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On the exponential cardinality of FDS for the ordered p-median problem☆

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Abstract

We study finite dominating sets (FDS) for the ordered median problem. This kind of problems allows to deal simultaneously with a large number of models. We show that there is no valid polynomial size FDS for the general multifacility version of this problem even on path networks.

Keywords: Location; Networks; Finite dominating sets

1. Introduction

Network location models have been widely studied in the literature as can be seen in several textbooks [1,4,8,12]. Since the seminal paper by Hakimi [7], much of this work has been devoted to identify finite sets of points where an optimal solution of a problem must belong to. These sets, called *finite dominating sets* (FDS), reduce the search for an optimal solution of a problem to a finite set of candidates.

In the last years, a new type of function has attracted the attention of locators: the ordered median objective function. The corresponding ordered median problem allows a common algebraic analysis for a wide range of location models since many of the classical problems in location theory can be formulated as some of its particular instances. In the literature of location analysis, we can find a number of results concerning the ordered median problem, for example in the continuous case, characterizations of the optimal solution set and some algorithms have been obtained in [5,6,14–18]. On networks, finite dominating sets are known for particular instances of the problem, see [10,11,13,19]. Recently, also the discrete version of this model has been studied in [2,3]. (This objective function was already introduced in [21] in the context of multi-criteria decision making.)

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Nevertheless, it has been an open problem whether polynomial size FDS exist for the general version of this problem even on path networks. In this paper, we show that such sets do not exist.

An overview of the literature, involving characterizations of FDS, shows a lot of papers that succeed in finding this type of sets for different versions of location models. Ref. [9] is an excellent paper on this subject that characterizes FDS for a large number of location problems. However, we are not aware of any paper that states a negative result concerning existence of a polynomial size FDS for a given problem.

The main result in this paper proves that there exists a path graph with n nodes satisfying the following property: there is a family of $O(n^n)$ ordered $\frac{n}{4}$-median problems defined on the above path graph, such that: (1) each problem has a unique optimal solution and (2) each optimal solution contains an element (facility) not included in any other solution. Therefore, in general, the multifacility ordered median problem cannot have polynomial size FDS.

To introduce the problem formally, some notation is needed. Let $G=(V,E)$ denote a path graph where $V=\{v_1,\dots,v_n\}$ is the set of nodes (demand points) and E the set of edges. Suppose without loss of generality that the nodes are points on the real line, satisfying $v_1 \leqslant \dots \leqslant v_n$. Therefore, we denote by $[v_i, v_i+1]$ the edge that joins the nodes v_i and v_i+1 for $i=1,\dots,n-1$. Let $A(G)$ be the interval $[v_1,v_n]$, then the distance from two points x and y in $A(G)$ is simply $d(x,y)=|x-y|$. In the same way, the distance from a node to a set with p points, $X_p=\{x_1,\dots,x_p\}\subseteq A(G)$, is defined as

$$d(v,X_p)=\min_{i=1,\dots,p} d(v,x_i)=\min_{i=1,\dots,p}|x_i-v|.$$

Notice that $A(G)$ is a metric space which distance function is induced by the edge lengths, see [20].

We consider a set of non-negative weights $\{w_1,\dots,w_n\}$, called w-weights, where the weight w_i is associated to the node v_i and represents the intensity of the demand at this node, for $i=1,\dots,n$.

Let β be a permutation of the set $\{1,\dots,n\}$ satisfying that

$$w_{\beta_1}d(v_{\beta_1},X_p)\leqslant w_{\beta_2}d(v_{\beta_2},X_p) \leqslant \dots \leqslant w_{\beta_n}d(v_{\beta_n},X_p).$$

For a given $\lambda=(\lambda_1,\dots,\lambda_n)$, vector of non-negative components, called λ-weights, the p-facility ordered median function (the ordered p-median function) on G is defined as

$$F_\lambda(X_p):=\sum_{i=1}^{n}\lambda_i w_{\beta_i} d(v_{\beta_i},X_p) :=\sum_{i=1}^{n}\lambda_{\sigma_i} w_i d(v_i,X_p), \qquad (1)$$

where σ is a permutation of $\{1,\dots,n\}$ such that $\sigma_j<\sigma_k$ if $w_j d(v_j,X_p)\leqslant w_k d(v_k,X_p)$ for all $j,k\in\{1,\dots,n\}$.

The λ-weights are the parameters that define the objective function and depending on the values of these parameters we obtain different problems. In fact, the ordered p-median problem allows to model the p-facility versions of the median ($\lambda_i=1,\ \forall i$), center ($\lambda_n=1, \lambda_i=0,\ \forall i\neq n$), α-centdian ($\lambda_n=1, \lambda_i=\alpha,\ \forall i\neq n$), k-centrum ($\lambda_i=1$, for $i=n-k+1,\dots,n$ and $\lambda_i=0$ for $i=1,\dots,n-k$), k-trimmed p-mean location model (we omit the $\frac{k}{2}$ smallest and $\frac{k}{2}$ largest weighted distances, to simplify assume k is even, $\lambda_1=\dots=\lambda_{\frac{k}{2}}=0$, $\lambda_{\frac{k}{2}+1}=\dots=\lambda_{n-\frac{k}{2}}=1$, $\lambda_{n-\frac{k}{2}+1}=\dots=\lambda_n=0$), etc. Notice that we do not impose any assumption on the monotonicity of the λ-weights, therefore we do not restrict to the convex nor the concave cases, see [13].

Although we have already used the concept of FDS, in what follows, we give its formal definition.

Definition 1.1. Let $G=(V,E)$ be a graph with n nodes and positive edge lengths. Let $w_1,\dots,w_n$ be non-negative reals and $\lambda=(\lambda_1,\dots,\lambda_n)$ a vector of non-negative components. A finite subset X of $A(G)$ is an FDS, for the multifacility ordered median problem, if for any integer p and w-weights associated to v_i for $i=1,\dots,n$, either w_i or 0 there is an optimal solution, X_p, of the respective ordered p-median problem, such that $X_p\subset X$.

Obtaining an FDS for this model allows the development of different types of algorithms to solve it. Therefore, recently much effort has been devoted to obtain FDS for the ordered median problems (see [10,11,13,19]). In the following, we recall several sets used to define FDS for particular instances of the problem.

A point $x \in A(G)$ is in equilibrium with respect to the nodes $v_k, v_l, v_k \neq v_l$, if: $w_k\, d(v_k, x) = w_l\, d(v_l, x)$. It is important to realize that there may exist subedges in equilibrium with respect to two nodes. We denote by EQ the set consisting of the nodes of G, the points in equilibrium which are isolated and the extreme points of the subedges in equilibrium. Moreover, we consider the following sets:

$$
\begin{aligned}
Y = \{ & y \in A(G):\; w_i d(v_i, y) = w_j d(v_j, z), \\
& v_i, v_j \in V,\; z \in EQ\}, \\
T = \{ & X_2 = (x_1, x_2) \in A(G) \times A(G):\; \exists v_r, v_s \\
& \text{served by } x_1 \text{ and } v_{r'}, v_{s'} \text{ served by } x_2, \text{ such} \\
& \text{that } w_r d(v_r, x_1) = w_{r'} d(v_{r'}, x_2) \text{ and } w_s \\
& d(v_s, x_1) = w_{s'} d(v_{s'}, x_2). \text{ Moreover, if } w_r = w_{r'} \\
& \text{and } w_s = w_{s'}, \text{ then the slopes of the functions} \\
& d(v_r, \cdot) \text{ and } d(v_s, \cdot), \text{ in the edge that} \\
& x_1 \text{belongs to, must have the same signs at } x_1 \\
& \text{and the slopes of the functions } d(v_{r'}, \cdot) \\
& \text{and } d(v_{s'}, \cdot), \text{ in the edge that } x_2 \text{ belongs to,} \\
& \text{must have different signs at } x_2\}.
\end{aligned}
$$

Ref. [13] proves that for $\lambda_1 \geqslant \ldots \geqslant \lambda_n \geqslant 0$ the node set V constitutes an FDS for the multifacility ordered median problem. For arbitrary non-negative λ-weights, it also obtains that EQ is an FDS for the single-facility ordered median problem.

Ref. [11] studies the multifacility ordered median problem where the λ-weights are defined as

$$a = \lambda_1 = \ldots = \lambda_k \neq \lambda_{k+1} = \ldots = \lambda_n = b,$$

for a fixed k, such that, $1 \leqslant k < n$. It proves that the set Y is an FDS for this problem.

Ref. [19] proves that the set $F = (EQ \times Y) \cup T \subset A(G) \times A(G)$ contains an optimal solution of the ordered 2-median problem in any network for any choice of λ-weights.

Ref. [10] gives an FDS for the single facility ordered median problem with general node weights (the w-weights can be negative). Moreover, for the case of a directed network with non-negative w-weights, it proves that there is always an optimal solution in V.

However, none of these papers deals with the general case of the multifacility ordered median problem. In fact, these papers impose very restrictive hypotheses such that their respective results cannot be extended any further. Indeed, only [13] and [11] consider p-facility problems for any $p > 2$, although for particular cases: [13] when the λ-weights are given in non increasing order and [11] when the λ-weights satisfy $a = \lambda_1 = \ldots = \lambda_k \neq \lambda_{k+1} = \ldots = \lambda_n = b$, for some k, $1 \leqslant k < n$.

2. On the exponential cardinality of FDS for the p-facility ordered median problem

In this section we prove that there is no polynomial size FDS for the general ordered p-median problem even on path networks. In order to do that we consider a path graph G where $V = \{v_1, \ldots, v_{2p}\}$, being p a fixed natural number, $v_1 = 0$, $v_2 = 2$, $v_{2i-1} = v_{2i-2} + M$ and $v_{2i} = v_{2i-1} + 2^i$, for $i = 2, \ldots, p$, and $M = 4\sum_{i=1}^{p} 2^i + 1$ (a sufficiently large number) (see Fig. 1).

The w-weights associated to the nodes are assumed to be equal to one and the λ-weights are given as follows:

$$
\begin{aligned}
& \lambda_1 = 0,\; \lambda_2 = \lambda_3 = 2p,\; \lambda_4 = p \quad \text{and} \\
& \lambda_i = \frac{2^{2p} + 1}{2^{2p+1}} (\lambda_{i-2} + \lambda_{i-1}), \\
& \quad \text{for } i = 5, \ldots, 2p.
\end{aligned} \tag{2}
$$

Under these conditions our goal is to find p points on $A(G)$, $X_p = \{x_1, \ldots, x_p\}$, solving the following problem:

$$\min_{X_p \subseteq A(G)} F(X_p) := \sum_{i=1}^{2p} \lambda_{\sigma_i} d(v_i, X_p), \tag{3}$$

where σ is a permutation of $\{1, \ldots, 2p\}$, such that, $\sigma_k < \sigma_l$ if $d(v_k, X_p) \leqslant d(v_l, X_p)$ for each $k, l \in \{1, \ldots, 2p\}$. (In this case, we say that the λ-weight λ_{σ_i} is assigned (allocated) to the node v_i.)

Remark 2.1. Notice that, the λ-weights defined in (2) satisfy the relationships:

$$
\begin{aligned}
& 2 \max\{\lambda_{i-2}, \lambda_{i-1}\} > 2\lambda_i > \lambda_{i-2} + \lambda_{i-1}, \\
& \quad \text{for all } i = 5, \ldots, 2p,
\end{aligned} \tag{4}
$$

$$2\lambda_4 = \lambda_2 = \lambda_3 > \lambda_1 = 0, \tag{5}$$

$$2\lambda_2 > \lambda_4 + \lambda_5 + \lambda_8. \tag{6}$$

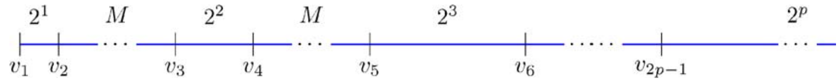

Fig. 1. Illustration of the graph in Section 2.

Moreover, the components of the vector $\lambda = (\lambda_1, \ldots, \lambda_{2p})$ satisfy the following chain of inequalities:

$$\lambda_2 = \lambda_3 > \ldots > \lambda_{2p-3} > \lambda_{2p-1} > \lambda_{2p} > \lambda_{2p-2} > \lambda_{2p-4} > \ldots > \lambda_4 > \lambda_1 = 0.$$

Therefore

(i) $\lambda_{2j+1} > \lambda_{2(j+1)+1}$, for $j = 1, \ldots, p-2$ (the sequence of λ-weights with odd indexes (>1) is decreasing).
(ii) $\lambda_{2(j+1)} > \lambda_{2j}$, for $j = 2, \ldots, p-1$ (the sequence of λ-weights with even indexes (>2) is increasing).
(iii) $\lambda_{2j-1} > \lambda_{2i}$, for any $i, j \in \{2, \ldots, p\}$ (a λ-weight with odd index (>1) is always greater than any other with an even index (>2)).
(iv) $\lambda_{2j} < \lambda_k$, if $k > 2j$, $j > 1$ and $\lambda_{2j+1} > \lambda_k$, if $k > 2j+1$, $j \geqslant 1$.

We will prove that the optimal policy to solve Problem (3) is to locate a service facility on each edge $[v_{2i-1}, v_{2i}]$ for $i = 1, \ldots, p$ and to assign λ_i to v_i for $i = 1, \ldots, 2p$.

Lemma 2.1. *If $X_p = \{x_1, \ldots, x_p\}$ is an optimal solution of Problem* (3) *then $x_i \in [v_{2i-1}, v_{2i}]$ for $i = 1, \ldots, p$.*

Proof. First, we prove that the nodes v_{2i} and v_{2i+1} for $i = 1, \ldots, p$, are not covered by the same service facility. Suppose on the contrary that there exists $j \in \{1, \ldots, p\}$, such that, v_{2j} and v_{2j+1} are served by the same service facility $x \in X_p$. This implies that the following terms would appear in the objective function:

$$\lambda_{\sigma_{2j}} d(v_{2j}, x) + \lambda_{\sigma_{2j+1}} d(v_{2j+1}, x).$$

Notice that $d(v_{2j}, x) + d(v_{2j+1}, x) \geqslant M$. Moreover

1. If both σ_{2j} and σ_{2j+1} are different from 1, we have that $\lambda_{\sigma_{2j}} \geqslant \lambda_4$, $\lambda_{\sigma_{2j+1}} \geqslant \lambda_4$, and at least one of these inequalities is strict.
2. If $\sigma_{2j}=1$, then $d(v_{2j+1}, x) \geqslant M/2$. In a similar way, the case $\sigma_{2j+1} = 1$ implies that $d(v_{2j}, x) \geqslant M/2$.

In all cases

$$\lambda_{\sigma_{2j}} d(v_{2j}, x) + \lambda_{\sigma_{2j+1}} d(v_{2j+1}, x) > \frac{M-1}{2}\lambda_4 = 2\lambda_4 \sum_{i=1}^{p} 2^i = \lambda_2 \sum_{i=1}^{p} 2^i.$$

The inequality above contradicts the optimality of X_p. Indeed, consider $X'_p = \{x'_1, \ldots, x'_p\}$ such that x'_i is located at the midpoint of the edge $[v_{2i-1}, v_{2i}]$ for $i = 1, \ldots, p$. Then, since $\lambda_2 \geqslant \lambda_i$ for $i = 1, \ldots, 2p$, we have that $F(X'_p) \leqslant \lambda_2 \sum_{i=1}^{p} 2^i$. Therefore, v_{2i} and v_{2i+1} for $i=1, \ldots, p$, can not be covered by the same service facility.

Hence, in what follows, we can assume without loss of generality that each service facility x_i covers the demand of v_{2i-1} and v_{2i} for $i = 1, \ldots, p$.

The fact that $x_i \in [v_{2i-1}, v_{2i}]$ follows directly from the isotonicity property of the ordered median objective with nonnegative λ-weights (see Theorem 1 in [5]). Indeed, if $x_i \notin [v_{2i-1}, v_{2i}]$ for some i, $i = 1, \ldots, p$, we move x_i to its closest node in the interval $[v_{2i-1}, v_{2i}]$, the new vector of distances of $\{v_1, \ldots, v_{2p}\}$ from the servers is smaller than the old vector. □

Remark 2.2. Since all the w-weights are equal to one, by symmetry arguments and without loss of generality, in what follows we only consider solutions of this problem satisfying that $d(v_{2i-1}, x_i) \leqslant d(v_{2i}, x_i)$ for $i=1, \ldots, p$, and consequently, by the structure of the graph, $d(v_{2i}, x_i) \leqslant d(v_{2i+2}, x_{i+1})$ for $i=1, \ldots, p-1$. Hence,

(i) $\sigma_{2i-1} < \sigma_{2i}$ for $i = 1, \ldots, p$,
(ii) $\sigma_{2i} < \sigma_{2i+2}$ for $i = 1, \ldots, p-1$.

The above assertions imply that $\sigma_{2p}=2p$. Moreover, by Lemma 2.1 and for the sake of the readability, we can represent the graph of Fig. 1 as a graph with only

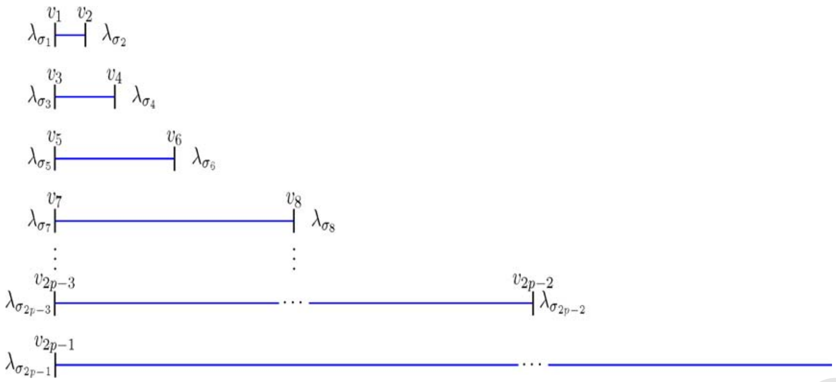

Fig. 2. The new representation of the graph where $\sigma_{2i} > \sigma_{2i-1}$, for $i = 1, \ldots, p$ and $\sigma_{2i+2} > \sigma_{2i}$, for $i = 1, \ldots, p-1$.

p edges where the edges with length M are omitted (see Fig. 2).

Theorem 2.1. *If X_p is an optimal solution of Problem* (3) *then $\lambda_{\sigma_i} = \lambda_i$ for $i = 1, \ldots, 2p$.*

Proof. First, we prove that λ_1 must be assigned to v_{2i-1} for some i, $i = 1, \ldots, p$. Indeed, if $\sigma_{2i} = 1$ for some i, $i = 1, \ldots, p$ then, by Remark 2.2.i, $\sigma_{2i-1} < 1$. However, this is impossible because $\sigma_{2i-1} \in \{1, \ldots, 2p\}$.

Second, by Lemmas A.1 and A.2, for $k = 1, \ldots, p$, we have that λ_{2k} is assigned to v_{2i} for some i, $i = 1, \ldots, p$. Moreover, Remark 2.2(ii), implies that $\sigma_{2i} = 2i$, (i.e. λ_{2i} is assigned to v_{2i}) for $i = 1, \ldots, p$. Therefore, using a recursive argument and Remark 2.2(i), we obtain that $\sigma_1 = 1$ (i.e. λ_1 is assigned to v_1), $\sigma_3 = 3$ and so on. Thus, the result follows. □

Remark 2.3. The above result has been proven assuming that $|v_{2i+1} - v_{2i+2}| = 2|v_{2i-1} - v_{2i}|$. However, the reader may notice that the result also holds whenever $|v_{2i+1} - v_{2i+2}| \geqslant 2|v_{2i-1} - v_{2i}|$.

In order to disprove the polynomial cardinality of any FDS for the multifacility ordered median problem, we consider the graph G of Fig. 1 (assume that p is even). Let $P = \{1, \ldots, p\}$ and $J = \{j_1, j_2, \ldots, j_{\frac{p}{2}-1}, j_{\frac{p}{2}}\} \subseteq P$, such that, $1 = j_1 < j_2 < \ldots < j_{\frac{p}{2}-1} < j_{\frac{p}{2}} = p$. On the graph G we formulate the following $(\frac{p}{2})$-facility ordered median problem:

$$\min_{X_{\frac{p}{2}} \subseteq A(G)} \sum_{i=1}^{2p} \lambda'_{\sigma_i} w'_i d(v_i, X_{\frac{p}{2}}), \tag{7}$$

where $\lambda' = (0, \ldots, 0, \lambda_1, \ldots, \lambda_p)$, such that, λ_i is defined by (2) for $i = 1, \ldots, p$, $w'_{2j-1} = w'_{2j} = 1$ for each $j \in J$ and $w'_{2j-1} = w'_{2j} = 0$ for each $j \in P \setminus J$.

Moreover, since $w'_{2j-1} d(v_{2j-1}, X_{\frac{p}{2}}) = w'_{2j}\, d(v_{2j}, X_{\frac{p}{2}}) = 0 \forall j \in P \setminus J$, the first p positions of the ordered sequence of weighted distances between each node and its service facility are given by $w'_{2j-1} d(v_{2j-1}, X_{\frac{p}{2}})$, $w'_{2j} d(v_{2j}, X_{\frac{p}{2}})$ with $j \in P \setminus J$. (Indeed, these positions are always zeros.) Thus, we can assume without loss of generality that the λ-weights allocated to v_{2j-1} and v_{2j} for any $j \in P \setminus J$ are the first p components of the vector λ', that is, 0.

Notice that the nodes v_{2j-1} and v_{2j} $\forall j \in P \setminus J$ are not really taken into account in the objective value because $w'_{2j-1} = w'_{2j} = 0$. Thus, using Lemma 2.1, this problem reduces to locate $p/2$ service facilities on a graph with $p/2$ edges. Indeed, if we consider

$$V' = \bigcup_{i=1,\, j_i \in J}^{\frac{p}{2}} \{v_{2j_i-1}, v_{2j_i}\} := \bigcup_{i=1}^{\frac{p}{2}} \{v'_{2i-1}, v'_{2i}\}$$

and the path graph G' induced by the set of nodes V', Problem (7) can be reformulated as

$$\min_{X_{\frac{p}{2}} \subseteq A(G')} \sum_{i=1}^{p} \lambda_{\sigma_i} d(v'_i, X_{\frac{p}{2}}). \tag{8}$$

Observe that the components of the vector $\lambda = (\lambda_1, \ldots, \lambda_p)$ coincide with the first p entries of (2) and therefore they satisfy (4)–(6). In addition, $\sigma_i \in \{1, \ldots, p\}$ and $\sigma_i < \sigma_j$ if $d(v'_i, X_{\frac{p}{2}}) \leqslant d(v'_j, X_{\frac{p}{2}})$ (the w-weights are all equal to one).

Theorem 2.2. *If $X_{p/2}$ is the optimal solution of Problem (7) then $x_i = v_{2j_i-1} + z_i$ with $z_1 = 1$ and $z_i = 2^{j_{i-1}} - z_{i-1}$ for $i = 2, \ldots, p/2$.*

Proof. To prove this result we consider the equivalent formulation of Problem (7) given in (8). Applying Theorem 2.1 and Remark 2.3, we get that the λ-weight allocated to the node v'_i is λ_i for $i = 1, \ldots, p$.

In addition, the solution $X_{\frac{p}{2}}$ satisfies the relationship $d(v'_{2i}, x_i) = d(v'_{2i+1}, x_{i+1})$ for $i = 1, \ldots, p/2-1$. Indeed, since λ_{2i} and λ_{2i+1} are assigned to v'_{2i} and v'_{2i+1}, respectively, for $i = 1, \ldots, p/2 - 1$, then $d(v'_{2i}, x_i) \leqslant d(v'_{2i+1}, x_{i+1})$. Moreover, x_{i+1} must be located as close as possible to v'_{2i+1} because $\lambda_{2i+1} > \lambda_{2i+2}$, which in turn implies that $d(v'_{2i}, x_i) = d(v'_{2i+1}, x_{i+1})$.

Next, we prove that $d(v'_1, x_1) = 1$. Notice that, by Remark 2.2, we have that $d(v'_1, x_1) \leqslant 1$. If $d(v'_1, x_1) < 1$ then we would move x_1 towards v'_2 a small enough amount, ξ. This movement would allow us to move x_i towards v'_{2i-1} for any even index $i = 2, \ldots, p/2$, and x_j towards v'_{2j} for any odd index $j = 2, \ldots, p/2$ by the same amount ξ; without any reassignment of the λ-weights. These movements would produce the following change in the objective function:

$$\xi \left(-(\lambda_2 - \lambda_1) + \sum_{k=2}^{\frac{p}{2}} (-1)^{k-1} (\lambda_{2k-1} - \lambda_{2k}) \right).$$

This amount is negative because $-(\lambda_2 - \lambda_1)$ is negative and $\{\lambda_{2k-1} - \lambda_{2k}\}_{k \geqslant 2}$ is a decreasing sequence of positive values, that is, $\lambda_{2k-1} - \lambda_{2k} > \lambda_{2k+1} - \lambda_{2k+2} > 0$ for $k = 1, \ldots, p/2$. However, this is not possible since $X_{\frac{p}{2}}$ is optimal. Therefore, we obtain that $x_1 = v'_1 + 1$ and that $X_{\frac{p}{2}}$ is the unique solution satisfying that $d(v_{2i-1}, x_i) \leqslant d(v_{2i}, x_i)$ for $i = 1, \ldots, p$.(See Remark 2.2.) Finally, since $d(v'_{2i}, x_i) = d(v'_{2i+1}, x_{i+1})$ for $i = 1, \ldots, p/2 - 1$, the result follows. □

Our next result proves that there is no polynomial size cardinality FDS for the multifacility ordered median problem. The proof consists of building a family of $O(n^n)$ problems on the same graph with different solutions (each solution contains at least one point not included in the remaining), n being the number of nodes.

Theorem 2.3. *There is no polynomial size FDS for the multifacility ordered median problem.*

Proof. Consider Problem (7), by Theorem 2.2 and Remark 2.2, for each choice of the set $J \subseteq P$, we have an unique optimal solution satisfying that $d(v_{2i-1}, x_i) \leqslant d(v_{2i}, x_i)$ for $i = 1, \ldots, p$, such that, the service facility located on the edge $[v_{2p-1}, v_{2p}]$ has a different location, recall that $j_{p/2} = p$. Thus, since there are $\binom{2p-2}{\frac{p}{2}-2}$ different choices of the set J, any FDS for the considered problem contains at least $\binom{2p-2}{\frac{p}{2}-2}$ elements.

Therefore, we have found a family of problems for which a valid FDS is at least of order $O(n^n)$ where n denotes the number of nodes (recall that in our case $n = 2p$). □

Remark 2.4. Problem (7) is formulated based on the concrete λ given in (2). Nevertheless, a detailed reading of the proofs shows that any λ satisfying (4)–(6) would be also valid.

3. Concluding remarks

This paper proves that polynomial size FDS cannot exist for the multifacility ordered median problem. However, it is still an open question whether polynomial size FDS may exist for the convex version of this problem (λ-weights given in non-decreasing order).

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Appendix A.

Lemma A.1. *If X_p is an optimal solution of Problem* (3) *then for $k = 2, \dots, p$, λ_{2k} is assigned to v_{2i} for some i, $i = 1, \dots, p$.*

Proof. Suppose on the contrary that λ_{2k} is assigned to v_{2j-1} for some j, $j = 1, \dots, p$. We can assume without loss of generality that $2k$ is the maximum possible even index of a λ-weight assigned to v_{2j-1} with $j = 1, \dots, p$. Recall that k must be less than p since $\sigma_{2p} = 2p$ (see Remark 2.2). In what follows we distinguish two cases depending on the type of node where λ_{2k+1} has been assigned to.

Case 1: λ_{2k+1} is assigned to $v_{2j'}$ for some j', $j' = 1, \dots, p$.

By Remark 2.2(i) we have that $\sigma_{2j} > \sigma_{2j-1} = 2k$. Thus, by Remark 2.1(iv), we must have that $\lambda_{\sigma_{2j}} > \lambda_{\sigma_{2j-1}} = \lambda_{2k}$. Hence, since X_p is optimal, x_j must be located as far as possible from v_{2j-1}. Besides, since $\sigma_{2j-1} = 2k < 2k + 1 = \sigma_{2j'}$ then $d(v_{2j-1}, x_j) \leqslant d(v_{2j'}, x_{j'})$. Therefore, we have that $d(v_{2j-1}, x_j) = d(v_{2j'}, x_{j'})$and we can reassign the λ-weights, so that λ_{2k} is assigned to $v_{2j'}$ and λ_{2k+1} to v_{2j-1}.

Case 2: λ_{2k+1} is assigned to $v_{2j'-1}$ for some j', $j' = 1, \dots, p$.

Assume that $d(v_{2j-1}, x_j) \neq d(v_{2j}, x_j)$ and $d(v_{2j'-1}, x_{j'}) \neq d(v_{2j'}, x_{j'})$. Under this assumption, we can move x_j and $x_{j'}$ towards v_{2j} and $v_{2j'}$, respectively, by the same small enough amount, ξ, without any reassignment of the λ-weights (see Fig. 3(a)).This is possible because $2k + 2 = \sigma_{2j''}$ for some j'', $j'' = 1, \dots, p$ (recallthat λ_{2k} is the maximum index of a λ-weight assigned to a node with odd index) and $d(v_{2j'-1}, x_{j'})$ as well as $d(v_{2j-1}, x_j)$ are strictly smaller than $d(v_{2j''}, x_{j''})$. These movements imply the following change in the objective function:

$$\xi(\lambda_{2k} + \lambda_{2k+1} - \lambda_{\sigma_{2j}} - \lambda_{\sigma_{2j'}}).$$

This amount is negative. Indeed, since $\sigma_{2j} > 2k + 1$ and $\sigma_{2j'} > 2k + 1$, by Remark 2.1(iv), we get $\lambda_{\sigma_{2j}} + \lambda_{\sigma_{2j'}} > 2\lambda_{2k+2}$ and, by (4), we have that $2\lambda_{2k+2} > \lambda_{2k} + \lambda_{2k+1}$. This is a contradiction because X_p was an optimal solution.

In what follows, we study the cases $d(v_{2j-1}, x_j) = d(v_{2j}, x_j)$ and $d(v_{2j'-1}, x_{j'}) = d(v_{2j'}, x_{j'})$.

Case 2.1: $d(v_{2j'-1}, x_{j'}) = d(v_{2j'}, x_{j'})$.

Since $\sigma_{2j'-1} = 2k + 1$ we can assume without loss of generality that $\sigma_{2j'} = 2k + 2$. Now, since λ_{2k+1} and λ_{2k+2} have been already assigned and, by Remark 2.2(i), $\sigma_{2j} > 2k$ we get that $\sigma_{2j} > 2k + 2 = \sigma_{2j'}$. This means, by Remark 2.2(ii), that $j > j'$. Moreover, since $\sigma_{2j} > 2k$ then, by Remark 2.1(iv), $\lambda_{\sigma_{2j}} > \lambda_{\sigma_{2j-1}} = \lambda_{2k}$. Hence, x_j must be located as far as possible from v_{2j-1} because X_p is optimal. Besides, the relationship $\sigma_{2j-1} = 2k < 2k + 1 = \sigma_{2j'-1}$ implies that $d(v_{2j-1}, x_j) \leqslant d(v_{2j'-1}, x_{j'})$. Therefore, we obtain that $d(v_{2j-1}, x_j) = d(v_{2j'-1}, x_{j'})$. This permits reassigning the λ-weights so that λ_{2k} is assigned to $v_{2j'-1}$, λ_{2k+1} to $v_{2j'}$ andλ_{2k+2} to v_{2j-1} (see Fig. 3(b)). However, this allocation induces a contradiction because $2k$ is the maximum even index of a λ-weight assigned to a node with odd index.

Case 2.2: $d(v_{2j-1}, x_j) = d(v_{2j}, x_j)$. The analysis of this case is analogous to the Case 2.1 and also induces a contradiction.

After this case analysis, we conclude that the optimal assignment of the λ-weights satisfies that each λ_{2k} for any $k = 2, \dots, p$, is allocated to v_{2i} for some i, $i = 1, \dots, p$. □

The result above describes the optimal assignment of the λ-weights with even index, $k > 2$. However, it is still missing the case λ_2. The following result analyzes this case:

Lemma A.2. *If X_p is an optimal solution of Problem* (3) *then λ_2 must be assigned to v_2.*

Proof. First, notice that if λ_2 were assigned to v_{2i} for some i, $i = 1, \dots, p$, then by Remark 2.2(ii) and since λ_1 is already assigned to a v_{2j-1} for some j, $j = 1, \dots, p$, we would have that $i = 1$.

In order to prove the result, we proceed by contradiction assuming that λ_2 is assigned to v_{2j-1} for some j, $j = 1, \dots, p$. Therefore, since by Lemma A.1, for $k = 2, \dots, p$, λ_{2k} is assigned to v_{2i} for some i with $i = 1, \dots, p$, and λ_2 is assigned to v_{2j-1} then there exists only one $j_o \in \{1, \dots, p\}$ such that λ_{2j_o-1} is assigned to a node v_{2i} for some i, $i = 1, \dots, p$. Depending on the value of j_o, we distinguish the following

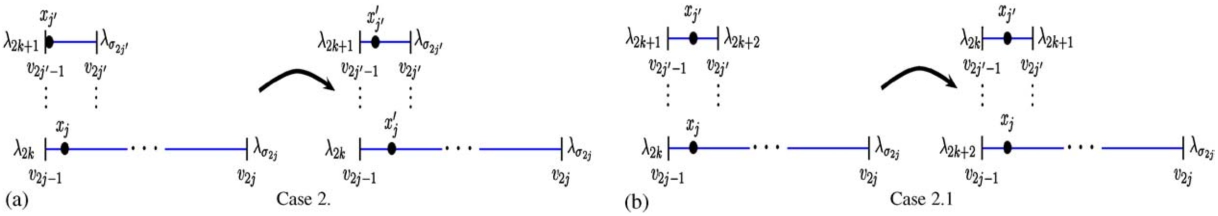

Fig. 3. Illustration of Lemma A.1.

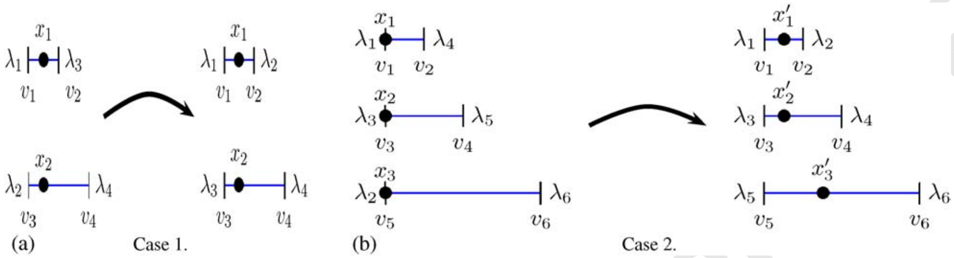

Fig. 4. Illustration of Lemma A.2.

cases:

Case 1: $j_o = 2$ (see Fig. 4(a)).

If λ_3 is assigned to v_{2i} for some i, $i = 1, \ldots, p$, then, by Remark 2.2(ii) and Lemma A.1, λ_3 must be assigned to v_2 $(i = 1)$ and λ_4 to v_4. By Remark 2.2(i), $\sigma_1 < 3$ and $\sigma_3 < 4$ then $\sigma_1 \leqslant 2$ and $\sigma_3 \leqslant 2$. Therefore, λ_1 is assigned either to v_1 or v_3 and the same occurs with λ_2. In any case, to minimize the objective function we must have that $d(v_1, x_1) = d(v_2, x_2) = d(v_3, x_3)$ and this implies that we can reassign the λ-weights such that λ_1 goes to v_1, λ_2 to v_2 and λ_3 to v_3. Since the objective value does not change, we get the thesis of the Lemma.

Case 2: $j_o = 3$ (see Fig. 4(b)).

If λ_5 is assigned to v_{2i} for some i, $i=1, \ldots, p$, then, by Remark 2.2(ii) and Lemma A.1, λ_4 must be assigned to v_2, λ_5 to v_4 and λ_{2i} to v_{2i} for any $i=3, \ldots, p$. Since λ_4, λ_5 and λ_6 have been already allocated and Remark 2.2(i) ensures that $\sigma_1 < 4$, $\sigma_2 < 5$ and $\sigma_3 < 6$ then $\sigma_1 \leqslant 3$, $\sigma_3 \leqslant 3$ and $\sigma_5 \leqslant 3$. Moreover

(i) Since $\lambda_4 < \lambda_5$ then $d(v_1, x_1) \leqslant d(v_3, x_2)$. (Otherwise the objective function may decrease). Indeed, if $d(v_1, x_1) > d(v_3, x_2)$ we move x_1 and x_2 towards v_1 and v_4, respectively, such that, x_1' and x_2', the new locations of x_1 and x_2, satisfy that $d(v_1, x_1') = d(v_3, x_2)$ and $d(v_3, x_2') = d(v_1, x_1)$. This movement induces the following change in the objective function:

$$(d(v_1, x_1) - d(v_3, x_2))(\lambda_4 - \lambda_5) < 0,$$

what contradicts the optimality of X_p.

(ii) Since $\sigma_3 \leqslant 3$, $\sigma_5 \leqslant 3$ and $\sigma_2 = 4$, X_p must satisfy that $d(v_2, x_1) \geqslant d(v_3, x_2)$ and $d(v_2, x_1) \geqslant d(v_5, x_3)$. In addition, we have by construction that $d(v_2, x_1) \leqslant 2$ then $d(v_3, x_2) \leqslant 2$ and $d(v_5, x_3) \leqslant 2$. This allows us to use the same arguments of Case 2(i) to prove that $d(v_3, x_2) \geqslant d(v_5, x_3)$ because $\lambda_5 > \lambda_6$.

Therefore, λ_1 must be assigned to v_1, λ_2 to v_5 and λ_3 to v_3. In addition, since λ_1, λ_2, λ_3, λ_4, λ_5, λ_6 have been already assigned and λ_8 is assigned to v_8; Remark 2.2(i) implies that λ_7 is assigned to v_7. Repeating this argument for any $i > 4$ we have that λ_{2i-1} is assigned to v_{2i-1}. Thus, $\sigma_{2i-1} = 2i - 1$ and $\sigma_{2i} = 2i$ for any $i = 4, \ldots, p$.

This assignment of the λ-weights implies that $d(v_{2i}, x_i) = d(v_{2i+1}, x_{i+1})$ for any $i = 3, \ldots, p - 1$. Indeed, since $\sigma_{2i} = 2i < 2i + 1 = \sigma_{2i+1}$ then $d(v_{2i}, x_i) \leqslant d(v_{2i+1}, x_{i+1})$, and since $\lambda_{2i+1} > \lambda_{2i+2}$ we deduce that x_{i+1} is located as close as possible

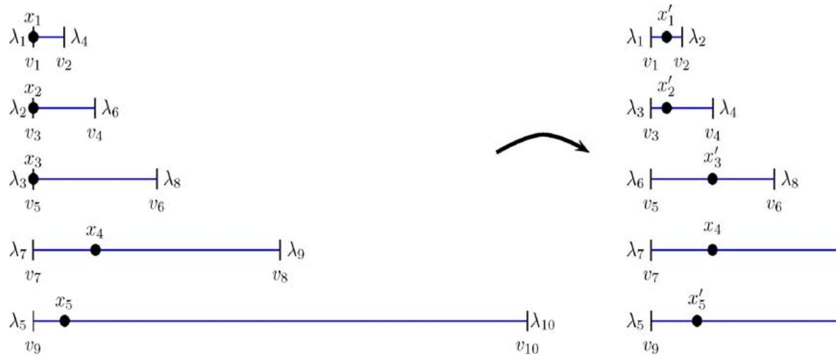

Fig. 5. Illustration of Case 4.

to v_{2i+1}, $i = 3, \ldots, p-1$. Hence, it implies that $d(v_{2i}, x_i) = d(v_{2i+1}, x_{i+1})$ for any $i = 3, \ldots, p-1$.

Moreover, with this assignment of the λ-weights and since, by (6), $2\lambda_2 > \lambda_4 + \lambda_5 + \lambda_8$ then the optimal location for x_1, x_2 and x_3 must be: $x_1 = v_1$, $x_2 = v_3$, $x_3 = v_5$.

However, this is a contradiction because we will prove that the above configuration of X_p does not provide an optimal solution of Problem (3). Indeed, move x_1, x_2 and x_3 to the new positions x'_1, x'_2 and x'_3, respectively, where $x'_1 = v_1 + 1$, $x'_2 = v_3 + 1$, and $x'_3 = v_5 + 3$. Using the condition $d(v_{2i}, x_i) = d(v_{2i+1}, x_{i+1})$ for any $i = 3, \ldots, p$, this movement allows us to displace three units length: (1) x_i towards v_{2i-1} for any even index $i = 4, \ldots, p$ and (2) x_j towards v_{2j} for any odd index $j = 4, \ldots, p$; without any reassignment of the λ-weights corresponding to these nodes. Therefore, these movements produce the following change in the objective function:

$$+\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 - \lambda_5 - 3\lambda_6 - 3(\lambda_7 - \lambda_8) + 3(\lambda_9 - \lambda_{10}) - 3(\lambda_{11} - \lambda_{12}) + \ldots .$$

We prove that this amount is negative. Indeed, by the definition of the λ-weights, see (2) and (4)–(6), they satisfy that $+\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 - \lambda_5 - 3\lambda_6$ is negative. Besides,

$$-3(\lambda_7 - \lambda_8) + 3(\lambda_9 - \lambda_{10}) - 3(\lambda_{11} - \lambda_{12}) + \ldots$$

is negative because the sequence $\lambda_7 - \lambda_8$, $\lambda_9 - \lambda_{10}$, ... is decreasing. This fact contradicts the optimality of X_p since the objective function decreases.

Case 3: $j_o = 4$. The proof is similar to the one in Case 2, and therefore it is omitted.

Case 4: $j_o > 4$ (see Figs. 5 and 6).

Using Remark 2.2(ii), Lemma A.1 and a similar argument to that used in Case 2(i), λ_1 must be assigned to v_1, λ_2 to v_3, λ_{2i+2} to v_{2i} for $i = 1, \ldots, j_o - 2$, λ_{2i-3} to v_{2i-1} for $i = 3, \ldots, j_o - 2$, λ_{2j_o-3} to v_{2j_o-3}, λ_{2j_o-5} to v_{2j_o-1}, λ_{2j_o-1} to v_{2j_o-2}, and λ_i to v_i for $i = 2j_o, \ldots, 2p$.

Moreover, notice that, $\lambda_{\sigma_{2i-1}} > \lambda_{\sigma_{2i}}$ for $i > 1$. Hence, following a similar argument to the one in Case 2, we obtain that $d(v_{2i}, x_i) = d(v_{2j-1}, x_j)$ whenever $\sigma_{2j-1} - \sigma_{2i} = 1$.

For this assignment of the λ-weights, using (2) and (4)–(6), the optimal location of x_1, x_2, x_3, x_4 and x_5 must be $x_1 = v_1$, $x_2 = v_3$, $x_3 = v_5$, and either

1. $x_4 = v_7 + 4$ and $x_5 = v_9 + 2$, if $j_o = 5$ (Fig. 5)

or

2. $x_4 = v_7 + 2$ and $x_5 = v_9 + 4$, if $j_o > 5$ (Fig. 6).

However, this is a contradiction because we will prove that the above configuration of X_p does not provide an optimal solution of Problem (3). Move x_1, x_2, x_3, x_4 and x_5 to the new positions x'_1, x'_2, x'_3, x'_4 and x'_5, respectively, where $x'_1 = v_1 + 1$, $x'_2 = v_3 + 1$, $x'_3 = v_5 + 4$, and

1. $x'_4 = v_7 + 4$ and $x'_5 = v_9 + 3$, if $j_o = 5$ (Fig. 5).
2. $x'_4 = v_7 + 3$ and $x'_5 = v_9 + 4$, if $j_o > 5$ (Fig. 6).

These displacements permit us to move x_i towards either v_{2i-1} or v_{2i} for $i = 6, \ldots, p$ without any re-

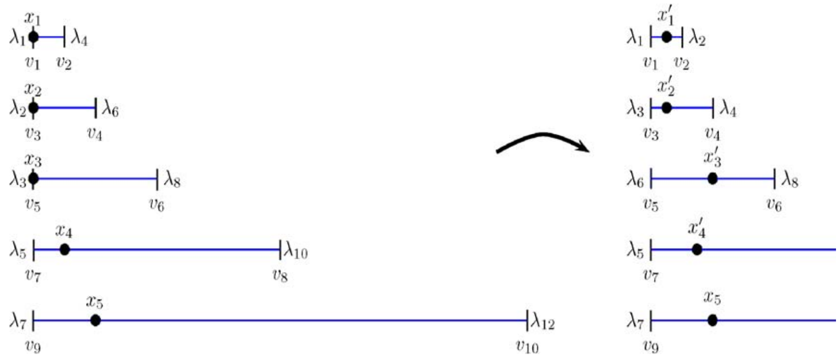

Fig. 6. Illustration of Case 4.

assignment of their corresponding λ-weights. (This is possible using the condition $d(v_{2i}, x'_i) = d(v_{2j-1}, x'_j)$ when $\sigma_{2j-1} - \sigma_{2i} = 1$.) The change of the objective function is as follows:

(i) If $j_o = 5$

$$+\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 - 4\lambda_8 - \lambda_{10} - (\lambda_{11} - \lambda_{12}) + (\lambda_{13} - \lambda_{14}) - (\lambda_{15} - \lambda_{16}) + \ldots.$$

By (2) and (4)–(6), we have that $+\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 - 4\lambda_8 - \lambda_{10}$ is negative (by the definition of the λ-weights). Besides, $-(\lambda_{11} - \lambda_{12}) + (\lambda_{13} - \lambda_{14}) - (\lambda_{15} - \lambda_{16}) + \ldots$ is negative because the sequence $\lambda_{11} - \lambda_{12}$, $\lambda_{13} - \lambda_{14}$, $\lambda_{15} - \lambda_{16}$,... is decreasing.

(ii) If $j_o > 5$

$$\begin{aligned}
&+\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 - 4\lambda_8 - \lambda_{10} \\
&+ \sum_{\{j \geqslant 0, 7+6j < 2j_o - 3\}} (-1)^{j+1} \cdot 0 \cdot (\lambda_{7+6j} - \lambda_{12+6j}) \\
&+ \sum_{\{j \geqslant 0, 9+6j < 2j_o - 3\}} (-1)^{j+1} 4(\lambda_{9+6j} - \lambda_{14+6j}) \\
&+ \sum_{\{j \geqslant 0, 11+6j < 2j_o - 3\}} (-1)^{j+1} (\lambda_{11+6j} - \lambda_{16+6j}) \\
&+ r(-1)^{j_r}(\lambda_{2j_o-3} - \lambda_{2j_o-1}) + \sum_{j=j_o}^{p-1} t(-1)^{j_t+(j-j_o)} \\
&\times (\lambda_{2j+1} - \lambda_{2j+2}),
\end{aligned}$$

where

$$t = \begin{cases} 0, & \text{if } \exists j_t \text{ such that } 2j_o - 3 = 7 + 6j_t, \\ 4, & \text{if } \exists j_t \text{ such that } 2j_o - 3 = 9 + 6j_t, \\ 1, & \text{if } \exists j_t \text{ such that } 2j_o - 3 = 11 + 6j_t. \end{cases}$$

$$r = \begin{cases} 0, & \text{if } \exists j_r \text{ such that } 2j_o + 1 = 7 + 6j_r, \\ 4, & \text{if } \exists j_r \text{ such that } 2j_o + 1 = 9 + 6j_r, \\ 1, & \text{if } \exists j_r \text{ such that } 2j_o + 1 = 11 + 6j_r. \end{cases}$$

In case (i) we proved that $+\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 - 4\lambda_8 - \lambda_{10}$ is negative. Moreover,

$$\begin{aligned}
&+ \sum_{\{j \geqslant 0, 9+6j < 2j_o - 3\}} (-1)^{j+1} 4(\lambda_{9+6j} - \lambda_{14+6j}) \\
&+ \sum_{\{j \geqslant 0, 11+6j < 2j_o - 3\}} (-1)^{j+1} (\lambda_{11+6j} - \lambda_{16+6j}) \\
&+ r(-1)^{j_r}(\lambda_{2j_o-3} - \lambda_{2j_o-1}) \\
&+ \sum_{j=j_o}^{p-1} t(-1)^{j_t+(j-j_o)}(\lambda_{2j+1} - \lambda_{2j+2})
\end{aligned}$$

is negative because we can decompose the expression above in different sums, where each one of them constitutes a decreasing sequence in absolute value with alternate signs and being its first element negative.

Since in all the possible cases we get a contradiction, the initial hypothesis that λ_2 is assigned to a vertex with odd index is inconsistent. Therefore, using Lemma A.1 we conclude that λ_2 can only be assigned to v_2. □

References

[1] M.O. Ball, T.L. Magnanti, C.L. Monma, G.L. Nemhauser (Eds.), Network Models, Handbooks in Operations Research and Management Science, vol. 7, North-Holland, Amsterdam; 1995.

[2] N. Boland, P. Domíguez-Marín, S. Nickel, J. Puerto, Exact procedures for solving the discrete ordered median problem, Ber. Fraunhofer ITWN 47 (2003).

[3] P. Domíguez-Marín, P. Hansen, N. Mladenović, S. Nickel, Heuristic procedures for solving the discrete ordered median problem, Ber. Fraunhofer ITWN 46 (2003).

[4] Z. Drezner, H.W. Hamacher (Eds.), Facility Location, Applications and Theory, Springer, New York, 2002.

[5] R.L. Francis, T.J. Lowe, A. Tamir, Aggregation error bounds for a class of location models, Oper. Res. 48 (2) (2000) 294–307.

[6] R.L. Francis, T.J. Lowe, A. Tamir, Worst-case incremental analysis for a class of *p*-facility location problems, Networks 39 (3) (2002) 139–143.

[7] S.L. Hakimi, Optimum locations of switching centers and the absolute centers and medians of a graph, Oper. Res. 12 (1964) 450–459.

[8] G.Y. Handler, P.B. Mirchandani, Location on Networks Theory and Algorithms, vol. 18, The MIT Press, Cambridge, 1979.

[9] J.N. Hooker, R.S. Garfinkel, C.K. Chen, Finite dominating sets for network location problems, Oper. Res. 39 (1) (1991) 100–118.

[10] J. Kalcsics, S. Nickel, J. Puerto, A. Tamir, Algorithmic results for ordered median problems, Oper. Res. Lett. 30 (3) (2002) 149–158.

[11] J. Kalcsics, S. Nickel, J. Puerto, Multifacility ordered median problems on networks: a further analysis, Networks 41 (1) (2003) 1–12.

[12] P.B. Mirchandani, R.L. Francis (Eds.), Discrete Location Theory, Wiley-Interscience Series in Discrete Mathematics and Optimization, Wiley, New York, 1990.

[13] S. Nickel, J. Puerto, A unified approach to network location problems, Networks 34 (4) (1999) 283–290.

[14] W. Ogryczak, On the distribution approach to location problems, Comput. Ind. Eng. 37 (1999) 595–612.

[15] J. Puerto, F.R. Fernández, The symmetrical single facility location problem, Preprint of the Mathematics Faculty, Universidad de Sevilla, 1995.

[16] J. Puerto, F.R. Fernández, Geometrical properties of the symmetrical single facility location problem, J. Nonlinear Convex Anal. 1 (3) (2000) 321–342.

[17] J. Puerto, A.M. Rodríguez-Chía, F. Fernández-Palacín, Ordered Weber problems with attraction and repulsion, Stud. Locational Anal. 11 (1997) 127–141.

[18] A.M. Rodríguez-Chía, S. Nickel, J. Puerto, F.R. Fernández, A flexible approach to location problems, Math. Methods Oper. Res. 51 (1) (2000) 69–89.

[19] A.M. Rodríguez-Chía, J. Puerto, D. Pérez-Brito, J.A. Moreno, The *p*-facility ordered median problem on networks, Working paper, 2003.

[20] A. Tamir, An $O(pn^2)$ algorithm for the *p*-median and related problems on tree graphs, Oper. Res. Lett. 19 (1996) 59–64.

[21] R. Yager, On ordered weighted averaging aggregation operators in multicriteria decision making, IEEE Trans. Systems Man Cybernet. 18 (1988) 183–190.