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MORSE DECOMPOSITION OF GLOBAL ATTRACTORS WITH INFINITE COMPONENTS

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Abstract. In this paper we describe some dynamical properties of a Morse decomposition with a countable number of sets. In particular, we are able to prove that the gradient dynamics on Morse sets together with a separation assumption is equivalent to the existence of an ordered Lyapunov function associated to the Morse sets and also to the existence of a Morse decomposition -that is, the global attractor can be described as an increasing family of local attractors and their associated repellers.

1. **Introduction.** The asymptotic behaviour of a system of (ordinary or partial) differential equations modeling real phenomena from different areas of Science is usually described by the analysis of their global attractors, a compact invariant set for the associated semigroups attracting (uniformly) bounded sets forwards in time. This subject has received much attention throughout the last decades (see, for instance, [4], [9], [12], [16], [19], [18] or [20]). We recall now the definition of global attractor associated to a semigroup.

First, let X be a metric space with metric $d : X \times X \to \mathbb{R}^+$, where $\mathbb{R}^+ = [0, \infty)$, and denote by $\mathscr{C}(X)$ the set of continuous maps from X into X. Given a subset $A \subset X$, the *ϵ-neighborhood* of A is the set $\mathcal{O}_\epsilon(A) := \{x \in X : d(x, a) < \epsilon \text{ for some } a \in A\}$.

Definition 1.1. *A family $\{T(t) : t \geq 0\} \subset \mathscr{C}(X)$ is a semigroup in a complete metric space X if:*
- $T(0) = I_X$*, with I_X being the identity map in X,*
- $T(t + s) = T(t)T(s)$*, for all $t, s \in \mathbb{R}^+$,*
- $\mathbb{R}^+ \times X \ni (t, x) \mapsto T(t)x \in X$ *is continuous.*

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The notion of invariance plays a fundamental role in the study of the asymptotic behavior of semigroups.

Definition 1.2. *A subset A of X is said invariant under the semigroup $\{T(t) : t \geq 0\}$ if $T(t)A = A$ for all $t \geq 0$.*

Given $A, B \subset X$, the Hausdorff semidistance from A to B is given by

$$d(A, B) := \sup_{a \in A} \inf_{b \in B} d(a, b).$$

Definition 1.3. *Given two subsets A, B of X we say that A attracts B under the action of the semigroup $\{T(t) : t \geq 0\}$ if $d(T(t)B, A) \stackrel{t \to \infty}{\longrightarrow} 0$.*

We are now in a position to define *global attractors.*

Definition 1.4. *A subset $\mathcal{A}$ of X is a* global attractor *for a semigroup $\{T(t) : t \geq 0\}$ if it is compact, invariant under the action of $\{T(t) : t \geq 0\}$ and for every bounded subset B of X we have that $\mathcal{A}$ attracts B under the action of $\{T(t) : t \geq 0\}$.*

Definition 1.5. *The semigroup $\{T(t) : t \geq 0\}$ is eventually dissipative if for any bounded set B there exists $t^* = t^*(B) \geq 0$ such that $\cup_{t \geq t^*} T(t)B$ is bounded.*

Remark 1.6. *It is obvious that if $T(t)$ possesses a global attractor, then it is eventually dissipative.*

One of the main properties in the study of attractors is referred to the description of their geometrical internal structure. Generically, a global attractor is characterized by a (finite or infinite) number of isolated invariant sets and the connecting orbits among them. This fact leads to a Morse decomposition of the global attractor in terms of a family of attracting-repeller pairs (see [8, 17, 11, 14, 15]). We now introduce this concept.

Definition 1.7. *Let $\{T(t) : t \geq 0\}$ be a semigroup on X. We say that an invariant set $E \subset X$ for the semigroup $\{T(t) : t \geq 0\}$ is an* isolated invariant set *if there is an $\epsilon > 0$ such that E is the maximal invariant subset of $\mathcal{O}_\epsilon(E)$.*

Definition 1.8. *A disjoint family of isolated invariant sets is a family $\{M_1, \cdots, M_n\}$ of isolated invariant sets with the property that*

$$\mathcal{O}_\epsilon(M_i) \cap \mathcal{O}_\epsilon(M_j) = \varnothing, \ 1 \leq i < j \leq n,$$

for some $\epsilon > 0$.

Definition 1.9. *A global solution for a semigroup $\{T(t) : t \geq 0\}$ is a continuous function $\xi : \mathbb{R} \to X$ with the property that $T(t)\xi(s) = \xi(t + s)$ for all $s \in \mathbb{R}$ and for all $t \in \mathbb{R}^+$. We say that $\xi : \mathbb{R} \to X$ is a global solution through $x \in X$ if it is a global solution with $\xi(0) = x$.*

It is also well known that the global attractor is the union of all bounded complete global solutions of the semigroup T.

Definition 1.10. *Let $\{T(t) : t \geq 0\}$ be a semigroup which possesses a disjoint family of isolated invariant sets $M = \{M_1, \cdots, M_n\}$. A* homoclinic structure *associated to M is a subset $\{M_{k_1}, \cdots, M_{k_p}\}$ of M $(p \leq n)$ together with a set of global solutions $\{\xi_1, \cdots, \xi_p\}$ such that*

$$M_{k_j} \stackrel{t \to -\infty}{\longleftarrow} \xi_j(t) \stackrel{t \to \infty}{\longrightarrow} M_{k_{j+1}}, \ 1 \leq j \leq p,$$

where $M_{k_{p+1}} := M_{k_1}$.

Remark 1.11. *Here,* $\xi(t) \overset{t\to\pm\infty}{\longrightarrow} M$ *means that* $d(\xi(t), M) \to 0$ *as* $t \to \pm\infty$.

We will study the dynamics of the semigroup inside the global attractor $\mathcal{A}$. We now define generalized dynamically gradient semigroups (see [6, 5]).

Definition 1.12. *Let* $\{T(t) : t \geq 0\}$ *be a semigroup with a global attractor* $\mathcal{A}$ *and a disjoint family of isolated invariant sets* $M = \{M_1, \cdots, M_n\}$ *in* $\mathcal{A}$. *We say that* $\{T(t) : t \geq 0\}$ *is a* generalized dynamically gradient semigroup *relative to* M *if:*

a) *For any global solution* $\xi : \mathbb{R} \to \mathcal{A}$ *there are* $1 \leq i, j \leq n$ *such that*
$$M_i \overset{t\to-\infty}{\longleftarrow} \xi(t) \overset{t\to\infty}{\longrightarrow} M_j.$$
b) *There is no homoclinic structure associated to* M.

Remark 1.13. *The concept of generalized dynamically gradient semigroup is the same as the concept of gradient-like semigroup as given in* [1], [5].

To introduce the notion of a Morse decomposition for the attractor $\mathcal{A}$ of a semigroup $\{T(t) : t \geq 0\}$ (see [8], [17] or [18]) we previously need the notion of attractor-repeller pair. We recall that the omega-limit set of $B \subset X$ is defined by
$$\omega(B) = \cap_{t\geq 0}\overline{\cup_{s\geq t} T(s)B}.$$

Definition 1.14. *Let* $\{T(t) : t \geq 0\}$ *be a semigroup with a global attractor* $\mathcal{A}$. *We say that a non-empty subset* A *of* $\mathcal{A}$ *is a* local attractor *if there is an* $\epsilon > 0$ *such that* $\omega(\mathcal{O}_\epsilon(A)) = A$. *The* repeller A^* *associated to a local attractor* A *is the set defined by*
$$A^* := \{x \in \mathcal{A} : \omega(x) \cap A = \varnothing\}.$$
The pair (A, A^*) *is called an* attractor-repeller pair *for* $\{T(t) : t \geq 0\}$.

Note that if A is a local attractor, then A^* is closed and invariant.

Definition 1.15. *Given an increasing family* $\varnothing = A_0 \subset A_1 \subset \cdots \subset A_n = \mathcal{A}$, *of* $n+1$ *local attractors, for* $j = 1, \cdots, n$, *define* $M_j := A_j \cap A^*_{j-1}$. *The ordered* n*-tuple* $M := \{M_1, M_2, \cdots, M_n\}$ *is called a* Morse decomposition *for* $\mathcal{A}$.

Definition 1.16. *We will say that a semigroup* $\{T(t) : t \geq 0\}$ *with a global attractor* $\mathcal{A}$ *and a disjoint family of isolated invariant sets* $M = \{M_1, \cdots, M_n\}$ *in* $\mathcal{A}$ *is a* gradient semigroup *with respect to* M, *if there exists a continuous function* $V : X \to \mathbb{R}$ *such that* $[0, \infty) \ni t \mapsto V(T(t)x) \in \mathbb{R}$ *is non-increasing for each* $x \in X \backslash M$, V *is constant in* M_i *for each* $1 \leq i \leq n$, *and* $V(T(t)x) = V(x)$ *for all* $t \geq 0$ *if and only if* $x \in \bigcup_{i=1}^{n} M_i$.
V *is called a* Lyapunov function *related to* M.

It has been proved in [1] that given a disjoint family of isolated invariant sets on the global attractor $M = \{M_1, \cdots, M_n\}$ for a semigroup $T(t)$, the dynamical property of being generalized dynamically gradient, the existence of an associated ordered family of local attractor-repellers, and the existence of a Lyapunov functional related to M, are equivalent properties. Many of the arguments in [1] make a precise use of the fact that the number of Morse sets is finite. The aim of this paper is to generalize this result to the case of a countable number of Morse sets.

Indeed, the general theory of Morse decomposition of invariant sets is generically adapted to the existence of a finite number of isolated Morse sets. However, it is not unusual to have an infinite number of invariants in a global attractor. For instance, consider the scalar differential equation

$$\frac{dy}{dt} = f(y)$$

with

$$f(y) = \begin{cases} -y, & \text{if } y \leq 0, \\ (1 - e^{-y}) \left| \sin\left(\frac{\pi}{y}\right) \right|, & \text{if } 0 < y \leq 1, \\ 1 - y, & \text{if } y \geq 1. \end{cases}$$

Note that the equation possesses the following fixed points:

$$y_1 = 1,\ y_2 = \frac{1}{2},\ y_3 = \frac{1}{3}, ...,\ y_k = \frac{1}{k}, ...,\ y_\infty = 0,$$

with their respective associated unstable manifolds (see Definition 4.2)

$$W^u(1) = 1,\ W^u\left(\frac{1}{2}\right) = [\frac{1}{2}, 1),\ ...,\ W^u\left(\frac{1}{k}\right) = [\frac{1}{k}, \frac{1}{k-1}), ...,\ W^u(0) = 0,$$

and as global attractor $\mathcal{A} = [0, 1]$. In [3] the authors study a multivalued version of the well-known Chafee-Infante equation, also leading to a global attractor with an infinite number of equilibria, which actually has motivated the necessity of developing the theory in this paper. We will consider this application in a subsequent work.

In Section 2 we recall some results on the dynamics related to an attractor-repeller pair. In Section 3 we will generalize Definitions 1.12, 1.15 and 1.16 to the case of an infinite number of disjoint isolated invariant sets $\mathbf{M}_\infty = \{M_i\}_{i=1}^\infty \cup M_\infty$ inside the global attractor. In Sections 4, 5 and 6 we prove the main result of this paper, the equivalence between a generalized dynamically gradient semigroup referred to $\mathbf{M}_\infty$ with a suitable separation assumption, the existence of an ordered Lyapunov function associated to $\mathbf{M}_\infty$, and the existence of a Morse decomposition on the global attractor. This is done in several steps: first, we prove that the property of the semigroup of being generalized dynamically gradient together with a separation assumption implies that a Morse decomposition can be constructed; then we prove that from a Morse decomposition related to $\mathbf{M}_\infty$ an ordered Lyapunov function can be defined; finally, we check that the existence of an ordered Lyapunov function implies that the semigroup is generalized dynamically gradient semigroup referred to $\mathbf{M}_\infty$ and that the separation assumption holds.

2. Preliminary results on attractor-repeller pairs.

The following results on the dynamics on attractor-repeller pairs are taken from [1].

We recall that local attraction of A in $\mathcal{A}$ is equivalent to local attraction in X, for which we firstly need the following result.

Lemma 2.1. *Let $\{T(t) : t \geq 0\}$ be a semigroup in X with a global attractor $\mathcal{A}$. If $A \subset \mathcal{A}$ is a compact invariant set for $\{T(t) : t \geq 0\}$ and there is an $\epsilon > 0$ such that A attracts $\mathcal{O}_\epsilon(A) \cap \mathcal{A}$, then given $\delta \in (0, \varepsilon)$ there is a $\delta' \in (0, \delta)$ such that $\gamma^+(\mathcal{O}_{\delta'}(A)) \subset \mathcal{O}_\delta(A)$, where $\gamma^+(\mathcal{O}_{\delta'}(A)) = \bigcup_{x \in \mathcal{O}_{\delta'}(A)} \bigcup_{t \geq 0} \{T(t)x\}$.*

The next result generalizes for semigroups a known result for groups given in [8] and shows that our definition of local attractor is equivalent to that one in [8, 17].

Lemma 2.2. *If $\{T(t) : t \geq 0\}$ is a semigroup in X with a global attractor $\mathcal{A}$ and $S(t) := T(t)|_{\mathcal{A}}$, clearly $\{S(t) : t \geq 0\}$ is a semigroup in the metric space $\mathcal{A}$. If A is a local attractor for $\{S(t) : t \geq 0\}$ in the metric space $\mathcal{A}$ (that is, there exists $\varepsilon > 0$ with $\omega(\mathcal{O}_\epsilon(A) \cap \mathcal{A}) = A$), and K is a compact subset of $\mathcal{A}$ such that $K \cap A^* = \varnothing$, then A attracts K. Furthermore A is a local attractor for $\{T(t) : t \geq 0\}$ in X.*

We now describe the dynamics on an attractor-repeller pair.

Lemma 2.3. *Let $\{T(t) : t \geq 0\}$ be a semigroup in X with a global attractor $\mathcal{A}$ and (A, A^*) an attractor-repeller for $\{T(t) : t \geq 0\}$. Then:*

(i) *If $\xi : \mathbb{R} \to X$ is a global bounded solution for $\{T(t) : t \geq 0\}$ through $x \notin A \cup A^*$, then $\xi(t) \stackrel{t\to\infty}{\longrightarrow} A$ and $\xi(t) \stackrel{t\to-\infty}{\longrightarrow} A^*$.*

(ii) *A global solution $\xi : \mathbb{R} \to X$ of $\{T(t) : t \geq 0\}$ with the property that $\xi(t) \in \mathcal{O}_\delta(A^*)$ for all $t \leq 0$ for some $\delta > 0$ such that $\mathcal{O}_\delta(A^*) \cap A = \varnothing$ must satisfy $d(\xi(t), A^*) \stackrel{t\to-\infty}{\longrightarrow} 0$.*

(iii) *If $x \in X \backslash \mathcal{A}$, then $T(t)x \stackrel{t\to\infty}{\longrightarrow} A \cup A^*$.*

Part (i) of the previous lemma is proved in Theorem 1.4 in [17]. Parts (ii) and (iii) can be found in [1].

3. **Generalized dynamically gradient semigroups.**

In this section we will introduce the concepts of generalized dynamically gradient semigroups and Morse decomposition for a countable set of isolated invariant sets.

Definition 3.1. *A disjoint (countable) family of invariant sets is a family $\mathbf{M}_\infty = \{M_i\}_{i=1}^\infty \cup M_\infty$ of invariant sets with the property that, given $j \in \mathbb{N}$, there exists δ_j such that*

$$\mathcal{O}_{\delta_j}(M_j) \cap \mathcal{O}_{\delta_j}(M_i) = \varnothing, \text{ for all } i \neq j,\ i \in \mathbb{N} \cup \{\infty\}. \tag{3.1}$$

Definition 3.2. *Let $\{T(t) : t \geq 0\}$ be a semigroup which possesses a disjoint family of invariant sets $\mathbf{M}_\infty = \{M_i\}_{i=1}^\infty \cup M_\infty$ with M_j isolated for each $j \in \mathbb{N}$. A homoclinic structure associated to $\mathbf{M}_\infty$ is a finite subset $\{M_{k_1}, \cdots, M_{k_p}\}$ of $\mathbf{M}_\infty$ together with a set of global solutions $\{\xi_1, \cdots, \xi_p\}$ such that*

$$M_{k_j} \stackrel{t\to-\infty}{\longleftarrow} \xi_j(t) \stackrel{t\to\infty}{\longrightarrow} M_{k_{j+1}},\ 1 \leq j \leq p,$$

where $M_{k_{p+1}} := M_{k_1}$.

Remark 3.3. *The set M_∞ is not assumed to be isolated. The reason is that typically in applications M_∞ is an accumulation set of the sequence M_n as $n \to \infty$. Hence, it is not isolated. This is the case in the example given in the introduction, and also, for instance, in the application for multivalued semiflows in [3].*

Definition 3.4. *Let $\{T(t) : t \geq 0\}$ be a semigroup with a global attractor $\mathcal{A}$ and a disjoint family of invariant sets $\mathbf{M}_\infty = \{M_i\}_{i=1}^\infty \cup M_\infty$ in $\mathcal{A}$ with M_j isolated for each $j \in \mathbb{N}$. We say that $\{T(t) : t \geq 0\}$ is a generalized dynamically gradient semigroup relative to $\mathbf{M}_\infty$ if for any global solution $\xi : \mathbb{R} \to \mathcal{A}$ such that $\xi(t_0) \notin M_k$, for some $t_0 \in \mathbb{R}$ and any $k \in \mathbb{N} \cup \infty$, it holds that*

$$M_j \stackrel{t\to-\infty}{\longleftarrow} \xi(t) \stackrel{t\to\infty}{\longrightarrow} M_i, \quad \text{for } 1 \leq i < j \leq \infty. \tag{3.2}$$

Remark 3.5. *It is obvious that condition (3.2) implies the following properties:*

- *For any global solution $\xi : \mathbb{R} \to \mathcal{A}$ there are $1 \leq i, j \leq \infty$ such that*

$$M_j \stackrel{t\to-\infty}{\longleftarrow} \xi(t) \stackrel{t\to\infty}{\longrightarrow} M_i.$$

- *There is no homoclinic structure associated to* $\mathbf{M}_\infty$*.*

When the number of sets M_i is finite, then it is proved in [2] that these last two properties imply (3.2) for a suitable rearrangement of the sets. In fact, Definition 3.4 is the way in which it is defined a Morse decomposition of a global attractor in [18].

Note that, in particular, (3.2) implies that there is no global solution $\xi(t) : \mathbb{R} \to \mathcal{A}$ with $\xi(t_0) \notin M_1$ for some $t_0 \in \mathbb{R}$ such that

$$\lim_{t\to-\infty} d(\xi(t), M_1) = 0.$$

The following lemma implies that an isolated invariant set inside a global attractor is compact.

Lemma 3.6. *Let M be an isolated invariant set which is relatively compact. Then M is compact.*

Proof. We need to prove that M is closed. Let $y_n \to y$, where $y_n \in M$. By the continuity of T we have that $T(t)y_n \to T(t)y$ for any $t > 0$. Hence, $T(t)y \in \overline{M}$. Thus, $T(t)\overline{M} \subset \overline{M}$ for all $t \geq 0$. On the other hand, as M is invariant, for any $t > 0$ there exists $z_n \in M$ such that $T(t)z_n = y_n$. Since M is relatively compact, passing to a subsequence we have $z_n \to z \in \overline{M}$, and then $T(t)z = y$. Therefore, $\overline{M} \subset T(t)\overline{M}$ for all $t > 0$. It follows that $\overline{M}$ is invariant. As M is an isolated invariant set, we get $M = \overline{M}$. □

As a consequence of the first statement in Lemma 2.3 we obtain the following.

Corollary 3.7. *If $\{T(t) : t \geq 0\}$ is a semigroup in X with a global attractor $\mathcal{A}$ and (A, A^*) is an attractor-repeller pair for $\{T(t) : t \geq 0\}$, then $\{T(t) : t \geq 0\}$ is a generalized dynamically gradient semigroup associated to the disjoint family of isolated invariant sets $\{A, A^*\}$.*

Note that (3.1) implies

$$M_i \cap M_\infty = \varnothing, \text{ for each } i \in \mathbb{N}. \tag{3.3}$$

Lemma 3.8. *Condition (3.2) implies that there is no global solution $\xi : \mathbb{R} \to \mathcal{A}$ with $\xi(t_0) \in \mathcal{A} \setminus M_\infty$ for some $t_0 \in \mathbb{R}$ such that*

$$\lim_{t\to+\infty} d(\xi(t), M_\infty) = 0. \tag{3.4}$$

Proof. It is obvious, as in (3.2), that the index i cannot be ∞. □

Lemma 3.9. *Let $\mathbf{M}_\infty = \{M_i\}_{i=1}^{\infty} \cup M_\infty$ be compact invariant sets such that $M_j \cap M_i = \varnothing$ for $i \neq j$, $i, j \in \mathbb{N} \cup \infty$, and also suppose that the invariant compact set $M_\infty \subset \mathcal{A}$ is such that*

$$\lim_{i\to\infty} d(M_i, M_\infty) = 0. \tag{3.5}$$

Then $\mathbf{M}_\infty$ is a disjoint family of invariant sets.

Proof. Take $j \in \mathbb{N}$ arbitrary. We have to check (3.1). There exists $\delta_1 > 0$ such that

$$\mathcal{O}_{\delta_1}(M_j) \cap \mathcal{O}_{\delta_1}(M_\infty) = \varnothing.$$

In view of (3.5) there is $N > j$ such that

$$M_i \subset \mathcal{O}_{\frac{\delta_1}{2}}(M_\infty) \text{ if } i > N.$$

Hence,

$$\mathcal{O}_{\delta_1}(M_j) \cap \mathcal{O}_{\frac{\delta_1}{2}}(M_i) = \varnothing \text{ if } i > N.$$

Obviously, there exists $\delta_2 > 0$ for which

$$\mathcal{O}_{\delta_2}(M_j) \cap \mathcal{O}_{\delta_2}(M_i) = \varnothing \text{ for } 1 \leq i \leq N,\, i \neq j.$$

Then the result follows for $\delta_j = \min\{\delta_1/2, \delta_2\}$. □

We can now introduce the concept of a Morse decomposition referred to $\mathbf{M}_\infty$.

Definition 3.10. *Given an increasing family $\varnothing = A_0 \subset A_1 \subset \cdots \subset A_n \subset \cdots A_\infty = \mathcal{A}$ of local attractors, for $j \in \mathbb{N}$ define $M_j := A_j \cap A^*_{j-1}$, $M_\infty = \cap_{j=0}^\infty A^*_j$. The ordered countable set $\mathbf{M}_\infty := \{M_i\}_{i=1}^\infty \cup M_\infty$ is called a Morse decomposition of $\mathcal{A}$.*

The following properties of the sets M_j follow.

Lemma 3.11. *$M_\infty \cap A_j = \varnothing$ for any $j \in \mathbb{N}$. Hence, $M_\infty \subset A_\infty \setminus \cup_{j=1}^\infty A_j$ and $M_\infty \cap M_j = \varnothing$ for all $j \in \mathbb{N}$.*

Proof. Let $y \in M_\infty$. Then $y \in A^*_j$, for any $j \in \mathbb{N}$, implies $y \notin A_j$ for all $j \in \mathbb{N}$. □

Lemma 3.12. *The sets M_j, $j \in \mathbb{N} \cup \infty$, are compact.*

Proof. Since $M_j \subset \mathcal{A}$, they are relatively compact. Also, as M_j are the intersection of closed sets, they are closed. □

We can also give the following characterization.

Proposition 3.13. *Let $\{T(t) : t \geq 0\}$ be a semigroup with the global attractor $\mathcal{A}$ and $\mathbf{M}_\infty = \{M_i\}_{i=1}^\infty \cup M_\infty$ a Morse decomposition for $\mathcal{A}$ with the family $\varnothing = A_0 \subset A_1 \subset \cdots \subset A_\infty = \mathcal{A}$ of local attractors. Then,*

$$\bigcap_{j=0}^{\infty}(A_j \cup A^*_j) = (\bigcup_{j=1}^{\infty} M_j) \cup M_\infty.$$

Proof. If $z \in \bigcup\limits_{j=1}^{\infty} M_j$, let $k \in \mathbb{N}$ be such that $z \in M_k = A_k \cap A^*_{k-1}$. Hence $z \in A_k \subset A_{k+1} \subset \cdots \subset A_\infty$ and $z \in A^*_{k-1} \subset A^*_{k-2} \subset \cdots \subset A^*_0$. Thus

$$z \in (\bigcap_{j=k}^{\infty} A_j) \cap (\bigcap_{j=0}^{k-1} A^*_j) \subset \left[\bigcap_{j=k}^{\infty}(A_j \cup A^*_j)\right] \cap \left[\bigcap_{j=0}^{k-1}(A_j \cup A^*_j)\right] = \bigcap_{j=0}^{\infty}(A_j \cup A^*_j),$$

proving that $\bigcup\limits_{j=1}^{\infty} M_j \subset \bigcap\limits_{j=0}^{\infty}(A_j \cup A^*_j)$. If $z \in M_\infty$, then $z \in \bigcap\limits_{j=0}^{\infty} A^*_j \subset \bigcap\limits_{j=0}^{\infty}(A_j \cup A^*_j)$.

Conversely, we take $z \in \bigcap\limits_{j=0}^{\infty}(A_j \cup A^*_j)$. If $z \in \bigcap\limits_{j=0}^{\infty} A^*_j$, then $z \in M_\infty$. Otherwise, $z \in A_j$ for some $j \in \mathbb{N}$. Denote $I := \{i_1, i_2, \cdots, i_k, \dots\}$ and $J := \{j_1, j_2, \cdots, j_l, \dots\}$ such that $I \cup J = \mathbb{Z}^+$ with $I \cap J = \varnothing$ and $z \in A_i$ for all $i \in I$ and $z \in A^*_j$ for all $j \in J$. Clearly, if $i := \min I$, necessarily $I = \{j \geq i\}$ and $J = \{0, 1, \cdots, i-1\}$, consequently $z \in A_i$ and $z \in A^*_{i-1}$. So, $z \in A_i \cap A^*_{i-1} = M_i$, from which $\bigcap\limits_{j=0}^{\infty}(A_j \cup A^*_j) \subset \bigcup\limits_{j=1}^{\infty} M_j$. □

4. **Construction of a Morse decomposition from the dynamics on $\mathbf{M}_\infty$.** In this section we describe the construction of a Morse decomposition of the global attractor $\mathcal{A}$ relative to the disjoint family of invariant sets $\mathbf{M}_\infty = \{M_i\}_{i=1}^\infty \cup M_\infty$ in $\mathcal{A}$ such that M_j is isolated if $j \in \mathbb{N}$ and satisfying (3.2). By Lemma 3.8 we have that (3.4) does not hold.

The following lemma will play an important role in what follows.

Lemma 4.1. *Let $\{T(t) : t \geq 0\}$ be a semigroup with a global attractor $\mathcal{A}$ and the disjoint family of invariant sets $\mathbf{M}_\infty = \{M_i\}_{i=1}^\infty \cup M_\infty = \{M_1, \ldots, M_n, \ldots; M_\infty\}$ in $\mathcal{A}$ be such that M_j are isolated for $j \in \mathbb{N}$. Assume that T is generalized dynamically gradient relative to $\mathbf{M}_\infty$. Then, M_1 is a local attractor for $\{T(t) : t \geq 0\}$.*

Proof. We firstly prove that for all $\delta \in (0, \delta_1)$ there exists $\delta' \in (0, \delta)$ such that

$$\gamma^+ \left(\mathcal{O}_{\delta'} \left(M_1\right)\right) \subset \mathcal{O}_\delta \left(M_1\right),$$

where δ_1 satisfies $\mathcal{O}_{\delta_1}(M_1) \cap \mathcal{O}_{\delta_1}(M_i) = \varnothing$ for $i > 1$ or $i = \infty$.

If not, there exist $0 < \delta < \delta_1$ and sequences $\{t_k\}_{k\in\mathbb{N}}$ of positive times and $\{x_k\}_{k\in\mathbb{N}}$ of points in X such that for all k

$$d(x_k, M_1) < \frac{1}{k},$$

$$d(T(t_k)x_k, M_1) = \delta$$

and

$$d(T(t)x_k, M_1) < \delta \text{ for } t \in [0, t_k).$$

Thus, if we define, for each k, $\xi_k(t) := T(t+t_k)x_k$ for $t \in [-t_k, \infty)$, as $t_k \underset{k\to\infty}{\to} \infty$, we conclude that there exists a global solution $\xi : \mathbb{R} \to X$ for $T(\cdot)$ such that $\xi_k \underset{k\to\infty}{\to} \xi$ uniformly in compact sets of times (see [7, Lemma 3.1]). Then, $d(\xi_k(t), M_1) \leq \delta$ for $t \in [-t_k, 0]$ implies

$$d(\xi(t), M_1) \leq \delta < \delta_1 \text{ for } t \leq 0.$$

But by (3.2) we have $\xi(t) \to M_j$, with $j > 1$, as $t \to -\infty$, a contradiction.

M_1 is the maximal invariant set in $\mathcal{O}_\varepsilon(M_1)$ for some $\varepsilon > 0$. Thus, for $\delta < \min\{\varepsilon, \delta_1\}$ take $\delta' \in (0, \delta)$ such that

$$\gamma^+(\mathcal{O}_{\delta'}(M_1)) \subset \mathcal{O}_\delta(M_1),$$

so that

$$\omega(\mathcal{O}_{\delta'}(M_1)) \subset \overline{\mathcal{O}_\delta(M_1)} \subset \mathcal{O}_\varepsilon(M_1),$$

and then, as $\omega(\mathcal{O}_{\delta'}(M_1))$ is invariant,

$$\omega(\mathcal{O}_{\delta'}(M_1)) \subset M_1.$$

The other inclusion is trivial, so that M_1 is a local attractor. □

For M_1 a local attractor, let $M_1^* = \{x \in \mathcal{A} : \omega(x) \cap M_1 = \varnothing\}$ be its associated repeller, so each M_i, with $i \geq 2$, is contained in M_1^* and more generally the orbit $\xi(\mathbb{R})$ of any global solution $\xi : \mathbb{R} \to \mathcal{A}$ that converges to M_i, $i \geq 2$, when $t \to +\infty$, is contained in M_1^*. Considering the restriction $\{T_1(t) : t \geq 0\}$ of $\{T(t) : t \geq 0\}$ to M_1^* we have that $\{T_1(t) : t \geq 0\}$ satisfies (3.2) in the space M_1^* with the invariant sets $\{M_i\}_{i=2}^\infty \cup M_\infty$ and we may assume, by the last lemma, that M_2 is a local attractor for the semigroup $\{T_1(t) : t \geq 0\}$ in M_1^*. If $M_{2,1}^*$ is the repeller associated to the local attractor M_2 for $\{T_1(t) : t \geq 0\}$ in M_1^* we may proceed and consider the restriction

$\{T_2(t) : t \geq 0\}$ of the semigroup $\{T_1(t) : t \geq 0\}$ to $M^*_{2,1}$ and then $\{T_2(t) : t \geq 0\}$ satisfies (3.2) in $M^*_{2,1}$ with the associated invariant sets $\{M_i\}_{i=3}^{\infty} \cup M_\infty$.

Setting $\mathcal{A} =: M^*_{0,-1}$ and $M^*_{1,0} := M^*_1$, for $j \geq 1$ we have that M_j is a local attractor for the restriction of $\{T(t) : t \geq 0\}$ to $M^*_{j-1,j-2}$ whose repeller will be indicated by $M^*_{j,j-1}$.

Definition 4.2. *Let $\{T(t) : t \geq 0\}$ be a semigroup. The* unstable set *of an invariant set M is defined by*

$$W^{\mathrm{u}}(M) := \{z \in X : \text{ there is a global solution } \xi : \mathbb{R} \to X \text{ such that } \xi(0) = z \text{ and } \lim_{t\to-\infty} d(\xi(t), M) = 0\}.$$

Define $A_0 := \varnothing$, $A_1 := M_1$ and for $j = 2, 3, \cdots$,

$$A_j := A_{j-1} \cup W^{\mathrm{u}}(M_j) = \bigcup_{i=1}^{j} W^{\mathrm{u}}(M_i). \tag{4.1}$$

Also, $A_\infty = \mathcal{A}$.

It is clear that $\mathcal{A} = \cup_{i=1}^{\infty} W^{\mathrm{u}}(M_i) \cup W^{\mathrm{u}}(M_\infty)$.

Lemma 4.3. *Assume the conditions of Lemma 4.1. Then $M_\infty = \cap_{j=0}^{\infty} A^*_j$.*

Proof. Let $z \in M_\infty$. Then as M_∞ is invariant, $\omega(z) \subset M_\infty$. Then z cannot be in $W^u(M_j)$ for $j \in \mathbb{N}$, as in such a case by (3.2) we would have $\omega(z) \cap M_i \neq \varnothing$ for some $i \leq j$, a contradiction. Thus, by (4.1) we have that $z \notin A_j$ for $j \in \mathbb{N}$. Hence, $\omega(z) \cap A_j = \varnothing$, so that $z \in \cap_{j=0}^{\infty} A^*_j$.

Conversely, let $z \in \cap_{j=0}^{\infty} A^*_j$. Then $\omega(z) \cap A_j = \varnothing$ for all $j \in \mathbb{N}$. If $z \notin M_\infty$, we take a global solution $\xi(\cdot)$ such that $\xi(0) = z$. Then by condition (3.2) we have that $\xi(t) \to M_i$ as $t \to +\infty$ for some $i \in \mathbb{N}$. But then $\omega(z) \cap A_i \neq \varnothing$, a contradiction. □

Lemma 4.4. *Assume the conditions of Lemma 4.1. Then the sets M_j, $j \in \mathbb{N} \cup \infty$, are compact.*

Proof. We note that $M_j \subset \mathcal{A}$ implies by Lemma 3.6 that the sets M_j are compact if $j \in \mathbb{N}$. Also, Lemma 4.3 implies that M_∞ is closed, and then $M_\infty \subset \mathcal{A}$ implies that it is compact. □

Lemma 4.5. *Assume the conditions of Lemma 4.1. Suppose that, given $j \in \mathbb{N}$, there exists δ_j such that*

$$W^u(M_j) \cap \mathcal{O}_{\delta_j}(\bigcup_{i=j+1}^{\infty} M_i \cup M_\infty)) = \emptyset. \tag{4.2}$$

Then,

$$A_j \cap \mathcal{O}_{\delta_j}((\bigcup_{i=j+1}^{\infty} M_i) \cup M_\infty) = \emptyset. \tag{4.3}$$

Proof. For $j = 1$ the result follows since $A_1 = M_1 = W^u(M_1)$. Suppose (4.3) is true for $j - 1$ and we will show it for j. If not, there exists a sequence $\{x_k\}_{k\in\mathbb{N}}$ in A_j such that for all k

$$d(x_k, (\bigcup_{i=j+1}^{\infty} M_i) \cup M_\infty) < \frac{1}{k}.$$

As $A_j := A_{j-1} \cup W^u(M_j)$ and we have (4.3) for $j - 1$, then $x_k \in W^u(M_j)$, from which, by hypothesis, we get a contradiction. □

Remark 4.6. *The separation condition (4.2) can be proved easily in the case of a finite number of elements M_j. It is interesting to study whether this assumption can be somehow avoided in the case of an infinte number of elements.*

Corollary 4.7. *Under the hypotheses of the previous lemma, given $j \in \mathbb{N}$, there exists δ_j such that*

$$\mathcal{O}_{\delta_j}(A_j) \cap ((\bigcup_{i=j+1}^{\infty} M_i) \cup M_\infty) = \emptyset. \tag{4.4}$$

Theorem 4.8. *Let $\{T(t) : t \geq 0\}$ be a semigroup with a global attractor $\mathcal{A}$ and consider the disjoint family of invariant sets $\mathbf{M}_\infty = \{M_i\}_{i=1}^{\infty} \cup M_\infty$ in $\mathcal{A}$ such that M_j is isolated if $j \in \mathbb{N}$. Assume that T is generalized dynamically gradient relative to $\mathbf{M}_\infty$ and such that (4.2) holds, so that each M_j is a local attractor for the restriction of $\{T(t) : t \geq 0\}$ to $M^*_{j-1,j-2}$. Then A_j defined in (4.1) is a local attractor for $\{T(t) : t \geq 0\}$ in X, and*

$$M_j = A_j \cap A^*_{j-1}. \tag{4.5}$$

As a consequence, $\mathbf{M}_\infty = \{M_i\}_{i=1}^{\infty} \cup M_\infty$ defines a Morse decomposition on the global attractor $\mathcal{A}$.

Proof. If we prove that for any $0 < \delta < \delta_j$, there is $\delta' < \delta$ such that $\gamma^+(\mathcal{O}_{\delta'}(A_j)) \subset \mathcal{O}_\delta(A_j)$, then $\omega(\mathcal{O}_{\delta'}(A_j))$ attracts $\mathcal{O}_{\delta'}(A_j)$ and (as $\omega(\mathcal{O}_{\delta'}(A_j))$ is invariant) is contained in A_j proving that A_j is a local attractor.

Suppose there is $j \in \mathbb{N}$ for which there exist $\delta \in (0, \delta_j)$ and sequences $(t_k)_{k\in\mathbb{N}}$ with $t_k \to \infty$ and $(x_k)_{k\in\mathbb{N}}$ in X such that

$$d(x_k, A_j) < \frac{1}{k},$$
$$d(T(t_k)x_k, A_j) = \delta.$$

and

$$d(T(t)x_k, A_j) < \delta \text{ for } t \in [0, t_k).$$

Then, as in Lemma 4.1, we get a global solution $\xi_0 : \mathbb{R} \to X$ satisfying

$$d(\xi_0(t), A_j) \leq \delta \text{ for all } t \leq 0 \tag{4.6}$$

with

$$d(\xi_0(0), A_j) = \delta. \tag{4.7}$$

For this global solution, there exists $M_i \in \mathbf{M}_\infty$ such that

$$\lim_{t\to-\infty} d(\xi_0(t), M_i) = 0,$$

and since $\delta \in (0, \delta_j)$, with δ_j satisfying (4.4), it holds that $i \leq j$, and so $\xi_0(0) \in W^u(M_i) \subset A_j$, which contradicts (4.7).

To prove that $M_j = A_j \cap A^*_{j-1}$ note that

$$A_j = \bigcup_{i=1}^{j} W^{\mathrm{u}}(M_i)$$

and $A^*_{j-1} = \{z \in \mathcal{A} : \omega(z) \cap A_{j-1} = \varnothing\}$. Hence, given $z \in A_j \cap A^*_{j-1}$ we have that any global solution $\xi : \mathbb{R} \to \mathcal{A}$ through z must satisfy that

$$\cup_{i=1}^{j} M_i \overset{t\to-\infty}{\longleftarrow} \xi(t) \overset{t\to\infty}{\longrightarrow} \cup_{i=j}^{\infty} M_i.$$

As a consequence of that and of the fact that $\{T(t) : t \geq 0\}$ satisfies (3.2) we obtain that $z \in M_j$. This shows that $A_j \cap A^*_{j-1} \subset M_j$. The other inclusion is immediate from the definition of A_j and A^*_{j-1}.

Finally, (4.5) and Lemma 4.3 imply that $\mathbf{M}_\infty = \{M_i\}_{i=1}^\infty \cup M_\infty$ defines a Morse decomposition on the global attractor $\mathcal{A}$. □

Remark 4.9. *As we suppose* (3.2) *for a dynamically gradient system, we get an order in Morse sets by an energy level decomposition of the global attractor in the sense of* [2], *in which the attractor is described by connecting global solutions among the different levels in a decreasing way.*

5. **A Lyapunov function for a Morse decomposition.** In this section we will construct a Lyapunov function for semigroups having a Morse decomposition with an infinite number of elements.

Definition 5.1. *We say that a semigroup $\{T(t) : t \geq 0\}$ with a global attractor $\mathcal{A}$ and a disjoint family of invariant sets $\mathbf{M}_\infty = \{M_i\}_{i=1}^\infty \cup M_\infty$ in $\mathcal{A}$ such that M_j are isolated for $j \in \mathbb{N}$ is a generalized gradient semigroup with respect to $\mathbf{M}_\infty$ if there is a continuous function $V : \mathcal{A} \to \mathbb{R}$ such that:*

(i) The real function $[0,\infty) \ni t \mapsto V(T(t)x) \in \mathbb{R}$ is non-increasing for each $x \in \mathcal{A} \setminus \cup_{i=1}^\infty M_i \cup M_\infty$,

(ii) V is constant in M_i for each $i \in \mathbb{N} \cup \infty$,

(iii) $V(T(t)x) = V(x)$ for all $t \geq 0$ if and only if $x \in \mathbf{M}_\infty$.

A function V with the properties above is called a Lyapunov function for the generalized gradient semigroup $\{T(t) : t \geq 0\}$ with respect to $\mathbf{M}_\infty$.

The following result, which is proved in [1, Proposition 3.3], gives the existence of a Lyapunov type functional for an attractor-repeller pair

Proposition 5.2. *Let $\{T(t) : t \geq 0\}$ be a nonlinear semigroup in a metric space (X, d) with the global attractor $\mathcal{A}$, and let (A, A^*) be an attractor-repeller pair in $\mathcal{A}$. Then, for any $\gamma > 0$ there exists a function $f : \mathcal{A} \to [0,1]$ satisfying the following:*

(i) $f : \mathcal{A} \to [0,1]$ is continuous in $\mathcal{A}$.
(ii) $f : \mathcal{A} \to [0,1]$ is non-increasing along solutions.
(iii) $f^{-1}(0) = A$ and $f^{-1}(1) = A^*$.
(iv) $f(T(t)z) = f(z)$, for all $t \geq 0$, if and only if $z \in A \cup A^*$.

We now prove that the existence of a Morse decomposition implies the existence of a Lyapunov function

Proposition 5.3. *Let $\{T(t) : t \geq 0\}$ be a semigroup with the global attractor $\mathcal{A}$ and a disjoint family of invariant sets $\mathbf{M}_\infty = \{M_i\}_{i=1}^\infty \cup M_\infty$ in $\mathcal{A}$ such that M_j are isolated for $j \in \mathbb{N}$. If $\mathbf{M}_\infty$ is a Morse decomposition, then $\{T(t) : t \geq 0\}$ is gradient in the sense of the Definition 5.1 with respect to $\mathbf{M}_\infty$. In addition, the Lyapunov function $V : \mathcal{A} \to \mathbb{R}$ may be chosen in such a way that $V(x) = 1 - \frac{1}{2^{k-1}}$, for $x \in M_k$, $k \in \mathbb{N}$, $V(x) = 1$, for $x \in M_\infty$.*

Proof. Let $\emptyset = A_0 \subset A_1 \subset \cdots \subset A_n \subset \ldots \mathcal{A}$ be the sequence of local attractors given in Definition 3.10 and $\emptyset = A^*_\infty \subset \ldots A^*_n \subset \cdots \subset A^*_0 = \mathcal{A}$ their corresponding repellers such that for each $j \in \mathbb{N}$ we have $M_j = A_j \cap A^*_{j-1}$, and $M_\infty = \cap_{j=0}^\infty A^*_j$.

Let $f_j : X \to \mathbb{R}$ be the function from Proposition 5.2 for the attractor-repeller pair (A_j, A^*_j), $j \in \mathbb{N}$.

Define the function $V : \mathcal{A} \to \mathbb{R}$ by

$$V(z) := \sum_{j=1}^{\infty} \frac{1}{2^j} f_j(z), \ z \in \mathcal{A}.$$

Then $V : \mathcal{A} \to \mathbb{R}$ is a Lyapunov function for the generalized gradient semigroup $\{T(t) : t \geq 0\}$ with respect to $\mathbf{M}_\infty$.

Indeed, since each $f_j : \mathcal{A} \to \mathbb{R}$, $j \geq 1$, is non-increasing along solutions of $\{T(t) : t \geq 0\}$, V is also non-increasing along solutions of $\{T(t) : t \geq 0\}$.

Now, if $z \in \mathcal{A}$ is such that $V(T(t)z) = V(z)$ for all $t \geq 0$, then, using that each f_j, $j \geq 0$, are non-increasing along solutions of $\{T(t) : t \geq 0\}$, we conclude that $f_j(T(t)z) = f_j(z)$ for all $t \geq 0$ and for each $j \in \mathbb{N}$. From part (iv) of Proposition 5.2, we have that $z \in (A_j \cup A_j^*)$, for each $j \in \mathbb{N}$; that is, $z \in \bigcap_{j=0}^{\infty} (A_j \cup A_j^*)$. From Lemma 3.13 we have that

$$\bigcap_{j=0}^{\infty} (A_j \cup A_j^*) = (\bigcup_{j=1}^{\infty} M_j) \cup M_\infty,$$

and so $z \in \bigcup_{j=1}^{\infty} M_j \cup M_\infty$.

If $k \in \mathbb{N}$ and $z \in M_k = A_k \cap A_{k-1}^*$, it follows that $z \in A_k \subset A_{k+1} \subset \cdots \subset A_\infty = \mathcal{A}$ and $z \in A_{k-1}^* \subset A_{k-2}^* \subset \cdots \subset A_0^* = \mathcal{A}$. Hence $f_j(z) = 0$ if $k \leq j$ and $f_j(z) = 1$ if $1 \leqslant j \leqslant k-1$. Hence,

$$V(z) = \sum_{j=1}^{\infty} f_j(z) = \sum_{j=1}^{k-1} f_j(z) + \sum_{j=k}^{\infty} f_j(z) = \sum_{j=1}^{k-1} \frac{1}{2^j} = 1 - \frac{1}{2^{k-1}}.$$

If $z \in M_\infty$, then $z \in \cap_{j=1}^{\infty} A_j^*$. Hence, $f_j(z) = 1$, for all $j \geq 1$, and then

$$V(z) = \sum_{j=1}^{\infty} \frac{1}{2^j} = 1.$$

Finally, we prove the continuity of V. Since $f_j(z) \in [0,1]$, for any $\varepsilon > 0$ there exists $N(\varepsilon) > 0$ such that

$$\sum_{j \geq N} \frac{1}{2^j} f_j(z) \leq \sum_{j \geq N} \frac{1}{2^j} \leq \varepsilon \text{ for all } z \in \mathcal{A}.$$

Then, as each f_j is continuous, it is standard to prove the continuity of V.

□

6. **Dynamically gradient semigroups via a Lyapunov function.** We now prove that the existence of an ordered Lyapunov function with respect to a family $\mathbf{M}_\infty = \{M_i\}_{i=1}^{\infty} \cup M_\infty$ in $\mathcal{A}$ implies that the semigroup is generalized dynamically gradient and that (4.2) holds. Hence, together with the previous results we will obtain the equivalence of generalized dynamically gradient semigroups referred to $\mathbf{M}_\infty$ satisfying (4.2), the existence of an ordered Lyapunov function associated to $\mathbf{M}_\infty$ and the existence of a Morse decomposition of the global attractor.

As before, $\{T(t) : t \geq 0\}$ is a semigroup with the global attractor $\mathcal{A}$ and we consider a disjoint family of isolated sets $\mathbf{M}_\infty = \{M_i\}_{i=1}^{\infty} \cup M_\infty$ in $\mathcal{A}$ such that M_j are isolated for $j \in \mathbb{N}$.

Definition 6.1. *We say that $\mathbf{M}_\infty$ is ordered with respect to the generalized Lyapunov function V, or that V is an ordered Lyapunov function for $\mathbf{M}_\infty$, if*

$$L_1 \leq L_2 \leq \cdots \leq L_n \leq \cdots < L_\infty,$$

where $L_j = V(z)$ for $z \in M_j$. Moreover, there cannot be an infinite number of sets M_j with the same value of V.

Remark 6.2. *If $L_n \to L_\infty$, then the last condition in Definition 6.1 holds. Also, if (3.5) is satisfied, from the continuity of V it follows that $L_n \to L_\infty$.*

Proposition 6.3. *Let $\{T(t) : t \geq 0\}$ be a semigroup with global attractor $\mathcal{A}$ and a disjoint family of invariant sets $\mathbf{M}_\infty = \{M_i\}_{i=1}^\infty \cup M_\infty$ in $\mathcal{A}$ such that M_j are isolated for $j \in \mathbb{N}$. Let $\mathbf{M}_\infty$ be ordered with respect to the generalized Lyapunov function V. Then for any complete bounded trajectory $\xi : \mathbb{R} \to X$,*

i) *either there exists $i \in \mathbb{N}$ such that $\xi(t) \in M_i$, for all $t \in \mathbb{R}$,*
ii) *or there exist $M_j, M_r \in \mathbf{M}_\infty$ with $r > j$ such that*

$$\lim_{t\to-\infty} d(\xi(t), M_r) = 0, \ \lim_{t\to+\infty} d(\xi(t), M_j) = 0.$$

Proof. Suppose that i) is not true. The function $t \mapsto V(\xi(t))$ is monotone. Since $\xi(t) \in \mathcal{A}$, it is also bounded. Hence, the following limits exist

$$L_- = \lim_{t\to-\infty} V(\xi(t)), \ L_+ = \lim_{t\to+\infty} V(\xi(t)).$$

Thus,

$$\begin{aligned} V(y) &= L_- \text{ for all } y \in \alpha(\xi), \\ V(y) &= L_+ \text{ for all } y \in \omega(\xi), \end{aligned}$$

where $\alpha(\xi)$ is the alfa-limit set $\alpha(\xi) = \cap_{t\leq 0}\overline{\cup_{s\leq t}\xi(s)}$. It is well known that the sets $\omega(\xi)$, $\alpha(\xi)$ are invariant and connected (see e.g. [18]).

As $\omega(\xi)$ is invariant, for any $y \in \omega(\xi)$ and $t \geq 0$ we have that $T(t)y \in \omega(\xi)$, and then $V(y) = V(T(t)y) = L_+$. Thus, $y \in M_j$ for some $j \in \mathbb{N} \cup \infty$.

In fact, we shall prove that $\omega(\xi) \subset M_j$. By contradiction assume that there exists $z \in M_i \cap \omega(\xi)$, $i \neq j$. This is not possible if $j = \infty$, as in such a case we have that $L_i < L_+ = L_\infty$. Assume then that $j < \infty$. The number of sets M_i such that $L_i = L_+$ is finite. Denote by $\widehat{E}_1, \ ..., \ \widehat{E}_m \in \mathbf{M}_\infty$ the sets such that $V(x) = L_+$ if $x \in \widehat{E}_k$ for some $k \in \{1, ..., m\}$. We can find $\varepsilon > 0$ for which $\mathcal{O}_\varepsilon\left(\widehat{E}_k\right) \cap \mathcal{O}_\varepsilon\left(\widehat{E}_r\right)$ for all $r \neq k \in \{1, ..., m\}$. Since $\omega(\xi)$ is connected, there exists $u \in \omega(\xi)$ such that $u \notin \cup_{k=1}^m \mathcal{O}_\varepsilon\left(\widehat{E}_k\right)$. But we have proved that any $u \in \omega(\xi)$ belongs to M_k for some $k \in \mathbb{N} \cup \infty$, and then $V(u) = L_+$ implies that $u \in \cup_{k=1}^m \mathcal{O}_\varepsilon\left(\widehat{E}_k\right)$, a contradiction.

Therefore, $\lim_{t\to+\infty} d(\xi(t), M_j) = 0$. Similarly, we prove $\alpha(\xi) \subset M_r$ for some $r \in \mathbb{N} \cup \{\infty\}$. Hence, $\lim_{t\to-\infty} d(\xi(t), M_r) = 0$.

Since $L_- \geq L_+$, it is clear that $r \geq j$. As we are in the case where i) does not hold, the fact that if V is constant on a global solution $\xi(t)$ implies that it belongs to a fixed M_i prevents that $r = j$. □

Corollary 6.4. *Assume the conditions of Proposition 6.3. Then the sets M_j, $j \in \mathbb{N} \cup \infty$, are compact.*

Proof. By Proposition 6.3 condition (3.2) is satisfied. Then the result follows from Lemma 4.4. □

The existence of a Lyapunov function associated to an infinite number of invariant sets gives, as in the case of a finite number of invariants, a characterization of the global attractor as follows.

Proposition 6.5. *Let $\{T(t) : t \geq 0\}$ be a semigroup with global attractor $\mathcal{A}$ and a disjoint family of invariant sets $\mathbf{M}_\infty = \{M_i\}_{i=1}^\infty \cup M_\infty$ in $\mathcal{A}$ such that M_j are isolated for $j \in \mathbb{N}$. Let $\mathbf{M}_\infty$ be ordered with respect to the generalized Lyapunov function V. Then*

$$\mathcal{A} = \cup_{j=1}^\infty W^u(M_j) \cup W^u(M_\infty).$$

Proof. If $x \in \mathcal{A}$, then x belongs to a bounded complete trajectory, so that Proposition 6.3 implies $x \in W^u(M_j)$ for some $j \in \mathbb{N} \cup \infty$. Thus, $\mathcal{A} \subset \cup_{j=1}^\infty W^u(M_j) \cup W^u(M_\infty)$.

Conversely, let $x \in W^u(M_j)$ for some $j \in \mathbb{N} \cup \infty$. Since M_j is bounded, there exists t_0 such that $\cup_{t \leq t_0} \xi(t)$ is bounded, where $\xi(\cdot)$ is a complete trajectory satisfying $\lim_{t \to -\infty} d(\xi(t), M_j) = 0$. From the definition of a complete trajectory and the fact that $T(t)$ is eventually dissipative (see Remark 1.6) it follows that $\cup_{t \geq t_0} \xi(t)$ is also bounded. Thus, $\xi(\cdot)$ is a bounded complete trajectory. But then $\xi(t) \in \mathcal{A}$ for all $t \in \mathbb{R}$. In particular, $x = \xi(0) \in \mathcal{A}$. □

Note that Lemma 4.5 is also a consequence of the existence of a Lyapunov functional.

Proposition 6.6. *Let $\{T(t) : t \geq 0\}$ be a semigroup with the global attractor $\mathcal{A}$ and a disjoint family of invariant sets $\mathbf{M}_\infty = \{M_i\}_{i=1}^\infty \cup M_\infty$ in $\mathcal{A}$ such that M_j are isolated for $j \in \mathbb{N}$. Let $\mathbf{M}_\infty$ be ordered with respect to the generalized Lyapunov function V. Then (4.2) holds, that is, for any $j \in \mathbb{N}$ there exists δ_j such that*

$$W^u(M_j) \cap \mathcal{O}_{\delta_j}(\cup_{i \geq j+1} M_i \cup M_\infty) = \varnothing.$$

Proof. We note that by Corollary 6.4 the sets M_j are compact for $j \in \mathbb{N} \cup \infty$.

First, let k_j be the first integer $k_j > j$ such that $L_{k_j} > L_j$. We shall prove the existence of δ_j' for which $W^u(M_j) \cap \mathcal{O}_{\delta_j'}(\cup_{i \geq k_j} M_i \cup M_\infty) = \varnothing$.

By contradiction assume the existence of $j \in \mathbb{N}$ and a sequence $x_n \in W^u(M_j)$ such that

$$d(x_n, \cup_{i \geq k_j} M_i \cup M_\infty) < \frac{1}{n}.$$

Then, there exists $y_n \in \cup_{i \geq k_j} M_i \cup M_\infty$ such that $d(x_n, y_n) < \frac{1}{n}$. Since $V(y_n) \geq L_{k_j} > L_j$, by the continuity of V there exist n, $\varepsilon > 0$ such that

$$V(x_n) \geq L_j + \varepsilon.$$

But $x_n \in W^u(M_j)$ implies the existence of a bounded complete trajectory $\xi(t)$ such that $\xi(0) = x_n$ and

$$\lim_{t \to -\infty} d(\xi(t), M_j) = 0.$$

By the definition of V we have that $V(\xi(t)) \geq L_j + \varepsilon$ for $t \leq 0$. We take then sequences $t_m \to -\infty$, $z_m \in M_j$ for which

$$\lim_{t_m \to -\infty} d(\xi(t_m), z_m) = 0.$$

Since M_j is compact, we can assume that $z_m \to z_0$ and then

$$\lim_{t_m \to -\infty} d\left(\xi\left(t_m\right), z_0\right) = 0.$$

Again, by the continuity of V we have that $V\left(z_0\right) \geq L_j + \varepsilon$ and $V(z_0) = L_j$, which is a contradiction.

Further, considering a j for which $k_j - 1 > j$, we will check that there exists δ_j'' such that $W^u\left(M_j\right) \cap \mathcal{O}_{\delta_j''}(\cup_{i=j+1}^{k_j-1} M_i) = \varnothing$. If not, then arguing as before we obtain sequences $x_n \in W^u\left(M_j\right),\ y_n \in \cup_{i=j+1}^{k_j-1} M_i$ such that $d\left(x_n, y_n\right) < \frac{1}{n}$. We can assume passing to a subsequence that $y_n \in M_k$ for all n and some $k \in \{j+1, ..., k_j - 1\}$. We take a bounded complete trajectory $\xi_n\left(t\right)$ such that $\xi_n\left(0\right) = x_n$ and

$$\lim_{t \to -\infty} d\left(\xi_n\left(t\right), M_j\right) = 0.$$

We choose $\varepsilon > 0$ satisfying

$$\mathcal{O}_\varepsilon\left(M_r\right) \cap \mathcal{O}_\varepsilon\left(M_i\right) = \varnothing \text{ for all } r, i \in \{j, ..., k_j - 1\},$$

and take n for which $\frac{1}{n} < \varepsilon$. Then $x_n \in O_\varepsilon(M_k)$. Since $t \mapsto \xi_n\left(t\right)$ is continuous, it follows the existence of $t_n > 0$ such that

$$d(\xi_n\left(-t_n\right), M_k) = \varepsilon,$$
$$d(\xi_n\left(t\right), M_k) < \varepsilon \text{ for all } t \in (-t_n, 0].$$

We define the functions $\overline{\xi}_n\left(t\right) = \xi_n\left(t - t_n\right)$. Then $d(\overline{\xi}_n\left(0\right), M_k) = \varepsilon$ and $\overline{\xi}_n\left(t_n\right) = x_n$. There exists a complete trajectory $\overline{\xi}\left(\cdot\right)$ (see [7, Lemma 3.1]) such that up to a subsequence $\overline{\xi}_n\left(t\right) \to \overline{\xi}\left(t\right)$ for all $t \in \mathbb{R}$. We note that

$$V(\overline{\xi}_n(t)) \leq L_j \text{ for all } t \in \mathbb{R},$$

and then by the continuity of V,

$$V(\overline{\xi}(t)) \leq L_j \text{ for all } t \in \mathbb{R}.$$

We note that $t_n \to +\infty$. Otherwise, if $t_n \to t_0$, then as M_k is compact, we have $\overline{\xi}\left(t_0\right) = \lim_{n \to \infty}\ x_n = x \in M_k$, so that $V\left(\overline{\xi}\left(t_0\right)\right) = L_j$. Hence,

$$V(\overline{\xi}\left(t\right) \geq V(\overline{\xi}\left(t_0\right)) = L_j \text{ for all } t \leq t_0.$$

By the last two inequalities, we have that $V(\overline{\xi}\left(t\right)) = L_j$ for all $0 \leq t \leq t_0$. Also, $\overline{\xi}(t) = T(t - t_0)\overline{\xi}(t_0)$ if $t \geq t_0$, so that $V(\overline{\xi}\left(t\right)) = L_j$ for all $t \geq t_0$ as well. From the definition of the Lyapunov function, we obtain that $\overline{\xi}\left(0\right) \in M_k$. But $d(\overline{\xi}_n\left(0\right), M_k) = \varepsilon$ and $\overline{\xi}_n\left(0\right) \to \overline{\xi}\left(0\right)$ imply that $d(\overline{\xi}\left(0\right), M_k) = \varepsilon$, a contradiction.

Hence, it is clear that

$$d(\overline{\xi}\left(t\right), M_k) \leq \varepsilon \text{ for any } t \geq 0.$$

On the other hand,

$$V\left(x_n\right) = V\left(\overline{\xi}_n\left(t_n\right)\right) \leq V\left(\overline{\xi}_n\left(t\right)\right) \leq L_j \text{ for any } 0 \leq t \leq t_n.$$

Since $x_n \to x \in M_k$, the continuity of V implies that $V\left(x_n\right) \to V\left(x\right) = L_j$ and $V\left(\overline{\xi}_n\left(t\right)\right) \to V\left(\overline{\xi}\left(t\right)\right)$. Thus,

$$V\left(\overline{\xi}\left(t\right)\right) = L_j \text{ for all } t \geq 0.$$

From the definition of a Lyapunov function, we have that $\overline{\xi}\left(t\right) \in M_k$ for any $t \geq 0$. This is a contradiction, as $d(\overline{\xi}_n\left(0\right), M_k) = \varepsilon$ and $\overline{\xi}_n\left(0\right) \to \overline{\xi}\left(0\right)$.

Taking $\mathcal{O}_{\delta_j} = \mathcal{O}_{\delta_j'} \cup \mathcal{O}_{\delta_j''}$ we obtain the required result. □

We can now conclude our main theorem.

Theorem 6.7. *Let $\{T(t) : t \geq 0\}$ be a semigroup with global attractor $\mathcal{A}$ and consider a disjoint family of invariant sets $\mathbf{M}_\infty = \{M_i\}_{i=1}^{\infty} \cup M_\infty$ in $\mathcal{A}$ such that M_j are isolated for $j \in \mathbb{N}$. Then, the following conditions are equivalent:*

1. *$\{T(t) : t \geq 0\}$ is a generalized gradient semigroup with respect to $\mathbf{M}_\infty$ in the sense of the Definition 5.1 and $\mathbf{M}_\infty$ is ordered with respect to the respective Lyapunov function.*
2. *$\{T(t) : t \geq 0\}$ is a generalized dynamically gradient semigroup with respect to $\mathbf{M}_\infty$ (as in Definition 3.4) satisfying (4.2).*
3. *$\mathbf{M}_\infty$ is a Morse decomposition of $\mathcal{A}$.*

Proof. It is a straightforward consequence of Theorem 4.8 and Propositions 5.3, 6.3 and 6.6. □

Corollary 6.8. *Let $\{T(t) : t \geq 0\}$ be a semigroup with global attractor $\mathcal{A}$ and consider a disjoint family of invariant sets $\mathbf{M}_\infty = \{M_i\}_{i=1}^{\infty} \cup M_\infty$ in $\mathcal{A}$ such that M_j are isolated for $j \in \mathbb{N}$. Assume that $\mathbf{M}_\infty$ is a Morse decomposition of $\mathcal{A}$. Then*

$$\mathcal{A} = \cup_{j=1}^{\infty} W^u(M_j) \cup W^u(M_\infty).$$

Proof. In view of Theorem 6.7, $\{T(t) : t \geq 0\}$ is a generalized gradient semigroup with respect to $\mathbf{M}_\infty$ in the sense of the Definition 5.1 and $\mathbf{M}_\infty$ is ordered with respect to the Lyapunov function. Hence, the result follows from Proposition 6.5. □

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