Stationary 2D and 3D results for the Oseen and Navier-Stokes problem with singular data

María Ángeles Rodríguez-Bellido

joint to Chérif Amrouche (LAMA, Université de Pau et des Pays de l'Adour, France)

Dpto. Ecuaciones Diferenciales y Análisis Numérico
Facultad de Matemáticas
Universidad de Sevilla, Spain

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Stokes equations

$$(S)\ \begin{cases} -\Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f}, \quad \nabla \cdot \boldsymbol{u} = h & \text{in } \Omega, \\ \boldsymbol{u} = \boldsymbol{g} & \text{on } \Gamma. \end{cases}$$

Oseen equations

$$(O)\ \begin{cases} -\Delta \boldsymbol{u} + \boldsymbol{v} \cdot \nabla \boldsymbol{u} + \nabla \pi = \boldsymbol{f}, \quad \nabla \cdot \boldsymbol{u} = h & \text{in } \Omega, \\ \boldsymbol{u} = \boldsymbol{g} & \text{on } \Gamma. \end{cases}$$

Navier-Stokes equations

$$(NS)\ \begin{cases} -\Delta \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla \pi = \boldsymbol{f}, \quad \nabla \cdot \boldsymbol{u} = h & \text{in } \Omega, \\ \boldsymbol{u} = \boldsymbol{g} & \text{on } \Gamma. \end{cases}$$

Generalized solutions for (NS)

- **For $h = 0$, we know (Leray, 1933) that if**

$$\boldsymbol{f} \in \mathbf{W}^{-1,p}(\Omega) \quad \text{and} \quad \boldsymbol{g} \in \mathbf{W}^{1-1/p,p}(\Gamma), \quad \text{with} \quad p \geq 2$$

and
$$\int_{\Gamma_i} \boldsymbol{g} \cdot \boldsymbol{n} \, d\sigma = 0, \qquad \forall i = 0, ..., I, \tag{1}$$

for Γ_i the connected components of the boundary Γ, $i = 0, \ldots, I$, then there exists a solution of (NS) with

$$(\boldsymbol{u}, \pi) \in \mathbf{W}^{1,p}(\Omega) \times L^p(\Omega)$$

- **Serre (1983) proved the existence of weak solution**

$$(\boldsymbol{u}, \pi) \in \mathbf{W}^{1,p}(\Omega) \times L^p(\Omega) \quad \text{for any} \quad \frac{3}{2} < p < 2$$

with the same hypotheses for h and $\boldsymbol{g}$.
- **Kim (2009) extended the existence result to the case $\frac{3}{2} \leq p < 2$, with connected Γ ($I = 0$) and for h and g small enough in a convenient norm.**

Very weak solutions for (NS)

- **The existence of very weak solutions $(\boldsymbol{u}, \pi) \in \mathbf{L}^3(\Omega) \times W^{-1,3}(\Omega)$, considering**

$$\boldsymbol{f} \in \mathbf{H}^{-1}(\Omega), \quad h = 0 \quad \text{and} \quad \boldsymbol{g} \in \mathbf{L}^2(\Gamma)$$

big enough and without the restriction of null-flux (1) was established by Marusic-Paloka (2000) for Ω connected and $\mathcal{C}^{1,1}$.

- **BUT the proof is only correct when condition (1) is satisfied or**

$$\sum_{i=0}^{i=I} |\langle \boldsymbol{g} \cdot \boldsymbol{n}, 1 \rangle_{\Gamma_i}| \leq \delta \quad \text{(general case when } h = 0). \quad (2)$$

- **Kim (2009) proved the same result for any $\boldsymbol{f} \in [\mathbf{W}_0^{1,3/2}(\Omega) \cap W^{2,3}(\Omega)]'$, and $h \in [W^{1,3/2}(\Omega)]'$ and $\boldsymbol{g} \in \mathbf{W}^{-1/3,3}(\Gamma)$ small enough, Γ connected $(I = 0)$. (Non-correct spaces)**

AIM:

- **Generalize the very weak solution theory**

$$(\boldsymbol{u}, \pi) \in \mathbf{L}^p(\Omega) \times W^{-1,p}(\Omega),$$

  with $1 < p < \infty$, for the stationary Stokes, Oseen and Navier-Stokes equations with non-homogeneous Dirichlet boundary conditions.
- **We need a rigorous definition of the traces of functions in $\mathbf{L}^p(\Omega)$ (see Amrouche-Girault (1994) ou Amrouche- Rodriguez-Bellido (2010,2011)).**
- **Regularity and uniqueness of very weak solutions.**
- **Solutions in fractional Sobolev spaces.**

For Stokes, Oseen and Navier-Stokes we follow the steps:

- **Existence of weak solution, that is $(\boldsymbol{u}, \pi) \in \mathbf{H}^1(\Omega) \times L^2(\Omega)$.**
- **Existence of strong solution, that is $(\boldsymbol{u}, \pi) \in \mathbf{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)$ for any $p > 1$.**
- **Existence of generalized solution, that is $(\boldsymbol{u}, \pi) \in \mathbf{W}^{1,p}(\Omega) \times L^p(\Omega)$ for any $p > 1$.**
- **Existence of very weak solution, that is $(\boldsymbol{u}, \pi) \in \mathbf{L}^p(\Omega) \times W^{-1,p}(\Omega)$ for any $p > 1$.**
- **Fractional Sobolev spaces results.**

Questions related to the three problems:

- **A trace theorem.**
- **The 2-dimensional case and the 3-dimensional case.**
- **Regularity demanded for the convective velocity v in the Oseen problem.**

The first work

- **The trace result.**
- **The 3-dimensional case.**
- **The convective velocity $v \in \mathbf{L}^s(\Omega)$ (for s depending on $\mathbf{L}^p$ space) and $\nabla \cdot v = 0$.**

The second work

- **The 2-dimensional case and 3-dimensional case.**
- **The convective velocity $v \in \mathbf{L}^3(\Omega)$ in the 3D case, $v \in \mathbf{L}^2(\Omega)$ in the 2D case and $\nabla \cdot v \neq 0$ is small (in some sense).**

Lemma 1 (tangential traces)

Let Ω be a bounded open set of $\mathbb{R}^3$ of class $\mathcal{C}^{1,1}$. Let $1 < p < \infty$ and $r > 1$ be such that $\frac{1}{r} \leq \frac{1}{p} + \frac{1}{3}$. The mapping $\gamma_\tau : \boldsymbol{v} \mapsto \boldsymbol{v}_\tau|_\Gamma$ on the space $\mathcal{D}(\overline{\Omega})^3$ can be extended by continuity to a linear and continuous mapping, still denoted by γ_τ, from $\mathbf{T}_{p,r}(\Omega)$ into $\mathbf{W}^{-1/p,p}(\Gamma)$, and the following Green formula holds

$$\langle \Delta \boldsymbol{v}, \boldsymbol{\psi} \rangle_{[\mathbf{X}_{r',p'}(\Omega)]' \times \mathbf{X}_{r',p'}(\Omega)} = \int_\Omega \boldsymbol{v} \cdot \Delta \boldsymbol{\psi} \, d\boldsymbol{x} - \left\langle \boldsymbol{v}_\tau, \frac{\partial \boldsymbol{\psi}}{\partial \boldsymbol{n}} \right\rangle_{\mathbf{W}^{-1/p,p}(\Gamma) \times \mathbf{W}^{1/p,p'}(\Gamma)}, \qquad (3)$$

for any $\boldsymbol{v} \in \mathbf{T}_{p,r}(\Omega)$ and $\boldsymbol{\psi} \in \mathbf{Y}_{p'}(\Omega)$.

We introduce the spaces:

$$\mathcal{D}_\sigma(\Omega) = \{\varphi \in \mathcal{D}(\Omega);\ \nabla \cdot \varphi = 0\},$$

$$\mathcal{D}_\sigma(\overline{\Omega}) = \{\varphi \in \mathcal{D}(\overline{\Omega})^3;\ \nabla \cdot \varphi = 0\},$$

- **For the test functions, we consider the space**

$$\mathbf{Y}_{p'}(\Omega) = \{\boldsymbol{\psi} \in \mathbf{W}^{2,p'}(\Omega);\ \boldsymbol{\psi}|_\Gamma = \mathbf{0},\ (\nabla \cdot \boldsymbol{\psi})|_\Gamma = 0\}$$

also be described (see Amrouche-Girault (94)) as:

$$\mathbf{Y}_{p'}(\Omega) = \{\boldsymbol{\psi} \in \mathbf{W}^{2,p'}(\Omega);\ \boldsymbol{\psi}|_\Gamma = \mathbf{0},\ \frac{\partial \boldsymbol{\psi}}{\partial \boldsymbol{n}} \cdot \boldsymbol{n}\Big|_\Gamma = 0\}. \quad (4)$$

- **Which is contained in the space:**

$$\mathbf{X}_{r,p}(\Omega) = \{\boldsymbol{\varphi} \in \mathbf{W}_0^{1,r}(\Omega);\ \nabla \cdot \boldsymbol{\varphi} \in W_0^{1,p}(\Omega)\}$$

Lemma 2

The space $\boldsymbol{\mathcal{D}}(\Omega)$ is dense in $\mathbf{X}_{r,p}(\Omega)$ and for all $q \in W^{-1,p}(\Omega)$ and $\boldsymbol{\varphi} \in \mathbf{X}_{r',p'}(\Omega)$, we have

$$\langle \nabla q,\ \boldsymbol{\varphi}\rangle_{[\mathbf{X}_{r',p'}(\Omega)]' \times \mathbf{X}_{r',p'}(\Omega)} = -\langle q,\ \nabla \cdot \boldsymbol{\varphi}\rangle_{W^{-1,p}(\Omega) \times W_0^{1,p'}(\Omega)}. \quad (5)$$

- **For the searched solution, we use the space:**

$$(\mathbf{X}_{r',p'}(\Omega))' = \left\{ \boldsymbol{f} = \nabla \cdot \mathbb{F}_0 + \nabla f_1;\ \mathbb{F}_0 \in \mathbb{L}^r(\Omega),\ f_1 \in W^{-1,p}(\Omega),\right.$$
$$\left. \text{with} \quad \mathbb{F}_0 = (f_{ij})_{1 \le i,j \le 3} \right\}. \tag{6}$$

We can prove that:

$$\mathbf{W}^{-1,r}(\Omega) \hookrightarrow (\mathbf{X}_{r',p'}(\Omega))' \hookrightarrow \mathbf{W}^{-2,p}(\Omega), \tag{7}$$

where the second embedding holds if $\frac{1}{r} \le \frac{1}{p} + \frac{1}{3}$.

The trace's space will be defined over the space:

$$\begin{aligned} \mathbf{T}_{p,r}(\Omega) &= \{\boldsymbol{v} \in \mathbf{L}^p(\Omega);\ \Delta \boldsymbol{v} \in (\mathbf{X}_{r',p'}(\Omega))'\}, \\ \mathbf{T}_{p,r,\sigma}(\Omega) &= \{\boldsymbol{v} \in \mathbf{T}_{p,r}(\Omega);\ \nabla \cdot \boldsymbol{v} = 0\}, \end{aligned}$$

endowed with the norm

$$\|\boldsymbol{v}\|_{\mathbf{T}_{p,r}(\Omega)} = \|\boldsymbol{v}\|_{\mathbf{L}^p(\Omega)} + \|\Delta \boldsymbol{v}\|_{[\mathbf{X}_{r',p'}(\Omega)]'}.$$

Lemma 3

i) **The space $\mathcal{D}(\overline{\Omega})$ is dense in $\mathbf{T}_{p,r}(\Omega)$ and in $\mathbf{T}_{p,r}(\Omega) \cap \mathbf{H}_{p,r}(\mathrm{div};\Omega)$ respectively.**

ii) **The space $\mathcal{D}_\sigma(\overline{\Omega})$ is dense in $\mathbf{T}_{p,r,\sigma}(\Omega)$.**

- **The tangential trace of functions $\boldsymbol{v}$ of $\mathbf{T}_{p,r,\sigma}(\Omega)$ belongs to the dual space of $\mathbf{Z}_{p'}(\Gamma)$, which is**

$$(\mathbf{Z}_{p'}(\Gamma))' = \{\boldsymbol{\mu} \in \mathbf{W}^{-1/p,p}(\Gamma);\ \boldsymbol{\mu} \cdot \boldsymbol{n} = 0\}.$$

We recall that we can decompose $\boldsymbol{v}$ into its tangential, $\boldsymbol{v}_\tau$, and normal parts: $\boldsymbol{v} = \boldsymbol{v}_\tau + (\boldsymbol{v} \cdot \boldsymbol{n})\,\boldsymbol{n}$.

- **We also introduce the spaces**

$$\mathbf{H}_p(\Omega) = \{\boldsymbol{v} \in \mathbf{L}^p(\Omega);\ \nabla \cdot \boldsymbol{v} = 0\},$$

$$\mathbf{H}_{p,r}(\mathsf{div};\Omega) = \{\boldsymbol{v} \in \mathbf{L}^p(\Omega);\ \nabla \cdot \boldsymbol{v} \in L^r(\Omega)\},$$

which is endowed with the graph norm.

Lemma 4

i) **The space $\mathcal{D}(\overline{\Omega})$ is dense in $\mathbf{T}_{p,r}(\Omega)$ and in $\mathbf{T}_{p,r}(\Omega) \cap \mathbf{H}_{p,r}(\mathsf{div};\Omega)$ respectively.**

ii) **The space $\mathcal{D}_\sigma(\overline{\Omega})$ is dense in $\mathbf{T}_{p,r,\sigma}(\Omega)$.**

The Stokes problem

We always assume the compatibility condition:

$$\int_\Omega h(\boldsymbol{x})\, d\boldsymbol{x} = \langle \boldsymbol{g}\cdot\boldsymbol{n}, 1\rangle_{W^{-1/p,p}(\Gamma)\times W^{1/p,p'}(\Gamma)}. \qquad (8)$$

- **Generalized solutions for Stokes system (Cattabriga (1961), Amrouche-Girault (1994)), that is,**

$$\boldsymbol{f} \in \mathbf{W}^{-1,p}(\Omega),\ h \in L^p(\Omega),\ \boldsymbol{g} \in \mathbf{W}^{1-1/p,p}(\Gamma)$$
$$\Rightarrow (\boldsymbol{u},\pi) \in \mathbf{W}^{1,p}(\Omega) \times L^p(\Omega)/\mathbb{R}$$

- **Strong solution, that is,**

$$\boldsymbol{f} \in \mathbf{L}^p(\Omega),\ h \in W^{1,p}(\Omega)\ \boldsymbol{g} \in \mathbf{W}^{2-1/p,p}(\Gamma)$$
$$\Rightarrow (\boldsymbol{u},\pi) \in \mathbf{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)$$

Definition (Very weak solution for the Stokes problem)

A pair

$$(\boldsymbol{u}, \pi) \in \mathbf{L}^p(\Omega) \times W^{-1,p}(\Omega)$$

is a very weak solution of (S) if the following equalities hold:

For any $\boldsymbol{\varphi} \in \mathbf{Y}_{p'}(\Omega)$ and $\chi \in W^{1,p'}(\Omega)$,

$$\int_\Omega \boldsymbol{u} \cdot \Delta\boldsymbol{\varphi}\, d\boldsymbol{x} - \langle \pi, \nabla \cdot \boldsymbol{\varphi} \rangle_{W^{-1,p}(\Omega) \times W_0^{1,p'}(\Omega)} = \langle \boldsymbol{f}, \boldsymbol{\varphi} \rangle_\Omega - \langle \boldsymbol{g}_\tau, \frac{\partial \boldsymbol{\varphi}}{\partial \boldsymbol{n}} \rangle_\Gamma,$$

$$\int_\Omega \boldsymbol{u} \cdot \nabla\chi\, d\boldsymbol{x} = -\int_\Omega h\, \chi\, d\boldsymbol{x} + \langle \boldsymbol{g} \cdot \boldsymbol{n}, \chi \rangle_\Gamma, \qquad (9)$$

- Very weak solution, that is:

$$\boldsymbol{f} \in (\mathbf{X}_{p'}(\Omega))', \; h \in L^p(\Omega), \; \boldsymbol{g} \in \mathbf{W}^{-1/p,p}(\Gamma)$$
$$\Rightarrow (\boldsymbol{u}, \pi) \in \mathbf{L}^p(\Omega) \times W^{-1,p}(\Omega)/\mathbb{R}$$

- Let $\boldsymbol{f} = \nabla \cdot \mathbb{F}_0 + \nabla f_1$, h, $\boldsymbol{g}$ be given satisfying (8) and

$$\mathbb{F}_0 \in \mathbb{L}^r(\Omega), \; f_1 \in W^{-1,p}(\Omega), \; h \in L^r(\Omega), \; \boldsymbol{g} \in \mathbf{W}^{1-1/r,r}(\Gamma).$$

  Then the previous solution $\boldsymbol{u}$ belongs to $\mathbf{W}^{1,r}(\Omega)$. If moreover $f_1 \in L^r(\Omega)$, then $\pi \in L^r(\Omega)$.

Corollary 5 (Solutions in fractionary Sobolev spaces)

Let s be a real number such that $0 \le s \le 1$.
i) Let $\boldsymbol{f} = \nabla \cdot \mathbb{F}_0 + \nabla f_1$, h and $\boldsymbol{g}$ satisfying (8) with

$$\mathbb{F}_0 \in \mathbf{W}^{s,r}(\Omega),\ f_1 \in W^{s-1,p}(\Omega),\ \boldsymbol{g} \in \mathbf{W}^{s-1/p,p}(\Gamma),\ h \in W^{s,r}(\Omega),$$

with $\frac{1}{r} \le \frac{1}{p} + \frac{1}{3}$ and $r \le p$. Then, the Stokes problem (S) has exactly one solution

$$(\boldsymbol{u},\ \pi) \in \mathbf{W}^{s,p}(\Omega) \times W^{s-1,p}(\Omega)/\mathbb{R}$$

satisfying the estimate

$$\|\boldsymbol{u}\|_{\mathbf{W}^{s,p}(\Omega)} + \|\pi\|_{W^{s-1,p}(\Omega)/\mathbb{R}} \le C\,(\|\mathbb{F}_0\|_{\mathbf{W}^{s,r}(\Omega)} + \|f_1\|_{W^{s-1,p}(\Omega)} + \\ + \|h\|_{W^{s,r}(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{s-1/p,p}(\Gamma)}).$$

Theorem 6 (Solutions in fractionary Sobolev spaces)

Let s be a real number such that $\frac{1}{p} < s \leq 2$. Let $\boldsymbol{f}$, h and $\boldsymbol{g}$ satisfy the compatibility condition (8) with

$$\boldsymbol{f} \in \mathbf{W}^{s-2,p}(\Omega), \quad h \in W^{s-1,p}(\Omega) \quad \text{and} \quad \boldsymbol{g} \in \mathbf{W}^{s-1/p,p}(\Gamma).$$

Then, the Stokes problem (S) has exactly one solution

$$(\boldsymbol{u},\ \pi) \in \mathbf{W}^{s,p}(\Omega) \times W^{s-1,p}(\Omega)/\mathbb{R}$$

satisfying the corresponding estimate.

The Oseen problem

Weak solution in the 3D case

- **For** $\boldsymbol{f} \in \mathbf{H}^{-1}(\Omega)$, $\boldsymbol{v} \in \mathbf{H}_3(\Omega)$, $h \in L^2(\Omega)$ **and** $\boldsymbol{g} \in \mathbf{H}^{1/2}(\Gamma)$, **with** h **and** $\boldsymbol{g}$ **verifying the compatibility condition (8) then the problem** (O) **has a unique solution**

$$(\boldsymbol{u}, \pi) \in \mathbf{H}^1(\Omega) \times L^2(\Omega)/\mathbb{R}$$

verifying the estimate:

$$\begin{aligned} \|\boldsymbol{u}\|_{\mathbf{H}^1(\Omega)} &\leq C\Big(\|\boldsymbol{f}\|_{\mathbf{H}^{-1}(\Omega)} \\ &+ \left(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\right) \left(\|h\|_{L^2(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{H}^{1/2}(\Gamma)}\right)\Big) \\ \|\pi\|_{L^2(\Omega)} &\leq C \left(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\right) \Big(\|\boldsymbol{f}\|_{\mathbf{H}^{-1}(\Omega)} \\ &+ \left(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\right) \left(\|h\|_{L^2(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{H}^{1/2}(\Gamma)}\right)\Big). \end{aligned}$$

The 2-dimensional case (sketch of the proof)

- **We lift the boundary and the divergence data using $\boldsymbol{u}_0 \in \mathbf{H}^1(\Omega)$ such that**

$$\nabla \cdot \boldsymbol{u}_0 = h \text{ in } \Omega, \quad \boldsymbol{u}_0 = \boldsymbol{g} \text{ on } \Gamma$$

and:

$$\|\boldsymbol{u}_0\|_{\mathbf{H}^1(\Omega)} \leq C \left(\|h\|_{L^2(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{H}^{1/2}(\Gamma)} \right). \qquad (10)$$

- **It remains to find $(\boldsymbol{z}, \pi) = (\boldsymbol{u} - \boldsymbol{u}_0, \pi)$ in $\mathbf{H}^1_0(\Omega) \times L^2(\Omega)$ such that:**

$$-\Delta \boldsymbol{z} + \boldsymbol{v} \cdot \nabla \boldsymbol{z} + \nabla \pi = \mathbf{F} \text{ and } \nabla \cdot \boldsymbol{z} = 0 \text{ in } \Omega, \quad \boldsymbol{z} = \mathbf{0} \text{ on } \Gamma. \qquad (11)$$

being $\widetilde{\boldsymbol{f}} = \boldsymbol{f} + \Delta \boldsymbol{u}_0 - (\boldsymbol{v} \cdot \nabla) \boldsymbol{u}_0$.

Since the space $\mathcal{D}_\sigma(\Omega) = \{\varphi \in \mathcal{D}(\Omega);\ \nabla\cdot\varphi = 0\}$ is dense in the space $\mathbf{V} = \{z \in \mathbf{H}_0^1(\Omega);\ \nabla\cdot z = 0\}$, the previous problem is equivalent to:

$$\begin{cases} \text{Find } z \in \mathbf{V} \text{ such that:} \\ \forall\varphi \in \mathbf{V}, \qquad \displaystyle\int_\Omega \nabla z\cdot\nabla\varphi\,dx + b(v,z,\varphi) = \langle\widetilde{f},\varphi\rangle_{\mathbf{H}^{-1}(\Omega)\times\mathbf{H}_0^1(\Omega)}, \end{cases}$$

where

the trilinear form

$$b(v,z,\varphi) = \int_\Omega (v\cdot\nabla)z\cdot\varphi\,dx$$

is an antisymmetric form with respect to the last two variables, well-defined for $v \in \mathbf{H}_2(\Omega)$, z, $\varphi \in \mathbf{H}_0^1(\Omega)$.

We consider $w \in \mathbf{L}^2(\mathbb{R}^2)$ the extension of v to $\mathbb{R}^2$ given by:

$$w = \begin{cases} v & \text{in } \Omega, \\ \nabla\theta & \text{in } \Omega' = \mathbb{R}^2 \setminus \Omega. \end{cases}$$

where $\theta \in H^1_0(\Omega')$ is the solution of the following problem:

$$\begin{cases} \Delta\theta = 0 & \text{in } \Omega', \\ \dfrac{\partial\theta}{\partial n} = -v \cdot n & \text{on } \Gamma, \end{cases}$$

satisfies

$$\|\theta\|_{H^1(\Omega')} \leq C\|v \cdot n\|_{\mathbf{H}^{-1/2}(\Gamma)}$$

and

$$\nabla \cdot w = 0 \text{ in } \mathbb{R}^2$$

with $\|w\|_{\mathbf{L}^2(\mathbb{R}^2)} \leq C\,\|v\|_{\mathbf{L}^2(\Omega)}$.

Considering $\widetilde{z}$ the extension by zero of z and $\widetilde{z} \in \mathbf{H}^1(\mathbb{R}^2)$, using the result:

Coifman-Lions-Meyer

Suppose $\boldsymbol{u} \in \mathbf{L}^p(\mathbb{R}^N)$ such that $\nabla \cdot \boldsymbol{u} = 0$ and $\boldsymbol{v} \in \mathbf{L}^q(\mathbb{R}^N)$ such that $\nabla^{\perp} \cdot \boldsymbol{v} = 0$ $(0 < p < +\infty$ and $0 < q < +\infty$ with $\frac{1}{p} + \frac{1}{q} < 1 + \frac{1}{N})$. Then, $\boldsymbol{u} \cdot \boldsymbol{v} \in \mathcal{H}^r(\mathbb{R}^N)$ with $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$,

we can deduce that:

$$\boldsymbol{w} \cdot \nabla \widetilde{\boldsymbol{z}} \in \mathcal{H}^1(\mathbb{R}^2)$$

and the bound:

$$\|\boldsymbol{w} \cdot \nabla \widetilde{\boldsymbol{z}}\|_{\mathcal{H}^1(\mathbb{R}^2)} \leq C \|\boldsymbol{w}\|_{\mathbf{L}^2(\mathbb{R}^2)} \|\nabla \widetilde{\boldsymbol{z}}\|_{\mathbf{L}^2(\mathbb{R}^2)} \leq C \|\boldsymbol{v}\|_{\mathbf{L}^2(\Omega)} \|\nabla \boldsymbol{z}\|_{\mathbf{L}^2(\Omega)}$$

Therefore,

- $\boldsymbol{v}\cdot\nabla\boldsymbol{z}\in\mathbf{H}^{-1}(\Omega)$ **because for** $\boldsymbol{\varphi}\in\boldsymbol{\mathcal{D}}(\Omega)$

$$\begin{aligned}|\langle\boldsymbol{v}\cdot\nabla\boldsymbol{z},\boldsymbol{\varphi}\rangle_{\boldsymbol{\mathcal{D}}'(\Omega)\times\boldsymbol{\mathcal{D}}(\Omega)}| &= |\int_{\mathbb{R}^2}\boldsymbol{w}\cdot\nabla\widetilde{\boldsymbol{z}}\cdot\widetilde{\boldsymbol{\varphi}}\,d\boldsymbol{x}| \\ &\leq \|\boldsymbol{w}\cdot\nabla\widetilde{\boldsymbol{z}}\|_{\mathcal{H}^1(\mathbb{R}^2)}\|\widetilde{\boldsymbol{\varphi}}\|_{BMO(\mathbb{R}^2)} \\ &\leq C\|\boldsymbol{v}\|_{\mathbf{L}^2(\Omega)}\|\nabla\boldsymbol{z}\|_{\mathbf{L}^2(\Omega)}\|\widetilde{\boldsymbol{\varphi}}\|_{\mathbf{H}^1(\mathbb{R}^2)} \\ &\leq C\|\boldsymbol{v}\|_{\mathbf{L}^2(\Omega)}\|\nabla\boldsymbol{z}\|_{\mathbf{L}^2(\Omega)}\|\boldsymbol{\varphi}\|_{\mathbf{H}^1(\Omega)}\end{aligned} \tag{12}$$

because

$$H^1(\mathbb{R}^2)\hookrightarrow VMO(\mathbb{R}^2)\hookrightarrow BMO(\mathbb{R}^2).$$

- **Also** $\langle\boldsymbol{v}\cdot\nabla\boldsymbol{z},\boldsymbol{z}\rangle_{\mathbf{H}^{-1}(\Omega)\times\mathbf{H}^1_0(\Omega)}=0$.

Theorem 7 (Strong solutions for $p \geq 6/5$)

Consider $p \geq \frac{6}{5}$, $\boldsymbol{f} \in \mathbf{L}^p(\Omega)$, $h \in W^{1,p}(\Omega)$, $\boldsymbol{g} \in \mathbf{W}^{2-1/p,p}(\Gamma)$ and $\boldsymbol{v} \in \mathbf{H}_s(\Omega)$ with

$$s = 3 \quad \text{if } p < 3, \qquad s = p \quad \text{if } p > 3 \quad \text{or} \quad s = 3 + \varepsilon \quad \text{if } p = 3,$$

for some arbitrary $\varepsilon > 0$ and satisfying the compatibility condition (8). Then, the unique solution of (O) verifies

$$(\boldsymbol{u}, \pi) \in \mathbf{W}^{2,p}(\Omega) \times W^{1,p}(\Omega).$$

Moreover, there exists a constant $C > 0$ such that

$$\|\boldsymbol{u}\|_{\mathbf{W}^{2,p}(\Omega)} + \|\pi\|_{W^{1,p}(\Omega)/\mathbb{R}} \leq C\left(1 + \|\boldsymbol{v}\|_{\mathbf{L}^s(\Omega)}\right)\Big(\|\boldsymbol{f}\|_{\mathbf{L}^p(\Omega)} + $$
$$+ \left(1 + \|\boldsymbol{v}\|_{\mathbf{L}^s(\Omega)}\right)\left(\|h\|_{W^{1,p}(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{2-1/p,p}(\Gamma)}\right)\Big).$$

Theorem 7 new (Strong solutions for $p \geq 6/5$)

Consider $p \geq \frac{6}{5}$, $\boldsymbol{f} \in \mathbf{L}^p(\Omega)$, $h \in W^{1,p}(\Omega)$, $\boldsymbol{g} \in \mathbf{W}^{2-1/p,p}(\Gamma)$ **and** $\boldsymbol{v} \in \mathbf{H}_3(\Omega)$ **and satisfying the compatibility condition (8). Then, the unique solution of (O) verifies**

$$(\boldsymbol{u}, \pi) \in \mathbf{W}^{2,p}(\Omega) \times W^{1,p}(\Omega).$$

Moreover, there exists a constant $C > 0$ **such that**

$$\|\boldsymbol{u}\|_{\mathbf{W}^{2,p}(\Omega)} + \|\pi\|_{W^{1,p}(\Omega)/\mathbb{R}} \leq C\left(1 + \|\boldsymbol{v}\|_{\mathbf{L}^s(\Omega)}\right)\Big(\|\boldsymbol{f}\|_{\mathbf{L}^p(\Omega)} + $$
$$+ \left(1 + \|\boldsymbol{v}\|_{\mathbf{L}^s(\Omega)}\right)\left(\|h\|_{W^{1,p}(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{2-1/p,p}(\Gamma)}\right)\Big).$$

As a consequence, $\boldsymbol{v} \cdot \nabla \boldsymbol{u} \in \mathbf{L}^p(\Omega)$ **for** $p \geq 3$.

Sketch of the proof:

1. **The result is true for $v \in \mathbf{H}_s(\Omega)$ with**

$$s = 3 \quad \text{if } p < 3, \qquad s = p \quad \text{if } p > 3 \quad \text{or} \quad s = 3 + \varepsilon \quad \text{if } p = 3,$$

2. **Suppose $v \in \mathbf{H}_3(\Omega)$ and its approximate function $v_\lambda \in \mathcal{D}_\sigma(\overline{\Omega}) \subset \mathbf{H}_s(\Omega)$.**
3. **We study the problem:**

$$(O_\lambda)\begin{cases} -\Delta u_\lambda + v_\lambda \cdot \nabla u_\lambda + \nabla \pi_\lambda = f, \quad \nabla \cdot u_\lambda = h \quad \text{in } \Omega, \\ u_\lambda = g \quad \text{on } \Gamma. \end{cases}$$

- **The problem (O_λ) in under conditions of (1)**
- **Its solution $(u_\lambda, \pi_\lambda) \in \mathbf{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)$ verifies an inequality independent of λ on the RHS.**

Theorem 8 (Generalized Solutions for the Oseen problem)

Let $\boldsymbol{f} \in \mathbf{W}^{-1,p}(\Omega)$, $\boldsymbol{v} \in \mathbf{H}_3(\Omega)$, $h \in L^p(\Omega)$, $\boldsymbol{g} \in \mathbf{W}^{1-1/p,p}(\Gamma)$ verify the compatibility condition (8). Then, the problem (O) has a unique solution

$$(\boldsymbol{u}, \pi) \in \mathbf{W}^{1,p}(\Omega) \times L^p(\Omega)/\mathbb{R}.$$

Moreover, $\exists\ C > 0$ such that,

$$\|\boldsymbol{u}\|_{\mathbf{W}^{1,p}(\Omega)} + \|\pi\|_{L^p(\Omega)/\mathbb{R}} \leq C(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}) \times (\|\boldsymbol{f}\|_{\mathbf{W}^{-1,p}(\Omega)} + (1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)})\|h\|_{L^p(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{1-1/p,p}(\Gamma)}). \tag{13}$$

Moreover, if $\boldsymbol{v} \cdot \boldsymbol{n} = 0$ on Γ, then the estimate (13) holds for any $1 < p < \infty$.

Sketch of the new proof:

For $1 < p < 2$, we use an argument from Amrouche-Meslami-Nečasová:

- **First, we consider $h = 0$ and $\boldsymbol{g} = \mathbf{0}$.**
- **We regularize $\boldsymbol{f}$ by $\boldsymbol{f}_\lambda = \nabla \cdot (\mathbb{G}_{t,\lambda}|_\Omega) \in \mathbf{W}^{-1,p}(\Omega)$ where $\|\mathbb{F}_\lambda - \mathbb{F}\|_{\mathbb{L}^p(\Omega)} \leq \lambda$, and $\mathbb{G}_{t,\lambda} = \rho_t \star \widetilde{\mathbb{F}}_\lambda$ (for $\widetilde{\mathbb{F}}_\lambda$ the extension by zero of $\mathbb{F}$ to $\mathbb{R}^3$)**
- **We study the problem:**

$$-\Delta \boldsymbol{u}_\lambda + \boldsymbol{v}_\lambda \cdot \nabla \boldsymbol{u}_\lambda + \nabla \pi_\lambda = \boldsymbol{f}_\lambda, \ \nabla \cdot \boldsymbol{u}_\lambda = 0 \text{ in } \Omega, \ \boldsymbol{u}_\lambda = \mathbf{0} \text{ on } \Gamma$$

- **By contradiction, we prove that**

$$\|\boldsymbol{u}_\lambda\|_{\mathbf{W}^{1,p}(\Omega)} + \|\pi_\lambda\|_{\mathbf{L}^p(\Omega)} \leq C \left(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\right) \|\boldsymbol{f}_\lambda\|_{\mathbf{W}^{-1,p}(\Omega)}$$

- **The case $h \neq 0$ and $\boldsymbol{g} \neq \mathbf{0}$ uses**

$$-\Delta \boldsymbol{u}_0 + \nabla \pi_0 = \mathbf{0}, \quad \nabla \cdot \boldsymbol{u}_0 = h, \quad \boldsymbol{u}_0|_\Gamma = \boldsymbol{g}$$

Corollary 9 (Strong solutions for $1 < p < 6/5$)

Consider $1 < p < 6/5$ and

$$\boldsymbol{f} \in \mathbf{L}^p(\Omega), \quad \boldsymbol{v} \in \mathbf{H}_3(\Omega), \quad h \in W^{1,p}(\Omega) \quad \text{and} \quad \boldsymbol{g} \in \mathbf{W}^{2-1/p,p}(\Gamma)$$

verifying the compatibility condition (8). Then, the solution given by Theorem 3 satisfies $(\boldsymbol{u}, \pi) \in \mathbf{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)$ and the following estimate holds:

$$\begin{aligned} \|\boldsymbol{u}\|_{\mathbf{W}^{2,p}(\Omega)} &+ \|\pi\|_{W^{1,p}(\Omega)/\mathbb{R}} \leq C\left(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\right)^2 \Big(\|\boldsymbol{f}\|_{\mathbf{L}^p(\Omega)} + \\ &+ \left(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\right)\left(\|h\|_{W^{1,p}(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{2-1/p,p}(\Gamma)}\right)\Big). \end{aligned}$$

Theorem 10 (Very weak solution of Oseen equations)

Let $\boldsymbol{f} \in (\mathbf{X}_{r',p'}(\Omega))'$, $h \in L^r(\Omega)$, $\boldsymbol{g} \in \mathbf{W}^{-1/p,p}(\Gamma)$, with $\frac{1}{r} = \frac{1}{p} + \frac{1}{s}$, satisfying (8), and $\boldsymbol{v} \in \mathbf{H}_s(\Omega)$, with
$s = 3$ if $p > 3/2$, $s = p'$ if $p < 3/2$, or $s = 3 + \varepsilon$ if $p = 3/2$.

Then, the Oseen problem (O) has a unique solution $(\boldsymbol{u}, \pi) \in \mathbf{T}_{p,r}(\Omega) \times W^{-1,p}(\Omega)/\mathbb{R}$ verifying the estimates

$$\begin{aligned}
\|\boldsymbol{u}\|_{\mathbf{T}_{p,r}(\Omega)} \leq\ & C \left(1 + \|\boldsymbol{v}\|_{\mathbf{L}^s(\Omega)}\right) \Big(\|\boldsymbol{f}\|_{[\mathbf{X}_{r',p'}(\Omega)]'} + \|h\|_{L^r(\Omega)} + \\
& + \|\boldsymbol{g}\|_{\mathbf{W}^{-1/p,p}(\Gamma)}\Big), \\
\|\pi\|_{W^{-1,p}(\Omega)/\mathbb{R}} \leq\ & C \left(1 + \|\boldsymbol{v}\|_{\mathbf{L}^s(\Omega)}\right)^2 \Big(\|\boldsymbol{f}\|_{[\mathbf{X}_{r',p'}(\Omega)]'} + \|h\|_{L^r(\Omega)} + \\
& + \|\boldsymbol{g}\|_{\mathbf{W}^{-1/p,p}(\Gamma)}\Big).
\end{aligned}$$

Definition 10 (Very weak solution for the Oseen problem)

Let $\boldsymbol{f} \in [\mathbf{X}_{r',p'}(\Omega)]'$ **for** $p \geq 3/2$ **and**

$$\boldsymbol{f} = \nabla \cdot \mathbb{F}_0 + \nabla f_1, \quad \mathbb{F}_0 \in \mathbb{L}^1(\Omega), \quad f_1 \in L^1(\Omega) \textbf{ if } p < 3/2,$$

$h \in L^r(\Omega)$, $\boldsymbol{g} \in \mathbf{W}^{-1/p,p}(\Gamma)$ **satisfying:**

$$r = r(p) = \begin{cases} 1 & \textbf{if } p < \frac{3}{2}, \\ 1+\varepsilon & \textbf{if } p = \frac{3}{2}, \\ r \quad \textbf{such that } \frac{1}{r} = \frac{1}{p} + \frac{1}{3} & \textbf{if } p > \frac{3}{2}. \end{cases} \tag{14}$$

the compatibility condition (8) and $\boldsymbol{v} \in \mathbf{H}_3(\Omega)$.
We say that $(\boldsymbol{u}, \pi) \in \mathbf{L}^p(\Omega) \times W^{-1,p}(\Omega)$ **is a very weak solution of** (O) **if the following equalities hold:**

For any $\varphi \in \mathbf{Y}_{p'}(\Omega)$ such that $\boldsymbol{v} \cdot \nabla \boldsymbol{\varphi} \in \mathbf{L}^{p'}(\Omega)$ and $\chi \in W^{1,p'}(\Omega)$,

$$\begin{aligned} \int_\Omega \boldsymbol{u} \cdot (-\Delta \boldsymbol{\varphi} - \boldsymbol{v} \cdot \nabla \boldsymbol{\varphi})\, d\boldsymbol{x} \quad &- \quad \langle \pi, \nabla \cdot \boldsymbol{\varphi} \rangle_{W^{-1,p}(\Omega) \times W_0^{1,p'}(\Omega)} \\ &= \quad \langle \boldsymbol{f}, \boldsymbol{\varphi} \rangle_\Omega - \langle \boldsymbol{g}_\tau, \frac{\partial \boldsymbol{\varphi}}{\partial \boldsymbol{n}} \rangle_\Gamma, \\ \int_\Omega \boldsymbol{u} \cdot \nabla \chi\, d\boldsymbol{x} = - \int_\Omega h\, \chi\, d\boldsymbol{x} \quad &+ \quad \langle \boldsymbol{g} \cdot \boldsymbol{n}, \chi \rangle_\Gamma, \end{aligned}$$

Remark

The condition $\boldsymbol{v} \cdot \nabla \boldsymbol{\varphi} \in \mathbf{L}^{p'}(\Omega)$ is necessary to guarantee that $\int_\Omega \boldsymbol{u} \cdot (-\Delta \boldsymbol{\varphi} - \boldsymbol{v} \cdot \nabla \boldsymbol{\varphi})\, d\boldsymbol{x}$ is well-defined for $p \leq \frac{3}{2}$.
The term $\int_\Omega h\, \chi\, d\boldsymbol{x}$ is well-defined from $r = r(p)$ in (14).

Theorem 10 new (Very weak solution of Oseen equations)

Let $\boldsymbol{f} \in (\mathbf{X}_{r',p'}(\Omega))'$ **for** $p \geq 3/2$ **and**

$$\boldsymbol{f} = \nabla \cdot \mathbb{F}_0 + \nabla f_1, \quad \mathbb{F}_0 \in \mathbb{L}^1(\Omega), \quad f_1 \in L^1(\Omega)\ (p < 3/2),$$

$h \in L^r(\Omega)$, $\boldsymbol{g} \in \mathbf{W}^{-1/p,p}(\Gamma)$ **satisfying (8),** $\boldsymbol{v} \in \mathbf{H}_3(\Omega)$ **and** $r = r(p)$ **defined in (14).**
Then, the Oseen problem (O) **has a unique solution** $(\boldsymbol{u}, \pi) \in \mathbf{T}_{p,r}(\Omega) \times W^{-1,p}(\Omega)/\mathbb{R}$ **with estimates of Theorem 10.**

Theorem 11 (Regularity for Oseen equations)

Consider $\sigma \in (1/p, 2]$. **Let** $\boldsymbol{f} \in \mathbf{W}^{\sigma-2,p}(\Omega)$, $h \in W^{\sigma-1,p}(\Omega)$, $\boldsymbol{g} \in \mathbf{W}^{\sigma-1/p,p}(\Gamma)$ **be given satisfying** (8), **and** $\boldsymbol{v} \in \mathbf{H}_3(\Omega)$. **Then, the Oseen problem (O) has a unique solution**

$$(\boldsymbol{u},\ \pi) \in \mathbf{W}^{\sigma,p}(\Omega) \times W^{\sigma-1,p}(\Omega)/\mathbb{R}$$

satisfying

$$\|\boldsymbol{u}\|_{\mathbf{W}^{\sigma,p}(\Omega)} + \|\pi\|_{W^{\sigma-1,p}(\Omega)/\mathbb{R}} \leq C\ \left(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\right)^{\alpha}$$

$$\times \left(\|\boldsymbol{f}\|_{\mathbf{W}^{\sigma-2,p}(\Omega)} + \|h\|_{W^{\sigma-1,p}(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{\sigma-1/p,p}(\Gamma)}\right)$$

with $\alpha = 3$ **if** $1 < p < \frac{6}{5}$ **and** $\alpha = 2$ **if** $p \geq \frac{6}{5}$.

Main steps:

- **Results must be rewritten when $\nabla \cdot \boldsymbol{v} \neq 0$.**
- **We consider**

$$\boldsymbol{v} \in \mathbf{L}^3(\Omega),\ \nabla \cdot \boldsymbol{v} \in L^{3/2}(\Omega), \quad i.e.\ \boldsymbol{v} \in \mathbf{H}_{3,3/2}(\mathsf{div}; \Omega)$$

- **We only focus in the obtention of weak solution for (O).**

- **We lift the data by $\boldsymbol{u}_0 \in \mathbf{H}^1(\Omega)$ such that $\nabla \cdot \boldsymbol{u}_0 = h$ and $\boldsymbol{u}_0|_\Gamma = \boldsymbol{g}$, satisfying:**

$$\|\boldsymbol{u}_0\|_{\mathbf{H}^1(\Omega)} \leq C \left(\|h\|_{L^2(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{H}^{1/2}(\Gamma)} \right).$$

- **The initial problem is equivalent to finding $\boldsymbol{z} = \boldsymbol{u} - \boldsymbol{u}_0 \in \mathbf{H}_0^1(\Omega)$ such that:**

$$\forall \boldsymbol{\varphi} \in \mathbf{V}, \quad a(\boldsymbol{z}, \boldsymbol{\varphi}) + b(\boldsymbol{v}, \boldsymbol{z}, \boldsymbol{\varphi}) = \langle \widetilde{\boldsymbol{f}}, \boldsymbol{\varphi} \rangle_{\mathbf{H}^{-1}(\Omega) \times \mathbf{H}_0^1(\Omega)},$$

with $\widetilde{\boldsymbol{f}} = \boldsymbol{f} + \Delta \boldsymbol{u}_0 - (\boldsymbol{v} \cdot \nabla) \boldsymbol{u}_0$,

$$a(\boldsymbol{z}, \boldsymbol{\varphi}) = \int_\Omega \nabla \boldsymbol{z} : \nabla \boldsymbol{\varphi} \, d\boldsymbol{x} - \frac{1}{2} \int_\Omega (\nabla \cdot \boldsymbol{v}) (\boldsymbol{z} \cdot \boldsymbol{\varphi}) \, d\boldsymbol{x}.$$

Observe that

$$\begin{aligned} a(\boldsymbol{z},\boldsymbol{z}) &= \|\nabla \boldsymbol{z}\|^2_{\mathbf{L}^2(\Omega)} - \frac{1}{2}\int_\Omega (\nabla\cdot\boldsymbol{v})\,|\boldsymbol{z}|^2\,d\boldsymbol{x} \\ &\geq \left(1 - \frac{C_0}{2}\,\|\nabla\cdot\boldsymbol{v}\|_{L^{3/2}(\Omega)}\right)\|\nabla \boldsymbol{z}\|^2_{\mathbf{L}^2(\Omega)} \end{aligned}$$

with C_0 is the product of the constant of the Sobolev embedding $\mathbf{H}^1(\Omega) \hookrightarrow \mathbf{L}^6(\Omega)$ and the Poincaré constant. If we chose v such that:

$$\|\nabla\cdot\boldsymbol{v}\|_{L^{3/2}(\Omega)} < \frac{1}{C_0}, \tag{15}$$

the bilinear form $a(\cdot,\cdot)$ is then coercive.

- **Moreover, b is a trilinear antisymmetric form with respect to the last two variables, well-defined for $\boldsymbol{v} \in \mathbf{L}^3(\Omega)$ with $\nabla \cdot \boldsymbol{v} \in L^{3/2}(\Omega)$, $\boldsymbol{z}$, $\boldsymbol{\varphi} \in \mathbf{H}_0^1(\Omega)$ because**

$$b(\boldsymbol{v}, \boldsymbol{z}, \boldsymbol{\varphi}) = \int_\Omega (\boldsymbol{v} \cdot \nabla)\boldsymbol{z} \cdot \boldsymbol{\varphi} \, d\boldsymbol{x} + \frac{1}{2} \int_\Omega (\nabla \cdot \boldsymbol{v}) \, \boldsymbol{z} \cdot \boldsymbol{\varphi} \, d\boldsymbol{x}.$$

- **By Lax-Milgram's Theorem, we can deduce the existence of a unique $\boldsymbol{z} \in \mathbf{H}_0^1(\Omega)$ verifying the estimate:**

$$\begin{aligned} \|\boldsymbol{z}\|_{\mathbf{H}^1(\Omega)} &\leq C \, \|\widetilde{\boldsymbol{f}}\|_{\mathbf{H}^{-1}(\Omega)} \\ &\leq C \left(\|\boldsymbol{f}\|_{\mathbf{H}^{-1}(\Omega)} + (1 + \|\boldsymbol{v}\|_{L^3(\Omega)}) \left(\|h\|_{L^2(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{H}^{1/2}(\Gamma)} \right) \right) \end{aligned}$$

The Navier-Stokes problem

Navier-Stokes equations

- **The existence of very weak solution is obtained in**

$$\mathbf{L}^3(\Omega) \times W^{-1,3}(\Omega),$$

- **First, for the small data case.**
- **Second, for arbitrary large $\boldsymbol{f}$ but h and $\boldsymbol{g}$ small enough.**

Main ideas

- **Apply Banach's fixed point theorem over the Oseen equations. Indeed, let**

$$T : \mathbf{H}_3(\Omega) \to \mathbf{H}_3(\Omega)$$

be the application defined as $v \mapsto Tv = u$, where u is the unique solution of (O) provided by Theorem 4. We set

$$\mathbf{B}_r = \{v \in \mathbf{H}_3(\Omega);\ \|v\|_{\mathbf{L}^3(\Omega)} \leq r\}.$$

- **To eliminate the smallness on $\boldsymbol{f}$, we decompose the problem in two ($\varepsilon > 0$):**

$$(1) -\Delta \boldsymbol{v}_\varepsilon + \boldsymbol{v}_\varepsilon \cdot \nabla \boldsymbol{v}_\varepsilon + \nabla q_\varepsilon^1 = \boldsymbol{f} - \boldsymbol{f}_\varepsilon,\ \nabla \cdot \boldsymbol{v}_\varepsilon = h - h_\varepsilon \text{ in } \Omega, \quad \boldsymbol{v}_\varepsilon = \boldsymbol{g} - \boldsymbol{g}_\varepsilon$$

$$(2) -\Delta \boldsymbol{z}_\varepsilon + \boldsymbol{z}_\varepsilon \cdot \nabla \boldsymbol{z}_\varepsilon + \boldsymbol{z}_\varepsilon \cdot \nabla \boldsymbol{v}_\varepsilon + \boldsymbol{v}_\varepsilon \cdot \nabla \boldsymbol{z}_\varepsilon + \nabla q_\varepsilon^2 = \boldsymbol{f}_\varepsilon,$$

$$\nabla \cdot \boldsymbol{z}_\varepsilon = h_\varepsilon \text{ in } \Omega, \quad \boldsymbol{z}_\varepsilon = \boldsymbol{g}_\varepsilon \quad \text{on } \Gamma$$

where

$$\boldsymbol{f}_\varepsilon \in \mathbf{H}^{-1}(\Omega), \quad h_\varepsilon \in L^2(\Omega) \quad \text{and} \quad \boldsymbol{g}_\varepsilon \in \mathbf{H}^{1/2}(\Gamma)$$

satisfy

$$\|\boldsymbol{f} - \boldsymbol{f}_\varepsilon\|_{[\mathbf{X}_{3,3/2}(\Omega)]'} + \|h - h_\varepsilon\|_{L^{3/2}(\Omega)} + \|\boldsymbol{g} - \boldsymbol{g}_\varepsilon\|_{\mathbf{W}^{-1/3,3}(\Gamma)} \leq \varepsilon$$

and

$$\|h_\varepsilon\|_{L^{3/2}(\Omega)} + \sum_{i=0}^{i=I} |\langle \boldsymbol{g}_\varepsilon \cdot \boldsymbol{n}, 1 \rangle_{\Gamma_i}| \leq 2\delta,$$

Finally, we use an extension of Hopf's lemma: for any $\alpha > 0$, there exists $\boldsymbol{y}_\varepsilon \in \mathbf{H}^1(\Omega)$, depending on α, such that for $C_1 > 0$ depending only on Ω,

$$\nabla \cdot \boldsymbol{y}_\varepsilon = h_\varepsilon \quad \text{in} \quad \Omega, \qquad \boldsymbol{y}_\varepsilon = \boldsymbol{g}_\varepsilon \quad \text{on}\Gamma$$

and for any $\boldsymbol{w} \in \mathbf{H}^1_0(\Omega)$, with $\nabla \cdot \boldsymbol{w} = 0$,

$$\begin{aligned}\left| \int_\Omega (\boldsymbol{w} \cdot \nabla)\boldsymbol{y}_\varepsilon \cdot \boldsymbol{w}\, d\boldsymbol{x} \right| &\leq \left(\alpha + \|h_\varepsilon\|_{L^{3/2}} + C \sum_{i=0}^{i=I} |\langle \boldsymbol{g}_\varepsilon \cdot \boldsymbol{n}, 1 \rangle_{\Gamma_i}|\right) \|\boldsymbol{w}\|^2_{\mathbf{H}^1(\Omega)} \\ &\leq (\alpha + 2C_1\delta)\|\boldsymbol{w}\|^2_{\mathbf{H}^1(\Omega)}. \qquad \blacksquare\end{aligned}$$

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Thank you!