PROCEEDINGS OF THE ROYAL SOCIETY A

Proc. R. Soc. A (2007) **463**, 2153–2164
doi:10.1098/rspa.2007.1867
Published online 26 June 2007

Time-periodic solutions for a generalized Boussinesq model with Neumann boundary conditions for temperature

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The aim of this work is to prove the existence of regular time-periodic solutions for a generalized Boussinesq model (with nonlinear diffusion for the equations of velocity and temperature). The main idea is to obtain higher regularity (of H^3 type) for temperature than for velocity (of H^2 type), using specifically the Neumann boundary condition for temperature. In fact, the case of Dirichlet condition for temperature remains as an open problem.

Keywords: regularity; time-periodic solutions; nonlinear diffusion; Navier–Stokes type equations

1. Introduction

Assume that $\Omega \subset \mathbb{R}^N$ ($N=2$ or 3) is a regular bounded domain. This paper is concerned with a partial differential problem governing the coupled mass and heat flow of a viscous incompressible fluid considering a generalized Boussinesq approximation by assuming that viscosity and heat conductivity are explicit functions depending on temperature (which is a much more natural condition that takes viscosity and heat conductivity as constants). The equations involved are

$$\begin{cases} \partial_t \boldsymbol{u} - \nabla\cdot(\nu(\theta)\nabla \boldsymbol{u}) + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p = \alpha \boldsymbol{g}\theta + \boldsymbol{f}, \\ \nabla\cdot \boldsymbol{u} = 0, \\ \partial_t\theta - \nabla\cdot(k(\theta)\nabla\theta) + (\boldsymbol{u}\cdot\nabla)\theta = 0 \end{cases} \tag{1.1}$$

in $\Omega\times[0, \infty)$, where $\boldsymbol{u}(x, t)\in\mathbb{R}^N$ is the velocity field at point $x\in\Omega$ and time $t\in[0, +\infty)$; $p(x, t)\in\mathbb{R}$ is the (hydrostatic) pressure; and $\theta(x, t)\in\mathbb{R}$ is the temperature. Data are $\boldsymbol{g}(x,t)\in\mathbb{R}^N$, the gravitational field with $\alpha>0$, a constant associated with the coefficient of volume expansion, $\boldsymbol{f}(x,t)\in\mathbb{R}^N$, the external forces, $\nu(\cdot):\mathbb{R}\to\mathbb{R}$, the kinematic viscosity, and $k(\cdot):\mathbb{R}\to\mathbb{R}$, the thermal conductivity.

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Received 14 March 2007
Accepted 14 May 2007

We will search for a triplet $\{\boldsymbol{u}, p, \theta\}$ regular periodic solution of (1.1) in $\Omega \times [0, \infty)$, together with the Dirichlet–Neumann boundary conditions,

$$\boldsymbol{u} = 0, \qquad \partial_n \theta = 0 \quad \text{on } [0, \infty) \times \partial\Omega. \tag{1.2}$$

(the first one is a non-slip condition and the second one is a null heat flux condition) and the time-periodic condition,

$$\boldsymbol{u}(0) = \boldsymbol{u}(T), \qquad \theta(0) = \theta(T) \quad \text{in } \Omega. \tag{1.3}$$

It is important to remark that stationary solutions are not valid here, because the external force f is time dependent.

Moreover, the problem with non-homogeneous boundary conditions can be treated in a similar manner, using adequate lifting functions, rewriting the problem (1.1)–(1.3) with a function f depending on boundary data.

The existence and uniqueness of the initial value problem related to (1.1), and with Dirichlet boundary conditions for velocity and temperature, were proved in the work of Lorca & Boldrini (1999). The stationary problem is studied in Lorca & Boldrini (1996) for bounded domains and Notte-Cuello & Rojas-Medar (1998) for exterior domains. On the other hand, the work of Moretti *et al.* (2002) is devoted to the existence of reproductive weak solutions in exterior domains. The classical Boussinesq model, where ν and k are positive constants, has been analysed in great extent (see, for instance, Óeda 1988; Morimoto 1992).

The arguments used in Lorca & Boldrini (1999) in order to obtain regular solutions (and uniqueness) are not valid to find reproductivity, since the initial conditions play a fundamental role. Our contribution in this paper is to obtain higher-order estimates for the temperature than in Lorca & Boldrini (1999); namely in Lorca & Boldrini (1999), $H^2(\Omega)$ regularity is obtained for velocity and temperature, but now we will arrive at $H^3(\Omega)$ regularity for the temperature. Consequently, a periodic condition for the time derivative of temperature also holds, i.e. $\partial_t\theta(0) = \partial_t\theta(T)$. In addition, the arguments used in this paper are remarkably simpler than the used ones in Lorca & Boldrini (1999). On the contrary, now the regularity obtained for the solution is not sufficient to prove uniqueness, because more regularity than $H^2(\Omega)$ for the velocity is necessary.

(*a*) *Notation*

- In general, the notation will be abridged. We set $L^p = L^p(\Omega)$, $p \geq 1$, $H_0^1 = H_0^1(\Omega)$, etc. If $X = X(\Omega)$ is a space of functions defined in the open set Ω, we denote by $L^p(X)$ the Banach space $L^p(0, T; X)$. In addition, boldface letters will be used for vectorial spaces, for instance $\boldsymbol{L}^2 = L^2(\Omega)^N$.
- The L^p norm is denoted by $|\cdot|_p$, $1 \leq p \leq \infty$ and the H^{m} norm by $\|\cdot\|_m$.
- We set $\mathcal{V}$ the space formed by all fields $\boldsymbol{v} \in C_0^\infty(\Omega)^N$ satisfying $\nabla \cdot \boldsymbol{v} = 0$. We denote by $\boldsymbol{H}$ (respectively $\boldsymbol{V}$) the closure of $\mathcal{V}$ in $\boldsymbol{L}^2$ (respectively $\boldsymbol{H}^1$). $\boldsymbol{H}$ and $\boldsymbol{V}$ are Hilbert spaces for the norms $|\cdot|_2$ and $\|\cdot\|_1$, respectively. Furthermore,

$$\boldsymbol{H} = \{\boldsymbol{u} \in \boldsymbol{L}^2;\ \nabla \cdot \boldsymbol{u} = 0,\ \ \boldsymbol{u} \cdot \boldsymbol{n} = 0 \text{ on } \partial\Omega\},$$
$$\boldsymbol{V} = \{\boldsymbol{u} \in \boldsymbol{H}^1;\ \nabla \cdot \boldsymbol{u} = 0,\ \ \boldsymbol{u} = 0 \text{ on } \partial\Omega\}.$$

— Let P be the orthogonal projection of $\boldsymbol{L}^2$ onto $\boldsymbol{H}$. A denotes the Stokes operator $A = -P\Delta$ defined in $\boldsymbol{V} \cap \boldsymbol{H}^2$.

— It is easy to deduce, from the convection–diffusion equation for θ, the equality $\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \theta(x,t) = 0$. Then, we can fix θ such that $\int_\Omega \theta = 0$. Therefore, let us consider the following spaces:

$$H_N^k = \left\{\theta \in H^k;\ \frac{\partial \theta}{\partial n} = 0 \text{ on } \partial\Omega\ ,\int_\Omega \theta = 0\right\},$$

where $k=2, 3$. Hence, H_N^k is a closed subspace of H^k. Consequently, $|\Delta\theta|_2$ is equivalent to $\|\theta\|_2$ in H_N^2 and $|\nabla\Delta\theta|_2$ is equivalent to $\|\theta\|_3$ in H_N^3 (Veiga 1983).

(b) *Some interpolation inequalities*

We will use the following classical interpolation and Sobolev inequalities (for three-dimensional domains):

$$|v|_6 \le C\|v\|_1, \qquad |v|_3 \le |v|_2^{1/2}\|v\|_1^{1/2} \quad \forall\, v \in H^1,$$

and

$$|v|_\infty \le C\|v\|_1^{1/2}\|v\|_2^{1/2} \quad \forall\, v \in H^2.$$

In this work, the following result (Lorca & Boldrini 1999) will be useful:

Lemma 1.1. *Let $\boldsymbol{u} \in \boldsymbol{V} \cap \boldsymbol{H}^2$ and consider the Helmholtz decomposition of $-\Delta\boldsymbol{u}$, i.e. $-\Delta\boldsymbol{u} = A\boldsymbol{u} + \nabla q$, where $q \in H^1$ is taken such that $\int_\Omega q\,\mathrm{d}x = 0$ and A is the Stokes operator. Then,*

$$\|q\|_1 \le C|A\boldsymbol{u}|_2.$$

Moreover, for every $\delta > 0$, there exists a positive constant C_δ (independent of $\boldsymbol{u}$), such that

$$|q|_2 \le C_\delta|\nabla\boldsymbol{u}|_2 + \delta|A\boldsymbol{u}|_2.$$

2. The main result

Definition 2.1. It will be said that $(\boldsymbol{u}, p, \theta)$ is a regular solution of (1.1)–(1.3) in $(0, T)$, if

$$\boldsymbol{u} \in L^2(\boldsymbol{H}^2) \cap L^\infty(\boldsymbol{H}^1), \qquad \partial_t\boldsymbol{u} \in L^2(\boldsymbol{L}^2) \quad \text{and} \quad p \in L^2(H^1),$$

$$\theta \in L^2(H_N^3) \cap L^\infty(H_N^2) \quad \text{and} \quad \partial_t\theta \in L^2(H_N^1)$$

satisfying (1.1) a.e. in $(0, T) \times \Omega$, boundary conditions (1.2) and time reproductivity conditions (1.3) in the sense of spaces $\boldsymbol{V}$ and H_N^2, respectively.

Note that we have imposed higher regularity for θ than for $\boldsymbol{u}$.

Theorem 2.1. *Let $T > 0$ and let Ω be a bounded domain in $\mathbb{R}^N$ ($N=2$ or 3) with a boundary of class $C^{2,1}$. Let the functions $\nu \in C^1(\mathbb{R})$ and $k \in C^2(\mathbb{R})$, such that they satisfy*

$$0 < \nu_{\min} \le \nu(s) \le \nu_{\max} \quad \text{and} \quad 0 < k_{\min} \le k(s) \le k_{\max} \quad \text{in } \mathbb{R},$$

$$|\nu'(s)| \le \nu'_{\max}, \quad |k'(s)| \le k'_{\max}, \quad |k''(s)| \le k''_{\max}.$$

Assume that $\boldsymbol{f} \in L^2(\boldsymbol{L}^2)$, $\boldsymbol{g} \in L^\infty(\boldsymbol{L}^2)$ *and* $\|\boldsymbol{f}\|_{L^2(0,T;\, L^2)}$ *are small enough, then there exists a regular (and small) time-periodic solution of (1.1)–(1.3) in* $(0, T)$. *Moreover, this solution also verifies* $\partial_t\theta(0) = \partial_t\theta(T)$.

Remark. More concretely, from the proof of theorem 2.1 (§5), the following hypothesis of smallness of $\boldsymbol{f}$ will be imposed:

$$\int_0^T |\boldsymbol{f}|_2^2 \mathrm{d}t \le \delta(1 - \mathrm{e}^{-\bar{C}\,T}),$$

where $\bar{C} = \bar{C}(\Omega, \nu, k) > 0$ and $\delta > 0$ is small enough. Moreover, the periodic solution obtained, $(\boldsymbol{u}, \theta)$, verifies $\|\boldsymbol{u}(0)\|_1^2 + \|\theta(0)\|_2^2 + \|\partial_t\theta(0)\|_0^2 \le \delta$.

Remark. The uniqueness of solutions furnished by theorem 2.1 remains open, because higher regularity for the velocity is necessary. To obtain H^3, regularity for the velocity seems complicated because the argument made in the proof of lemma 4.2 in order to get H^3 regularity is based on the Neumann condition, but we have Dirichlet condition for $\boldsymbol{u}$.

The proof of theorem 2.1 will be given in §5. The method is based on the Galerkin approximation with spectral basis (defined in §3) and some differential inequalities in regular norms given in §4.

3. The Galerkin initial-boundary problem

Let $\{\phi_i\}_{i\ge1}$ and $\{\varphi_i\}_{i\ge1}$ be the 'special' basis of $\boldsymbol{V}$ and $\boldsymbol{H}_0^1(\Omega)$, respectively, formed by eigenfunctions of the Stokes and the Poisson problems as follows:

$$\begin{cases} A\phi_i = \lambda_i\phi_i & \text{in } \Omega \\ \phi_i = 0 & \text{on } \partial\Omega \end{cases} \qquad \begin{cases} -\Delta\varphi_i = \mu_i\varphi_i & \text{in } \Omega \\ \partial_n\varphi_i = 0 & \text{on } \partial\Omega \end{cases}$$

with $\|\phi_{\rm i}\|_1 = 1$, $\|\varphi_{\rm i}\|_1 = 1$, for all i and $\int_\Omega \varphi_i = 0$. Let $\boldsymbol{V}^m$ and W^m be the finite-dimensional subspaces spanned by $\{\phi_1, \phi_2, \dots, \phi_m\}$ and $\{\varphi_1, \varphi_2, \dots, \varphi_m\}$, respectively.

For each $m \ge 1$, given $\boldsymbol{u}_{0m} \in \boldsymbol{V}^m$ and $\theta_{0m} \in W^m$, we seek an approximate solution $(\boldsymbol{u}_m, \theta_m)$, with $\boldsymbol{u}_m : [0, T] \mapsto \boldsymbol{V}^m$ and $\theta_m : [0, T] \mapsto W^m$, verifying the following variational formulation a.e. in $t \in (0, T)$:

$$\begin{cases} (\partial_t \boldsymbol{u}_m(t), \boldsymbol{v}_m) + ((\boldsymbol{u}_m(t)\cdot\nabla)\boldsymbol{u}_m(t), \boldsymbol{v}_m) + (\nu(\theta_m(t))\nabla\boldsymbol{u}_m(t), \nabla\boldsymbol{v}_m) \\ -(\alpha\theta_m(t)\boldsymbol{g}, \boldsymbol{v}_m) - (\boldsymbol{f}, \boldsymbol{v}_m) = 0 \ \ \forall\, \boldsymbol{v}_m \in \boldsymbol{V}^m \\ (\partial_t\theta_m(t), e_m) + ((\boldsymbol{u}_m(t)\cdot\nabla)\theta_m(t), e_m) + (k(\theta_m(t))\nabla\theta_m(t), \nabla e_m) = 0 \ \ \forall\, e_m \in W^m \\ \boldsymbol{u}_m(0) = \boldsymbol{u}_{0m}, \ \ \theta_m(0) = \theta_{0m}. \end{cases} \tag{3.1}$$

If we put

$$\boldsymbol{u}_m(t) = \sum_{j=1}^m \xi_{i,m}(t)\phi_i \ \text{ and } \ \theta_m(t) = \sum_{j=1}^m \zeta_{i,m}(t)\varphi_i,$$

then (3.1) can be rewritten as a first-order ordinary differential system (in normal form) associated with the unknowns $(\xi_{i,m}(t), \zeta_{i,m}(t))$. Then, one has the existence of a maximal solution (defined in some interval $[0, \tau_m)\subset[0, T]$) of the related Cauchy problem. Moreover, from *a priori* estimates (independent of m) that will be obtained below, in particular, one has that $\tau_m = T$. Finally, using regularity of the chosen spectral basis, uniqueness of approximate solution holds (Climent-Ezquerra *et al.* 2006).

4. Differential inequalities in regular norms

In the sequel, γ and ε will denote some constants sufficiently small. By C, we will denote different constants, independent of data, and γ and ε.

Lemma 4.1. *For each γ, $\varepsilon>0$ sufficiently small, there exists a constant $K=K(\gamma, \varepsilon)>0$, such that*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (\nu(\theta_m)+1)|\nabla \boldsymbol{u}_m|^2 + \nu_{\min}\|\boldsymbol{u}_m\|_2^2 + |\partial_t \boldsymbol{u}_m|_2^2 \le \gamma\|\partial_t\theta_m\|_1^2 + \varepsilon\|\boldsymbol{u}_m\|_2^2\|\theta_m\|_2 + K\Big(\|\boldsymbol{u}_m\|_1^6 + \|\boldsymbol{u}_m\|_1^2\|\theta_m\|_2^4 + \|\boldsymbol{g}\|_{L^\infty(L^2)}^2\|\theta_m\|_2^2 + |\boldsymbol{f}|_2^2\Big). \tag{4.1}$$

Proof. Lemma 1.1 is crucial in this proof because the nonlinear diffusion term, $\nabla\cdot(\nu(\theta_m)\nabla\boldsymbol{u}_m)$, is decomposed in $\nu(\theta_m)\Delta\boldsymbol{u}_m$ and $\nu'(\theta_m)\nabla\theta\nabla\boldsymbol{u}_m$, and the control on the difference between $A\boldsymbol{u}_m$ and $-\Delta\boldsymbol{u}_m$ is necessary in the treatment of the Laplacian term, $\nu(\theta_m)\Delta\boldsymbol{u}_m$.

First, taking $\boldsymbol{v}=A\boldsymbol{u}_m$ as test function in the $\boldsymbol{u}_m$-system of (3.1) (A is the Stokes operator mentioned in lemma 1.1), one has

$$(\partial_t\boldsymbol{u}_m, A\boldsymbol{u}_m) - (\nabla\cdot(\nu(\theta_m)\nabla\boldsymbol{u}_m), A\boldsymbol{u}_m) + ((\boldsymbol{u}_m\cdot\nabla)\boldsymbol{u}_m, A\boldsymbol{u}_m) - \alpha(\boldsymbol{g}\theta_m, A\boldsymbol{u}_m) = (\boldsymbol{f}, A\boldsymbol{u}_m). \tag{4.2}$$

We can write the first term as

$$(\partial_t\boldsymbol{u}_m, A\boldsymbol{u}_m) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{u}_m\|_1^2.$$

The second term of (4.2) is split as follows (using the Helmholtz decomposition $\Delta\boldsymbol{u}_m = -A\boldsymbol{u}_m + \nabla q_m$),

$$-(\nabla\cdot(\nu(\theta_m)\nabla\boldsymbol{u}_m), A\boldsymbol{u}_m) = (\nu(\theta_m)A\boldsymbol{u}_m, A\boldsymbol{u}_m) + (\nu(\theta_m)\nabla q_m, A\boldsymbol{u}_m) - (\nu'(\theta_m)\nabla\theta_m\nabla\boldsymbol{u}_m, A\boldsymbol{u}_m).$$

Taking into account that

$$\begin{aligned}(\nu(\theta_m)\nabla q_m, A\boldsymbol{u}_m) &= -(q_m, \nabla\cdot(\nu(\theta_m)A\boldsymbol{u}_m)) \\ &= -(q_m, \nu'(\theta_m)\nabla\theta_m A\boldsymbol{u}_m) - (q_m, \nu(\theta_m)\nabla\cdot A\boldsymbol{u}_m) \\ &= -(q_m, \nu'(\theta_m)\nabla\theta_m A\boldsymbol{u}_m),\end{aligned}$$

since $\nabla\cdot A\boldsymbol{u}_m=0$, hence the second term of (4.2) becomes

$$-(\nabla\cdot(\nu(\theta_m)\nabla\boldsymbol{u}_m),A\boldsymbol{u}_m)=(\nu(\theta_m)A\boldsymbol{u}_m,A\boldsymbol{u}_m)-(q_m,\nu'(\theta_m)\nabla\theta_m A\boldsymbol{u}_m)\\ -(\nu'(\theta_m)\nabla\theta_m\nabla\boldsymbol{u}_m,A\boldsymbol{u}_m).$$

Then, (4.2) can also be written as follows (using $\nu(\theta_m)\geq\nu_{\min}>0$):

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{u}_m\|_1^2+\nu_{\min}\|\boldsymbol{u}_m\|_2^2&\leq-((\boldsymbol{u}_m\cdot\nabla)\boldsymbol{u}_m,A\boldsymbol{u}_m)+\alpha(\boldsymbol{g}\theta_m,A\boldsymbol{u}_m)\\&+(q_m,\nu'(\theta_m)\nabla\theta_m A\boldsymbol{u}_m)+(\nu'(\theta_m)\nabla\theta_m\nabla\boldsymbol{u}_m,A\boldsymbol{u}_m)+(\boldsymbol{f},A\boldsymbol{u}_m)\\&=I_1+I_2+I_3+I_4+I_5.\end{aligned}\tag{4.3}$$

The first two terms and the last term on the right-hand side of (4.3) are bounded respectively by

$$I_1\leq\gamma\|\boldsymbol{u}_m\|_2^2+C_\gamma\|u_m\|_1^6,\quad I_2\leq\gamma\|\boldsymbol{u}_m\|_2^2+C_\gamma|\boldsymbol{g}|_2^2\|\theta_m\|_2^2,$$

and

$$I_5\leq\gamma\|\boldsymbol{u}_m\|_2^2+C_\gamma|\boldsymbol{f}|_2^2.$$

In order to estimate the third term, we use lemma 1.1 (and $|\nu'(\theta_m)|\leq\nu'_{\max}$)

$$\begin{aligned}I_3&\leq\nu'_{\max}|q_m|_3|\nabla\theta_m|_6|A\boldsymbol{u}_m|_2\leq C|q_m|_2^{1/2}\|q_m\|_1^{1/2}\|\theta_m\|_2\|\boldsymbol{u}_m\|_2\\&\leq C\Big(C_\varepsilon\|\boldsymbol{u}_m\|_1^{1/2}+\varepsilon\|\boldsymbol{u}_m\|_2^{1/2}\Big)\|\boldsymbol{u}_m\|_2^{3/2}\|\theta_m\|_2\leq C_\varepsilon\|\boldsymbol{u}_m\|_1^{1/2}\|\boldsymbol{u}_m\|_2^{3/2}\|\theta_m\|_2\\&\quad+\varepsilon\|\boldsymbol{u}_m\|_2^2\|\theta_m\|_2\leq\gamma\|\boldsymbol{u}_m\|_2^2+C_{\varepsilon,\gamma}\|\boldsymbol{u}_m\|_1^2\|\theta_m\|_2^4+\varepsilon\|\boldsymbol{u}_m\|_2^2\|\theta_m\|_2.\end{aligned}$$

While for the fourth term,

$$\begin{aligned}I_4&\leq\nu'_{\max}|\nabla\theta_m|_6|\nabla\boldsymbol{u}_m|_3|A\boldsymbol{u}_m|_2\leq C\|\theta_m\|_2\|\boldsymbol{u}_m\|_1^{1/2}\|\boldsymbol{u}_m\|_2^{3/2}\\&\leq\gamma\|\boldsymbol{u}_m\|_2^2+C_\gamma\|\boldsymbol{u}_m\|_1^2\|\theta_m\|_2^4.\end{aligned}$$

Consequently, choosing γ small enough, from (4.3) we arrive at

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{u}_m\|_1^2+\nu_{\min}\|\boldsymbol{u}_m\|_2^2\leq C_\varepsilon\big(\|\boldsymbol{u}_m\|_1^6+\|\boldsymbol{u}_m\|_1^2\|\theta_m\|_2^4+|\boldsymbol{g}|_2^2\|\theta_m\|_2^2+|\boldsymbol{f}|_2^2\big)\\+\varepsilon\|\boldsymbol{u}_m\|_2^2\|\theta_m\|_2.\tag{4.4}$$

On the other hand, using $\partial_t\boldsymbol{u}_m$ as a test function in the $\boldsymbol{u}_m$-system of (3.1), one obtains

$$\begin{aligned}&(\partial_t\boldsymbol{u}_m,\partial_t\boldsymbol{u}_m)+(\nu(\theta_m)\nabla\boldsymbol{u}_m,\partial_t\nabla\boldsymbol{u}_m)+((\boldsymbol{u}_m\cdot\nabla)\boldsymbol{u}_m,\partial_t\boldsymbol{u}_m)\\&\quad=\alpha(\boldsymbol{g}\theta_m,\partial_t\boldsymbol{u}_m)+(\boldsymbol{f},\partial_t\boldsymbol{u}_m).\end{aligned}\tag{4.5}$$

By taking into account that the second term of the left-hand side of (4.5) can be written as

$$(\nu(\theta_m)\nabla\boldsymbol{u}_m,\partial_t\nabla\boldsymbol{u}_m)=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\nu(\theta_m)\nabla\boldsymbol{u}_m,\nabla\boldsymbol{u}_m)-\frac{1}{2}(\partial_t(\nu(\theta_m))\nabla\boldsymbol{u}_m,\nabla\boldsymbol{u}_m),$$

we deduce from (4.5) that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega}\nu(\theta_m)|\nabla \boldsymbol{u}_m|^2 + |\partial_t \boldsymbol{u}_m|_2^2 \le -((\boldsymbol{u}_m\cdot\nabla)\boldsymbol{u}_m, \partial_t \boldsymbol{u}_m) + \alpha(\boldsymbol{g}\theta_m, \partial_t \boldsymbol{u}_m) + \frac{1}{2}(\partial_t(\nu(\theta_m))\nabla \boldsymbol{u}_m, \nabla \boldsymbol{u}_m) + (\boldsymbol{f}, \partial_t \boldsymbol{u}_m) = J_1 + J_2 + J_3 + J_4. \tag{4.6}$$

The first two terms and the last term on the right-hand side of (4.6) are bounded, respectively, by

$$J_1 \le \gamma\big(|\partial_t \boldsymbol{u}_m|_2^2 + \|\boldsymbol{u}_m\|_2^2\big) + C_\gamma\|\boldsymbol{u}_m\|_1^6, \qquad J_2 \le \gamma|\partial_t \boldsymbol{u}_m|_2^2 + C_\gamma|\boldsymbol{g}|_2^2\|\theta_m\|_2^2$$

and

$$J_4 \le \gamma|\partial_t \boldsymbol{u}_m|_2^2 + C_\gamma|\boldsymbol{f}|_2^2.$$

Lastly, we go into detail for the third term,

$$J_3 = \frac{1}{2}(\nu'(\theta_m)\partial_t\theta_m\nabla \boldsymbol{u}_m, \nabla \boldsymbol{u}_m) \le \frac{1}{2}\nu'_{\max}|\partial_t\theta_m|_6|\nabla \boldsymbol{u}_m|_3|\nabla \boldsymbol{u}_m|_2 \le C\|\partial_t\theta_m\|_1\|\boldsymbol{u}_m\|_1^{3/2}\|\boldsymbol{u}_m\|_2^{1/2} \le \gamma\big(\|\partial_t\theta_m\|_1^2 + \|\boldsymbol{u}_m\|_2^2\big) + C_\gamma\|\boldsymbol{u}_m\|_1^6.$$

Consequently, choosing γ small enough,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega}\nu(\theta_m)|\nabla \boldsymbol{u}_m|^2 + |\partial_t \boldsymbol{u}_m|_2^2 \le \gamma\big(\|\partial_t\theta_m\|_1^2 + \|\boldsymbol{u}_m\|_2^2\big) + C_\gamma\big(\|\boldsymbol{u}_m\|_1^6 + |\boldsymbol{g}|_2^2\|\theta_m\|_2^2 + |\boldsymbol{f}|_2^2\big). \tag{4.7}$$

Finally, (4.4) and (4.7) prove the lemma.

Lemma 4.2. *For each $\gamma > 0$ small enough, there exists $C_\gamma > 0$ such that*

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\|\theta_m\|_2^2 + |\partial_t\theta_m|_2^2\big) + k_{\min}\big(\|\theta_m\|_3^2 + \|\partial_t\theta_m\|_1^2\big) \le \gamma|\partial_t u_m|_2^2 + C_\gamma\big(\|\theta_m\|_2^6 + \|\theta_m\|_2^4|\partial_t\theta_m|_2^2 + \|\theta_m\|_2^2\|u_m\|_1^4\big). \tag{4.8}$$

■

Proof. Differentiating with respect to the time the θ_m-equation of (3.1) and multiplying by $\partial_t\theta_m$ as test function, using that $(\boldsymbol{u}_m\cdot\nabla\partial_t\theta_m, \partial_t\theta_m) = 0$, one obtains

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\partial_t\theta_m|_2^2 + (\partial_t(k(\theta_m)\nabla\theta_m), \partial_t\nabla\theta_m) + (\partial_t \boldsymbol{u}_m\cdot\nabla\theta_m, \partial_t\theta_m) = 0. \tag{4.9}$$

By taking into account that the second term in (4.9) can be split as

$$(\partial_t(k(\theta_m)\nabla\theta_m), \partial_t\nabla\theta_m) = (k'(\theta_m)\partial_t\theta_m\nabla\theta_m, \partial_t\nabla\theta_m) + (k(\theta_m)\partial_t\nabla\theta_m, \partial_t\nabla\theta_m),$$

we deduce from (4.9) that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\partial_t\theta_m|_2^2 + k_{\min}|\partial_t\nabla\theta_m|_2^2 \le -(k'(\theta_m)\partial_t\theta_m\nabla\theta_m, \partial_t\nabla\theta_m) - (\partial_t \boldsymbol{u}_m\cdot\nabla\theta_m, \partial_t\theta_m). \tag{4.10}$$

Bounding both terms on the right-hand side of (4.10) ($k'_{\max} = \max|k'|$),

$$-(k'(\theta_m)\partial_t\theta_m\nabla\theta_m, \partial_t\nabla\theta_m) \le k'_{\max}|\nabla\theta_m|_6|\partial_t\theta_m|_3|\partial_t\nabla\theta_m|_2 \le C\|\theta_m\|_2|\partial_t\theta_m|_2^{1/2}\|\partial_t\theta_m\|_1^{3/2} \le \gamma\|\partial_t\theta_m\|_1^2 + C_\gamma\|\theta_m\|_2^4|\partial_t\theta_m|_2^2,$$

and

$$-(\partial_t \boldsymbol{u}_m \cdot \nabla\theta_m, \partial_t\theta_m) \leq |\partial_t \boldsymbol{u}_m|_2 |\nabla\theta_m|_6 |\partial_t\theta_m|_3 \leq C|\partial_t \boldsymbol{u}_m|_2 \|\theta_m\|_2 |\partial_t\theta_m|_2^{1/2} \|\partial_t\theta_m\|_1^{1/2}$$
$$\leq \gamma\big(\|\partial_t\theta_m\|_1^2 + |\partial_t u_m|_2^2\big) + C_\gamma \|\theta_m\|_2^4 |\partial_t\theta_m|_2^2,$$

we obtain, for γ small enough,

$$\frac{\mathrm{d}}{\mathrm{d}t}|\partial_t\theta_m|_2^2 + k_{\min}\|\partial_t\theta_m\|_1^2 \leq \gamma|\partial_t \boldsymbol{u}_m|_2^2 + C_\gamma\|\theta_m\|_2^4|\partial_t\theta_m|_2^2. \tag{4.11}$$

Now, using $\Delta^2\theta_m$ as test function ($\Delta^2\theta_m \in W^m$ thanks to the choice of spectral basis) and integrating by parts in all terms (boundary terms vanish since $(\nabla\Delta\theta_m \cdot \boldsymbol{n})|_{\partial\Omega} = 0$), one obtains

$$-(\partial_t\nabla\theta_m, \nabla\Delta\theta_m) + (\nabla[\nabla\cdot(k(\theta_m)\nabla\theta_m)], \nabla\Delta\theta_m) - (\nabla(\boldsymbol{u}\cdot\nabla\theta_m), \nabla\Delta\theta_m) = 0. \tag{4.12}$$

Note that if Dirichlet boundary condition is imposed for the temperature θ, the boundary terms do not vanish in the integration by parts and we cannot obtain the previous inequalities.

Integrating by parts the first term of (4.12) (again the boundary term vanishes since $(\partial_t\nabla\theta_m\cdot\boldsymbol{n})|_{\partial\Omega} = 0$), the term becomes $\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\Delta\theta_m|_2^2$. The second term is

$$(\nabla[\nabla\cdot(k(\theta_m)\nabla\theta_m)], \nabla\Delta\theta_m) = (k''(\theta_m)(\nabla\theta_m)^3, \nabla\Delta\theta_m) + 2(k'(\theta_m)\nabla^2\theta_m\nabla\theta_m, \nabla\Delta\theta_m)$$
$$+ (k'(\theta_m)\nabla\theta_m\Delta\theta_m, \nabla\Delta\theta_m) + (k(\theta_m)\nabla\Delta\theta_m, \nabla\Delta\theta_m).$$

Hence, we deduce from (4.12) that ($|k''(\theta_m)| \leq k''_{\max} = \max|k''|$)

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\Delta\theta_m|_2^2 + k_{\min}|\nabla\Delta\theta_m|_2^2 \leq k''_{\max}|((\nabla\theta_m)^3, \nabla\Delta\theta_m)| + 2k'_{\max}|(\nabla^2\theta_m\nabla\theta_m, \nabla\Delta\theta_m)|$$
$$+ k'_{\max}|(\nabla\theta_m\Delta\theta_m, \nabla\Delta\theta_m)| + |(\nabla\boldsymbol{u}_m\nabla\theta_m, \nabla\Delta\theta_m)| + |(\boldsymbol{u}_m\nabla^2\theta_m, \nabla\Delta\theta_m)|$$
$$= L_1 + L_2 + L_3 + L_4 + L_5.$$

Replacing in the above inequality the following estimations,

$$L_1 \leq C|\nabla\theta_m|_6^3|\nabla\Delta\theta_m|_2 \leq \gamma\|\theta_m\|_3^2 + C_\gamma\|\theta_m\|_2^6,$$

$$L_2 \leq C|\nabla^2\theta_m|_3|\nabla\theta_m|_6|\nabla\Delta\theta_m|_2 \leq C\|\theta_m\|_2^{3/2}\|\theta_m\|_3^{3/2} \leq \gamma\|\theta_m\|_3^2 + C_\gamma\|\theta_m\|_2^6,$$

$$L_3 \leq C|\nabla\theta_m|_6|\Delta\theta_m|_3|\nabla\Delta\theta_m|_2 \leq C\|\theta_m\|_2^{3/2}\|\theta_m\|_3^{3/2} \leq \gamma\|\theta_m\|_3^2 + C_\gamma\|\theta_m\|_2^6,$$

$$L_4 \leq C|\nabla\boldsymbol{u}_m|_2|\nabla\theta_m|_\infty|\nabla\Delta\theta_m|_2 \leq C\|\boldsymbol{u}_m\|_1\|\theta_m\|_2^{1/2}\|\theta_m\|_3^{3/2} \leq \gamma\|\theta_m\|_3^2$$
$$+ C_\gamma\|\boldsymbol{u}_m\|_1^4\|\theta_m\|_2^2,$$

$$L_5 \leq C|\boldsymbol{u}_m|_6|\nabla^2\theta_m|_3|\nabla\Delta\theta_m|_2 \leq C\|\boldsymbol{u}_m\|_1\|\theta_m\|_2^{1/2}\|\theta_m\|_3^{3/2} \leq \gamma\|\theta_m\|_3^2$$
$$+ C_\gamma\|\boldsymbol{u}_m\|_1^4\|\theta_m\|_2^2,$$

we get, taking γ small enough,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\theta_m\|_2^2 + k_{\min}\|\theta_m\|_3^2 \le C_\gamma\left(\|\theta_m\|_2^6 + \|\boldsymbol{u}_m\|_1^4\|\theta_m\|_2^2\right). \tag{4.13}$$

Finally, (4.11) added to (4.13) proves the lemma.

5. Proof of theorem 2.1

If we denote

$$\Phi_m(t) = \int_\Omega (\nu(\theta_m)+1)|\nabla \boldsymbol{u}_m|^2 + \|\theta_m\|_2^2 + |\partial_t\theta_m|_2^2,$$
$$\Psi_m(t) = \|\boldsymbol{u}_m\|_2^2 + |\partial_t \boldsymbol{u}_m|_2^2 + \|\theta_m\|_3^2 + \|\partial_t\theta_m\|_1^2,$$

taking an adequate balance between inequalities (4.1) and (4.8) (from lemmas 4.1 and 4.2, respectively) in order to eliminate the term $K\|\boldsymbol{g}\|^2_{L^\infty(L^2)}\|\theta_m\|_2^2$ from the right-hand side of (4.1) (more concretely, adding (4.1) and (4.8), multiplied by $(2K/k_{\min})\|\boldsymbol{g}\|^2_{L^\infty(L^2)}$), one has

$$\begin{cases} \Phi'_m + C\Psi_m \le \varepsilon\Psi_m\Phi_m^{1/2} + C_0|\boldsymbol{f}|_2^2 + D\Phi_m^3, \\ \Phi_m(0) = \Phi_{m0}, \end{cases} \tag{5.1}$$

where C, D, $C_0>0$ are constants.

Let $\delta>0$ be a small enough constant that we will specify below.

First step. If $\Phi_m(0)\le\delta$ and $\|\boldsymbol{f}\|_{L^2(0,T;L^2)}\le\delta/C_0$, then $\Phi_m(t)<2\delta$, $\forall t\in[0, T]$.

Indeed, by an absurd argument, let T^* be the first value in $[0, T]$ such that $\Phi_m(T^*)=2\delta$, hence

$$\Phi_m(T^*) = 2\delta \quad \text{and} \quad \Phi_m(s) < 2\delta \quad \forall\, s\in[0,T^*).$$

Moreover, there exists a Poincaré constant $C_p<0$ such that $\Phi_m(t)\le C_p\Psi_m(t)$. Then for ε small enough, we have

$$C\Psi_m - \varepsilon\Psi_m\Phi_m^{1/2} \ge C\Psi_m - \varepsilon\Psi_m(2\delta)^{1/2} \ge \bar{C}\Psi_m \ge \frac{\bar{C}}{C_p}\Phi_m \equiv \tilde{C}\Phi_m.$$

The above inequality together with (5.1) leads to

$$\begin{cases} \Phi'_m + \tilde{C}\Phi_m \le C_0|\boldsymbol{f}|_2^2 + D\Phi_m^3 \\ \Phi_m(0) = \Phi_{m0} \end{cases}, \tag{5.2}$$

in $[0, T^*]$. Then, $\Phi'_m + \tilde{C}\Phi_m \le C_0|\boldsymbol{f}|_2^2 + 4\delta^2 D\Phi_m$ in $[0, T^*]$. We can find $\delta>0$ such that $\tilde{C} - 4\delta^2 D \ge \bar{C}$ being a positive constant (for instance, $\bar{C}=\tilde{C}/2$). Therefore,

$$\Phi'_m + \bar{C}\Phi_m \le C_0|\boldsymbol{f}|_2^2 \quad \text{in } [0,T^*],$$

hence, integrating in $[0, T^*]$ with a Gronwall's technique, one finds

$$\Phi_m(T^*) \le \delta\mathrm{e}^{-\bar{C}T^*} + C_0\int_0^{T^*}|\boldsymbol{f}|_2^2.$$

We can choose $\int_0^{T^*} C_0|\boldsymbol{f}|_2^2$ small enough, such that the right-hand side is smaller than 2δ (e.g. $\int_0^{T^*} |\boldsymbol{f}|_2^2 \le \delta/C_0$). Thus, we arrive at a contradiction.

Second step. Under the conditions of the first step and assuming that

$$\int_0^T |\boldsymbol{f}|_2^2 \le \delta(1-\mathrm{e}^{-\bar{C}\,T})/C_0,$$

then $\Phi_m(T) \le \Phi_m(0)$.

Now, since $\Phi_m(t) < 2\delta\ \forall\, t \in [0,T]$, we can repeat the above argument obtaining (5.2) in $[0,\ T]$ and arrive at

$$\Phi_m(T) \le \Phi_m(0)\mathrm{e}^{-\bar{C}\,T} + C_0 \int_0^T |\boldsymbol{f}|_2^2,$$

hence $\Phi_m(T) \le \Phi_m(0)$ using the additional hypothesis $\int_0^T |\boldsymbol{f}|_2^2 \le \delta(1-\mathrm{e}^{-\bar{C}\,T})/C_0$).

Third step. Existence of approximate periodic solution

Given $(\boldsymbol{u}_{m0}, \theta_{m0}) \in V^m \times W^m$, we define the map

$$\begin{aligned} L^m &: [0,\ T] \mapsto \mathbb{R}^m \times \mathbb{R}^m \\ t &\mapsto (\xi_{1m}(t), \ldots, \xi_{mm}(t), \zeta_{1m}(t), \ldots, \zeta_{mm}(t)) \end{aligned},$$

where $(\xi_{1m}(t), \ldots, \xi_{mm}(t))$ and $(\zeta_{1m}(t), \ldots, \zeta_{mm}(t))$ are coefficients of $\boldsymbol{u}_m(t)$ and $\theta_m(t)$ with respect to V^m and W^m, respectively, $(\boldsymbol{u}_m(t),\ \theta_m(t))$ being the (unique) approximate solution of (3.1) corresponding to the initial data $(\boldsymbol{u}_{m0},\ \theta_{m0})$.

Now, varying the initial data $(\boldsymbol{u}_{m0},\ \theta_{m0})$, we are going to define a new map

$$\mathcal{R}^m : \bar{B} \subset \mathbb{R}^m \times \mathbb{R}^m \mapsto \mathbb{R}^m \times \mathbb{R}^m,$$

as follows: given $L_0^m \in \mathbb{R}^m \times \mathbb{R}^m$, we define $\mathcal{R}^m(L_0^m) = L^m(T)$, where $L^m(t)$ is related to the solution of problem (3.1) with initial data $L_0^m(=L^m(0))$ and

$$\bar{B} = \{(\xi_{1m}, \ldots, \xi_{mm}, \zeta_{1m}, \ldots, \zeta_{mm}) = L_0^m : \Phi_m(0) \le \delta\}.$$

By the uniqueness of approximate solution of problem (3.1), this map is well defined. Moreover, using regularity of the corresponding ordinary differential system (equivalent to (3.1)), this map is continuous. By the second step, $\mathcal{R}^m$ maps $\bar{B}$ into $\bar{B}$ and $\bar{B}$ is a closed, convex and compact set. Consequently, Brouwer's theorem implies the existence of fixed point of $\mathcal{R}^m$, which gives us the existence of periodic Galerkin solution.

Fourth step. Pass to the limit in periodic approximate solutions

If the data of the problem are small, thanks to the first step, we have

$$\Phi_m(t) = \int_\Omega (\nu(\theta_m)+1)|\nabla \boldsymbol{u}_m|^2 + \|\theta_m\|_2^2 + |\partial_t\theta_m|_2^2 \le 2\delta.$$

Therefore, the following bounds hold uniformly:

$$\begin{gathered} (\boldsymbol{u}_m, \theta_m) \text{ in } L^\infty(\boldsymbol{H}^1 \times H_N^2) \cap L^2(\boldsymbol{H}^2 \times H_N^3), \\ (\partial_t \boldsymbol{u}_m) \text{ in } L^2(\boldsymbol{L}^2), \\ (\partial_t\theta_m) \text{ in } L^\infty(L^2) \cap L^2(H^1). \end{gathered}$$

Using compactness results for time spaces with values in Banach spaces with the compact embedding of H^2 into H^1, one has

$$(\boldsymbol{u}_m, \theta_m) \text{ is relatively compact in } L^2(\boldsymbol{H}^1 \times H^2).$$

In fact, this compactness is sufficient in the pass to the limit in (3.1) in order to control the nonlinear terms.

Now, we go to pass to the limit in periodic conditions. From the estimations of θ_m in $L^\infty(H^2)$ and $\partial_t\theta_m$ in $L^2(H^1)$ and using the compact embedding of H^2 into H^1, one has that θ_m is relatively compact in $C([0, T];H^1)$, hence $\theta_m(T)\to\theta(T)$ and $\theta_m(0)\to\theta(0)$ strongly in $H^1(\Omega)$. Since $\theta_m(T)=\theta_m(0)$, then $\theta(T)=\theta(0)$ in $H^1(\Omega)$. Finally, since $\theta_m(T)$ and $\theta_m(0)$ are bounded in $H^2(\Omega)$, we have that $\theta(T)=\theta(0)$ in $H^2(\Omega)$.

The argument for $\boldsymbol{u}$ is similar, hence one deduces $\boldsymbol{u}(T)=\boldsymbol{u}(0)$ in $\boldsymbol{H}^1(\Omega)$.

To prove $\partial_t\theta(0)=\partial_t\theta(T)$, we are going to consider the orthogonal projector $P_m: H^1\to W^m$ defined as $P_m(g)=\sum_{k=1}^m(\nabla g,\nabla\varphi_i)\varphi_i$, for all $g\in H^1$. One has $\|P_m\|_{\mathcal{L}(H^1,H^1)}\le 1$ and $\|P_m\|_{\mathcal{L}((H^1)',(H^1)')}\le 1$ (Lions 1969). Since $P_m(g)=\sum_{k=1}^m \mu_i(g,\varphi_i)\varphi_i$, for all $g\in H^1$, then P_m is also the orthogonal projector from L^2 to W^m with respect to the L^2 inner product. Therefore,

$$(P_m(g),\varphi_i) = (g,\varphi_i) \quad \forall\, i, \quad \forall g\in H^1(\Omega),$$

hence the Galerkin equation for θ_m can be written as

$$\partial_t\theta_m = P_m(-\boldsymbol{u}_m\cdot\nabla\theta_m + \nabla\cdot(k(\theta_m)\nabla\theta_m)).$$

Differentiating with respect to the time,

$$\partial_{tt}\theta_m = P_m(-\partial_t\boldsymbol{u}_m\cdot\nabla\theta_m - \boldsymbol{u}_m\cdot\nabla\partial_t\theta_m + \nabla\cdot(k'(\theta_m)\partial_t\theta_m\nabla\theta_m + k(\theta_m)\partial_t\nabla\theta_m)).$$

In particular,

$$\begin{aligned}\|\partial_{tt}\theta_m\|_{(H^1)'} &\le \|-\partial_t\boldsymbol{u}_m\cdot\nabla\theta_m - \boldsymbol{u}_m\cdot\nabla\partial_t\theta_m\|_{(L^6)'} + \|\nabla\cdot(k'(\theta_m)\partial_t\theta_m\nabla\theta_m \\ &\quad + k(\theta_m)\partial_t\nabla\theta_m)\|_{(H^1)'} \le |\partial_t\boldsymbol{u}_m|_2|\nabla\theta_m|_3 + |\boldsymbol{u}_m|_3|\nabla\partial_t\theta_m|_2 \\ &\quad + |k'(\theta_m)\partial_t\theta_m\nabla\theta_m + k(\theta_m)\partial_t\nabla\theta_m|_2.\end{aligned}$$

The terms on the right-hand side of the previous inequality will be bounded in $L^2(0, T)$. Indeed, $\partial_t\theta_m$ is bounded in $L^2(\boldsymbol{L}^2)$ and $\nabla\theta_m$ in $L^\infty(\boldsymbol{H}^1)$, hence

$$|\partial_t\boldsymbol{u}_m|_2|\nabla\theta_m|_3 \quad \text{is bounded in} \quad L^2(0,T).$$

Using that $\boldsymbol{u}_m$ is bounded in $L^\infty(\boldsymbol{H}^1)$ and $\partial_t\nabla\theta_m$ in $L^2(\boldsymbol{L}^2)$, one has

$$|\boldsymbol{u}_m|_3|\nabla\partial_t\theta_m|_2 \quad \text{is bounded in} \quad L^2(0,T).$$

Using that $\partial_t\theta_m$ and $\nabla\theta_m$ are bounded in $L^4(L^3)$ and $L^\infty(H^1)$, respectively, one has that $k'(\theta_m)\partial_t\theta_m\nabla\theta_m$ is bounded in $L^2(L^2)$. Finally, $k(\theta_m)\partial_t\nabla\theta_m$ is bounded in $L^2(L^2)$.

Consequently, $\|\partial_{tt}\theta_m\|_{(H^1)'}$ is uniformly bounded in $L^2(0, T)$. This along with $\partial_t\theta_m$ which is uniformly bounded in $L^\infty(L^2)$ gives that $\partial_t\theta_m$ is relatively compact in $C([0, T];(\mathrm{H}^1)')$, which suffices to prove $\partial_t\theta(0)=\partial_t\theta(T)$.

The authors have been partially supported by the project BFM2003-06446-C02-01 (Spain) and the project PHB2005-0042-PC/2137-05-4 (Spain/Brazil). Moreover, M.A.R.-M. is also supported by the project CNPq (Brazil) 301354/03-0.

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