NON BAIRE MEASURE SPACES

by

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ABSTRACT.

Let Σ be an infinite σ-field of subsets of X. It is shown that there exists a finite real-valued nonnegative finitely additive measure μ on Σ such that the semimetric space (Σ, ρ) is not Baire, where $\rho(A, B) = \mu(A \Delta B)$.

1. INTRODUCTION.

The best known proof of the Vitali-Hahn-Saks theorem [4, p. 158] is based on the validity of the Baire category theorem for the semimetric space $M(\Sigma, \mu)$ associated with the measure space (X, Σ, μ). This proof is due to Saks [9]. Ando proved that the Vitali-Hahn-Saks theorem holds for finitely additive scalar - valued measures, but he returned to the more primitive "sliding hump" arguments formerly used by Lebesgue, Hahn and Nikodým.

In [3, p. 35] it is raised the problem wether $M(\Sigma, \mu)$ is a Baire space for every finite nonnegative finitely additive measure μ on a σ-field Σ. Armstrong and Prikry [2] solved this problem in the negative. In this paper it is proved, using a different construction, that, on every infinite σ-field Σ, there exits a finite nonnegative finitely additive measure μ such that $M(\Sigma, \mu)$ is not a Baire space.

2. NOTATIONS.

We denote by ω the least non zero limit ordinal, by N the set of all positive integers and by c the least ordinal of cardinality of the continuum.

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If Σ is a σ-field and μ a finite real-valued nonnegative finitely additive measure μ on Σ, we denote by $M(\Sigma, \mu)$ the semimetric space that is obtained endowing Σ with the semimetric $d(A, B) = \mu(A \Delta B)$ and by $M(\Sigma, \mu)$ the metric space determined by the canonical equivalence relation: $A \sim B$ if and only if

$$d(A, B) = 0.$$

We consider the two elements space $2 = \{0,1\}$ endowed with the discrete topology, which is a compact space. Let λ be the measure defined on $\{0,1\}$ by $\lambda(\{0\}) = \lambda(\{1\}) = 1/2$, and let μ be the product measure defined on the σ-field $\mathcal{B}(2^c)$ of Borel subsets of 2^c. Since μ is countably additive, $M(\mathcal{B}(2^c), \mu)$ is a complete metric space. We also have that $M(\mathcal{B}(2^c), \mu)$ can be endowed with a standard structure of complete Boolean algebra [10, S 17], whose operations are denoted by $\wedge$, $\vee$, $*$, its orderin by $\leqslant$ and its two designated elements by 0 and 1. The measure induced on $M(\mathcal{B}(2^c), \mu)$ by μ will be denoted by μ too.

If s is a function such that dom(s) is a finite subset of c and $\mathrm{ran}(p) \subset \{0,1\}$, we shall denote by I_s the equivalence class of the Borel subset B_s of 2^c defined by

$$B_s = \{ < x_\alpha : \alpha < c > : \; x_\alpha = s(\alpha) \text{ for every } \alpha \in \mathrm{dom}(s) \}.$$

The sets I_s will be called cylinders and the class of all cylinders in

$$M(\mathcal{B}(2^c), \mu)$$

will be denoted by $\mathcal{C}$.

For every cilinder I_s, we shall denote by $l(I_s)$ the cardinal number of dom(s) and we shall write $\mathrm{dom}(I_s) = \mathrm{dom}(s)$. If $K \in M(\mathcal{B}(2^c), \mu)$, we define $l(K)$ as the least $n < \omega$ satisfying that there exists a cylinder $I_s \leqslant K$ and $l(I_s) = n$. We shall write $l(K) = \omega$ if there not exists a cylinder $I_s \leqslant K$. For a cylinder I_s, the notation $l(I_s)$ is unambigous.

3. Main Results.

Recall that a family Γ of subsets of ω is said to be independent [12, p. 43] if, for any mutually distinct sets $X_1, \ldots, X_n, Y_1, \ldots, Y_m$ in Γ, we have

$$X_1 \cap \ldots \cap X_n \cap (\omega \setminus Y_1) \cap \ldots \cap (\omega \setminus Y_m) \neq \phi.$$

In order to determine the cardinality of the dual of the Banach space L^{∞} [0,1], Fichtenholz and Kantorovitch [5] proved that there exists an independent family of subsets of ω, whose cardinality is c. In Hausdorff [6] and in Kuratowski, Mostowski [8, p. 303] there appears a more simple proof of this fact. Using that result, we can easily prove that there exists a maximal family $\{A_{\alpha}: \alpha < c\}$ of independent subsets of $\Sigma = \mathcal{P}(\omega)$.

Set $f(A_{\alpha}) = I_{s^{\alpha}}$, where $\mathrm{dom}(s^{\alpha}) = \{\alpha\}$ and $s^{\alpha}(\alpha) = 0$. According to [12, 12.2], we can extend f to an homomorphism from the subalgebra of Σ generated by $\{A_{\alpha}: \alpha < c\}$ to the algebra $M(\mathcal{B}(2^{c}), \mu)$. This extension will be denoted by f too.

By the Sikorski extension theorem (cf. [11], [12, 33.1]), f can be extended to a Boolean algebra homomorphism, denoted by f again, from Σ to $\dot{M}(\mathcal{B}(2^{c}), \mu)$ as $\dot{M}(\mathcal{B}(2^{c}), \mu)$ is a complete Boolean algebra. Then a positive finitely additive measure can be defined on Σ by $\upsilon(A) = \mu(f(A))$.

Since the family $\{A_{\alpha}: \alpha < c\}$ is maximal, for every $B \in \Sigma$, there exist ordinal numebrs $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m < c$ such that either

$$B \supset A_{\alpha_1} \cap \ldots \cap A_{\alpha_n} \cap (\omega \setminus A_{\beta_1}) \cap \ldots \cap (\omega \setminus A_{\beta_m})$$

or

$$\omega \setminus B \supset A_{\alpha_1} \cap \ldots \cap A_{\alpha_n} \cap (\omega \setminus A_{\beta_1}) \cap \ldots \cap (\omega \setminus A_{\beta_m}).$$

Hence there exists a cylinder I_s such that either $I_s \leqslant f(B)$ or $I_s \leqslant f(B)^*$.

With these notations, we have:

Proposition. $M(\Sigma, \upsilon)$ is not a Baire space.

Proof. It is sufficient to prove that $\dot{M}(\Sigma, \upsilon)$ is not a Baire space. Since $\dot{M}(\Sigma, \upsilon)$ is isometric to $f(\Sigma)$ and $f(\Sigma)$ is dense in $\dot{M}(\mathcal{B}(2^{c}), \mu)$, we only need to prove that $f(\Sigma)$ is of first category in $\dot{M}(\mathcal{B}(2^{c}), \mu)$.

Denote by $\mathcal{A}$ the set of all $K \in \dot{M}(\mathcal{B}(2^{c}), \mu)$ for which there exists $I \in \mathcal{C}$ verifying either $I \leqslant K$ or $I \leqslant K^*$. We shall prove that $\mathcal{A}$ is of first category in $M(\mathcal{B}(2^{c}), \mu)$.

Let $\mathcal{V}$ be the set of $V \in M(\mathcal{B}(2^{c}), \mu)$ that can be represented by an open set in 2^{c}. Put

$$\mathfrak{D} = \{K \in \dot{M}(\mathcal{B}(2^c), \mu): (\forall m \in N)(\exists \epsilon > 0)(\forall V \in \upsilon)$$

$$[\mu(K \Delta V) < \epsilon \to l(V) > m]\}$$

and $\mathfrak{D}^* = \{K \in \dot{M}(\mathcal{B}(2^c), \mu): K^* \in \mathfrak{D}\}$.

Since $\mathcal{A} \cap \mathfrak{D} \cap \mathfrak{D}^* = \phi$, in order to prove that $\mathcal{A}$ is of first category we shall show that $\mathfrak{D} \cap \mathfrak{D}^*$ is residual. As the mapping $*$ is an isometry, it is sufficient to prove that $\mathfrak{D}$ is residual.

Since $\mathfrak{D} = \bigcap_m \bigcup_n \mathfrak{D}_{n,m}$, where, for every pair n, m of natural number,

$$\mathfrak{D}_{n,m} = \{K \in M(\mathcal{B}(2^c), \mu): (\forall V \in \upsilon)(\mu(K \Delta V) < \frac{1}{n} \to l(V) > m)\},$$

we shall see that, for every $m \in N$, $\bigcup_n \mathfrak{D}_{n,m}$ is a dense open set.

1. $\bigcup_n \mathfrak{D}_{n,m}$ is open: Suppose that $K \in \bigcup_n \mathfrak{D}_{n,m}$. Then $K \in \mathfrak{D}_{n,m}$ for some $n \in N$. If $\mu(K \Delta L) < \frac{1}{2n}$, we have $L \in \mathfrak{D}_{2n,m}$ because $\mu(L \Delta V) < \frac{1}{2n}$ implies $\mu(K \Delta V) < \frac{1}{2n} + \frac{1}{2n} < \frac{1}{n}$ and, since $K \in \mathfrak{D}_{n,m}$ $l(V) > m$. Therefore the ball in $\dot{M}(\mathcal{B}(2^c), \mu)$ with center K and radius $\frac{1}{2n}$ is contained in $\bigcup_n \mathfrak{D}_{n,m}$.

2. $\bigcup_n \mathfrak{D}_{n,m}$ is dense: Since the set of all finite unions of cylinders is dense in $\dot{M}(\mathcal{B}(2^c), \mu)$, we have only to prove that given $\epsilon > 0$ and $A = \bigcup_{i=1}^{k} I_i$, where $I_i \in \mathcal{C}$ for $1 \leqslant i \leqslant k$, there exists $K \in \bigcup_n \mathfrak{D}_{n,m}$ such that $\mu(A \Delta K) < \epsilon$. We can assume that $\mathrm{dom}(I_i) = \mathrm{dom}(I_j)$ and $l(I_i) = l(I_j) = r$ for every $i, j \leqslant k$.

Let $s > m$ be a natural number such that $\binom{s}{m} 2^{m-s} < \epsilon$. and let $a \subset c$ be a finite set satisfying $a \cap \mathrm{dom}(I_i) = \phi$ and $\mathrm{card}(a) = s$.

If $I \in \mathcal{C}$, $\mathrm{dom}(I) \subset a$ and $l(I) = m$, we choose an $\alpha(I) \in \mathcal{C}$ such that

$$\mathrm{dom}(\alpha(I)) = a \text{ and } \alpha(I) \leqslant I.$$

Put $B = \bigvee_I \alpha(I)$ where the join is taken over all $I \in \mathcal{C}$ such that $l(I) = n$ and dom(I) $\subset$ a. Let $K = A \wedge B^*$. As $\mu(K \,\Delta\, A) \leqslant \Sigma\, \mu(\alpha(I)) \leqslant \binom{s}{m} 2^m 2^{-s} < \epsilon$, it remains to show that $K \in \bigcup_n \mathfrak{D}_{n,m}$.

Let $n \in N$ and $1/n < 2^{-s}$. We shall prove that $K \in \mathfrak{D}_{n,m}$. Suppose

$$\mu(K \,\Delta\, V) < 1/n < 2^{-s},\ J \in \mathcal{C}$$

and $J \leqslant V$. We have $J \wedge B \leqslant J \wedge K^* \leqslant V \wedge K^* \leqslant V \,\Delta\, K$, so $\mu(J \wedge B) < 2^{-s}$. Nevertheless, if card(dom(J) $\cap$ a) $= m_1 \leqslant m < s$, we have

$$\mu(J \wedge B) \geqslant \frac{1}{2^{l(J) - m_1}} 2^{m - m_1} \frac{1}{2^s} = 2^{m - l(J) - s}.$$

Hence $m < l(J)$.

Remark. Since in the space $M(\mathcal{B}(2^c), \mu)$ we have $\mathfrak{D} \cap \mathfrak{D}^* \neq \phi$, there exists a $K \in M(\mathcal{B}(2^c), \mu)$ such that, for every $I \in \mathcal{C}$, $0 < \mu(K \wedge I) < \mu(I)$. This result can not be obtained by the method used by Kirk [7] because the metric space $M(\mathcal{B}(2^c), \mu)$ is not separable.

Theorem. Let Σ be an infinite σ-field of subsets of the set X. There exists a finite real valued nonnegative finitely additive measure μ on Σ such that $M(\Sigma, \mu)$ is not a Baire space.

Proof. Using the Marczewski indicator $\Sigma\, 3^{-n} \chi_{A_n}$ for a sequence of mutually distinct elements of Σ, it is easy to construct a sequence $< B_n : n < \omega >$ in Σ of pairwise disjoint sets. In this case, Σ contains a sub-σ-field isomorphic to $\mathcal{P}(\omega)$. Therefore the construction of the proposition can be carried out in this case.

REFERENCES

1. T. Ando, Convergent sequences of finitely additive measures, Pacific J. Math. 11 (1961), 395-404.

2. T. E. Armstrong and K. Prikry, On the semimetric on a Boolean algebra induced by a finitely additive probability measure, Pacific J. Math. 99 (1982), 249-264.

3. J. Diestel and J. J. Uhl, Vector measures, Amer. Math. Soc., Providence, Rhode Island, (1977).

4. N. Dunford and J. T. Schwartz, Linear operators I (General Theory), Interscience, New York, (1958).

5. G. Fichtenholz and L. Kantorovitch, Sur les opérations linéaires dans l'espace des fonctions bornées, Studia Math. 5 (1934), 70-98.

6. F. Hausdorff, Uber zwei Sätze von G. Fichtenholz und L. Kantorovitch, Studia Marh. 6 (1936), 18-19.

7. R. B. Kirk, Sets which split families of measurable sets, Amer. Math, Monthly, 79 (1972), 884-886.

8. K. Kuratowski and A. Mostowski, Set theory, North Holland, Amsterdam, (1968).

9. S. Saks, Addition to the note on some functionals, Trans, Amer. Math. Soc. 35 (1933), 965-970.

10. Z. Semadeni, Banach spaces of continuous functions I, Polish Scientific Publishers, Warszawa, (1971).

11. R. Sikorski, A Theorem on extension of homomorphism, Ann. Soc. Pol. Math. 21 (1948), 332-335.

12. R. Sikorski, Boolean algebras (2^{nd} edition), Springer, Berlin, (1964).

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