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Attractors for non-autonomous retarded lattice dynamical systems

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Abstract: In this paper we study a non-autonomous lattice dynamical system with delay. Under rather general growth and dissipative conditions on the nonlinear term, we define a non-autonomous dynamical system and prove the existence of a pullback attractor for such system as well. Both multivalued and single-valued cases are considered.

Keywords: lattice dynamical systems, non-autonomous systems, differential equations with delay, set-valued dynamical systems, pullback attractor

MSC: 34K05, 34K31, 35B40, 35B41, 35K55, 35K40, 37L30, 58C06

1 Introduction

Lattice dynamical systems often arise as an approximative system of infinite differential equations of a partial differential equation in an unbounded domain, although they also appear as models of a variety of phenomena such as image processing, pattern recognition, brain science, among others.

In the last years many authors have been interested in the asymptotic behaviour of solutions of such systems. As a result, a sheer number of papers have been published concerning the existence and properties of global attractors in the autonomous, nonautonomous and stochastic cases; with or without uniqueness; in weighted or unweighted spaces. Usually, the models under consideration are obtained by a spatial discretization of a parabolic or a hyperbolic equation (see e.g. [1], [2], [4], [5], [8], [11] [12], [15], [16], [19], [20], [22], [23], [26], [28], [29]).

The addition of a delay in the system, which appears naturally in real models, gives rise to new difficulties. Retarded autonomous lattice dynamical systems were studied from the point of view of dynamical systems in [25], [27], [24]. These results were improved later on by Caraballo et. al. [13].

Our main aim in this paper is to analyze the asymptotic behavior of the following nonautonomous retarded lattice differential equation

$$\begin{cases} \dfrac{du_i}{dt} - (u_{i-1} - 2u_i + u_{i+1}) + \lambda u_i + f_i(t, u_{it}) = 0, \ t > \tau, \ i \in \mathbb{Z}, \\ \\ u_i(s) = \psi_i(s), \ \forall s \in [\tau - h, \tau], \end{cases} \tag{1}$$

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where $\lambda \in \mathbb{R}$. This model is obtained after a spatial discretization of the scalar retarded reaction-diffusion equation:

$$\begin{cases} \dfrac{du}{dt} - \dfrac{\partial^2 u}{\partial x^2} + \lambda u + f(t, u_t) = 0, \ t > \tau, x \in \mathbb{R}, \\ \\ u(s) = \psi(s), \forall s \in [\tau - h, \tau]. \end{cases}$$

Here $u = (u_i)_{i \in \mathbb{Z}} \in \ell^2$, $\mathbb{Z}$ denotes the integers set and for a continuous function $u : [\tau - h, T] \to Y$ (where Y is some space), u_t denotes the segment of the solution, i.e., the element in $C\left([-h, 0], Y\right)$ defined by $u_t(s) = u(t+s)$, $s \in [-h, 0]$.

The existence and uniqueness of solutions for problem (1) were addressed in [13]. It is worth pointing out that rather general assumptions on the nonlinear functions f_i (just continuity and growth conditions) are imposed, not ensuring any kind of compactness properties in the space ℓ^2 for the corresponding Nemytskii operator, which are necessary in order to apply the solvability results stated in other papers (see [14], [17], [21]). Also, in the autonomous case, when f does not depend explicitely on t, the existence of global attractors was established in both the multivalued and single-valued settings for a particular type of functions f_i.

In the present paper we extend the results carried out in [13] to the nonautonomous case. For this aim we apply the well-known theory of pullback attractors [7], [9].

The paper is organized in two parts. In Section 2 we recall briefly the general solvability theorems proved in [13] and apply them to problem (1) under rather general assumptions on the nonlinear term f. In Section 3 we consider the particular case of a lattice dynamical system with a nonlinear term of the form

$$f_i(t, u_{it}) = F_{0,i}(u_i(t)) + F_{1,i}\left(u_i\left(t - \rho(t)\right)\right) + \int_{-h}^{0} b_i(t, s, u_i(t+s))\, ds,$$

with $\rho(\cdot) \in C^1(\mathbb{R})$ and $\rho(t) \in [0, h]$ for all $t \in \mathbb{R}$. Under some dissipative and sublinear growth conditions on the maps $F_{0,i}$, $F_{1,i}$, b_i, we define for this problem a multivalued process U and prove the existence of a pullback attractor. Additionally, with extra Lipschitz conditions we obtain uniqueness of the Cauchy problem, so that U is in fact a single-valued process.

2 Existence of solutions of a lattice differential equation with delay

2.1 Some results on the existence of solutions of differential equations with delay in Banach spaces

Let us first recall some abstract results which were proved in [13] and which will be useful in the present case.

Let E be a real Banach space with dual $E^{\star}$, and let $E_0 = C([-h, 0], E)$, with norms $\|\cdot\|$, $\|\cdot\|_{\star}$ and $\|\cdot\|_{E_0}$, respectively, where $\|\varphi\|_{E_0} = \max_{t \in [-h,0]} \|\varphi(t)\|$. Also,

$$B_X(y_0, r) = \{y \in X : \|y - y_0\|_X \le r\},$$

where $X = E$ or E_0, and $(\cdot, \cdot)$ will denote the pairing between E and $E^{\star}$.

Let us consider the following Cauchy problem for a functional differential equation in a Banach space:

$$\begin{cases} \dfrac{du}{dt} = F(t, u_t), \\ \\ u_\tau = \psi \in E_0, \end{cases} \tag{2}$$

where $F : \mathbb{R} \times E_0 \to E$. Also, for any $u \in C([\tau - h, +\infty), E)$, the function $u_t \in E_0$, $t \ge \tau$, is defined by $u_t(s) = u(t+s)$, $s \in [-h, 0]$.

Let E_w be the space E endowed with the weak topology. We consider the space $E_{0,w} = C([-h, 0], E_w)$. Let $u_n, u \in E_{0,w}$. We say that $u_n \to u$ in $E_{0,w}$ if

$$u_n(s_n) \to u(s) \text{ in } E_w \text{ for all } s_n \to s \in [-h, 0].$$

We will say that the function F is sequentially weakly continuous in bounded sets if $t_n \to t$, $u_n \to u$ in $E_{0,w}$ and $\|u_n\|_{E_0} \le M$, for all n, imply $F(t_n, u_n) \to F(t, u)$ in E_w.

On the other hand, we will say that the function F is bounded if it maps bounded subsets of $\mathbb{R} \times E_0$ onto bounded subsets of E.

Definition 1. *The map $u : [\tau - h, T] \to E$ is called a solution of problem (2) if $u_\tau = \psi$, $u(\cdot)$ is continuous, once weakly continuously differentiable in $[\tau, T]$ and satisfies*

$$u(t) = u(\tau) + \int_\tau^t f(s, u_s)\, ds, \quad \text{for all } t \in [\tau, T].$$

Remark 2. *It follows from this definition that for any solution u of (2), the map $t \mapsto u_t \in E_0$ is continuous.*

Remark 3. *We note that if $F : \mathbb{R} \times E_0 \to E$ is sequentially weakly continuous in bounded sets and the map $t \mapsto u_t \in E_0$ is continuous, then $t \mapsto F(t, u_t)$ is weakly continuous, hence weakly measurable. If E is separable, we obtain that $t \mapsto F(t, u_t)$ is strongly measurable. If we assume, moreover, that the map F is bounded, then we have that $F(\cdot, u_\cdot) \in L^1(\tau, T; E)$.*

If $F : \mathbb{R} \times E_0 \to E$ and $t \mapsto u_t \in E_0$ are continuous, then the map $t \mapsto F(t, u_t)$ is continuous, hence strongly measurable. If we assume, moreover, that the map F is bounded, then we have that $F(\cdot, u_\cdot) \in L^1(\tau, T; E)$.

Then, we recall now some results ensuring the existence and uniqueness of solutions for problem (2), which were proved in [13].

Theorem 4. *Assume that E is reflexive and separable. Let $f : \mathbb{R} \times E_0 \to E$ be sequentially weakly continuous in bounded sets, and let F be a bounded map. Then, for each $r > 0$, there exists $a(r) > 0$ such that if $\psi \in E_0$ and $\|\psi\|_{E_0} \le r$, problem (2) possesses at least one solution defined on $[0, a(r)]$. Moreover, $u(\cdot)$ is a.e. differentiable and $\frac{du}{dt} = f(t, u_t)$ for a.a. $t \in (0, a(r))$.*

If we assume additionally that $f : \mathbb{R} \times E_0 \to E$ is continuous, then $u \in C^1([0, a]; E)$ and the separability of E is not needed.

Theorem 5. *Assume the conditions of Theorem 4. If a solution $u(\cdot)$ of (2) has a maximal interval of existence $[0, b)$ and there exists $K > 0$ such that $\|u(t)\| \le K$, for all $t \in [0, b)$, then $b = +\infty$, that is, $u(\cdot)$ is a globally defined solution.*

Let $J : E \to 2^{E^*}$ be the duality map, i.e. $J(y) = \{\xi \in E^* : (y, \xi) = \|y\|^2 = \|\xi\|_*^2\}$, $\forall y \in E$. We state a result concerning uniqueness of solutions.

Theorem 6. *Assume the hypotheses of Theorem 4. Also, suppose that, for any $M > 0$, there exists $\beta(\cdot, M) \in L^1_{loc}(\mathbb{R})$ such that $\beta(t, M) \ge 0$ for a.a. $t \in \mathbb{R}$ and the following inequality holds:*

$$(f(t, v) - f(t, w), j) \le \beta(t, M)\, \|v - w\|_{E_0}^2, \tag{3}$$

for all $j \in J(v(0) - w(0))$, all $v, w \in E_0$ with $\|v\|_{E_0}, \|u\|_{E_0} \le M$, and a.a. $t \in \mathbb{R}$. Then, for each $r > 0$, there exists $a(r) > 0$ such that if $\psi \in E_0$ and $\|\psi\|_{E_0} \le r$, problem (2) has a unique solution defined on $[0, a(r)]$.

2.2 Lattice dynamical systems with delay: setting of the problem

For a given $\tau \in \mathbb{R}$, consider the following first order lattice dynamical system with finite delay

$$\begin{cases} \dfrac{du_i}{dt} - (u_{i-1} - 2u_i + u_{i+1}) + \lambda u_i + f_i(t, u_{it}) = 0, \ t > \tau, \ i \in \mathbb{Z}, \\ \\ u_i(s) = \psi_i(s - \tau), \ \forall s \in [\tau - h, \tau], \end{cases} \tag{4}$$

where $\lambda \in \mathbb{R}$.

We consider the separable Hilbert space $\ell^2 = \{v = (v_i)_{i\in\mathbb{Z}} : \sum_{i\in\mathbb{Z}} v_i^2 < \infty\}$ with norm $\|v\| = \sqrt{\sum_{i\in\mathbb{Z}} v_i^2}$ and scalar product $(w, v) = \sum_{i\in\mathbb{Z}} w_i v_i$, and also the Banach space $\ell^\infty = \{v = (v_i)_{i\in\mathbb{Z}} : \sup_{i\in\mathbb{Z}} |v_i| < \infty\}$ with norm $\|v\|_\infty = \sup_{i\in\mathbb{Z}} |v_i|$.

Further, we shall use the notation $E = \ell^2$, $E_0 = C([-h, 0], \ell^2)$, $E_1 = C([-h, 0], \mathbb{R})$, with the norms $\|u\|_{E_0} = \max_{s\in[-h,0]} \|u(s)\|$, $\|u\|_{E_1} = \max_{s\in[-h,0]} |u(s)|$. Also, put $E_\infty = C([-h, 0], \ell^\infty)$ with norm $\|u\|_{E_\infty} = \max_{s\in[-h,0]} \|u(s)\|_\infty$. We note that $E_0 \subset E_\infty$, as

$$\|u(t) - u(s)\|_\infty = \sup_{i\in\mathbb{Z}} |u_i(t) - u_i(s)| \leq \sqrt{\sum_{i\in\mathbb{Z}} |u_i(t) - u_i(s)|^2} = \|u(t) - u(s)\|, \forall t, s \in [-h, 0],$$

and

$$\|u\|_{E_\infty} = \max_{s\in[-h,0]} \sup_{i\in\mathbb{Z}} |u_i| \leq \max_{s\in[-h,0]} \sqrt{\sum_{i\in\mathbb{Z}} |u_i|^2} = \|u\|_{E_0}.$$

We consider the following conditions:

$(H1)$ The operator $f : \mathbb{R} \times E_0 \to E$ given by the rule $(f(t, v))_i = f_i(t, v_i)$, $i \in \mathbb{Z}$, is well defined and bounded.
$(H2)$ The maps $f_i : \mathbb{R} \times C([-h, 0], \mathbb{R}) \to \mathbb{R}$ are continuous.

We shall first prove the existence of solutions for problem (4). For this aim we shall rewrite it in an abstract form. We define the operator $A : E \to E$ by

$$(Av)_i := -v_{i-1} + 2v_i - v_{i+1}, \ i \in \mathbb{Z}.$$

Also, we define the operators $B, B^\star : E \to E$ by

$$(Bv)_i := v_{i+1} - v_i, \ \left(B^\star v\right)_i := v_{i-1} - v_i.$$

It is easy to check that

$$A = B^\star B = BB^\star,$$
$$\left(B^\star w, v\right) = (w, Bv).$$

Then the operator $F : \mathbb{R} \times E_0 \to E$ is defined by

$$F(t, v) = -Av(0) - f(t, v) - \lambda v(0)$$

and (4) can be rewritten as

$$\begin{cases} \dfrac{du}{dt} = F(t, u_t), \ t > \tau, \\ \\ u_\tau = \psi, \ \text{i.e. } u(s) = \psi(s - \tau), \ \forall s \in [\tau - h, \tau]. \end{cases} \tag{5}$$

Lemma 7. *Let (H1)-(H2) hold. Then the map $f : \mathbb{R} \times E_0 \to E$ is sequentially weakly continuous in bounded sets. Also, the map $A : E \to E$ is weakly continuous.*

Proof. Let $t_n \to t$ in $\mathbb{R}$, and $v^n \to v \in E_{0,w}$, with $\|v^n\|_{E_0} \leq M_1$ for all n, and let $w \in \ell^2$ be arbitrary. For any $\varepsilon > 0$ we take $K_0(\varepsilon) > 0$ such that $\sum_{|i|\geq K_0} |w_i|^2 < \varepsilon$. Since f is bounded, there exists $M_2 > 0$ such that

$\|f(t_n, v^n)\| \le M_2$, $\|f(t,v)\| \le M_2$, for all n. Also, as $t_n \to t$ and $v_i^n \to v_i$ in $C([-h,0],\mathbb{R})$, for all i, (H2) implies the existence of $N(K_0, \varepsilon)$ such that $\sum_{|i|<K_0} \left|f_i(t_n, v_i^n) - f_i(t, v_i)\right|^2 < \varepsilon^2$ if $n \ge N$. Hence,

$$\begin{aligned} \left|\left(f(t_n, v^n) - f(t,v), w\right)\right| &\le \sqrt{\sum_{|i|<K_0} \left|f_i(t_n, v_i^n) - f_i(t, v_i)\right|^2}\, \|w\| + \left(\|f(t,v)\| + \left\|f(t_n, v^n)\right\|\right) \sqrt{\sum_{|i|\ge K_0} |w_i|^2} \\ &\le \varepsilon \|w\| + 2M_2\varepsilon. \end{aligned}$$

The result for the operator A can be proved similarly. This completes the proof. □

Theorem 8. *Let (H1)-(H2) hold. For each $r > 0$ there exists $a(r) > 0$ such that if $\psi \in E_0$ and $\|\psi\|_{E_0} \le r$, then problem (4) has at least one solution defined on $[\tau, \tau + a(r)]$. Moreover, $u(\cdot)$ is a.e. differentiable and $\frac{du}{dt} = F(t, u_t)$ for a.a. $t \in (\tau, \tau + a(r))$.*

Proof. Lemma 7 implies that the operator F is sequentially weakly continuous in bounded sets. Since f is bounded, F is also bounded. The result follows from Theorem 4. □

In order to obtain that the map f is continuous, we need an assumption which is stronger than (H1).

($H3$) The operator $f : \mathbb{R}\times E_0 \to E$ given by $(f(t,v))_i = f_i(t, v_i)$, $i \in \mathbb{Z}$, is well defined, and for any $(t,v) \in \mathbb{R}\times E_0$, we have

$$\sum_{|i|\ge K} |f_i(t, v_i)|^2 \le C\left(\|v\|_{E_0}\right)\left(\max_{s\in[-h,0]} \sum_{|i|\ge K} v_i^2(s) + b_K(t)\right), \text{ for all } K \in \mathbb{Z}^+,$$

where $b_K(t) \to 0^+$ as $K \to \infty$ uniformly in compact sets, and $C(\cdot) \ge 0$ is a continuous non-decreasing function.

Remark 9. *Condition (H3) implies that the map f is bounded.*

Lemma 10. *Let (H2)-(H3) hold. Then, the map $f : \mathbb{R} \times E_0 \to E$ is continuous.*

Proof. Let $t_n \to t$ in $\mathbb{R}$, and $v^n \to v$ in E_0. Then for any $\varepsilon > 0$ there exists $K(\varepsilon)$ such that

$$\max_{s\in[-h,0]} \sum_{|i|\ge K} \left|v_i^n(s)\right|^2 < \varepsilon, \quad \max_{s\in[-h,0]} \sum_{|i|\ge K} |v_i(s)|^2 < \varepsilon.$$

Then by (H3) one can choose $K_1(\varepsilon) \ge K(\varepsilon)$ such that

$$\sum_{|i|\ge K_1} \left|f_i(t_n, v_i^n)\right|^2 \le R\varepsilon, \quad \sum_{|i|\ge K_1} |f_i(t, v_i)|^2 \le R\varepsilon,$$

for some $R > 0$. On the other hand, by (H2) we obtain the existence of $N(\varepsilon, K)$ such that

$$\sum_{|i|<K_1} \left|f_i(t_n, v_i^n) - f_i(t, v_i)\right|^2 < \varepsilon \text{ if } n \ge N.$$

Thus,

$$\begin{aligned} \sum_{i\in\mathbb{Z}} \left|f_i(t_n, v_i^n) - f_i(t, v_i)\right|^2 &\le \sum_{|i|<K_1} \left|f_i(t_n, v_i^n) - f_i(t, v_i)\right|^2 + 2\sum_{|i|\ge K_1} \left|f_i(t_n, v_i^n)\right|^2 + 2\sum_{|i|\ge K_1} |f_i(t, v_i)|^2 \\ &\le \varepsilon + 2R\varepsilon, \text{ if } n \ge N. \end{aligned}$$

□

Corollary 11. *Under conditions (H2)-(H3), the solution given in Theorem 8 belongs to the space $C^1([\tau, \tau + a]; E)$.*

In order to obtain the uniqueness of solutions we need an additional Lipschitz assumption.

$(H4)$ For any $M > 0$ there exists $\beta(t, M) \geq 0$ such that $\beta(\cdot, M) \in L^1(\mathbb{R})$ and

$$\left(f(t,z) - f(t,v), z(0) - v(0)\right) \geq -\beta(t,M) \|z - v\|_{E_0}^2,$$

if $\|z\|_{E_0}, \|v\|_{E_0} \leq M$, $t \in \mathbb{R}$.

Theorem 12. *Assume (H1)-(H2) and (H4). Then the solution given in Theorem 8 is unique.*

Proof. Let $z, v \in E_0$, $\|z\|_{E_0}, \|v\|_{E_0} \leq M$, and $w = z - v$. It follows from (H4) and $\left(Aw(0), w(0)\right) = \left(Bw(0), Bw(0)\right) \geq 0$ that

$$\begin{aligned}\left(F(t,z) - F(t,v), z(0) - v(0)\right) &= -\left(Aw(0), w(0)\right) - \lambda \|w\|_{E_0} - \left(f(t,z) - f(t,v), w(0)\right) \\ &\leq \beta(t,M) \|w\|_{E_0}.\end{aligned}$$

Then the result follows from Theorem 6. □

We now aim to study the asymptotic behaviour of solutions for problem (4). In particular, we will show the existence of a non-autonomous attractor. When conditions (H1)-(H2), (H4) hold, if we assume that every solution is global (this is true if we obtain an estimate of the solutions by Theorem 5), then we can define the map $U : \mathbb{R}^d \times E_0 \to E_0$, $\mathbb{R}_d^2 = \{(t, \tau) \in \mathbb{R}^2 : t \geq \tau\}$ by

$$U(t, \tau, \psi) = u_t,$$

where $u(\cdot)$ is the unique solution to (4) with $u_\tau = \psi$. Moreover, it is easy to prove, using (3) and Gronwall's lemma, that the map $\psi \mapsto U(t, \tau, \psi)$ is continuous for any $\tau \leq t$. The map U is a process, that is, $U(\tau, \tau, \psi) = \psi$ and

$$U(t, \tau, \psi) = U(t, r, U(r, \tau, \psi)) \quad \text{for all } \tau \leq r \leq t \text{ and } \psi \in E_0. \tag{6}$$

On the other hand, if we assume only (H1)-(H2) and that every solution is global, then we can define a multivalued semiflow by $U : \mathbb{R}_d^2 \times E_0 \to \mathcal{P}(E_0)$ ($\mathcal{P}(E_0)$ is the set of all non-empty subsets of E_0) by

$$U(t, \tau, \psi) = \{u_t : u(\cdot) \text{ is a solution of (4) with } u_\tau = \psi\}. \tag{7}$$

Since we do not have uniqueness of the Cauchy problem, this map is in general multivalued. In a similar way to the autonomous case [19, Lemma 13] one can prove that it is a multivalued process, that is:

1. $U(\tau, \tau, \cdot) = Id$ (the identity map);
2. $U(t, \tau, \psi) \subset U(t, r, U(r, \tau, \psi))$ for all $\psi \in E_0$, $\tau \leq r \leq t$.

Moreover, it is strict, that is, $U(t, \tau, \psi) = U(t, r, U(r, \tau, \psi))$ for all $\psi \in E_0$, $\tau \leq r \leq t$.

Now, we will recall the main results from the theory of pullback attractors. First, let us consider the case of a single-valued process [9], [10] (see also [18]).

Let X be a complete metric space. Suppose that $\mathcal{D}$ is a nonempty class of parameterized sets $\widehat{D} = \{D(t);\ t \in \mathbb{R}\} \subset \mathcal{P}(X)$, where $\mathcal{P}(X)$ denotes the family of all nonempty subsets of X.

Definition 13. *The process U is said to be pullback $\mathcal{D}$-asymptotically compact if for any $t \in \mathbb{R}$, any $\widehat{D} \in \mathcal{D}$, any sequence $\tau_n \to -\infty$, and any sequence $y_n \in U(t, \tau_n, D(\tau_n))\}$ is relatively compact in X.*

Definition 14. *It is said that $\widehat{B} \in \mathcal{D}$ is pullback $\mathcal{D}$-absorbing for the process U if for any $t \in \mathbb{R}$ and any $\widehat{D} \in \mathcal{D}$, there exists a $\tau_0(t, \widehat{D}) \leq t$ such that*

$$U(t, \tau, D(\tau)) \subset B(t) \quad \text{for all } \tau \leq \tau_0(t, \widehat{D}).$$

Definition 15. *The family $\widehat{A} = \{A(t);\ t \in \mathbb{R}\} \subset \mathcal{P}(X)$ is said to be a pullback $\mathcal{D}$-attractor for $U(\cdot, \cdot)$ if:*

1. *$A(t)$ is compact for all $t \in \mathbb{R}$,*
2. *$\widehat{A}$ is pullback $\mathcal{D}$-attracting, i.e.,*

$$\lim_{\tau \to -\infty} \operatorname{dist}(U(t,\tau,D(\tau)), A(t)) = 0,$$

   for all $\widehat{D} \in \mathcal{D}$, and all $t \in \mathbb{R}$,
3. *$\widehat{A}$ is invariant, i.e.,*

$$U(t,\tau,A(\tau)) = A(t), \quad \text{for } -\infty < \tau \le t < +\infty.$$

We have the following result.

Theorem 16. *Suppose that the map $\psi \mapsto U(t,\tau,\psi)$ is continuous for any $\tau \le t$ and that the process U is pullback $\mathcal{D}$-asymptotically compact. Let $\widehat{B} \in \mathcal{D}$ be a family of pullback $\mathcal{D}$-absorbing sets for $U(\cdot,\cdot)$. Then, the family $\widehat{A} = \{A(t);\ t \in \mathbb{R}\} \subset \mathcal{P}(X)$ defined by $A(t) = \Lambda(\widehat{B}, t)$, $t \in \mathbb{R}$, where*

$$\Lambda(\widehat{D}, t) = \bigcap_{s \le t} \left(\overline{\bigcup_{\tau \le s} U(t,\tau,D(\tau))} \right), \text{ for each } \widehat{D} \in \mathcal{D},$$

is a pullback $\mathcal{D}$-attractor for $U(\cdot,\cdot)$ which satisfies in addition that

$$A(t) = \overline{\bigcup_{\widehat{D} \in \mathcal{D}} \Lambda(\widehat{D}, t)}, \text{ for } t \in \mathbb{R}.$$

Futhemore, $\widehat{A}$ is minimal in the sense that if $\widehat{C} = \{C(t);\ t \in \mathbb{R}\} \subset \mathcal{P}(X)$ is a family of closed sets such that $\lim_{\tau \to -\infty} \operatorname{dist}(U(t,\tau,B(\tau)), C(t)) = 0$, *then $A(t) \subset C(t)$.*

The family $\mathcal{D}$ is said to be inclusion-closed if $\widehat{D} \in \mathcal{D}$ and $\varnothing \neq B(t) \subset D(t)$, for all $t \in \mathbb{R}$, implies $\widehat{B} \in \mathcal{D}$. If the family is inclusion-closed and the absorbing set $\widehat{B} \in \mathcal{D}$ satisfies that the sets $B(t)$ are closed, then $A(t) \subset B(t)$ implies that the attractor $\widehat{A}$ belongs to $\mathcal{D}$.

Let us consider now the case of a multivalued process. The following result is proved in [7] (see also [6] for a more general non-autonomous and random framework).

The definitions of pullback $\mathcal{D}$-asymptotically compactness, pullback $\mathcal{D}$-absorbing family and pullback $\mathcal{D}$-attraction are the same as in the single-valued case. For fixed $\tau \le t$ the mapping $U(t,\tau,\cdot)$ is said to be upper-semicontinuous if for any $x_0 \in X$ and for every neighborhood $\mathcal{N}$ in X of the set $U(t,\tau,x_0)$, there exists $\delta > 0$ such that $U(t,\tau,y) \subset \mathcal{N}$ whenever $d_X(x_0, y) < \delta$.

Definition 17. *A family $\widehat{A} = \{A(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ is said to be a global pullback $\mathcal{D}$-attractor for the MNDS U if $A(t)$ is compact for any $t \in \mathbb{R}$, $\widehat{A}$ is pullback $\mathcal{D}$-attracting, and $\widehat{A}$ is negatively invariant, i.e.,*

$$A(t) \subset U(t,\tau,A(\tau)), \text{ for any } (t,\tau) \in \mathbb{R}^2_d.$$

$\widehat{A}$ is said to be a strict global pullback $\mathcal{D}$-attractor if the invariance property in the third item is strict, i.e.,

$$A(t) = U(t,\tau,A(\tau)), \text{ for } (t,\tau) \in \mathbb{R}^2_d.$$

Theorem 18. *Assume that the map $\psi \mapsto U(t,\tau,\psi)$ is upper-semicontinuous and possesses closed values. Let $\widehat{B} = \{B(t) : t \in \mathbb{R}\} \in \mathcal{D}$ be pullback $\mathcal{D}$-absorbing and such that U is asymptotically compact with respect to $\widehat{B}$.*

Then, the set $\widehat{A}$ given by

$$A(t) := \Lambda\left(\widehat{B}, t\right) = \bigcap_{s \le t} \overline{\bigcup_{\tau \le s} U(t,\tau,B(\tau))} \quad t \in \mathbb{R}, \tag{8}$$

is a pullback $\mathcal{D}$-attractor for the MNDS U.

Moreover, suppose that $\mathcal{D}$ is inclusion-closed and that $B(t)$ is closed in X for any $t \in \mathbb{R}$. Then the family $\widehat{A}$ defined by (8) *belongs to $\mathcal{D}$, and is the unique pullback $\mathcal{D}$-attractor with this property. In addition, in this case, if U is a strict MNDS, then $\widehat{A}$ is strictly invariant.*

3 A lattice system with sublinear non-autonomous retarded terms

We shall consider a function $f : \mathbb{R} \times E_0 \to E$ given by the rule $(f(t,v))_i = f_i(t, v_i)$ and

$$f_i(t,v_i) = F_{0,i}(t, v_i(0)) + F_{1,i}\left(t, v_i\left(-\rho(t)\right)\right) + \int_{-h}^{0} b_i(t,s,v_i(s))\, ds,$$

where $\rho(\cdot) \in C^1(\mathbb{R})$ and $\rho(t) \in [0,h]$ for all $t \in \mathbb{R}$, that is, putting $v = u_t = u(t+\cdot)$, problem (4) can be rewritten as

$$\begin{cases} \dfrac{du_i}{dt} - (u_{i-1} - 2u_i + u_{i+1}) + \lambda u_i + F_{0,i}(t, u_i(t)) + F_{1,i}\left(t, u_i\left(t-\rho(t)\right)\right) \\ \qquad + \int_{-h}^{0} b_i(t,s,u_i(t+s))\, ds = 0, \ t > \tau, \ i \in \mathbb{Z}, \\ \qquad u_i(s) = \psi_i(s-\tau), \ \forall s \in [\tau - h, \tau]. \end{cases} \tag{9}$$

We consider the following conditions:

(C1) $\lambda > 0$.

(C2) $F_{0,i} : \mathbb{R}^2 \to \mathbb{R}$ are continuous and satisfy that $F_{0,i}(x)\, x \geq -C_{0,i}(t)$, $C_0 \in C(\mathbb{R}; \ell^1)$ and

$$\int_{-\infty}^{t} \|C_0(s)\|_{\ell^1} e^{\delta s} ds < \infty, \text{ for all } t \in \mathbb{R} \text{ and } \delta > 0.$$

(C3) $\left|F_{0,i}(t,x)\right| \leq H(|x|)\,|x| + C_{1,i}(t)$, for all $x \in \mathbb{R}$, where $C_1 \in C(\mathbb{R}; \ell^2)$, and $H(\cdot) \geq 0$ is a continuous and non-decreasing function.

(C4) $F_{1,i} : \mathbb{R}^2 \to \mathbb{R}$ are continuous and satisfy that $\left|F_{1,i}(t,x)\right| \leq K_1 |x| + C_{2,i}(t)$, for all $x \in \mathbb{R}$, where $C_2 \in C(\mathbb{R}; \ell^2)$, $K_1 > 0$ and

$$\int_{-\infty}^{t} \|C_2(s)\|_{\ell^2}^2 e^{\delta s} ds < \infty, \text{ for all } t \in \mathbb{R} \text{ and } \delta > 0.$$

(C5) $|b_i(t,s,x)| \leq m_{0,i}(t,s) + m_{1,i}(s)\,|x|$, for all $x \in \mathbb{R}$ and a.a. $s \in (-h, 0)$, where b_i are Caratheodory in the sense that it is measurable in s and continuous in (t,x).
Also, $m_{0,i}(t,\cdot)$, $m_{1,i}(\cdot) \in L^1(-h,0)$, $m_{0,i}(t,s)$, $m_{1,i}(s) \geq 0$ and defining $M_{0,i}(t) = \int_{-h}^{0} m_{0,i}(t,s)\, ds$ and $M_{1,i} = \int_{-h}^{0} m_{1,i}(s)\, ds$ we assume that $M_1 := \sqrt{\sum_{i\in\mathbb{Z}} M_{1,i}^2} < \infty$, $M_0(t) := \sqrt{\sum_{i\in\mathbb{Z}} M_{0,i}^2(t)} < \infty$, $M_0 \in C(\mathbb{R}; \mathbb{R}^+)$ and

$$\int_{-\infty}^{t} (M_0(s))^2 e^{\delta s} ds < \infty, \text{ for all } t \in \mathbb{R} \text{ and } \delta > 0.$$

(C6) $\rho \in C^1\left(\mathbb{R}, [0,h]\right)$ and $\rho'(t) \leq \rho^* < 1$.

Let us check conditions (H1)-(H3). First, in order to obtain (H1) we prove that f is well defined and bounded. We note that

$$|f_i(t,v_i)| \leq \left|F_{0,i}(t, v_i(0))\right| + \left|F_{1,i}\left(t, v_i\left(-\rho(t)\right)\right)\right| + \int_{-h}^{0} |b_i(t,s,v_i(s))|\, ds. \tag{10}$$

For the first term we have by (C3) that

$$\begin{aligned} \left|F_{0,i}(t, v_i(0))\right|^2 &\leq 2\left(H^2(|v_i(0)|)\,|v_i(0)|^2 + C_{1,i}^2(t)\right) \\ &\leq 2\chi(\|v\|_{E_0})\,|v_i(0)|^2 + 2C_{1,i}^2(t), \end{aligned} \tag{11}$$

where $\chi(\|v\|_{E_0}) = \max_{i\in\mathbb{Z}}\left(H^2(|v_i(0)|)\right)$, which exists because $H(\cdot)$ is non-decreasing and $v \in E_0$. Then,

$$\sum_{i\in\mathbb{Z}} \left|F_{0,i}(t, v_i(0))\right|^2 \leq 2\chi(\|v\|_{E_0})\,\|v\|_{E_0}^2 + 2\,\|C_1(t)\|^2. \tag{12}$$

As for the second term we obtain thanks to (C4) that

$$\sum_{i\in\mathbb{Z}} \left|F_{1,i}\left(t, v_i\left(-\rho(t)\right)\right)\right|^2 \leq 2K_1^2 \sum_{i\in\mathbb{Z}} \left|v_i\left(-\rho(t)\right)\right|^2 + 2\left\|C_2(t)\right\|^2$$
$$\leq 2K_1^2 \|v\|_{E_0}^2 + 2\left\|C_2(t)\right\|^2. \quad (13)$$

Now, for the term with the integral delay, taking into account (C5), we proceed as follows:

$$\int_{-h}^{0} \left|b_i(t, s, v_i(s))\right| ds \leq \int_{-h}^{0} \left(m_{0,i}(s) + m_{1,i}(s)\left|v_i(s)\right|\right) ds$$
$$\leq M_{0,i}(t) + \|v\|_{E_\infty} M_{1,i}.$$

Then

$$\sum_{i\in\mathbb{Z}} \left(\int_{-h}^{0} \left|b_i(t, s, v_i(s))\right| ds\right)^2 \leq 2\sum_{i\in\mathbb{Z}} M_{0,i}^2 + 2\|v\|_{E_\infty}^2 \sum_{i\in\mathbb{Z}} M_{1,i}^2$$
$$\leq 2M_0^2(t) + 2\|v\|_{E_0}^2 M_1^2. \quad (14)$$

Using (12)-(14) in (10) we obtain that f is well defined and bounded.

Now, we check (H2), i.e., that the maps $f_i : \mathbb{R} \times C([-h, 0], \mathbb{R}) \to \mathbb{R}$ are continuous. We consider $t_n \in \mathbb{R}$, $\{v^n\}_{n\in\mathbf{N}} \subset C([-h, 0], \mathbb{R})$ and $t_0 \in \mathbb{R}$, $v^0 \in C([-h, 0], \mathbb{R})$ such that $t_n \to t_0$, $v^n \to v^0$ in $C([-h, 0], \mathbb{R})$. Now, we have

$$\left|f_i(t_n, v^n) - f_i\left(t_0, v^0\right)\right| \leq \left|F_{0,i}\left(t_n, v^n(0)\right) - F_{0,i}\left(t_0, v^0(0)\right)\right|$$
$$+ \left|F_{1,i}\left(t_n, v^n\left(-\rho(t_n)\right)\right) - F_{1,i}\left(t_0, v^0\left(-\rho(t_0)\right)\right)\right|$$
$$+ \left|\int_{-h}^{0} b_i\left(t_n, s, v^n(s)\right) ds - \int_{-h}^{0} b_i\left(t_0, s, v^0(s)\right) ds\right|.$$

From (C2) and (C4), $F_{0,i}$ and $F_{1,i}$ are continuous functions. Also, from (C5) and Lebesgue's theorem, the last term converges to 0. Thus, the continuity of f_i follows.

To check (H3) we observe that

$$\sum_{|i|\geq K} \left(\int_{-h}^{0} \left|b_i(t, s, v_i(s))\right| ds\right)^2 \leq 2\sum_{|i|\geq K} \left(\int_{-h}^{0} m_{0,i}(s)\, ds\right)^2 + 2\sum_{|i|\geq K} \left(\int_{-h}^{0} m_{1,i}(s)\left|v_i(s)\right| ds\right)^2$$
$$\leq 2\sum_{|i|\geq K} M_{0,i}^2(t) + 2\|v\|_{E_0}^2 \sum_{|i|\geq K} M_{1,i}^2.$$

Also, by (10), (11) and (C4) we have

$$\sum_{|i|\geq K} \left|f_i(t, v_i)\right|^2 \leq R\left(\chi(\|v\|_{E_0}) \sum_{|i|\geq K} \left|v_i(0)\right|^2 + \sum_{|i|\geq K} C_{1,i}^2(t) + K_1^2 \sum_{|i|\geq K} \left|v_i\left(-\rho(t)\right)\right|^2\right.$$
$$\left. + \sum_{|i|\geq K} C_{2,i}^2(t) + \sum_{|i|\geq K} M_{0,i}^2(t) + \|v\|_{E_0}^2 \sum_{|i|\geq K} M_{1,i}^2\right)$$
$$\leq C\left(\|v\|_{E_0}\right)\left(\max_{s\in[-h,0]} \sum_{|i|\geq K} v_i^2(s) + b_K(t)\right),$$

where $b_K \to 0^+$ as $K \to \infty$ uniformly in compact sets, and $C(\cdot) \geq 0$ is a continuous non-decreasing function. Thus, (H3) holds.

Then Theorem 8 and Corollary 11 imply that for any $\psi \in E_0$ there exists, at least, one solution $u(\cdot) \in C^1\left([\tau, \alpha), E\right)$ in a maximal interval $[\tau, \alpha)$. In order to obtain that every solution is globally defined we need to prove some estimates. This will be done in the next section.

3.1 Estimate of solutions

Now, we shall obtain some estimates of solutions, which will imply the existence of a pullback $\mathcal{D}$-absorbing for a suitable class of sets $\mathcal{D}$.

Proposition 19. *Assume (C1)-(C5). Also, let*

$$2M_1 eh < 1, \tag{15}$$

$$2K_1^2 < e^{-\eta h}\lambda(\lambda-\eta)\left(1-\rho^\star\right), \tag{16}$$

where $\eta \in (\eta_0, \eta_1)$ and η_j are the two solutions of the equation $\eta e^{-\eta h} = 2M_1$.

Then, every solution $u(\cdot)$ with $u_\tau = \psi \in E_0$ satisfies

$$\|u_t\|_{E_0}^2 \leq 2\hat{C}\,\|\psi\|_{E_0}^2\, e^{(L-\eta)t} e^{(\eta-L)\tau} + R(\tau,t), \forall t \in [\tau, T^\star), \tag{17}$$

where $T^\star$ is the maximal time of existence and

$$L = 2M_1 e^{\eta h} \tag{18}$$

$$\hat{C} := e^{\eta h} + \frac{2K_1^2}{\lambda\eta\left(1-\rho^\star\right)} e^{2\eta h}, \tag{19}$$

$$\beta(t) = e^{\eta h}\left(\frac{2\,\|C_2(t)\|^2}{\lambda} + \frac{(M_0(t))^2}{\hat{\epsilon}} + 2\,\|C_0(t)\|_{\ell^1}\right). \tag{20}$$

$$R(\tau,t) = e^{-\eta t}\int_\tau^t e^{\eta s}\beta(s)\,ds + e^{(L-\eta)t}\int_\tau^t e^{(\eta-L)s}\beta(s)\,ds, \tag{21}$$

where $\hat{\epsilon} > 0$ is a small constant depending on the parameters of the problem.

Remark 20. *We note that (15) implies that $\eta e^{-\eta h} > 2M_1$ if $\eta \in (\eta_0, \eta_1)$, so that $\eta > L$. Also, (16) implies that $\lambda > \eta$.*

Proof. We multiply (9) by $u = (u_i)_{i\in\mathbb{Z}}$ in ℓ^2. Then

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 + (Au,u) + \lambda\,\|u(t)\|^2 = -\sum_{i\in\mathbb{Z}} F_{0,i}(t,u_i(t))\,u_i(t) - \sum_{i\in\mathbb{Z}} F_{1,i}(t,u_i(t-\rho(t)))\,u_i(t)$$
$$-\sum_{i\in\mathbb{Z}}\int_{-h}^{0} b_i(t,s,u_i(t+s))\,ds\,u_i(t). \tag{22}$$

Multiplying (22) by $e^{\eta t}$, and using $(Au,u) = \|Bu\|^2$ and (C1)-(C4), we have, for any $\epsilon > 0$ to be determined later on,

$$\frac{d}{dt}\left(e^{\eta t}\,\|u(t)\|^2\right) \leq (\eta - 2\lambda + \epsilon)\,e^{\eta t}\,\|u(t)\|^2 + 2e^{\eta t}\,\left\|C_0(t)\right\|_{\ell^1}$$
$$+ 2\frac{e^{\eta t}}{\epsilon}\left(K_1^2\left\|u\left(t-\rho(t)\right)\right\|^2 + \|C_2(t)\|^2\right)$$
$$-2e^{\eta t}\sum_{i\in\mathbb{Z}}\int_{-h}^{0} b_i(t,s,u_i(t+s))\,ds\,u_i(t). \tag{23}$$

Now, integrating the last inequality over (τ, t) we obtain

$$e^{\eta t}\|u(t)\|^2 \le e^{\eta\tau}\|u(\tau)\|^2 + (\eta - 2\lambda + \epsilon)\int_\tau^t e^{\eta s}\|u(s)\|^2\,ds + 2\int_\tau^t e^{\eta s}\|C_0(s)\|_{\ell^1}\,ds$$
$$+\frac{2}{\epsilon}\int_\tau^t e^{\eta s}\left\|C_2(s)\right\|^2 ds + \frac{2K_1^2}{\epsilon}\int_\tau^t e^{\eta s}\|u(s-\rho(s))\|^2\,ds$$
$$-2\int_\tau^t e^{\eta s}\left(\sum_{i\in\mathbb{Z}}\int_{-h}^0 b_i(s,r,u_i(s+r))\,dr\,u_i(s)\right)ds. \tag{24}$$

We proceed to estimate the two last terms in (24). First, using $\frac{1}{1-\rho'(t)} \le \frac{1}{1-\rho^*}$ we have

$$\int_\tau^t e^{\eta s}\|u(s-\rho(s))\|^2\,ds \le \int_{\tau-h}^t \frac{e^{\eta(l+h)}}{1-\rho^*}\|u(l)\|^2\,dl$$
$$= \frac{e^{\eta h}}{1-\rho^*}\int_{\tau-h}^\tau e^{\eta l}\|u(l)\|^2\,dl + \frac{e^{\eta h}}{1-\rho^*}\int_\tau^t e^{\eta l}\|u(l)\|^2\,dl$$
$$\le \frac{e^{\eta h}}{\eta\left(1-\rho^*\right)}\|\psi\|_{E_0}^2\left(e^{\eta\tau} - e^{\eta(\tau-h)}\right) + \frac{e^{\eta h}}{1-\rho^*}\int_\tau^t e^{\eta l}\|u(l)\|^2\,dl. \tag{25}$$

Next, we analyze the last term in (24). By (C5),

$$\left|\sum_{i\in\mathbb{Z}}\int_{-h}^0 b_i(t,s,u_i(t+s))\,ds\,u_i(t)\right| \le \sum_{i\in\mathbb{Z}}\int_{-h}^0\left(m_{0,i}(t,s)|u_i(t)|\right)ds + \sum_{i\in\mathbb{Z}}\int_{-h}^0\left(m_{1,i}(s)|u_i(t+s)||u_i(t)|\right)ds. \tag{26}$$

Now, we estimate the two terms in (26) separately. On the one hand,

$$\sum_{i\in\mathbb{Z}}\int_{-h}^0\left(m_{0,i}(t,s)|u_i(t)|\right)ds = \sum_{i\in\mathbb{Z}} M_{0,i}(t)|u_i(t)| \le \|u(t)\|\,M_0(t). \tag{27}$$

On the other hand,

$$\sum_{i\in\mathbb{Z}}\int_{-h}^0\left(m_{1,i}(s)|u_i(t+s)||u_i(t)|\right)ds \le \|u_t\|_{E_\infty}\sum_{i\in\mathbb{Z}}\left(\int_{-h}^0\left(m_{1,i}(s)\right)ds\right)|u_i(t)|$$
$$\le \|u_t\|_{E_\infty} M_1\,\|u(t)\|$$
$$\le \|u_t\|_{E_0}^2\,M_1. \tag{28}$$

Now, using (27) and (28), we have

$$\left|2\int_\tau^t e^{\eta s}\left(\sum_{i\in\mathbb{Z}}\int_{-h}^0 b_i(t,r,u_i(s+r))\,dr\,u_i(s)\right)\right|ds$$
$$\le 2\int_\tau^t e^{\eta s}\left(\|u(s)\|\,M_0(s) + \|u_s\|_{E_0}^2\,M_1\right)ds$$
$$\le \hat{\epsilon}\int_\tau^t e^{\eta s}\|u(s)\|^2\,ds + \frac{1}{\hat{\epsilon}}\int_\tau^t e^{\eta s}M_0^2(s)ds + 2M_1\int_\tau^t e^{\eta s}\|u_s\|_{E_0}^2\,ds, \tag{29}$$

with $\hat{\epsilon} > 0$ arbitrary. Using (25) and (29) in (24) we obtain

$$e^{\eta t}\left\|u(t)\right\|^2 \le e^{\eta\tau}\left\|u(\tau)\right\|^2 + \left(\eta - 2\lambda + \epsilon + \hat{\epsilon} + \frac{2K_1^2 e^{\eta h}}{\epsilon(1-\rho^*)}\right)\int_\tau^t e^{\eta s}\left\|u(s)\right\|^2 ds$$
$$+\int_\tau^t e^{\eta s}\left(\frac{2\left\|C_2(s)\right\|^2}{\epsilon} + \frac{M_0^2(s)}{\hat{\epsilon}} + 2\left\|C_0(s)\right\|_{\ell^1}\right)ds$$
$$+\frac{2K_1^2 e^{\eta h}}{\epsilon\eta\left(1-\rho^*\right)}\left\|\psi\right\|_{E_0}^2\left(e^{\eta\tau} - e^{\eta(\tau-h)}\right) + 2M_1\int_\tau^t e^{\eta s}\left\|u_s\right\|_{E_0}^2 ds.$$

Taking $\epsilon = \lambda$, condition (16) implies that $\eta - \lambda + \hat{\epsilon} + \frac{2K_1^2 e^{\eta h}}{\lambda(1-\rho^*)} < 0$ for $\hat{\epsilon}$ small enough. Then

$$e^{\eta t}\left\|u(t)\right\|^2 \le e^{\eta\tau}\left\|u(\tau)\right\|^2 + \int_\tau^t e^{\eta s}\left(\frac{2\left\|C_2(s)\right\|^2}{\lambda} + \frac{M_0^2(s)}{\hat{\epsilon}} + 2\left\|C_0(s)\right\|_{\ell^1}\right)ds \tag{30}$$
$$+\frac{2K_1^2 e^{\eta h}}{\lambda\eta\left(1-\rho^*\right)}\left\|\psi\right\|_{E_0}^2\left(e^{\eta\tau} - e^{\eta(\tau-h)}\right) + 2M_1\int_\tau^t e^{\eta s}\left\|u_s\right\|_{E_0}^2 ds.$$

Let $\theta \in [-h, 0]$. Replacing t by $t + \theta$ in (30), using that $\left\|u(t+\theta)\right\| = \left\|\psi(t+\theta)\right\| \le \left\|\psi\right\|_{E_0}$ if $t + \theta < 0$, and multiplying by $e^{-\eta(t+\theta)}$ we have

$$\left\|u(t+\theta)\right\|^2 \le e^{-\eta(t+\theta)}e^{\eta\tau}\left\|\psi\right\|_{E_0}^2 + e^{-\eta(t+\theta)}\int_\tau^{t+\theta} e^{\eta s}\left(\frac{2\left\|C_2(s)\right\|^2}{\lambda} + \frac{M_0^2(s)}{\hat{\epsilon}} + 2\left\|C_0(s)\right\|_{\ell^1}\right)ds$$
$$+ e^{-\eta(t+\theta)}\frac{2K_1^2 e^{\eta h}}{\lambda\eta\left(1-\rho^*\right)}\left\|\psi\right\|_{E_0}^2\left(e^{\eta\tau} - e^{\eta(\tau-h)}\right) + 2M_1 e^{-\eta(t+\theta)}\int_\tau^{t+\theta} e^{\eta s}\left\|u_s\right\|_{E_0}^2 ds.$$

Using that $\theta \in [-h, 0]$ and neglecting the negative terms we get

$$e^{\eta t}\left\|u_t\right\|_{E_0}^2 \le e^{\eta h}e^{\eta\tau}\left\|\psi\right\|_{E_0}^2 + e^{\eta h}\int_\tau^t e^{\eta s}\left(\frac{2\left\|C_2(s)\right\|^2}{\lambda} + \frac{M_0^2(s)}{\hat{\epsilon}} + 2\left\|C_0(s)\right\|_{\ell^1}\right)ds$$
$$+\frac{2K_1^2 e^{2\eta h}}{\lambda\eta\left(1-\rho^*\right)}\left\|\psi\right\|_{E_0}^2 e^{\eta\tau} + 2M_1 e^{\eta h}\int_\tau^t e^{\eta s}\left\|u_s\right\|_{E_0}^2 ds.$$

We can rewrite this expression as

$$e^{\eta t}\left\|u_t\right\|_{E_0}^2 \le \hat{C}\left\|\psi\right\|_{E_0}^2 e^{\eta\tau} + \int_\tau^t e^{\eta s}\beta(s)\,ds + L\int_\tau^t e^{\eta s}\left\|u_s\right\|_{E_0}^2 ds, \tag{31}$$

where we have used the notation given in (18)-(20). Applying Gronwall's inequality, Fubini's theorem and using $\eta - L > 0$ (see Remark 20) yields

$$e^{\eta t}\left\|u_t\right\|_{E_0}^2 \le \hat{C}\left\|\psi\right\|_{E_0}^2 e^{\eta\tau} + \int_\tau^t e^{\eta s}\beta(s)\,ds + Le^{Lt}\int_\tau^t\left(\int_\tau^s e^{\eta r}\beta(r)\,dr + \hat{C}_2\left\|\psi\right\|_{E_0}^2 e^{\eta\tau}\right)e^{-Ls}ds$$
$$= \hat{C}\left\|\psi\right\|_{E_0}^2 e^{\eta\tau} + \int_\tau^t e^{\eta s}\beta(s)\,ds + \hat{C}\left\|\psi\right\|_{E_0}^2\left(e^{Lt}e^{-L\tau} - 1\right)e^{\eta\tau} + e^{Lt}\int_\tau^t e^{\eta r}\beta(r)\left(e^{-Lr} - e^{-Lt}\right)dr$$
$$\le \hat{C}\left\|\psi\right\|_{E_0}^2 e^{\eta\tau} + \hat{C}\left\|\psi\right\|_{E_0}^2 e^{Lt}e^{(\eta-L)\tau} + \int_\tau^t e^{\eta s}\beta(s)\,ds + e^{Lt}\int_\tau^t e^{(\eta-L)s}\beta(s)\,ds,$$

and then

$$\|u_t\|_{E_0}^2 \le \hat{C}\,\|\psi\|_{E_0}^2\, e^{-\eta t} e^{\eta \tau} + \hat{C}\,\|\psi\|_{E_0}^2\, e^{(L-\eta)t} e^{(\eta-L)\tau} + e^{-\eta t}\int_{\tau}^{t} e^{\eta s}\beta(s)\,ds + e^{(L-\eta)t}\int_{\tau}^{t} e^{(\eta-L)s}\beta(s)\,ds. \tag{32}$$

From here (17) follows. □

Let $\mathcal{R}_\eta$ be the set of all functions $r:\mathbb{R}\to(0,+\infty)$ such that

$$\lim_{t\to-\infty} e^{(\eta-L)t} r^2(t) = 0.$$

Denote by $\mathcal{D}_\eta$ the class of all families $\widehat{D} = \{D(t) : t\in\mathbb{R}\} \subset \mathcal{P}(\ell^2)$ such that $D(t) \subset \overline{B}(0, r_{\widehat{D}}(t))$ for some $r_{\widehat{D}} \in \mathcal{R}_\eta$, where $\overline{B}(0, r_{\widehat{D}}(t))$ denotes the closed ball in ℓ^2 centered at zero with radius $r_{\widehat{D}}(t)$. The class $\mathcal{D}_\eta$ is inclusion-closed.

Corollary 21. *Assuming the conditions of Proposition 19, Theorem 5 implies that every local solution of (4) can be defined globally. Also, the map U defined by (7) is a strict multivalued process.*

Corollary 22. *The balls $B_\eta(t) = \overline{B}_{l^2}(0, R_\eta(t))$, where $R_{\lambda_1}(t)$ is the nonnegative number given for each $t\in\mathbb{R}$ by*

$$R_\eta^2(t) = e^{-\eta t}\int_{-\infty}^{t} e^{\eta s}\beta(s)\,ds + e^{(L-\eta)t}\int_{-\infty}^{t} e^{(\eta-L)s}\beta(s)\,ds + 1, \tag{33}$$

form a family $\widehat{B}_\eta$ which is pullback $\mathcal{D}_\eta$-absorbing for the process U.

We are interested in proving that $\widehat{B}_\eta \in \mathcal{D}_\eta$. For this aim we will need and additional assumption on the function $\beta(t)$ (that is, on the functions $C_2,\ M_0,\ C_0$).

Lemma 23. *In addition to the conditions of Proposition 19, assume that*

$$\lim_{t\to-\infty} e^{-\delta_1 t}\int_{-\infty}^{t} e^{\delta_2 s}\beta(s)\,ds = 0 \text{ for all } 0<\delta_1<\delta_2. \tag{34}$$

Then $\widehat{B}_\eta \in \mathcal{D}_\eta$.

Remark 24. *Condition (34) is satisfied if*

$$\int_{-\infty}^{t} \beta(s)ds < \infty \text{ for all } t\in\mathbb{R}.$$

Indeed, for $t\to-\infty$ we have

$$e^{-\delta_1 t}\int_{-\infty}^{t} e^{\delta_2 s}\beta(s)\,ds \le \int_{-\infty}^{t} e^{-\delta_2(t-s)}\beta(s)ds \le \int_{-\infty}^{t}\beta(s)ds \to 0.$$

Proof. By (34), $\eta > L$ and $\int_{-\infty}^{t} e^{(\eta-L)s}\beta(s)\,ds < \infty$ we have

$$\begin{aligned}\lim_{t\to-\infty} e^{(\eta-L)t}R_\eta^2(t) &= \lim_{t\to-\infty} e^{(\eta-L)t}\left(e^{-\eta t}\int_{-\infty}^{t} e^{\eta s}\beta(s)\,ds + e^{(L-\eta)t}\int_{-\infty}^{t} e^{(\eta-L)s}\beta(s)\,ds + 1\right)\\ &= \lim_{t\to-\infty} e^{-Lt}\int_{-\infty}^{t} e^{\eta s}\beta(s)\,ds + \int_{-\infty}^{t} e^{(\eta-L)s}\beta(s)\,ds + e^{(\eta-L)t} = 0.\end{aligned}$$

□

3.2 Estimate of the tails

In order to obtain the existence of a pullback attractor we need to obtain an estimate of the tails of solutions.

Lemma 25. *We assume the conditions of Lemma 23. Let $\widehat{B}_\eta \in \mathcal{D}_\eta$ be the pullback $\mathcal{D}_\eta$-absorbing family given above. Then, for any $\epsilon > 0$ and $t_1 \leq t_2$ there exist $T\left(\epsilon, t_1, t_2, \widehat{B}_\eta\right) \leq t_1$, $K\left(\epsilon, t_1, t_2, \widehat{B}_\eta\right) \geq 1$ such that*

$$\max_{s\in[-h,0]} \sqrt{\sum_{|i|>2K} |u_i(t+s)|^2} < \epsilon, \forall \tau \leq T,\ t \in [t_1, t_2], \tag{35}$$

for any solution $u(\cdot)$ with $u_\tau \in B_\eta(\tau)$.

Proof. Define a smooth function θ satisfying

$$\theta(s) = \begin{cases} 0,\ 0 \leq s \leq 1, \\ 0 \leq \theta(s) \leq 1,\ 1 \leq s \leq 2, \\ 1,\ s \geq 2. \end{cases}$$

Obviously $\left|\theta'(s)\right| \leq C$, for all $s \in \mathbb{R}^+$. For any solution $u(\cdot)$, let $v(t) := (v_i(t))_{i\in\mathbb{Z}}$ be given by $v_i(t) = \rho_{K,i} u_i(t)$, where $\rho_{K,i} := \theta\left(\frac{|i|}{K}\right)$. We multiply (9) by v. We note that $u(\cdot) \in C^1([0,\infty), E)$ implies

$$\frac{1}{2}\frac{d}{dt}\sum_{i\in\mathbb{Z}} \rho_{K,i}|u_i|^2 = \sum_{i\in\mathbb{Z}} \frac{du_i(t)}{dt} v_i(t),\ \forall t > \tau.$$

Following now the arguments in [19, p.571] there exists a constant R_1 (depending on the parameters of the problem) such that

$$(Au(t), v(t)) \geq -\frac{R_1}{K}\left\|u(t)\right\|^2,\ \forall\, \tau \leq t \leq t_2.$$

Note that (17), $\eta - L > 0$ and $\widehat{B}_\eta \in \mathcal{D}_\eta$ imply the existence of $R_2(t_2)$ (independent of τ) such that

$$\left\|u(t)\right\|^2 \leq R_2(t_2),\ \forall\, \tau \leq t \leq t_2.$$

Hence,

$$\frac{1}{2}\frac{d}{dt}\sum_{i\in\mathbb{Z}} \rho_{K,i}|u_i(t)|^2 \leq -\lambda \sum_{i\in\mathbb{Z}} \rho_{K,i}|u_i(t)|^2 - \sum_{i\in\mathbb{Z}} \rho_{K,i} f_i(u_{it}) u_i(t) + \frac{C(t_2)}{K}.$$

Then, arguing as in the proof of Proposition 19 we have

$$\begin{aligned} \frac{d}{dt}\left(e^{\eta t}\sum_{i\in\mathbb{Z}} \rho_{K,i}|u_i(t)|^2\right) &\leq e^{\eta t}(\eta - 2\lambda + \epsilon)\sum_{i\in\mathbb{Z}} \rho_{K,i}|u_i(t)|^2 \\ &+ 2e^{\eta t}\sum_{i\in\mathbb{Z}} \rho_{K,i} C_{0,i}(t) + \frac{2C(t_2)}{K}e^{\eta t} \\ &+ \frac{2}{\epsilon}e^{\eta t}\sum_{i\in\mathbb{Z}} \rho_{K,i} C_{2,i}^2(t) + \frac{2K_1^2}{\epsilon}\sum_{i\in\mathbb{Z}} \rho_{K,i}|u_i(t-\rho(t))|^2 \\ &+ 2e^{\eta t}\sum_{i\in\mathbb{Z}} \rho_{K,i}\int_{-h}^{0} |b_i(t,s,u_i(t+s))|\, ds\ |u_i(t)|. \end{aligned} \tag{36}$$

Integrating over the interval (τ, t) we get

$$e^{\eta t}\left(\sum_{i\in\mathbb{Z}}\rho_{K,i}\left|u_i(t)\right|^2\right) \le e^{\eta\tau}\sum_{i\in\mathbb{Z}}\rho_{K,i}\left|u_i(\tau)\right|^2 + (\eta-2\lambda+\epsilon)\int_\tau^t e^{\eta s}\sum_{i\in\mathbb{Z}}\rho_{K,i}\left|u_i(s)\right|^2 ds \tag{37}$$
$$+ 2\int_\tau^t e^{\eta s}\left(\sum_{i\in\mathbb{Z}}\rho_{K,i}C_{0,i}(s)\,ds + \frac{C(t_2)}{K} + \frac{1}{\epsilon}\sum_{i\in\mathbb{Z}}\rho_{K,i}C_{2,i}^2(s)\right) ds$$
$$+ \frac{2K_1^2}{\epsilon}\int_\tau^t e^{\eta s}\sum_{i\in\mathbb{Z}}\rho_{K,i}\left|u_i(s-\rho(s))\right|^2 ds$$
$$+ 2\int_\tau^t e^{\eta s}\sum_{i\in\mathbb{Z}}\rho_{K,i}\int_{-h}^0 \left|b_i(s,r,u_i(s+r))\right| dr\ \left|u_i(s)\right| ds\,.$$

Next, we estimate the last two terms in (37). The first one, arguing as in (25), is estimated by

$$\int_\tau^t e^{\eta s}\sum_{i\in\mathbb{Z}}\rho_{K,i}\left|u_i(s-\rho(s))\right|^2 ds \le \frac{e^{\eta h}}{\eta(1-\rho^\star)}\left\|\rho_K^{\frac{1}{2}}\psi\right\|_{E_0}^2\left(e^{\eta\tau}-e^{\eta(\tau-h)}\right)$$
$$+ \frac{e^{\eta h}}{1-\rho^\star}\int_\tau^t e^{\eta s}\sum_{i\in\mathbb{Z}}\rho_{K,i}\left|u_i(s)\right|^2 ds. \tag{38}$$

As for the second term, using assumption (C5), in a similar way to that in (27)-(28), we have

$$\int_\tau^t e^{\eta s}\sum_{i\in\mathbb{Z}}\rho_{K,i}\left|u_i(s)\right|\int_{-h}^0 m_{0,i}(s,r)\,dr\,ds \le \frac{\hat\epsilon}{2}\int_\tau^t e^{\eta s}\left\|\rho_K^{\frac{1}{2}}u(s)\right\|^2 ds + \frac{1}{2\hat\epsilon}\int_\tau^t e^{\eta s}\sum_{i\in\mathbb{Z}}\rho_{K,i}M_{0,i}^2(s)\,ds,$$

$$\int_\tau^t e^{\eta s}\sum_{i\in\mathbb{Z}}\rho_{K,i}\left|u_i(s)\right|\int_{-h}^0 m_{1,i}(r)\left|u_i(s+r)\right| dr\,ds \le \int_\tau^t e^{\eta s}\left\|\rho_{K,i}^{\frac{1}{2}}u_s\right\|_{E_\infty}\sum_{i\in\mathbb{Z}}\rho_{K,i}^{\frac{1}{2}}\left|u_i(s)\right| M_{1,i}\,ds$$
$$\le M_1\int_\tau^t e^{\eta s}\left\|\rho_K^{\frac{1}{2}}u_s\right\|_{E_0}^2 ds,$$

and

$$\int_\tau^t e^{\eta s}\sum_{i\in\mathbb{Z}}\rho_{K,i}\int_{-h}^0 \left|b_i(r,u_i(s+r))\right| dr\ \left|u_i(s)\right| ds \le \frac{\hat\epsilon}{2}\int_\tau^t e^{\eta s}\left\|\rho_K^{\frac{1}{2}}u(s)\right\|^2 ds + \frac{1}{2\hat\epsilon}\int_\tau^t e^{\eta s}\sum_{i\in\mathbb{Z}}\rho_{K,i}M_{0,i}^2(s)\,ds \tag{39}$$
$$+ M_1\int_\tau^t e^{\eta s}\left\|\rho_K^{\frac{1}{2}}u_s\right\|_{E_0}^2 ds.$$

Consequently,

$$e^{\eta t}\sum_{i\in\mathbb{Z}}\rho_{K,i}\left|u_i(t)\right|^2 \le e^{\eta\tau}\sum_{i\in\mathbb{Z}}\rho_{K,i}\left|u_i(\tau)\right|^2 + \left(\eta-2\lambda+\epsilon+\hat\epsilon+\frac{2K_1^2}{\epsilon(1-\rho^\star)}e^{\eta h}\right)\int_\tau^t e^{\eta s}\sum_{i\in\mathbb{Z}}\rho_{K,i}\left|u_i(s)\right|^2 ds$$
$$\int_\tau^t e^{\eta s}\left(\frac{2\left\|\rho_K^{\frac{1}{2}}C_2(s)\right\|^2}{\epsilon} + \frac{\sum_{i\in\mathbb{Z}}\rho_{K,i}M_{0,i}^2(s)}{\hat\epsilon} + 2\left\|\rho_K C_0(s)\right\|_{\ell^1}\right) ds + \frac{2}{\eta}\left(e^{\eta t}-e^{\eta\tau}\right)\frac{C(t_2)}{K}$$
$$+ \frac{2K_1^2}{\lambda}\frac{e^{\eta h}}{\eta(1-\rho^\star)}\left\|\rho_K^{\frac{1}{2}}\psi\right\|_{E_0}^2\left(e^{\eta\tau}-e^{\eta(\tau-h)}\right) + 2M_1\int_\tau^t e^{\eta s}\left\|\rho_K^{\frac{1}{2}}u_s\right\|_{E_0}^2 ds.$$

Taking $\epsilon = \lambda$ and using condition (16), we have

$$e^{\eta t}\sum_{i\in\mathbb{Z}}\rho_{K,i}\left|u_i(t)\right|^2 \leq e^{\eta\tau}\sum_{i\in\mathbb{Z}}\rho_{K,i}\left|u_i(\tau)\right|^2 + \frac{2}{\eta}\left(e^{\eta t}-e^{\eta\tau}\right)\frac{C(t_2)}{K}$$
$$+\int_\tau^t e^{\eta s}\left(\frac{2\left\|\rho_K^{\frac{1}{2}}C_2(s)\right\|^2}{\lambda}+\frac{\sum_{i\in\mathbb{Z}}\rho_{K,i}M_{0,i}^2(s)}{\hat{\epsilon}}+2\left\|\rho_K C_0(s)\right\|_{\ell^1}\right)ds$$
$$+\frac{2K_1^2}{\lambda}\frac{e^{\eta h}}{\eta(1-\rho^*)}\left\|\rho_K^{\frac{1}{2}}\psi\right\|_{E_0}^2\left(e^{\eta\tau}-e^{\eta(\tau-h)}\right)+2M_1\int_\tau^t e^{\eta s}\left\|\rho_K^{\frac{1}{2}}u_s\right\|_{E_0}^2 ds. \tag{40}$$

Let $\theta\in[-h,0]$. We replace t by $t+\theta$ in (40), and use that $\left\|\rho_K^{\frac{1}{2}}u(t+\theta)\right\| = \left\|\rho_K^{\frac{1}{2}}\psi(t+\theta)\right\| \leq \left\|\rho_K^{\frac{1}{2}}\psi\right\|_{E_0}$ if $t+\theta<\tau$; multiplying by $e^{-\eta(t+\theta)}$ we obtain

$$\sum_{i\in\mathbb{Z}}\rho_{K,i}\left|u_i(t+\theta)\right|^2 \leq e^{\eta\tau}e^{-\eta(t+\theta)}\sum_{i\in\mathbb{Z}}\rho_{K,i}\left|u_i(\tau)\right|^2$$
$$+e^{-\eta(t+\theta)}\int_\tau^{t+\theta} e^{\eta s}\left(\frac{2\left\|\rho_K^{\frac{1}{2}}C_2(s)\right\|^2}{\lambda}+\frac{\left\|\rho_K^{\frac{1}{2}}M_0(s)\right\|^2}{\hat{\epsilon}}+2\left\|\rho_K C_0(s)\right\|_{\ell^1}\right)ds$$
$$+\frac{2}{\eta}\left(e^{\eta(t+\theta)}-e^{\eta\tau}\right)\frac{C(t_2)}{K}e^{-\eta(t+\theta)}$$
$$+\frac{2K_1^2}{\lambda}\frac{e^{\eta h}e^{-\eta(t+\theta)}}{\eta(1-\rho^*)}\left\|\rho_K^{\frac{1}{2}}\psi\right\|_{E_0}^2\left(e^{\eta\tau}-e^{\eta(\tau-h)}\right)+2M_1 e^{-\eta(t+\theta)}\int_\tau^{t+\theta} e^{\eta s}\left\|\rho_K^{\frac{1}{2}}u_s\right\|_{E_0}^2 ds,$$

and

$$e^{\eta t}\left\|\rho_K^{\frac{1}{2}}u_t\right\|_{E_0}^2 \leq \left(e^{\eta h}+\frac{2K_1^2e^{2\eta h}}{\lambda\eta(1-\rho^*)}\right)\left\|\rho_K^{\frac{1}{2}}\psi\right\|_{E_0}^2 e^{\eta\tau}$$
$$+e^{\eta h}\int_\tau^t e^{\eta s}\left(\frac{2\left\|\rho_K^{\frac{1}{2}}C_2(s)\right\|^2}{\lambda}+\frac{\sum_{i\in\mathbb{Z}}\rho_{K,i}M_{0,i}^2(s)}{\hat{\epsilon}}+2\left\|\rho_K C_0(s)\right\|_{\ell^1}\right)ds$$
$$+\frac{2}{\eta}\frac{C(t_2)}{K}e^{\eta t}+2M_1e^{\eta h}\int_\tau^t e^{\eta s}\left\|\rho_K^{\frac{1}{2}}u_s\right\|_{E_0}^2 ds.$$

We can rewrite this expression as

$$e^{\eta t}\left\|\rho_K^{\frac{1}{2}}u_t\right\|_{E_0}^2 \leq \frac{2}{\eta}\frac{C(t_2)}{K}e^{\eta t}+\hat{C}\left\|\rho_K^{\frac{1}{2}}\psi\right\|_{E_0}^2 e^{\eta\tau}+\int_\tau^t e^{\eta s}\beta_{\rho_K}(s)\,ds+L\int_\tau^t e^{\eta s}\left\|\rho_K^{\frac{1}{2}}u_s\right\|_{E_0}^2 ds, \tag{41}$$

where we have used the notation

$$\hat{C} := e^{\eta h}+\frac{2K_1^2}{\lambda\eta(1-\rho^*)}e^{2\eta h},$$
$$\beta_{\rho_K}(t) := e^{\eta h}\left(\frac{2\left\|\rho_K^{\frac{1}{2}}C_2(t)\right\|^2}{\lambda}+\frac{\sum_{i\in\mathbb{Z}}\rho_{K,i}M_{0,i}^2(s)}{\hat{\epsilon}}+2\left\|\rho_K C_0(t)\right\|_{\ell^1}\right),$$
$$L := 2M_1e^{\eta h}.$$

Now, proceeding in a similar way to (32) and using $\eta - L > 0$ (see Remark 20) we obtain

$$\left\|\rho_K^{\frac{1}{2}}u_t\right\|_{E_0}^2 \leq 2e^{(L-\eta)t}e^{(\eta-L)\tau}\hat{C}\left\|\rho_K^{\frac{1}{2}}\psi\right\|_{E_0}^2+\frac{2C(t_2)}{K(\eta-L)}+e^{-\eta t}\int_\tau^t e^{\eta s}\beta_{\rho_K}(s)\,ds+e^{(L-\eta)t}\int_\tau^t e^{(\eta-L)s}\beta_{\rho_K}(s)\,ds. \tag{42}$$

Now, it is convenient to keep in mind the definition of β_{ρ_k}, and its dependence on C_0, M_0 and C_2. It follows that

$$\beta_{\rho_K}(s) \to 0 \text{ as } K \to \infty \text{ for any } s.$$

Hence, Lebesgue's Dominated Convergence Theorem implies that

$$\int_{-\infty}^{t} e^{\delta s}\beta_{\rho_K}(s)\,ds \to 0, \text{ as } K \to \infty \text{ for any } t \in [t_1, t_2],\ \delta > 0.$$

Thus, there exist $T\left(\epsilon, t_1, t_2, \widehat{B}_\eta\right) \le t_1$, $K\left(\epsilon, t_1, t_2, \widehat{B}_\eta\right) \ge 1$ such that

$$\begin{aligned}\max_{s\in[-h,0]}\sqrt{\sum_{|i|\ge 2K}(u_i(t+s))^2} &\le \max_{s\in[-h,0]}\sqrt{\sum_{i\in\mathbb{Z}}\rho_{K,i}(u_i(t+s))^2}\\ &= \left\|\rho_K^{\frac{1}{2}}u_t\right\|_{E_0}\\ &\le \epsilon, \text{ if } \tau \le T,\ t \in [t_1, t_2].\end{aligned}$$

□

3.3 Existence of the pullback attractor: general case

We know that under the assumptions of Proposition 19, the map U given by (7) is a strict multivalued process. For any initial data $\psi \in E_0$ we denote

$$\mathcal{D}_\tau(\psi) = \{u(\cdot) \text{ is a global solution of (9) with initial data } u_\tau = \psi\}.$$

We will prove that the map $\psi \mapsto U(t, \tau, \psi)$ is upper-semicontinuous and has closed values, and also that U is asymptotically compact with respect to the pullback $\mathcal{D}_\eta$-absorbing family $\widehat{B}_\eta$ defined in Corollary 22.

First, we obtain an auxiliary lemma.

Lemma 26. *We assume the conditions of Lemma 23. Let $\psi^n \to \psi$ in E_0. Then:*

1. *For arbitrary $\epsilon > 0$, $\tau \le T$ there exists $K(\epsilon, \tau, T)$ such that for any $u^n(\cdot) \in \mathcal{D}_\tau(\psi^n)$,*

$$\max_{s\in[-h,0]}\sqrt{\sum_{|i|\ge 2K}\left|u_i^n(t+s)\right|^2} \le \epsilon, \forall t \in [\tau, T]. \tag{43}$$

2. *Let $u^n(\cdot) \in \mathcal{D}_\tau(\psi^n)$. Then there exists $u(\cdot) \in \mathcal{D}_\tau(\psi)$ and a subsequence $\{u^{n_k}\}$ of $\{u^n\}$ such that*

$$u^{n_k} \to u \text{ in } \mathcal{C}([\tau, T], E) \text{ for all } T > \tau. \tag{44}$$

Proof. It follows from $\psi^n \to \psi$ in E_0 the existence of $K_1(\epsilon) > 0$ such that

$$\sum_{i\in\mathbb{Z}}\rho_{K,i}\left|\psi_i^n(s)\right|^2 < \epsilon, \forall n,\ s \in [-h, 0]$$
$$\sum_{i\in\mathbb{Z}}\rho_{K,i}\left|\psi_i^0(s)\right|^2 < \epsilon, \forall s \in [-h, 0],$$

if $K \ge K_1$. Now, from (42) we obtain the existence of $K(\epsilon, \tau, T) \ge K_1$ such that

$$\left\|\rho_K^{\frac{1}{2}}u_t^n\right\|_{E_0}^2 \le 2e^{(L-\eta)t}e^{(\eta-L)\tau}\hat{C}\left\|\rho_K^{\frac{1}{2}}\psi^n\right\|_{E_0}^2 + \frac{2C(T)}{K(\eta-L)} + e^{-\eta t}\int_\tau^t e^{\eta s}\beta_{\rho_K}(s)\,ds + e^{(L-\eta)t}\int_\tau^t e^{(\eta-L)s}\beta_{\rho_K}(s)\,ds \le \epsilon,$$

for all $t \in [\tau, T]$, where we have used $\beta_{\rho_K}(s) \to 0$ as $K \to \infty$ for any s and the Lebesgue Dominated Convergence Theorem. Therefore,

$$\max_{s\in[-h,0]} \sqrt{\sum_{|i|\geq 2K} \left(u_i^n(t+s)\right)^2} \leq \max_{s\in[-h,0]} \sqrt{\sum_{i\in\mathbb{Z}} \rho_{K,i} \left(u_i^n(t+s)\right)^2} = \left\|\rho_K^{\frac{1}{2}} u_t^n\right\|_{E_0} \leq \epsilon,$$

which proves (43). Next, from Proposition 19 we have that u_t^n is bounded in E_0. Then, using (43) one can prove in a standard way (see [13, p.71] for the details) that $\{u^n(t)\}$ is precompact in E for any $t \in [\tau, T]$. After that, following the same lines as in [13, p.71], we can obtain the existence of $u(\cdot) \in \mathcal{D}_\tau(\psi)$ and a subsequence such that $u^n(\cdot) \to u(\cdot)$ in $\mathcal{C}([\tau, T], E)$ for all $T > \tau$. □

As a direct consequence we have the following result. The proof is rather similar to that in [13, p.72].

Corollary 27. *Assume the conditions of Lemma 23. Then, the multivalued map $\psi \mapsto G(t, \tau, \psi)$ possesses closed graph and is upper semicontinous. Moreover, it has compact values.*

Lemma 28. *Assume the conditions of Lemma 23. Then, the multivalued process U is pullback $\mathcal{D}_\eta$-asymptotically compact. In particular, it is pullback asymptotically compact with respect to the pullback $\mathcal{D}_\eta$-absorbing family $\widehat{B}_\eta$.*

Proof. We consider $\xi^n = u_t^n \in U(t, \tau_n, \psi^n)$, where $u^n(\cdot) \in \mathcal{D}_{\tau_n}(\psi^n)$, $\psi^n \in D(\tau_n)$, and $\widehat{D} = \{D(t)\} \in \mathcal{D}_\eta$. In view of Corollary 22, for n large enough we have $u_t^n \in B_\eta(t)$. Hence,

$$\left\|u_t^n(s)\right\| \leq C, \forall s \in [-h, 0],$$

for some $C > 0$. For fixed $s \in [-h, 0]$ we can find a subsequence (denoted again as u^n) such that

$$u^n(t+s) \to \omega_s \text{ in } E_w.$$

Using a similar argument as in [13, p.71] (with the help of Lemma 25) we obtain that $u^n(t_n + s) \to \omega_s$ in E. Therefore, $\{u_t^n(s)\}$ is a precompact sequence for any $s \in [-h, 0]$. In order to apply the Ascoli-Arzelà theorem, we need to obtain the equicontinuity property. Using Proposition 19, the boundedness of the sequence $\|\psi^n\|_{E_0}^2 e^{(\eta-L)\tau_n}$, the fact that the operator F is bounded and the integral representation of solution we can obtain that

$$\left\|u^n(t+s_2) - u^n(t+s_1)\right\| \leq \int_{t+s_1}^{t+s_2} \left\|F\left(r, u_r^n\right)\right\| dr \leq K(s_2 - s_1), \text{ if } -h \leq s_1 < s_2 \leq 0.$$

Then, the Ascoli-Arzelà theorem implies that ξ^n is relatively compact in E_0. Since by Lemma 23 we have that $\widehat{B}_\eta \in \mathcal{D}_\eta$, U is pullback asymptotically compact with respect to this family as well. □

The existence of the pullback attractor follows now from Proposition 19, Lemma 28, Corollaries 22, 27 and Theorem 18.

Theorem 29. *Assume the conditions of Lemma 23. Then, the multivalued process U possesses a unique pullback $\mathcal{D}_\eta$-attractor $\widehat{A}$, which belongs to $\mathcal{D}_\eta$. Moreover, it is strictly invariant.*

3.4 Existence of the pullback attractor: case of uniqueness

We can prove uniqueness of the solution of the Cauchy problem (9) if we assume the following extra assumption:

(C6) For any $x, y \in \mathbb{R}$ and $s \in [-h, 0]$ we have

$$\left|F_{0,i}(t,x) - F_{0,i}(t,y)\right| \le k_0(t) C_3(|x|, |y|) |x-y|,$$

$$\left|F_{1,i}(t,x) - F_{1,i}(t,y)\right| \le k_1(t) C_4(|x|, |y|) |x-y|,$$

$$|b_i(t,s,x) - b_i(t,s,y)| \le k_2(t) k_3(s) C_5(|x|, |y|) |x-y|,$$

where $C_j(\cdot,\cdot) \ge 0$ are continuous and non-decreasing functions in both variables and $k_i(\cdot) \in L^2_{loc}(\mathbb{R})$ for $i = 0, 1, 2$, $k_3(\cdot) \in L^2(-h, 0)$.

Lemma 30. *If (C6) holds, the map $f : \mathbb{R} \times E_0 \to E$ satisfies the local Lipschitz assumption $(H4)$.*

Proof. Let $v, w \in E_0$ be such that $\|v\|_{E_0}, \|w\|_{E_0} \le M$. On the one hand, we have that

$$\begin{aligned}\sum_{i\in\mathbb{Z}} \left|F_{0,i}(t, v_i(0)) - F_{0,i}(t, w_i(0))\right|^2 &\le k_0^2(t) \left(\max_{i\in\mathbb{Z}}(C_3(|v_i(0)|, |w_i(0)|))\right)^2 \sum_{i\in\mathbb{Z}} |v_i(0) - w_i(0)|^2 \\ &\le k_0^2(t) \chi_3^2\left(\|v\|_{E_0}, \|w\|_{E_0}\right) \|v-w\|_{E_0}^2,\end{aligned}$$

$$\begin{aligned}\sum_{i\in\mathbb{Z}} \left|F_{1,i}(v_i(-h_1)) - F_{1,i}(w_i(-h_1))\right|^2 &\le k_1^2(t) \left(\max_{i\in\mathbb{Z}}(C_4(|v_i(-h_1)|, |w_i(-h_1)|))\right)^2 \sum_{i\in\mathbb{Z}} |v_i(-h_1) - w_i(-h_1)|^2 \\ &\le k_1^2(t) \chi_4^2\left(\|v\|_{E_0}, \|w\|_{E_0}\right) \|v-w\|_{E_0}^2,\end{aligned}$$

where $\chi_j\left(\|v\|_{E_0}, \|w\|_{E_0}\right) = \max_{i\in\mathbb{Z}, s\in[-h,0]}(C_j(|v_i(s)|, |w_i(s)|))$. On the other hand,

$$\begin{aligned}&\sum_{i\in\mathbb{Z}} \left(\int_{-h}^{0} |b_i(t,s,v_i(s)) - b_i(t,s,w_i(s))|\, ds\right)^2 \\ &\le k_2^2(t) \left(\max_{i\in\mathbb{Z}, s\in[-h,0]}(C_5(|v_i(s)|, |w_i(s)|))\right)^2 \sum_{i\in\mathbb{Z}} \left(\int_{-h}^{0} k_3(s) |v_i(s) - w_i(s)|\, ds\right)^2 \\ &\le k_2^2(t) \chi_5^2\left(\|v\|_{E_0}, \|w\|_{E_0}\right) \sum_{i\in\mathbb{Z}} \int_{-h}^{0} k_3^2(s)\, ds \int_{-h}^{0} |v_i(s) - w_i(s)|^2\, ds \\ &= k_2^2(t) \chi_5^2\left(\|v\|_{E_0}, \|w\|_{E_0}\right) \int_{-h}^{0} k_3^2(s)\, ds \int_{-h}^{0} \sum_{i\in\mathbb{Z}} |v_i(s) - w_i(s)|^2\, ds \\ &\le k_2^2(t) \chi_5^2\left(\|v\|_{E_0}, \|w\|_{E_0}\right) h \int_{-h}^{0} k_3^2(s)\, ds\, \|v-w\|_{E_0}^2,\end{aligned}$$

where $\chi_5\left(\|v\|_{E_0}, \|w\|_{E_0}\right) = \max_{i\in\mathbb{Z}, s\in[-h,0]}(C_5(|v_i(s)|, |w_i(s)|))$. The fact that the sum and the integral can be exchanged follows easily using Lebesgue's theorem. Thus, there exist $K(M)$, $\beta(\cdot) \in L^1_{loc}(\mathbb{R})$ such that

$$\|f(t,v) - f(t,w)\|^2 \le \beta(t) K(M) \|v-w\|_{E_0}^2,$$

which proves the result. □

Then, if we assume conditions (C1)-(C6) and (15)-(16), Theorems 5, 8, 12, Corollary 11 and Proposition 19 imply that for any $\psi \in E_0$ there exists a unique global solution $u(\cdot) \in C^1([\tau, \infty), E)$ with $u(\tau) = \psi$.

Hence, as shown in Section 2.2, we can define the process U by putting $U(t, \tau, \psi) = u_t$, where $u(\cdot)$ is the unique solution to (9) with $\psi = u_0$. Moreover, this map is continuous with respect to the initial data ψ.

We obtain now the existence of a pullback attractor.

Theorem 31. *Assume conditions (C1)-(C6) and (15)-(16), (34). Then, the process U possesses a pullback* $\mathcal{D}_\eta$*-attractor* $\widehat{A}$*, which belongs to* $\mathcal{D}_\eta$.

Proof. Proposition 19, Lemma 28, Corollary 22 and Theorem 16 imply the existence of the pullback $\mathcal{D}_\eta$-attractor $\widehat{A}$. Since the sets $B_\eta(t)$ of the absorbing family are closed and $B_\eta \in \mathcal{D}_\eta$, we obtain that $\widehat{A} \in \mathcal{D}_\eta$. □

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