x+\sigma\phi_x z^*(\theta_t\omega))\right]_x \\ &+\sigma z^*(\theta_t\omega)g(x),\end{aligned} \tag{12}$$

where for simplicity we have set:

$$g(x):=g(x,\phi,\varepsilon,\sigma)=\phi(x)-\varepsilon^2\phi_{xxxx}(x).$$

Once at this point, it is possible to establish a result ensuring the existence and uniqueness of solution of the initial value problems associated to (12), as well as the continuity with respect to the initial data. However, we will only comment on the main differences between this case and the multiplicative noise one. Thus, we will show how to obtain inequalities concerning $\|v\|$, $\|v_x\|$ and $\|v_{xx}\|$ similar to those of the previous sections. We note that obtaining the estimates requires more technicalities in our computations.
We start with the absorption in $L^2(I)$. Multiplying equation (12) by v in $L^2(I)$ we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|v\|^2+\varepsilon^2\|v_{xx}\|^2 \\ &\quad=\frac{1}{2}\int_I\left[W''(v_x+\sigma z^*(\theta_t\omega)\phi_x)\right](v_{xx}+\sigma\phi_{xx}z^*(\theta_t\omega))vdx \\ &\quad+\sigma z^*(\theta_t\omega)\int_I gvdx,\end{aligned}$$

and, in more details,

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|v\|^2+\varepsilon^2\|v_{xx}\|^2+2\|v_x\|_4^4 \\ &\quad=6\sigma Y\int_I v_x^2 v\phi_{xx}dx+6\sigma^2Y^2\int_I v_{xx}v\phi_x^2 dx \\ &\quad+6\sigma^3Y^3\int_I v\phi_x^2\phi_{xx}dx \\ &\quad+12\sigma Y\int_I v_{xx}v_x v\phi_x dx \\ &\quad+12\sigma^2Y^2\int_I v_x v\phi_x dx+\sigma Y\int_I gvdx,\end{aligned}$$

where we have set, for notational simplicity, $Y:=z^*(\theta_t\omega)$.
We observe that using integration by parts we have:

$$\begin{aligned}&6\sigma Y\int_I v_x^2 v\phi_{xx}dx+12\sigma Y\int_I v_{xx}v_x v\phi_x dx \\ &\quad=-6\sigma Y\int_I v_x^3\phi_x dx \\ &\quad\le 6\sigma|Y|\|\phi_x\|_\infty\|v_x\|_3^3 \\ &\quad\le \|v_x\|_4^4+9\sigma^2Y^2\|\phi_x\|^2\|v_x\|^2 \\ &\quad\le \|v_x\|_4^4+9\sigma^2Y^2\|\phi_x\|^2\|v\|\|v_{xx}\| \\ &\quad\le \|v_x\|_4^4+\frac{\varepsilon^2}{4}\|v_{xx}\|^2+\frac{\sigma^4Y^4}{\varepsilon^2}\|\phi_x\|_\infty^4\|v\|^2.\end{aligned}$$

Then,

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|v\|^2+\varepsilon^2\frac{3}{4}\|v_{xx}\|^2+\|v_x\|_4^4 \\ &\quad\le\frac{\sigma^4Y^4}{\varepsilon^2}\|\phi_x\|_\infty^4\|v\|^2+6\sigma^2Y^2\|v_{xx}\|\|v\|\|\phi_x\|_\infty^2 \\ &\quad+6\sigma^3|Y^3|\|\phi_x\|_\infty^2\|\phi_{xx}\|\|v\| \\ &\quad+12\sigma^2Y^2\|v\|\|v_x\|\|\phi_x\|_\infty+\sigma|Y|\|g\|\|v\|,\end{aligned}$$

from which

$$\frac{1}{2}\frac{d}{dt}\|v\|^2+\frac{\varepsilon^2}{2}\|v_{xx}\|^2+\|v_x\|_4^4\leq f_1(t)\|v_x\|^2+f_2(t),$$

where

$$f_1(t):=2+12\sigma^2Y^2\|\phi_x\|_\infty+37\frac{\sigma^4Y^4}{\varepsilon^2}\|\phi_x\|_\infty^4,$$

and

$$f_2(t):=\frac{\sigma^2Y^4}{4}\|g\|^2+9\sigma^6Y^6\|\phi_x\|_\infty^4\|\phi_{xx}\|.$$

Finally, using $\|v_x\|^2\leq|I|^{\frac{1}{2}}\|v_x\|_4^2$, we conclude that

$$\frac{d}{dt}\|v\|^2+\varepsilon^2\|v_{xx}\|^2\leq h(t),$$

where

$$h(t):=2\left(\frac{|I|}{4}f_1^2(t)+f_2(t)\right).$$

Thus, we have obtained a similar inequality to the one in the previous section and, for this reason, we do not repeat the details while we pass to obtain the inequality for $\|v_x\|$. By multiplying equation (12) by v_{xx} in $L^2(I)$ we arrive at

$$\frac{1}{2}\frac{d}{dt}\|v_x\|^2+\varepsilon^2\|v_{xxx}\|^2$$
$$+\frac{1}{2}\int_I\left[W''(v_x+\sigma\phi_xz^*(\theta_t\omega))\right](v_{xx}+\sigma z^*(\theta_t\omega)\phi_{xx})v_{xx}dx$$
$$+\sigma z^*(\theta_t\omega)\int_I gv_{xx}dx=0.$$

In details

$$\frac{1}{2}\int_I\left[W''(v_x+\sigma\phi_xz^*(\theta_t\omega))\right](v_{xx}+\sigma z^*(\theta_t\omega)\phi_{xx})v_{xx}dx$$
$$=6\|v_xv_{xx}\|^2+6\sigma z^*(\theta_t\omega)\int_I v_x^2v_{xx}\phi_{xx}dx$$
$$+6\sigma^2[z^*(\theta_t\omega)]^2\int_I\phi_x^2v_{xx}^2dx$$
$$+6\sigma^3[z^*(\theta_t\omega)]^3\int_I\phi_x^2\phi_{xx}v_{xx}dx$$
$$+12\sigma[z^*(\theta_t\omega)]\int_I v_xv_{xx}^2\phi_xdx$$
$$+12\sigma^2[z^*(\theta_t\omega)]^2\int_I v_xv_{xx}\phi_x\phi_{xx}dx$$
$$-2\|v_{xx}\|^2-2\sigma z^*(\theta_t\omega)\int_I\phi_{xx}v_{xx}dx.$$

Then

$$\frac{1}{2}\frac{d}{dt}\|v_x\|^2+\varepsilon^2\|v_{xxx}\|^2+6\|v_xv_{xx}\|^2$$
$$\leq\sigma|Y|\|g\|\|v_{xx}\|+6\sigma|Y|\|\|\phi_{xx}\|_\infty\|v_x\|\|v_xv_{xx}\|$$
$$+6\sigma^3|Y^3|\|\phi_x\|_\infty^2\|\phi_{xx}\|\|v_{xx}\|$$
$$+12\sigma|Y|\|\phi_{xx}\|_\infty\|v_{xx}\|\|v_xv_{xx}\|$$
$$+12\sigma^2Y^2\|\phi_x\|_\infty\|\phi_{xx}\|_\infty\|v_x\|\|v_{xx}\|$$
$$+2\|v_{xx}\|^2+2\sigma|Y|\|\phi_{xx}\|\|v_{xx}\|.$$

Simply using $2ab\leq a^2+b^2$ we obtain

$$\frac{1}{2}\frac{d}{dt}\|v_x\|^2+\varepsilon^2\|v_{xxx}\|^2\leq f_3(t)\|v_{xx}\|^2+f_4(t),$$

where

$$f_3(t):=5+9\sigma^2Y^2\|\phi_{xx}\|_\infty^2+36\sigma^2Y^2\|\phi_x\|_\infty^2$$
$$+12\sigma^2Y^2\|\phi_x\|_\infty\|\phi_{xx}\|_\infty$$

and

$$f_4(t):=\frac{\sigma^2Y^2}{4}\|g\|^2+\sigma^2Y^2\|\phi_{xx}\|^2+9\sigma^6Y^6\|\phi_x\|_\infty^4\|\phi_{xx}\|^2.$$

Thus we conclude

$$\frac{d}{dt}\|v_x\|^2\leq\frac{2f_3^2(t)}{\varepsilon^2}\|v_x\|^2+f_4(t).$$

We note that the use of the uniform Gronwall lemma is possible using similar arguments to those of the previous section such as the ergodic theorem.
We conclude with the inequality involving $\|v_{xx}\|$. Multiplying equation (12) by v_{xxxx} in H,

$$\frac{1}{2}\frac{d}{dt}\|v_{xx}\|^2+\varepsilon^2\|v_{xxxx}\|^2$$
$$=\frac{1}{2}\int_I W''(v_x+\sigma z^*(\theta_t\omega)\phi_x)(v_{xx}+\sigma z^*(\theta_t\omega)\phi_{xx})v_{xxxx}dx$$
$$+\sigma z^*(\theta_t\omega)\int_I g_{xx}v_{xx}dx.$$

The term involving the potential can be written in details:

$$6\int_I v_x^2v_{xx}v_{xxxx}dx+6\sigma z^*(\theta_t\omega)\int_I v_x^2\phi_{xx}\phi_{xxxx}dx$$
$$+6\sigma^2[z^*(\theta_t\omega)]^2\int_I\phi_x^2v_{xx}v_{xxxx}dx$$
$$+6\sigma^3[z^*(\theta_t\omega)]^3\int_I\phi_x^2\phi_{xx}v_{xxxx}dx$$
$$+12\sigma z^*(\theta_t\omega)\int_I\phi_xv_xv_{xx}v_{xxxx}dx$$
$$+12\sigma^2[z^*(\theta_t\omega)]^2\int_I\phi_x\phi_{xx}v_xv_{xxxx}dx$$
$$-2\int_I v_{xx}v_{xxxx}dx-2\sigma z^*(\theta_t\omega)\int_I\phi_{xx}v_{xxxx}dx,$$

and can be estimated by

$$6\|v_x\|_\infty^2\|v_{xx}\|\|v_{xxxx}\|+6\sigma Y\|\phi_{xx}\|\|\phi_{xxxx}\|_\infty\|v_{xx}\|^2$$
$$+6\sigma^2Y^2\|\phi_x\|_\infty^2\|v_{xx}\|\|v_{xxxx}\|$$
$$+6\sigma^3Y^3\|\phi_x\|_\infty^2\|\phi_{xx}\|\|v_{xxxx}\|$$
$$+12\sigma Y\|\phi_x\|_\infty\|v_x\|_\infty\|v_{xx}\|\|v_{xxxx}\|$$
$$+12\sigma^2Y^2\|\phi_x\|_\infty\|\phi_{xx}\|_\infty\|v_{xx}\|\|v_{xxxxx}\|$$
$$+2\|v_{xxx}\|^2+2\sigma Y\|\phi_{xx}\|\|v_{xxxx}\|.$$

Then, by using the estimate in $H_0^1(I)$, that is $\|v_x\|\leq R_x$, the interpolating inequalities

$$\|u_x\|_\infty\leq\|u_x\|^{1/2}\|u_{xx}\|^{1/2},\qquad\|v_{xxx}\|^2\leq\|v_{xx}\|\|v_{xxxx}\|,$$

and $2ab \leq a^2 + b^2$, we have

$$\frac{d}{dt}\|v_{xx}\|^2 \leq f_5(t)\|v_{xx}\|^4 + f_6(t)\|v_{xx}\|^2 + f_7(t),$$

where

$$f_5(t) = \frac{144R_x^2}{\varepsilon^2}\left(1 + 4\sigma^2 Y^2 \|\phi_x\|_\infty^2\right),$$

$$f_6(t) = 2\left(1 + 6\sigma Y\|\phi_{xx}\|\|\phi_{xxxx}\|_\infty + \frac{72}{\varepsilon^2}\sigma^4 Y^4\|\phi_x\|_\infty^4 + \frac{288}{\varepsilon^2}\sigma^4 Y^4\|\phi_x\|_\infty^2\|\phi_{xx}\|_\infty^2 + \frac{8}{\varepsilon^2}\right),$$

and

$$f_7(t) = 2\left(\frac{\sigma^2 Y^2}{4}\|g_{xx}\|^2 + \frac{72}{\varepsilon^2}\sigma^6 Y^6\|\phi_x\|_\infty^4\|\phi_{xx}\|^2 + \frac{8\sigma^2 Y^2}{\varepsilon^2}\|\phi_{xx}\|^2\right).$$

Again we have obtained a similar inequality to that of the case of multiplicative noise. Then, it is not difficult to obtain the existence of the random attractor also in this additive case.

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