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A REVIEW ON REPRODUCTIVITY AND TIME PERIODICITY FOR INCOMPRESSIBLE FLUIDS

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Abstract

In this article, our aims is to review some of the results that are currently available concerning the existence, uniqueness and regularity of reproductive and time periodic solutions of the Navier-Stokes equations and some variants. By the way, we present some open problems.

Key words: *Reproductive and time periodic solutions, Navier-Stokes type equations, regularity of solutions*
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1 Introduction

We study some problems related with time periodic solutions for models of incompressible fluids.

We start recalling the main ideas to prove the existence of reproductive weak solutions (i.e. weak solutions defined in the time interval $(0, T)$ taking the same initial and final values in time) for the Navier-Stokes equations and some variants where these ideas are applicable, such as Boussinesq, micropolar and magneto-micropolar models. This proof relies on the obtention of time periodic Galerkin approximations via Leray-Schauder point fixed argument.

Moreover, in the case of $2D$ domains, using the uniqueness of weak solutions, the regularizing property of the system and the existence of global regular solutions when data are regular, one has that the periodic in time weak solutions defined as extension of reproductive solutions to the whole time interval $(0, +\infty)$ will be regular solutions. An extension of these results to the $3D$ case is possible imposing small enough external force, using the so called "weak/strong uniqueness" and the global strong solutions for small enough data (see Section 5).

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Also, we study in Section 4 some coupled models for velocity and pressure dynamic variables with another variable where the maximum principle holds, such as the generalized Boussinesq model (with temperature-dependent viscosity) and a nematic liquid crystal model with a Ginzburg-Landau penalization. In these cases one has, thanks to an adequate reformulation of the problem by truncation, existence of reproductive weak solutions as limit of time periodic Galerkin approximations. It is important to remark that Galerkin approximations do not verify the maximum principle but their limit does.

Finally, we will see that, for these models related with the maximum principle, the argument to prove regularity of reproductive solutions in the Navier-Stokes framework (see Section 5 below) are not valid in general. The particular case of generalized Boussinesq model with Neumann boundary condition for the temperature can be solved with other arguments, but the case of nematic liquid crystal model remains as an open problem.

2 Navier-Stokes equations

The modern theory of the Navier-Stokes equations began in the 1930s with Leray's pioneering work ([10]).

Let $\Omega \subset \mathbb{R}^d$ ($d = 2$ or 3) a bounded and regular enough domain filled by the fluid, and $[0,T]$ the time interval. We denote $Q = (0,T)\times\Omega$ and $\Sigma = (0,T)\times\partial\Omega$.

In the case where the fluid is subject to the action of a body force $\boldsymbol{f}$, the Navier-Stokes equations can be written as follows

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \nu\Delta\boldsymbol{u} + \nabla p = \boldsymbol{f}, \qquad \text{div } \boldsymbol{u} = 0\,, \tag{1}$$

where $\boldsymbol{u} = \boldsymbol{u}(x,t)$ is the velocity field evaluated at the point $\boldsymbol{x}\in\Omega$ and at time $t\in[0,T]$, $p = p(\boldsymbol{x},t)$ is the pressure field and $\nu > 0$ is the coefficient of kinematical viscosity (which is taken constant). This system can be completed with several boundary conditions. For simplicity, we fix the following non-slip boundary conditions:

$$\boldsymbol{u}(t,\boldsymbol{x}) = 0, \;\; \boldsymbol{x}\in\partial\Omega, \;\; t>0 \tag{2}$$

Finally, supplementary conditions in time must be considered. The more classical is the initial condition:

$$\boldsymbol{u}(0,\boldsymbol{x}) = \boldsymbol{u}_0(\boldsymbol{x}), \quad \boldsymbol{x}\in\Omega \tag{3}$$

Other possibility is to change this initial condition by the following time-periodic condition:

$$\boldsymbol{u}(0,\boldsymbol{x}) = \boldsymbol{u}(T,\boldsymbol{x}), \quad \boldsymbol{x}\in\Omega. \tag{4}$$

Mathematical properties for system (1) have been deeply investigated over the years and are still the object of profound researches.

We introduce some space functions. Let $\mathcal{V}$ the vectorial space formed by all fields $\boldsymbol{v}\in C_0^\infty(\Omega)^d$ satisfying $\nabla\cdot\boldsymbol{v} = 0$. We consider the Hilbert spaces $\boldsymbol{H}$

(respectively $\boldsymbol{V}$) as the closure of $\mathcal{V}$ in $\boldsymbol{L}^2$ (respectively $\boldsymbol{H}^1$). Furthermore, one has

$$\begin{aligned} \boldsymbol{H} &= \{\boldsymbol{u} \in \boldsymbol{L}^2;\ \nabla \cdot \boldsymbol{u} = 0,\ \boldsymbol{u} \cdot \boldsymbol{n} = 0 \text{ on } \partial\Omega\}, \\ \boldsymbol{V} &= \{\boldsymbol{u} \in \boldsymbol{H}^1;\ \nabla \cdot \boldsymbol{u} = 0,\ \boldsymbol{u} = \boldsymbol{0} \text{ on } \partial\Omega\} \end{aligned}$$

We denote $L^2_0(\Omega) = \left\{ p \in L^2(\Omega) : \int_\Omega p\, dx = 0 \right\}$.

2.1 Main classical results for the initial-boundary problem

Definition 1 *Given $\boldsymbol{u}_0 \in H$ and $\boldsymbol{f} \in L^2(0,T;\boldsymbol{H}^{-1}(\Omega))$, it will said that $\boldsymbol{u}$ is a weak solution of the problem (1),(2), (3) in $(0,T)$, if*

$$\boldsymbol{u} \in L^2(0,T;\boldsymbol{V}) \cap L^\infty(0,T;\boldsymbol{H}),$$

and verifies (3) and the variational formulation

$$\int_0^T \int_\Omega \Big\{ -\boldsymbol{u}(t)\boldsymbol{v}'(t) + \nabla \boldsymbol{u}(t) : \nabla \boldsymbol{u}(t) - (\boldsymbol{u}(t)\cdot\nabla)\boldsymbol{v}(t)\boldsymbol{u}(t) - \boldsymbol{f}(t)\boldsymbol{v}(t) \Big\} dx dt = 0,$$

for all $\boldsymbol{v} \in C^1([0,T];\boldsymbol{H}) \cap C([0,T];\boldsymbol{V})$, with compact support contained in $(0,T)$.

In addition, if $\boldsymbol{u}_0 \in \boldsymbol{V}$ and $f \in L^2(0,T;\boldsymbol{L}^2(\Omega))$ any weak solution will be a strong solution if

$$\boldsymbol{u} \in L^2(0,T;\boldsymbol{H}^2 \cap \boldsymbol{V}) \cap L^\infty(0,T;\boldsymbol{V}), \quad \boldsymbol{u}_t \in L^2(0,T;\boldsymbol{H}), \quad p \in L^2(0,T;H^1 \cap L^2_0(\Omega))$$

and verifies the system (1) pointwise a.e. in $(0,T)\times\Omega$.

Remark 1 *The previous definition can be extend to the case of final time $T = \infty$ changing the regularity $L^2(0,T)$ by $L^2_{\rm loc}(0,+\infty)$.*

The following results hold.

Theorem 1 *[22] For any $\boldsymbol{u}_0 \in \boldsymbol{H}$ and $\boldsymbol{f} \in L^2(0,T;\boldsymbol{H}^{-1}(\Omega))$, the problem (1)-(2) has (at least) a weak solution. If $\Omega \subset \mathbb{R}^2$, one has uniqueness of weak solutions.*

Theorem 2 *[22] For any $\boldsymbol{u}_0 \in \boldsymbol{V}$ and $\boldsymbol{f} \in L^\infty(0,\infty;\boldsymbol{L}^2(\Omega))$, the problem (1)-(2) has a unique strong solution $(\boldsymbol{u},p)$ local in time, defined in $(0,T^\star)$ with $T^\star > 0$ small enough. In fact, if a solution has the strong regularity, it coincides with any weak solution associated with the same data (this property is called weak/strong uniqueness). Moreover, this strong solution is global in time, defined in the whole time interval $(0,\infty)$ if either $\Omega \subset \mathbb{R}^2$ or $\Omega \subset \mathbb{R}^3$ and data $(\boldsymbol{u}_0,\boldsymbol{f})$ are small enough in their respective spaces $\boldsymbol{V} \times L^\infty(0,\infty;\boldsymbol{L}^2(\Omega))$.*

2.2 On the time-periodic weak solutions

Theorem 3 *[8] For any $\boldsymbol{f} \in L^2(0,T;\boldsymbol{H}^{-1}(\Omega))$, there exists a weak solution of (1)-(2) and (4), (i.e. the weak solution $\boldsymbol{u}$ has the so-called reproductive property: $\boldsymbol{u}(0,x) = \boldsymbol{u}(T,x)$).*

Notice that the time periodic extension, $\widetilde{\boldsymbol{u}}$, of any weak reproductive solution $\boldsymbol{u}$ to the whole time interval $(0,+\infty)$ is a periodic weak solution of (1)-(2) corresponding to the data, $\widetilde{\boldsymbol{f}}$, defined as the time periodic extension of $\boldsymbol{f}$.

Main ideas of the proof of Theorem 3

Let $\boldsymbol{u}^k$ the unique approximate solution of the Galerkin initial-boundary problem of Navier-Stokes in the finite-dimensional subspace $\boldsymbol{V}^k$, spanned by the first k elements of the "spectral" basis of $\boldsymbol{V}$ (orthogonal in $\boldsymbol{V}$ and orthonormal in $\boldsymbol{H}$), associated to a initial discrete data $\boldsymbol{u}_0^k \in \boldsymbol{V}^k$.

Since $\boldsymbol{V} \hookrightarrow \boldsymbol{H}$, there exists a Poincaré constant $c_1 > 0$ such that

$$c_1 \|\boldsymbol{u}^k\|_{L^2}^2 \le \|\nabla \boldsymbol{u}^k\|_{L^2}^2,$$

thus, from energy inequality, we have

$$\frac{d}{dt}\|\boldsymbol{u}^k\|_{L^2}^2 + c_1\|\boldsymbol{u}^k\|_{L^2}^2 \le C\,\|\boldsymbol{f}\|_{H^{-1}}^2, \tag{5}$$

or equivalently

$$\frac{d}{dt}(e^{c_1 t}\|\boldsymbol{u}^k\|_{L^2}^2) \le C\,e^{c_1 t}\|\boldsymbol{f}\|_{H^{-1}}^2.$$

Integrating from 0 to T, we have

$$e^{c_1 T}\|\boldsymbol{u}^k(T)\|_{L^2}^2 \le \|\boldsymbol{u}^k(0)\|_{L^2}^2 + C\int_0^T e^{c_1 t}\,\|\boldsymbol{f}(t)\|_{H^{-1}}^2 dt. \tag{6}$$

Now, we define the operator $L^k : [0,T] \to \mathbb{R}^k$ as follows

$$L^k(t) = (c_1^k(t), \ldots, c_k^k(t))$$

where $c_i^k(t),\ i = 1,\ldots,k$, are the coefficients of the expansion of $\boldsymbol{u}^k(t)$ in V^k.

Note that

$$\|L^k(t)\|_{\mathbb{R}^k} = \|\boldsymbol{u}^k\|_{L^2},$$

because we have choose the (orthonormal in L^2) spectral basis in $\boldsymbol{V}$.

We define the operator $\Phi^k : \mathbb{R}^k \to \mathbb{R}^k$ as follows: Given $L_0^k \in \mathbb{R}^k$, we define $\Phi^k(L_0^k) = L^k(T)$, where $L^k(t)$ are the coefficients of the Galerkin solution with initial value with coefficients L_0^k. It is easy to see that Φ^k is continuous and we want to prove that Φ^k has a fixed point.

For this, thanks to the Leray-Schauder Theorem, it suffices to show that for all $\lambda \in [0,1]$, the possible solutions of the equation

$$L_0^k(\lambda) = \lambda\Phi^k(L_0^k(\lambda)), \tag{7}$$

are bounded independently of λ.

Since $L_0^k(0) = 0$, it suffices to consider $\lambda \in (0,1]$. In this case, (7) is equivalent to $\Phi^k(L_0^k(\lambda)) = \frac{1}{\lambda} L_0^k(\lambda)$. Moreover, by the definition of Φ^k and (6), one obtains

$$e^{c_1 T} \| \frac{1}{\lambda} L_0^k(\lambda) \|_{\mathbb{R}^k}^2 \leq \|L_0^k(\lambda)\|_{\mathbb{R}^k}^2 + c \int_0^T e^{c_1 T} \|\boldsymbol{f}(t)\|_{H^{-1}}^2 dt,$$

which implies

$$\|L_0^k(\lambda)\|_{\mathbb{R}^k}^2 \leq \frac{c \int_0^T e^{c_1 T} \|\boldsymbol{f}(t)\|_{H^{-1}}^2 dt}{e^{c_1 T} - 1} = M,$$

for each $\lambda \in (0,1]$. This bound is independent of $\lambda \in [0,1]$ and k. Consequently, Leray-Shauder Theorem implies the existence of at least one fixed point of Φ^k, that is the existence of reproductive Galerkin solution.

Thus, since previous estimates are independent of k, one has the same estimates for these reproductive Galerkin solutions.

Finally, the convergence of a subsequence to a reproductive solution of (1),(2), (4) hold.

2.3 Relation between weak periodic solutions and global solutions

Assume $\boldsymbol{f} \colon [0,+\infty) \to \boldsymbol{H}^{-1}(\Omega)$ and T-time periodic.

Navier-Stokes $2D$

One has (see Theorem 1) uniqueness of weak solution for the initial-boundary problem (associated to any initial data $\boldsymbol{u}_0$). Consequently, given a reproductive solution $\boldsymbol{u}$ associated to $\boldsymbol{u}(0) = \boldsymbol{u}(T) := \boldsymbol{u}_0$, then $\boldsymbol{u}$ is the (unique) solution of the initial-boundary problem associated to the initial data $\boldsymbol{u}_0$, which is defined for all time $t \in (0,\infty)$. Moreover, this solution is T-periodic, because in $(T, 2T)$ must be equal to the reproductive solution defined as $\bar{\boldsymbol{u}}(t) = \boldsymbol{u}(t-T)$ (which verifies $\boldsymbol{u}(T) = \boldsymbol{u}(2T) = \boldsymbol{u}_0$) and so on.

Finally, using regularity of solution $\boldsymbol{u}$ for strictly positive times (see [5]), it is easy to prove that every periodic solution is regular.

Navier-Stokes $3D$

Since uniqueness of weak solution is not known, it is possible that the reproductive solution $\boldsymbol{u}$ and the global weak solution $\tilde{\boldsymbol{u}}$ associated to the initial data $\boldsymbol{u}_0 := \boldsymbol{u}(0) = \boldsymbol{u}(T)$ are different in $(0,T)$, although they coincide locally in time, near of the initial time $t = 0$.

2.4 Open problems

Navier-Stokes with large Reynolds number and a reaction term adding energy

Previous arguments of the proof of reproductive solutions are based on (exponential) decreasing of energy (thanks to dissipative terms). Naturally, the same argument, is applicable to models with energy strictly decreasing in finite time. But this is not always possible. For instance, we consider the following Navier-Stokes system with large Reynolds number and a reaction term adding energy:

$$\begin{aligned} &\partial_t \boldsymbol{u} - \varepsilon\Delta \boldsymbol{u} - \boldsymbol{u} + \nabla p = \boldsymbol{f}, \quad \nabla\cdot\boldsymbol{u} = 0, \\ &\boldsymbol{u}(0) = \boldsymbol{u}(T), \quad \boldsymbol{u}_{|\Sigma} = 0. \end{aligned} \tag{8}$$

The energy inequality is

$$\partial_t \|\boldsymbol{u}\|^2_{L^2} + \varepsilon\|\nabla \boldsymbol{u}\|^2_{L^2} \le C(\|\boldsymbol{f}\|^2_{H^{-1}} + \|\boldsymbol{u}\|^2_{L^2}).$$

Assuming ε small enough such that $\|\boldsymbol{u}\|^2_{L^2} \not< \varepsilon\|\nabla\boldsymbol{u}\|^2_{L^2}$, the strictly decreasing in time of $\|\boldsymbol{u}\|^2_{L^2}$ is not clear. Consequently, the existence of time-periodic weak solutions of (8) remains as an open problem.

Exterior domains

Assume Ω is an exterior domain where the Poincaré inequality is not true. Then, to show the existence of reproductive solutions one could use the "embedding domain technique" together with the Galerkin Method, obtaining reproductive solutions in a sequence of (bounded) truncated domains, see for instance [6, 18, 17]. However, since Poincaré imbedding is not applicable, it is not clear the controll to the pass to the limit from truncated domains to the whole domain.

Some partial results are known. For example, the existence of strong periodic solutions for the Navier-Stokes equations in the following unbounded domains, either Ω is the whole space $\mathbb{R}^n$ or the half-space $\mathbb{R}^n_+$ has been investigated by Kozono and Nakao [9] and Taniuchi [21] using the semigroup approach. By using potential theory, Maremonti [15] proved the existence of a unique time periodic solution on the whole space $\mathbb{R}^3$ for small external force. The problem, in the half-space $\mathbb{R}^3_+$, was considered in [16]. Kozono and Nakao [9], making use of $\mathbf{L}^p - \mathbf{L}^r$ estimates for the semigroup generated by the Stokes operator, constructed time-periodic solutions for small time-periodic forces and the stability of these solutions was considered in [21]. Yamazaki [23] analyzed the same problem of [9] in Morrey spaces.

3 Some variants of Navier-Stokes equations

We can apply the argument to find reproductive solutions done for Navier-Stokes in the precedent Section, for some variants:

3.1 Boussinesq equations

The Boussinesq system of hydrodynamics equations (see Joseph [7]) arise from zero order approximation to the coupling between the Navier-Stokes equation and the thermodynamic equation. Such a mathematical model reads:

Find the field $\boldsymbol{u} : Q \to \mathbb{R}^3$, the scalar functions $(\theta, p) : Q \to \mathbb{R}^2$ which satisfy the system of equations:

$$\begin{aligned} \frac{\partial \boldsymbol{u}}{\partial t} - \nu\Delta \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p &= \alpha\theta\boldsymbol{g} + \boldsymbol{f} \quad \text{in } Q,\\ \nabla\cdot\boldsymbol{u} &= 0 \quad \text{in } Q,\\ \frac{\partial \theta}{\partial t} - \chi\Delta\theta + (\boldsymbol{u}\cdot\nabla)\theta &= 0 \quad \text{in } Q. \end{aligned} \tag{9}$$

with $\partial_n\theta = 0$ on $\partial\Omega$ and $\int_\Omega \theta = 0$

Here $\boldsymbol{u}, p, \theta$ denote the velocity, the pressure and the temperature, respectively. $\boldsymbol{g}$ denotes the gravitational field, $\alpha > 0$ is a constant associated to the coefficient of volume expansion and $\boldsymbol{f}$ is a field of external forces. Again, $\nu > 0$ is the viscosity coefficient. Finally, $\chi > 0$ is the thermal conductivity coefficient.

This system is completed with the boundary conditions (for instance) $\boldsymbol{u}_{|\Sigma} = 0$, $\theta_{|\Sigma} = 0$ and the time-periodic conditions $\boldsymbol{u}(0) = \boldsymbol{u}(T)$, $\theta(0) = \theta(T)$ in Ω.

By taking $\boldsymbol{u}$ and θ as test function in the $\boldsymbol{u}$-system and θ-equation of (9) respectively, adding the resulting equalities considering an adequate balance (in order to eliminate the term that contains $\boldsymbol{g}$), we obtain

$$\frac{d}{dt}\|\boldsymbol{u}\|^2_{L^2} + \beta\frac{d}{dt}\|\theta\|^2_{L^2} + \nu\|\nabla\boldsymbol{u}\|^2_{L^2} + \beta\chi\|\nabla\theta\|^2_{L^2} \le \|\boldsymbol{f}\|^2_{H^{-1}}, \tag{10}$$

where β is a big enough number depending on α and $\|\boldsymbol{g}\|_{L^\infty}$. This together with the Poincaré inequality gives an inequality of type (5). Indeed, it suffices to consider a Galerkin approximation for both variables, velocity and temperature, and to follow the proof of Theorem 2.3, changing $\boldsymbol{u}^k$ by $(\boldsymbol{u}^k, \theta^k)$.

Another boundary conditions are possible: Neumann, mixed, etc, whenever an inequality for $(\boldsymbol{u}, \theta)$ similar to (5) holds.

3.2 Micropolar equations

The equations that describes the motion of a incompressible viscous and micropolar fluids in Q are given by (see [12])

$$\begin{aligned} &\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - (\nu+\nu_r)\Delta\boldsymbol{u} + \nabla p = 2\nu_r \text{ rot } \mathbf{w} + \boldsymbol{f},\\ &\text{div } \boldsymbol{u} = 0,\\ &\frac{\partial \mathbf{w}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\mathbf{w} - (c_a + c_d)\Delta\mathbf{w} - (c_0 + c_d - c_a)\nabla \text{ div } \mathbf{w} + 4\nu_r\mathbf{w}\\ &\qquad = 2\nu_r \text{ rot } \boldsymbol{u} + \mathbf{g}. \end{aligned} \tag{11}$$

The functions $\boldsymbol{u} : Q \to \mathbb{R}^3$, $\boldsymbol{w} : Q \to \mathbb{R}^3$ and $p : Q \to \mathbb{R}$ denote the liner velocity, the angular velocity (of rotation of particles) and the pressure of the fluid, respectively. The functions $\boldsymbol{f} : Q \to \mathbb{R}^3$ and $\boldsymbol{g} : Q \to \mathbb{R}^3$ denote external sources of linear and angular momentum, respectively. The positive constants ν, ν_r, c_0, c_a and c_d are viscosities, such that $c_0 + c_d > c_a$.

This system is completed with the boundary conditions $\boldsymbol{u}_{|\Sigma} = 0$, $\boldsymbol{w}_{|\Sigma} = 0$ (for instance) and the time-periodic conditions $\boldsymbol{u}(0) = \boldsymbol{u}(T)$, $\boldsymbol{w}(0) = \boldsymbol{w}(T)$ in Ω.

By taking $\boldsymbol{u}$ and $\boldsymbol{w}$ as test function in the $\boldsymbol{u}$-system and $\boldsymbol{w}$-system of (11) respectively, adding the resulting equalities, taking into account that $2\nu_r(\text{ rot}\,\boldsymbol{w}, \boldsymbol{u}) + 2\nu_r(\text{ rot }\,\boldsymbol{u}, \boldsymbol{w}) = 4\nu_r(\text{ rot }\,\boldsymbol{u}, \boldsymbol{w})$ and $|\nabla \boldsymbol{u}|^2 = |\text{ rot}\boldsymbol{u}|^2$, we obtain

$$\frac{d}{dt}\|\boldsymbol{u}\|_{L^2}^2 + \|\boldsymbol{w}\|_{L^2}^2 + \nu\|\nabla \boldsymbol{u}\|_{L^2}^2 + (c_a + c_d)\|\nabla \boldsymbol{w}\|_{L^2}^2 + (c_0 + c_d - c_a)\|\text{ div }\boldsymbol{w}\|_{L^2}^2 \\ \leq C(\|\boldsymbol{f}\|_{H^{-1}}^2 + \|\boldsymbol{g}\|_{H^{-1}}^2).$$

Starting from this inequality, the argument follows as in previous section.

3.3 Other models

Other fluid models where one has existence of reproductive solutions are: magnetohydrodynamic model [14], Magneto-micropolar fluid motion [20], a convection-diffusion model describing binary alloy solidification processes [3], etc.

4 Reproductivity and maximum principle

Given $\boldsymbol{u} : Q \to \mathbb{R}^3$ such that $\nabla \cdot \boldsymbol{u} = 0$ in Q and $\boldsymbol{u} \cdot \mathbf{n} = 0$ on $\partial\Omega$, we consider the (reproductive) diffusion-advection problem for the unknown $c : Q \to \mathbb{R}$ (a concentration):

$$\partial_t c - \Delta c + \boldsymbol{u} \cdot \nabla c = 0, \quad c_{|\Sigma} = c_\Sigma, \quad c(0) = c(T),$$

where $0 < \underline{c} \leq c_\Sigma \leq \overline{c}$ on Σ, for some constants $\underline{c}$ and $\overline{c}$. In particular,

$$\partial_t(c - \overline{c}) - \Delta(c - \overline{c}) + (\boldsymbol{u} \cdot \nabla)(c - \overline{c}) = 0 \quad \text{in } Q.$$

Multiplying by $(c - \overline{c})_+$ and integrating in Ω (notice that $(c - \overline{c})_+ = 0$ on Σ), one has

$$\frac{d}{dt}\int_\Omega |(c - \overline{c})_+|^2 + \int_\Omega |\nabla(c - \overline{c})_+|^2 \leq 0.$$

Integrating in $t \in (0, T)$ and using the periodic condition $c(0) = c(T)$, one arrives at

$$\int_0^T \|\nabla(c - \overline{c})_+\|_{L^2}^2 = 0.$$

Hence $c \leq \overline{c}$ in Q hold. Similarly $c \geq \underline{c}$ in Q hold.

Therefore, one has the following conclusion: The reproductive solution conserve the maximum principle.

In the following models, the maximum principle has an important role.

4.1 Generalized Boussinesq system, with diffusion depending on temperature

When the viscosity and heat conductivity are temperature dependent funtions in the Boussinesq system, one has the following system:

$$\begin{cases} \partial_t \boldsymbol{u} - \nabla \cdot (\nu(\theta)\nabla \boldsymbol{u}) + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nabla p = \alpha\theta \mathbf{g} + \boldsymbol{f}, \\ \nabla \cdot \boldsymbol{u} = 0, \\ \partial_t \theta - \nabla \cdot (k(\theta)\nabla\theta) + (\boldsymbol{u} \cdot \nabla)\theta = 0, \end{cases} \tag{12}$$

where $\nu : \mathbb{R} \to \mathbb{R}^+$ and $k : \mathbb{R} \to \mathbb{R}^+$ are strictly positive continuous functions (the kinematic viscosity and the thermal conductivity respectively).

The problem is to find a regular solution $\{\boldsymbol{u}, \theta, p\}$ of (12) in $\Omega \times [0, T]$, together the following boundary Dirichlet data:

$$\boldsymbol{u} = 0, \qquad \theta = \theta_{\partial\Omega} \qquad \text{on } \partial\Omega \times [0, T), \tag{13}$$

and time-periodic conditions:

$$\boldsymbol{u}(0) = \boldsymbol{u}(T), \qquad \theta(0) = \theta(T) \qquad \text{in } \Omega. \tag{14}$$

We define

$$\theta_{\min} = \min \theta_{\partial\Omega} \qquad \theta_{\max} = \max \theta_{\partial\Omega}.$$

Thanks to the maximum principle, one has $\theta_{\min} \le \theta \le \theta_{\max}$ in Q. Then, there exists $\nu_{\min} > 0$, $k_{\min} > 0$, $\nu_{\max} > 0$ and $k_{\max} > 0$ such that

$$\nu_{\min} \le \nu(s) \le \nu_{\max} \quad \text{and} \quad k_{\min} \le k(s) \le k_{\max}, \quad \forall\, s \in [\theta_{\min}, \theta_{\max}].$$

One can proves the existence of reproductive solution in the same way that in the classical Boussinesq case (see Section 3.1), considering the equivalent problem that result changing ν by $\widetilde{\nu}$ and k by $\widetilde{k}$, where

$$\widetilde{\nu}(\theta) = \begin{cases} \nu(\theta_{\min}) & \text{if} \quad \theta < \theta_{\min}, \\ \nu(\theta) & \text{if} \quad \theta_{\min} \le \theta \le \theta_{\max}, \\ \nu(\theta_{\max}) & \text{if} \quad \theta > \theta_{\max}, \end{cases}$$

$$\widetilde{k}(\theta) = \begin{cases} k(\theta_{\min}) & \text{if} \quad \theta < \theta_{\min}, \\ k(\theta) & \text{if} \quad \theta_{\min} \le \theta \le \theta_{\max}, \\ k(\theta_{\max}) & \text{if} \quad \theta > \theta_{\max}. \end{cases}$$

4.2 Penalized Nematic liquid crystal model

We assume the following nematic liquid crystal model in $(0, T) \times \Omega$, where $\Omega \subset \mathbb{R}^N$ for $N = 2$ or 3 is an open bounded domain:

$$\begin{cases} \partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} - \mu\Delta \boldsymbol{u} + \nabla p = -\lambda \nabla \cdot (\nabla \boldsymbol{d}^t \nabla \boldsymbol{d}), \quad \nabla \cdot \boldsymbol{u} = 0, \\ \partial_t \boldsymbol{d} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{d} = \gamma(\Delta \boldsymbol{d} - \boldsymbol{f}_\varepsilon(\boldsymbol{d})). \end{cases} \tag{15}$$

The positive constants ν, λ and γ, are the fluid viscosity, the elasticity constant and the relaxation time, respectively.

In this penalized model, the constraint $|\boldsymbol{d}| = 1$ (where $|\cdot|$ is the punctual euclidean norm) is partially conserved to $|\boldsymbol{d}| \leq 1$ as consequence of the maximum principle for the Ginzburg-Landau equation considering the penalization function

$$\boldsymbol{f}_\varepsilon(\boldsymbol{d}) = \varepsilon^{-2}(|\boldsymbol{d}|^2 - 1)\boldsymbol{d}$$

where $\varepsilon > 0$ is the penalization parameter. There exists a potential function

$$\boldsymbol{F}_\varepsilon(\boldsymbol{d}) = \frac{1}{4\varepsilon^2}(|\boldsymbol{d}|^2 - 1)^2$$

such that $\boldsymbol{f}_\varepsilon(\boldsymbol{d}) = \nabla_{\boldsymbol{d}}(\boldsymbol{F}_\varepsilon(\boldsymbol{d}))$ for each $\boldsymbol{d} \in \mathbb{R}^N$.

The problem (15) is completed with the (Dirichlet) boundary conditions

$$\boldsymbol{u} = 0, \qquad \boldsymbol{d} = \boldsymbol{h} \qquad \text{on } \partial\Omega \times (0,T) \tag{16}$$

and the time-periodic conditions:

$$\boldsymbol{u}(0) = \boldsymbol{u}(T), \qquad \boldsymbol{d}(0) = \boldsymbol{d}(T) \qquad \text{in } \Omega. \tag{17}$$

It is important to remark that reproductive solution with the following boundary data independent of time $\boldsymbol{d}(x,t)_{|\partial\Omega\times(0,T)} = \boldsymbol{d}_0(x)$ has the trivial stationary (static) solution:

$$\begin{aligned}
&\boldsymbol{u} \equiv 0,\\
&\boldsymbol{d} \quad \text{solution of the elliptic problem:} \quad -\Delta\boldsymbol{d} + \boldsymbol{f}_\varepsilon(\boldsymbol{d}) = 0 \quad \text{in } \Omega, \quad \boldsymbol{d}_{|\partial\Omega} = \boldsymbol{d}_0,\\
&p = -\lambda\left(\frac{|\nabla\boldsymbol{d}|^2}{2} + \boldsymbol{F}_\varepsilon(\boldsymbol{d})\right).
\end{aligned}$$

The expression of p is due to the momentum equation reduces to

$$\nabla p = -\lambda\nabla\cdot(\nabla\boldsymbol{d}^t\nabla\boldsymbol{d}) = -\lambda\nabla\left(\frac{|\nabla\boldsymbol{d}|^2}{2} + \boldsymbol{F}_\varepsilon(\boldsymbol{d})\right) + \lambda\nabla\boldsymbol{d}^t(\boldsymbol{f}_\varepsilon(\boldsymbol{d}) - \Delta\boldsymbol{d}).$$

Therefore, since $-\Delta\boldsymbol{d} + \boldsymbol{f}_\varepsilon(\boldsymbol{d}) = 0$, one has $\nabla p = -\lambda\nabla\left(\frac{|\nabla\boldsymbol{d}|^2}{2} + \boldsymbol{F}_\varepsilon(\boldsymbol{d})\right)$.

Therefore, in this work will be fundamental assume time-dependent boundary data for $\boldsymbol{d}$.

In order to obtain the maximum principle for $|\boldsymbol{d}|^2$, we multiply the $\boldsymbol{d}$-system by $\boldsymbol{d}$ getting

$$\frac{1}{2}\partial_t|\boldsymbol{d}|^2 + \frac{1}{2}\boldsymbol{u}\cdot\nabla|\boldsymbol{d}|^2 - \gamma\Delta|\boldsymbol{d}|^2 + \gamma|\nabla\boldsymbol{d}|^2 + \gamma\boldsymbol{f}_\varepsilon(\boldsymbol{d})\cdot\boldsymbol{d} = 0,$$

whence the following differential inequality holds for $c = |\boldsymbol{d}|^2$:

$$\partial_t c + \boldsymbol{u}\cdot\nabla c - 2\gamma\Delta c + 2\gamma\frac{1}{\epsilon^2}(c-1)c \leq 0.$$

Notice that, if $c = |\boldsymbol{d}|^2 \geq 1$ then $\frac{1}{\epsilon^2}(c-1)c = \boldsymbol{f}_\varepsilon(\boldsymbol{d}) \cdot \boldsymbol{d} \geq 0$. Therefore, assuming $|\boldsymbol{h}| \leq 1$, we obtain $c \leq 1$ in $\partial\Omega$. Then, we can apply the maximum principle argument obtaining $c \leq 1$ in Ω, i.e. $|\boldsymbol{d}| \leq 1$ in Ω.

This maximum principle is fundamental in order to obtain solution of the (15)-(17) problem because we can consider a equivalent problem changing $\boldsymbol{f}_\varepsilon$ by $\widetilde{\boldsymbol{f}}_\varepsilon$, the auxiliary function

$$\widetilde{\boldsymbol{f}}_\varepsilon(\boldsymbol{d}) = \begin{cases} \boldsymbol{f}_\varepsilon(\boldsymbol{d}) & \text{if} \quad |\boldsymbol{d}| \leq 1, \\ 0 & \text{if} \quad |\boldsymbol{d}| > 1. \end{cases}$$

Indeed, if $(\boldsymbol{u}, p, \boldsymbol{d})$ is a solution of (15)-(17) with $\widetilde{\boldsymbol{f}}_\varepsilon$, in particular $|\boldsymbol{d}| \leq 1$ (because the maximum principle is also verified, since $\widetilde{\boldsymbol{f}}_\varepsilon(\boldsymbol{d}) \cdot \boldsymbol{d} \geq 0$ as $|\boldsymbol{d}| > 1$), then $(\boldsymbol{u}, p, \boldsymbol{d})$ is also a solution of (15)-(17) with $\boldsymbol{f}_\varepsilon$. The inverse statement is easy to verify.

Now, the key is that $|\widetilde{\boldsymbol{f}}_\varepsilon(\boldsymbol{d})| \leq \frac{1}{\varepsilon^2}$ $\forall \boldsymbol{d} \in \mathbb{R}^3$. Then, existence of weak reproductive solution of this model is proved in [1]. The main steps of the proof are to prove existence and uniqueness of solution for a Galerkin initial-boundary problem, to obtain the reproductivity of approximate solution with the argument of Theorem 3 and to pass to the limit. More concretely, using the lifting function $\widetilde{\boldsymbol{d}}(t)$ as the solution of Laplace-Dirichlet problem

$$\begin{cases} -\Delta \widetilde{\boldsymbol{d}} = 0 & \text{in } \Omega, \\ \widetilde{\boldsymbol{d}}|_{\partial\Omega} = \boldsymbol{h}(t) & \text{on } \partial\Omega, \end{cases},$$

defining $\widehat{\boldsymbol{d}}(t) = \boldsymbol{d}(t) - \widetilde{\boldsymbol{d}}(t)$ and taking $\boldsymbol{u}$ and $-\lambda\Delta\widehat{\boldsymbol{d}} = -\lambda\Delta\boldsymbol{d}$ as test functions in the equations for $\boldsymbol{u}$ and $\widehat{\boldsymbol{d}}$ of (15) respectively, one has the energy inequality:

$$\begin{aligned} \frac{d}{dt}\left(\|\boldsymbol{u}\|_{L^2}^2 + \lambda\|\nabla\widehat{\boldsymbol{d}}\|_{L^2}^2\right) + 2\mu\|\nabla\boldsymbol{u}\|_{L^2}^2 + \lambda\gamma\|\Delta\widehat{\boldsymbol{d}}\|_{L^2}^2 \\ \leq C\left(\|\boldsymbol{f}_\varepsilon(\boldsymbol{d})\|_{L^2}^2 + \|\partial_t\widetilde{\boldsymbol{d}}\|_{L^2}^2\right), \end{aligned} \tag{18}$$

where the right hand side is bounded in $L^1(0,T)$ if $\boldsymbol{f}_\varepsilon$, $\partial_t\widetilde{\boldsymbol{d}} \in L^2(\boldsymbol{L}^2)$.

Remark 2 *An interesting open problem in this context is the asymptotic behavior as $\varepsilon \to 0$ of the reproductive solutions of this liquid crystal model (15)-(17). For the initial-boundary problem, this asymptotic behavior is studied in [4], for time independent boundary data.*

5 Regularity of periodic solutions via regularity of reproductive solutions

We consider the time-periodic boundary problem associated to 3D Navier Stokes model with data $\boldsymbol{f}$.

Let $\boldsymbol{u}$ be a reproductive solution in $[0,T]$ (given in Theorem 3). The problem is to obtain regularity for this solution. A possible argument is to prove that there exists at least one time $t_\star \in [0,T]$ such that $\|\boldsymbol{u}(t_\star)\|_{H^1}$ is small enough. In fact, we can find that $t_\star$ exists, integrating in $(0,T)$ the energy inequality

$$\frac{d}{dt}\|\boldsymbol{u}(t)\|_{L^2}^2 + \nu\|\nabla\boldsymbol{u}(t)\|_{L^2}^2 \le \frac{1}{\nu}\|\boldsymbol{f}\|_{H^{-1}}^2$$

and applying the reproductive condition $u(0) = u(T)$, arriving at

$$\nu\int_0^T \|\nabla\boldsymbol{u}(t)\|_{L^2}^2 \le \frac{1}{\nu}\int_0^T ||\boldsymbol{f}(t)||_{H^{-1}}^2.$$

Assuming external forces $\boldsymbol{f}$ small enough in the $L^\infty(0,\infty;\mathbf{L}^2)$ norm, in particular $\frac{1}{\nu^2}\int_0^T ||\boldsymbol{f}(t)||_{H^{-1}}^2 \le \varepsilon T$ for some ε small enough, hence $\int_0^T \|\nabla\boldsymbol{u}(t)\|_{L^2}^2 \le \varepsilon T$. From integral mean value theorem, there exists $t_\star \in [0,T]$ such that $\|\nabla\boldsymbol{u}(t_\star)\|_{L^2}^2 \le \varepsilon$.

On the other hand, let $\overline{\boldsymbol{u}}$ be the unique regular strong solution (see Theorem 2) with initial data $\boldsymbol{u}(t_\star)$ and the same force $\boldsymbol{f}$. Moreover, following the proof of this type of global in time results with small data (see for instance ([22]), one has $\|\nabla\overline{\boldsymbol{u}}(t)\|_{L^2}^2 \le 2\,\varepsilon$ for each $t \ge t_\star$ (here $\boldsymbol{f}$ small enough in the $L^\infty(0,\infty;\mathbf{L}^2)$ norm is necessary).

By uniqueness of weak-strong solution (Theorem 2), one has $\overline{\boldsymbol{u}} \equiv \boldsymbol{u}$ in $[t_\star,T]$ and therefore $\boldsymbol{u}$ is regular in $[t_\star,T]$. In particular, $\|\nabla\boldsymbol{u}(T)\|_{L^2}^2 = \|\nabla\overline{\boldsymbol{u}}(T)\|_{L^2}^2 \le 2\,\varepsilon$. Therefore $\|\nabla\boldsymbol{u}(0)\|_{L^2}^2 \le 2\,\varepsilon$, hence $\boldsymbol{u}$ is a strong solution in $[0,T]$. Finally, in $[T,2T]$, $\boldsymbol{u}(t-T) \equiv \overline{\boldsymbol{u}}(t)$ and so on. The precedent argument is used, for instance in [13].

Therefore, we arrive at the following conclusion: The periodic extension of a reproductive solution $\boldsymbol{u}$ is a regular solution in $[0,+\infty)$ assuming small enough external forces $\boldsymbol{f}$. This conclusion is also valid for the models presented in Section 3.

Previous argument is based on to obtain $\int_0^T \|\boldsymbol{u}(t)\|_{H^1}^2 dt$ small enough, only assuming force $\boldsymbol{f}$ small enough (in particular, $\|\boldsymbol{u}(t_\star)\|_{H^1}^2$ is small for some $t_\star \in [0,T]$). But, there are some fluids models, where this is not always possible to obtain. For example, we will see below the $3D$ penalized nematic liquid crystal (in Subsection 5.1) and the generalized Boussinesq model (in Subsection 5.2).

5.1 $3D$ penalized nematic liquid crystal (15)-(17)

Considering an adequate lifting function $\widetilde{\boldsymbol{d}}$ (the solution of problem: $-\Delta\widetilde{\boldsymbol{d}} = 0$ in Ω, $\widetilde{\boldsymbol{d}} = \boldsymbol{h}$ on $\partial\Omega$), and denoting $\widehat{\boldsymbol{d}} = \boldsymbol{d} - \widetilde{\boldsymbol{d}}$, testing the $\boldsymbol{u}$-system by $\boldsymbol{u}$ and the

$\boldsymbol{d}$-system by $-\Delta\widehat{\boldsymbol{d}}$, one has the energy inequality

$$\frac{d}{dt}\left(\|\boldsymbol{u}\|_{L^2}^2+\lambda\|\nabla\widehat{\boldsymbol{d}}\|_{L^2}^2\right)+2\mu\|\nabla\boldsymbol{u}\|_{L^2}^2+\lambda\gamma\|\Delta\widehat{\boldsymbol{d}}\|_{L^2}^2 \le C\left(\lambda\gamma\|\boldsymbol{f}_\varepsilon(\boldsymbol{d})\|_{L^2}^2+\|\partial_t\widetilde{\boldsymbol{d}}\|_{L^2}^2\right). \tag{19}$$

Notice that for $2D$ domains, with similar arguments as for Navier-Stokes case in $2D$ domains, applying now the existence and uniqueness of weak solutions of (15)-(17) ([11]), one has that the extension by periodicity of $(\boldsymbol{u},\boldsymbol{d})$ is a global solution defined in $[0,+\infty)$ and it is regular (if boundary data $\boldsymbol{h}$ is regular).

But, for $3D$ domains, if we intend to apply small data argument done before, smallness of the right hand-side of energy inequality (19) cannot be assured, because of the term $\|\boldsymbol{f}(\boldsymbol{d})\|_{L^2}^2$ (which is bounded by not small). Indeed, integrating (19) in $(0,T)$ and using the reproductivity, one has

$$\lambda\gamma\int_0^T\|\Delta\widehat{\boldsymbol{d}}\|_{L^2}^2\le C\int_0^T\left(\lambda\gamma\|\boldsymbol{f}_\varepsilon(\boldsymbol{d})\|_{L^2}^2+\|\partial_t\widetilde{\boldsymbol{d}}\|_{L^2}^2\right)\le\bar\varepsilon+C\lambda\gamma\int_0^T\|\boldsymbol{f}_\varepsilon(\boldsymbol{d})\|_{L^2}^2$$

but this bound is not necessary small. It is only small for the penalty parameter ε big enough, which is not a physical interesting case.

Another possibility is to start from the energy equality:

$$\frac{d}{dt}\left(\|\boldsymbol{u}\|_{L^2}^2+\lambda\|\nabla\widehat{\boldsymbol{d}}\|_{L^2}^2+2\,\lambda\int_\Omega F_\varepsilon(\boldsymbol{d})\right)+2\,\mu\|\nabla\boldsymbol{u}\|_{L^2}^2+\lambda\gamma\|\Delta\widehat{\boldsymbol{d}}-\boldsymbol{f}_\varepsilon(\boldsymbol{d})\|_{L^2}^2 \le\frac{\lambda}{\gamma}\int_0^T\|\partial_t\tilde{\boldsymbol{d}}\|_{L^2}^2+\frac{2\lambda}{\varepsilon^2}\int_0^T\|\partial_t\tilde{\boldsymbol{d}}\|_{L^1} \tag{20}$$

where $F_\varepsilon(\boldsymbol{d})=\dfrac{1}{4\varepsilon^2}(|\boldsymbol{d}|^2-1)^2$. Indeed, testing the $\boldsymbol{u}$-system by $\boldsymbol{u}$ and the $\boldsymbol{d}$-system by $\lambda(-\Delta\widehat{\boldsymbol{d}}+\boldsymbol{f}_\varepsilon(\boldsymbol{d}))$, one has

$$\frac{d}{dt}\left(\|\boldsymbol{u}\|_{L^2}^2+\lambda\|\nabla\widehat{\boldsymbol{d}}\|_{L^2}^2\right)+2\lambda\int_\Omega\partial_t\widehat{\boldsymbol{d}}\cdot\boldsymbol{f}_\varepsilon(\boldsymbol{d})+2\mu\|\nabla\boldsymbol{u}\|_{L^2}^2 +\lambda\gamma\|\Delta\widehat{\boldsymbol{d}}-\boldsymbol{f}_\varepsilon(\boldsymbol{d})\|_{L^2}^2\le C\|\partial_t\widetilde{\boldsymbol{d}}\|_{L^2}^2,$$

Then, using $\nabla_{\boldsymbol{d}}F_\varepsilon(\boldsymbol{d})=\boldsymbol{f}_\varepsilon(\boldsymbol{d})$, one has

$$\int_\Omega\partial_t\widehat{\boldsymbol{d}}\cdot\boldsymbol{f}_\varepsilon(\boldsymbol{d})=\frac{d}{dt}\int_\Omega F_\varepsilon(\boldsymbol{d})+\int_\Omega\partial_t\widetilde{\boldsymbol{d}}\cdot\boldsymbol{f}_\varepsilon(\boldsymbol{d})$$

Therefore, if we bound the last term by $\dfrac{1}{\varepsilon^2}\|\partial_t\widetilde{\boldsymbol{d}}\|_{L^1}$, since $\|\boldsymbol{f}_\varepsilon(\boldsymbol{d})\|_{L^\infty}\le\dfrac{1}{\varepsilon^2}$, inequality (20) is proven.

In this case, for $\varepsilon>0$ fixed, smallness for $\int_0^T\|\Delta\widehat{\boldsymbol{d}}-\boldsymbol{f}_\varepsilon(\boldsymbol{d})\|_{L^2}^2$ is obtained (using that $\int_0^T\|\partial_t\widetilde{\boldsymbol{d}}\|_{L^2}^2$ and $\int_0^T\|\partial_t\widetilde{\boldsymbol{d}}\|_{L^1}$ are small), but this does not give sufficient information to prove the smallness in the H^2 norm of $\widehat{\boldsymbol{d}}$ (again the term $\|\boldsymbol{f}_\varepsilon(\boldsymbol{d})\|_{L^2}^2$ appears).

In conclusion, the regularity of the reproductive solutions for the $3D$ penalized nematic liquid crystal model (15)-(17) is an open problem.

5.2 Generalized Boussinesq model with Neumann boundary conditions for temperature

If we apply the same argument done in this section to prove regularity of the time-periodic solution for the 3D Navier Stokes model, now we obtain from energy inequality (10) that

$$\int_0^T \left\{ \|\boldsymbol{u}(t)\|_{H^1}^2 + \|\theta(t)\|_{H^1}^2 \right\} \leq C \int_0^T \|\boldsymbol{f}(t)\|_{H^{-1}}^2$$

Hence assuming $\boldsymbol{f}$ small enough in the $L^\infty(0,+\infty;\mathbf{H}^{-1})$-norm, then there exists $t_\star \in [0,T]$ such that $\|\boldsymbol{u}(t_\star)\|_{H^1}^2 + \|\theta(t_\star)\|_{H^2}^2$ is small. To continue the argument, due to the highly nonlinear second order terms $\nabla \cdot (\nu(\theta)\nabla \boldsymbol{u})$ and $\nabla \cdot (k(\theta)\nabla\theta)$, also smallness in $\|\theta(t_\star)\|_{H^2}^2$ must be assured, in order to prove global and small regular solution $(\boldsymbol{u},\theta)$, but to obtain smallness of $\|\theta(t_*)\|_{H^2}$ for some t_*, is not clear.

Nevertheless a more direct argument could be considered. For instance, this argument works when the involved equations are (12), together with the Dirichlet-Neumann boundary conditions:

$$\boldsymbol{u} = 0, \qquad \partial_n \theta = 0 \qquad \text{on } [0,\infty) \times \partial\Omega, \tag{21}$$

and the time reproductive condition (14) as is proved in [2]. Indeed, assuming $\boldsymbol{f}$ small enough (but no the function $\boldsymbol{g}$ depending on the gravity), $H^2(\Omega)$ regularity for velocity and $H^3(\Omega)$ regularity for the temperature can be obtained ([2]) and consequently, a regular (and small) reproductive solution of (12), (14), (21) in $(0,T)$ exists (which a reproductive condition for time derivative of temperature also holds, i.e. $\partial_t\theta(0) = \partial_t\theta(T)$). The main ideas in the proof are, to obtain some differential inequalities in regular norms ($H^2(\Omega)$ for velocity and $H^3(\Omega)$ for temperature) and to use an argument of global solution for small data jointly with the argument of regular time periodic solution (see [2]).

Notice that, the uniqueness of regular time periodic solutions remains open, because higher regularity for the velocity (for instance of the H^3 type) is necessary in order to get uniqueness of the model (12), (14), (21). To obtain H^3 regularity for the velocity is not clear because the argument made in order to get H^3 regularity for θ is based in the Neumann condition, but for the velocity we have Dirichlet condition.

Remark 3 *When Dirichlet boundary conditions for $\boldsymbol{u}$ and θ are assumed, it is not clear how to obtain appropriate differential inequalities in H^2 for velocity and H^3 for temperature. In conclusion, the regularity of the time-periodic solution for the model* (12), (13), (14) *is an open problem.*

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