Weak pullback attractors of nonautonomous difference inclusions*

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Abstract *Weak pullback attractors are defined for nonautonomous difference inclusions and their existence and upper semi continuous convergence under perturbation is established. Unlike strong pullback attractors, invariance and pullback attraction here are required only for (at least) a single trajectory rather than all trajectories at each starting point. The concept is thus useful, in particular, for discrete time control systems.*

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1 Introduction

A nonautonomous difference inclusion

$$x_{t+1} \in F_t(x_t) \tag{1}$$

arises naturally in a variety of ways. An important source of applications is based on single valued control systems of the form

$$x_{t+1} = f_t(x_t, u_t) \tag{2}$$

with controls u_t taking values in a nonempty compact set U_t, thus (2) generates a nonautonomous difference inclusion of the form (1) with $F_t(x) := f_t(x, U_t)$. Other sources of examples are the discretization or time–1 mappings of differential control systems or differential equations without uniqueness [3, 7, 8, 9].

The mappings F_t, which are usually assumed to be compact valued and upper semi continuous, may vary in some regular or completely arbitrarily fashion. The discrete–time system generated by (1) is thus nonautonomous and no longer enjoys a setvalued semigroup property, so many of the concepts of autonomous systems are either too restrictive or inappropriate for an investigation of their asymptotic behaviour. The concept of a nonautonomous pullback attractor, which consists of a family of nonempty compact subsets rather than a single subset and "pull-back" attracts from asymptotically earlier starting times, was introduced in [7] for nonautonomous difference inclusion. In the autonomous case this pullback attractor reduces to as single set which is attracting in the usual forward sense. Szegö and Treccani [10] call it a strong attractor for their continuous time setvalued semi-groups. They also distinguish another type of attractor for autonomous setvalued systems, which they call a weak attractor. The difference is that only one or more trajectories for each starting point must be attracted to or remain in the weak attractor rather than all trajectories in the case of the strong attractors. This situation is of particular interest in control systems.

Our aim in this paper is to introduce and investigate a pullback version of a weak attractor for setvalued difference processes generated by nonautonomous difference inclusions. We define these setvalued difference processes in Section 2 and recall the definitions and basic results of strong autonomous and nonautonomous attractors in Section 3. Then in Section 4 we introduce the concept of a weak pullback attractor in terms of weakly invariant and weakly pullback attracting families of nonempty compact subsets of the state space. We state our first main result in Section 5, that the existence of a weak pullback attractor follows from that of a weakly positively invariant weakly pullback absorbing family of nonempty compact subsets, and indicate why other seemingly more natural constructions of the components subsets are not appropriate. Our second main result on the upper semi continuous convergence of weak pullback attractors is stated in Section 6. We then give five examples in Section 7 to illustrate some of the features and peculiarities of weak pullback attractors. The proofs of our main results and some supporting lemmata are given at the end of the paper in Section 8.

We require the following definitions and terminology [1]. The distance of a point $x \in \mathbb{R}^d$ from a nonempty compact set A is defined by

$$\mathrm{dist}(x, A) = \min_{a \in A} \|x - a\|.$$

The Hausdorff separation $H^*(A, B)$ of nonempty compact subsets A, B of $\mathbb{R}^d$ is defined by

$$H^*(A, B) := \max_{a \in A} \mathrm{dist}(a, B) = \max_{a \in A} \min_{b \in B} \|a - b\|$$

and $H(A, B) = \max\{H^*(A, B), H^*(B, A)\}$ denotes the Hausdorff metric on the space $\mathcal{H}(\mathbb{R}^d)$ of nonempty compact subsets of $\mathbb{R}^d$. An open ϵ–neighbourhood of $A \in \mathcal{H}(\mathbb{R}^d)$ is defined by $N_\epsilon(A) = \{x \in \mathbb{R}^d \ : \ \mathrm{dist}(x, A) < \epsilon\}$ and a closed ϵ–neighbourhood of A by $N_\epsilon[A] = \{x \in \mathbb{R}^d \ : \ \mathrm{dist}(x, A) \leq \epsilon\}$.

A mapping $F : \mathbb{R}^d \mapsto \mathcal{H}(\mathbb{R}^d)$ is upper semi continuous at x_0 if for all $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon, x_0) > 0$ such that $F(x) \subset N_\varepsilon(F(x_0))$ for all $x \in N_\delta(\{x_0\})$ or alternatively if

$$\lim_{x_n \to x_0} H^*(F(x_n), F(x_0)) = 0$$

for all sequences $x_n \to x_0$. Denote the space of all upper semi continuous mappings $F : \mathbb{R}^d \mapsto \mathcal{H}(\mathbb{R}^d)$ by $USC(\mathbb{R}^d, \mathcal{H}(\mathbb{R}^d))$ and define $\mathbb{Z}^2_+ = \{(i, j) \in \mathbb{Z}^2 \,|i \geq j\}$.

For any $A \in \mathcal{H}(\mathbb{R}^d)$ define $F(A) := \cup_{a \in A} F(a)$ and define the set composition of two mappings F, $G : \mathbb{R}^d \mapsto \mathcal{H}(\mathbb{R}^d)$ as $F \circ G(x) := F(G(x))$ for all $x \in \mathbb{R}^d$. Note that $F \circ G \in USC(\mathbb{R}^d, \mathcal{H}(\mathbb{R}^d))$ if F, $G \in USC(\mathbb{R}^d, \mathcal{H}(\mathbb{R}^d))$.

For simplicity we shall present our results for a Euclidean state space $\mathbb{R}^d$, though they are in fact valid for more general metric or Banach state spaces since we do not use the local compactness property of $\mathbb{R}^d$ in our proofs.

2 Setvalued difference processes

As in the singlevalued case, a natural nonautonomous generalization of an autonomous system defined in terms of a semigroup of mappings is a two–parameter semigroup or process.

Definition 2.1 *A mapping* $\Phi : \mathbb{Z}_+^2 \times \mathbb{R}^d \mapsto \mathcal{H}\left(\mathbb{R}^d\right)$ *is called a* setvalued difference process *on* $\mathbb{R}^d$ *if* $\Phi(t, t_0, \cdot) \in USC(\mathbb{R}^d, \mathcal{H}(\mathbb{R}^d))$ *for all* $(t, t_0) \in \mathbb{Z}_+^2$ *and*

$$\Phi(t_0, t_0, x) = \{x\}, \tag{3}$$

$$\Phi(t_2, t_0, x) = \Phi\left(t_2, t_1, \Phi(t_1, t_0, x)\right), \tag{4}$$

for all $t_0 \leq t_1 \leq t_2$ *in* $\mathbb{Z}$ *and all* $x \in \mathbb{R}^d$.

Thus nonautonomous difference inclusion (1) with mappings $F_t \in USC(\mathbb{R}^d, \mathcal{H}(\mathbb{R}^d))$ for $t \in \mathbb{Z}$ generates a setvalued difference process with the mappings $\Phi(t, t_0, \cdot) \in USC(\mathbb{R}^d, \mathcal{H}(\mathbb{R}^d))$ defined by

$$\Phi(t_0, t_0, x) := \{x\} \quad \text{and} \quad \Phi(t, t_0, x) := F_{t-1} \circ \cdots \circ F_{t_0}(x)$$

for all $x \in \mathbb{R}^d$ and $t_0 < t$ in $\mathbb{Z}$. Conversely, a setvalued difference process Φ generates a nonautonomous difference inclusion (1) with mappings F_n defined by $F_t(x) := \Phi(t+1, t, x)$ for all $x \in \mathbb{R}^d$ and $t \in \mathbb{Z}$.

A trajectory of a setvalued difference process Φ is a single valued mapping $\phi : [T_0, T_1] \cap \mathbb{Z} \mapsto \mathbb{R}^d$, for some $T_0 < T_1$ in $\mathbb{Z}$, which satisfies

$$\phi(t) \in \Phi(t, t_0, \phi(t_0)) \quad \text{for all} \quad T_0 \leq t_0 \leq t \leq T_1.$$

Note that, due to (4) , the concatenation of trajectories on adjacent time sets $[T_0, T_1] \cap \mathbb{Z}$ and $[T_1, T_2] \cap \mathbb{Z}$ forms a trajectory on the union $[T_0, T_2] \cap \mathbb{Z}$ of these times sets.

A trajectory defined on all of $\mathbb{Z}$ is called an entire trajectory.

3 Attractors of setvalued difference processes

An attractor for an autonomous difference inclusion, i.e., (1) with $F_t \equiv F$, is a nonempty compact subset A of $\mathbb{R}^d$ which is invariant, i.e., satisfies $F(A) = A$, and is attracting in the sense that

$$\lim_{n \to \infty} H^*(F^n(D), A) = 0$$

for every nonempty bounded set D of $\mathbb{R}^d$; here F^n denotes the n fold composition of F with itself. As for singlevalued dynamical systems, the existence of an attractor is implied by that of a more easily determinable absorbing set, i.e., a nonempty compact subset B of $\mathbb{R}^d$ such that for every nonempty bounded set D of $\mathbb{R}^d$ there exists a nonnegative integer N_D such that $F^n(D) \subseteq B$ for all $n \geq N_D$. The following theorem is a setvalued generalization of a well known result for singlevalued semigroups.

Theorem 3.1 *Let $F \in USC(\mathbb{R}^d, \mathcal{H}(\mathbb{R}^d))$ and suppose that the autonomous difference inclusion with mapping F has an absorbing set $B \in \mathcal{H}(\mathbb{R}^d)$. Then it has a unique attractor A defined by*

$$A = \bigcap_{m \geq 0} \overline{\bigcup_{n \geq m} F^n(B)}.$$

The concepts of an absorbing set and attractor are somewhat more complicated in the nonautonomous difference case, with the obvious generalisations being too restrictive for most situations. As in the case of singlevalued difference equations [5], families of sets rather than individual sets should be considered.

Definition 3.2 *A family $\mathcal{B} = \{B_t,\, t \in \mathbb{Z}\}$ of nonempty compact subsets of $\mathbb{R}^d$ is called a pullback absorbing family for a setvalued difference process Φ on $\mathbb{R}^d$ if for every $t_0 \in \mathbb{Z}$ and every nonempty bounded subset D of $\mathbb{R}^d$ there exists an $N_{t_0,D} \in \mathbb{Z}^+$ such that*

$$\Phi\left(t_0, t_0 - n, D\right) \subseteq B_{t_0}$$

for all $n \geq N_{t_0,D}$.

Definition 3.3 *A family $\mathcal{A} = \{A_t, t \in \mathbb{Z}\}$ of nonempty compact subsets of $\mathbb{R}^d$ is called a pullback attractor of a setvalued difference process Φ on $\mathbb{R}^d$ if it is strictly invariant, i.e.,*

$$\Phi\left(t, t_0, A_{t_0}\right) = A_t \quad \text{for any} \quad t \geq t_0, \tag{5}$$

and pullback attracts bounded sets, i.e.,

$$\lim_{n \to \infty} H^*\left(\Phi(t_0, t_0 - n, D), A_{t_0}\right) = 0 \tag{6}$$

for every $t_0 \in \mathbb{Z}$ and every bounded subset D of $\mathbb{R}^d$.

Property (5) is a generalisation of the positive invariance property of a semigroup. Note that the pullback convergence property (6) does not describe the convergence of $\Phi(t, t_0, D)$ as $t \to \infty$. See [4, 5, 6] for a discussion on these properties in the context of singlevalued processes. The following theorem from [7, 9] is a generalisation of Theorem 3.1 to the nonautonomous case.

Theorem 3.4 *Let Φ be a setvalued difference process with a positive invariant and uniformly bounded pullback absorbing family $\mathcal{B} = \{B_t, t \in \mathbb{Z}\}$. Then there exists the minimal negatively invariant (strictly if Φ is lower semicontinuous) pullback attractor $\mathcal{A} = \{A_t, t \in \mathbb{Z}\}$ which is determined by*

$$A_{t_0} = \bigcap_{m \geq 0} \overline{\bigcup_{n \geq m} \Phi\left(t_0, t_0 - n, B_{t_0 - n}\right)} \tag{7}$$

for each $t_0 \in \mathbb{Z}$.

If the pullback absorbing family $\mathcal{B}$ is not assumed or known to be positively invariant then one can also obtain the minimal negatively invariant attractor for each $t_0 \in \mathbb{Z}$ by

$$\overline{\bigcup_{\substack{\text{bounded} \\ D \subset \mathbb{R}^d}} \bigcap_{m \geq 0} \overline{\bigcup_{n \geq m} \Phi\left(t_0, t_0 - n, D\right)}} \tag{8}$$

It was shown in [4] for singlevalued difference equations that a pullback attractor always has a forward invariant pullback absorbing family. The proof can be adapted to the difference inclusion case under consideration here.

4 Weak attractors of difference inclusion processes

The preceding concepts of invariance and attraction for setvalued difference processes are nonautonomous counterparts of what Szegö and Treccani [10] called strong invariance and attractors of continuous time autonomous setvalued systems or semigroups. Essentially, the invariance and attraction holds with respect to all possible trajectories emanating from each starting point.

For setvalued systems arising from control systems, one is often interested in situations where just one, or a few, rather than all trajectories emanating from each starting point satisfy a given property. This is also of interest for systems generated by differential equations without uniqueness such as $x' = x^{1/3}$, for which the set $\{0\}$ is only "weakly" positively invariant due to the nonuniqueness of solutions with the initial value $x(0) = 0$.

Szegö and Treccani also introduced corresponding concepts of weak invariance and weak attraction for such continuous time autonomous setvalued systems. In the discrete time case under consideration here these read as follows: A nonempty compact subset A is weakly positively invariant if for each $x_0 \in A$ there exists a trajectory ϕ with $\phi(0) = x_0$ such that $\phi(t) \in A$ for all $t \geq 0$. A nonempty compact subset A is weakly attracting if for each $x_0 \in \mathbb{R}^d$, there exists a trajectory ϕ with $\phi(0)$

$= x_0$ such that $\text{dist}(\phi(t), A) \to 0$ as $t \to \infty$. Finally, a nonempty compact subset A is called a weak attractor if it is weakly positively invariant and weakly attracting.

Our aim in this paper is to introduce and investigate pullback versions of these weak concepts for setvalued difference processes. As with the strong concepts of invariance and attraction above, it is also less restrictive here to consider families of sets rather than individual sets.

Definition 4.1 *A family $\mathcal{A} = \{A_t, t \in \mathbb{Z}\}$ of nonempty compact subsets of $\mathbb{R}^d$ is said to be weakly positively invariant for a setvalued difference process Φ on $\mathbb{R}^d$ if for every $t_0 \in \mathbb{Z}$ and every $x_0 \in A_{t_0}$ there exists a trajectory $\phi : [t_0, \infty) \cap \mathbb{Z} \to \mathbb{R}^d$ of Φ with $\phi(t_0) = x_0$ such that $\phi(t) \in A_t$ for all $t \geq t_0$. The family $\mathcal{A} = \{A_t, t \in \mathbb{Z}\}$ is said to be weakly invariant if for every $t_0 \in \mathbb{Z}$ and every $x_0 \in A_{t_0}$ there exists an entire trajectory $\phi : \mathbb{Z} \to \mathbb{R}^d$ of Φ with $\phi(t_0) = x_0$ such that $\phi(t) \in A_t$ for all $t \in \mathbb{Z}$.*

Definition 4.2 *A weakly invariant family $\mathcal{A} = \{A_t, t \in \mathbb{Z}\}$ of nonempty compact subsets of $\mathbb{R}^d$ is called a weak pullback attractor of a setvalued difference process Φ on $\mathbb{R}^d$ if it is weakly pullback attracting, i.e., for any $t_0 \in \mathbb{Z}$, any nonempty bounded subset D of $\mathbb{R}^d$ and any sequence $d_n \in D$ there exist sequences of integers $t_n \to \infty$ as $n \to \infty$ and trajectories $\phi_n : [t_0 - t_n, t_0] \cap \mathbb{Z} \to \mathbb{R}^d$ of Φ with $\phi_n(t_0 - t_n) = d_n$ such that*

$$\lim_{n \to \infty} \text{dist} \left(\phi_n(t_0), A_{t_0} \right) = 0. \tag{9}$$

Note that a strong pullback attractor, if it exists, is also a weak pullback attractor.

5 Existence of weak pullback attractors

Our first main result is to show that the existence of a weak pullback attractor follows from that of a more easily determined weak pullback absorbing family.

Definition 5.1 *A weakly positively invariant family $\mathcal{B} = \{B_t, t \in \mathbb{Z}\}$ of nonempty compact subsets of $\mathbb{R}^d$ is called a weak pullback absorbing family of a setvalued difference process Φ on $\mathbb{R}^d$ if for $t_0 \in \mathbb{Z}$ and any bounded subset D of $\mathbb{R}^d$ there exists an integer $N_{t_0,D}$ such that for each $n \geq N_{t_0,D}$ and $d_n \in D$ there exists a trajectory $\phi_n : [t_0 - n, t_0] \cap \mathbb{Z} \to \mathbb{R}^d$ of Φ with $\phi_n(t_0 - n) = d_n$ and $\phi_n(t_0) \in B_{t_0}$.*

Note, by the weak positive invariance of $\mathcal{B}$ the trajectories ϕ_n can be extended to remain in $\mathcal{B}$ for $t \geq t_0$, i.e., with $\phi_n(t) \in B_t$ for each $t \geq t_0$.

Theorem 5.2 *Let Φ be a setvalued difference process with a weak pullback absorbing family $\mathcal{B} = \{B_t, t \in \mathbb{Z}\}$. Then Φ has a maximal weak pullback attractor $\mathcal{A} = \{A_t, t \in \mathbb{Z}\}$ relative to $\mathcal{B}$, which is uniquely determined by*

$$\begin{aligned} A_{t_0} \;=\; \Big\{ a_0 \in \mathbb{R}^d \; ; \; \exists \;\; t_n \to \infty \;\; \textit{as } n \to \infty \;\; \textit{and trajectories} \\ \phi_n : [t_0 - t_n, t_0] \cap \mathbb{Z} \to \mathbb{R}^d \quad \textit{such that} \quad \phi_n(t) \in B_t \\ \textit{for} \;\; t \in [t_0 - t_n, t_0] \cap \mathbb{Z} \quad \textit{and} \quad \lim_{n\to\infty} \phi_n(t_0) = a_0 \Big\} \end{aligned} \tag{10}$$

for each $t_0 \in \mathbb{Z}$.

The maximal weak pullback attractor $\mathcal{A} = \{A_t, t \in \mathbb{Z}\}$ here apparently consists of the entire trajectories that “move” or remain in $\mathcal{B}$ for the entire time set $\mathbb{Z}$, i.e., satisfy $\phi(t) \in B_t$ for each $t \in \mathbb{Z}$; we will prove this in Lemma 8.1. It is thus the maximal weakly invariant family which is contained in $\mathcal{B}$ and is unique in this sense. However, as our examples in Section 7 will show, a setvalued inclusion process may have several different pullback absorbing families either with overlapping or with disjoint component sets. Each of these absorbing families contains a maximal weak pullback attractor relative to itself, the component sets of which may overlap or be disjoint. Uniqueness of weak pullback attractors is thus not a universal property as in the case of a strong pullback attractor, the uniqueness of the weak pullback attractor being only relative to its given absorbing family. In particular, this means that steady state or periodic trajectories need not be contained in a given weak pullback attractor. Some of our examples are in fact an autonomous system, so this is a peculiarity of weak attractors in general rather than a characteristic of nonautonomity.

Our assumption that a weakly pullback absorbing family is weakly positively invariant is an interesting distinction between strong and weak pullback absorbing families. As we saw above, the strong positive invariance of a strong pullback absorbing is not essential to ensure the existence of a strong pullback attractor. In the weak case we need it to ensure the existence of a weak pullback attractor which is weakly invariant. A closely related issue is our construction (10) of the component sets of the weak pullback attractor, with other seemingly natural definitions failing to be weakly invariant with or without the assumed weak positively of the weakly pullback absorbing family.

For instance, suppose that the weak pullback absorbing family $\mathcal{B} = \{B_t, t \in \mathbb{Z}\}$ is weakly positive invariant. Then the family $\mathcal{A}^* = \{A_t^*, t \in \mathbb{Z}\}$ defined by

$$A_{t_0}^* := \bigcap_{m\geq 0} \left(\overline{\bigcup_{n\geq m} \Phi(t_0, t_0 - n, B_{t_0-n})} \bigcap B_{t_0} \right) \tag{11}$$

is weakly pullback attracting with $A_t \subset A_t^* \subset B_t$ for all $t \in \mathbb{Z}$, where A_t is defined by (10). The positive weak invariance of $\mathcal{A}^*$ follows from this inclusion, but the inclusion $A_t^* \subset \Phi(t, t_0, A_{t_0}^*) \bigcap B_t$ is generally false, so negative weak invariance usually does not hold, cf. Lemma 8.1.

On the other hand, if we do not assume that $\mathcal{B} = \{B_t, t \in \mathbb{Z}\}$ is positively weakly invariant, then we might define the family $\mathcal{A}^* = \{A_t^*, t \in \mathbb{Z}\}$ by

$$A_{t_0}^* = \overline{\bigcup_{\substack{\text{bounded} \\ D \subset \mathbb{R}^d}} \bigcap_{m \geq 0} \left(\overline{\bigcup_{n \geq m} \Phi(t_0, t_0 - n, D)} \bigcap B_{t_0} \right)}.$$

This family $\mathcal{A}^*$ weakly pullback attracts all bounded subsets of $\mathbb{R}^d$ and satisfies $A_t^* \subset B_t$ for all $t \in \mathbb{Z}$, but need not be either positively or negatively weakly invariant. (A sufficient condition for the positive weak invariance is that $\Phi(t, t_0, \cdot)$ be lower semi continuous for any pair $(t, t_0) \in \mathbb{Z}_+^2$, see e.g. [2]).

6 Upper semi continuity of weak pullback attractors

Our second main result is to establish the upper semi continuous dependence of weak pullback attractors under perturbation. For this we consider a perturbed nonautonomous difference inclusion

$$x_{t+1} \in F_t^\epsilon(x_t) \tag{12}$$

with the $F_t^\epsilon \in USC(\mathbb{R}^d, \mathcal{H}(\mathbb{R}^d))$ such that

$$H^*\left(F_t^\epsilon(x), F_t(x)\right) \leq \epsilon \tag{13}$$

for all $x \in \mathbb{R}^d$ and $n \in \mathbb{Z}$. Let Φ^ϵ be the setvalued difference process generated by the perturbed nonautonomous difference inclusion (12).

Theorem 6.1 *Suppose that the setvalued difference process Φ generated by the unperturbed nonautonomous difference inclusion (1) has a weakly positive invariant weakly pullback absorbing family $\mathcal{B} = \{B_t, t \in \mathbb{Z}\}$ and suppose that the perturbed setvalued difference process Φ^ϵ generated by the perturbed nonautonomous difference inclusion (12) satisfying (13) has a weakly positive invariant weakly pullback absorbing family $\mathcal{B}^\epsilon = \{B_t^\epsilon, t \in \mathbb{Z}\}$ such that*

$$\lim_{\epsilon \to 0} H^*\left(B_{t_0}^\epsilon, B_{t_0}\right) = 0 \tag{14}$$

for all $t_0 \in \mathbb{Z}$. Then the maximal weak pullback attractor $\mathcal{A}^\epsilon = \{A_t^\epsilon, t \in \mathbb{Z}\}$ of Φ^ϵ relative to $\mathcal{B}^\epsilon$ converges upper semi continuously to the maximal weak pullback attractor $\mathcal{A} = \{A_t, t \in \mathbb{Z}\}$ of Φ relative to $\mathcal{B}$ in the sense that

$$\lim_{\epsilon \to 0} H^* \left(A_{t_0}^\epsilon, A_{t_0} \right) = 0. \tag{15}$$

for each $t_0 \in \mathbb{Z}$.

The following structural conditions on the unperturbed nonautonomous difference inclusion (1) provide simple conditions ensuring the existence of a nearby weakly positively invariant weak pullback absorbing family. (The conditions need to be strengthened by, say, the compactness or asymptotic compactness of the set-valued inclusion process when the state space is a Banach space instead of just $\mathbb{R}^d$).

Let K be a nonempty compact subset of $\mathbb{R}^d$ for which there exists a $\gamma \in [0, 1)$ such that

$$\min_{y \in F_t(x)} \mathrm{dist}(y, K) \leq \gamma \; \mathrm{dist}(x, K)$$

for all $x \in \mathbb{R}^d$ and $t \in \mathbb{Z}$. We can take $K^\epsilon = N_\epsilon[K] := \{x \in \mathbb{R}^d \; : \; \mathrm{dist}(x, K) \leq \epsilon\}$ for a sufficiently small ϵ. Then the family $\mathcal{B} = \{B_t, t \in \mathbb{Z}\}$ with $B_t \equiv K^\epsilon$ for all $t \in \mathbb{Z}$ is both weakly positively invariant and weakly absorbing uniformly in both the forward and pullback sense.

More generally, given a family $\mathcal{K} = \{K_t, t \in \mathbb{Z}\}$ of nonempty compact sets we can obtain a weakly positively invariant and weakly pullback absorbing family $\mathcal{K}^\epsilon = \{K_t^\epsilon, t \in \mathbb{Z}\}$ with appropriately defined K_t^ϵ if we have

$$\min_{y \in F_t(x)} \mathrm{dist}(y, K_t) \leq \gamma_{t,t-1} \; \mathrm{dist}(x, K_{t-1})$$

for a sequence of positive constants $\{\gamma_{t,t-1}, \, t \in \mathbb{Z}\}$ such that

$$\rho(t, t_0) \sup_{x \in D} \mathrm{dist}(x, K_{t_0}) \to 0, \qquad t_0 \to -\infty$$

for all fixed $t \in \mathbb{Z}$ and bounded subsets set D of $\mathbb{R}^d$, where $\rho(t, t_0) = \gamma_{t,t-1} \cdot \ldots \cdot \gamma_{t_0+1,t_0}$.

7 Examples

We consider five examples which illustrate the properties and some of the peculiarities of weak invariance and weak pullback attractors.

Our first three examples are, in fact, autonomous difference inclusions, i.e., of the form $x_{t+1} \in F(x_t)$, in which case pullback attraction coincides with the usual forward attraction and a weak pullback attractor is a weak attractor in the sense of Szegö and Treccani [10].

For our first example, we consider

$$F(x) := \begin{cases} [0,1] & \text{if } x = 0 \\ \{x+1\} & \text{otherwise} \end{cases}, \qquad x \in \mathbb{R},$$

which is motivated by the time–1 mapping of the solution of the differential equation without uniqueness $x' = x^{1/3}$. The set $\{0\}$ here is weakly invariant but not strongly invariant. It is neither strongly nor weakly attracting.

For our second example, we take

$$F(x) = x + [-1,1], \qquad x \in \mathbb{R}.$$

Here any set of the form $B = [a,b]$ with finite $a \leq b$ is weakly positively invariant and weakly absorbing. The weak maximal attractor A relative to B is the set B itself. If we take two disjoint sets B_1 and B_2 of this form, then we have two disjoint maximal weak attractors relative to these absorbing sets, namely $A_1 = B_1$ and $A_2 = B_2$. Alternatively, if $B_1 \subset B_2$, then we have $A_1 \subset A_2$. This system thus has many possible weak attractors, each of which is maximal relative to its absorbing set. Moreover, some of these weak attractors may be disjoint.

For our third example, we take

$$F(x) = \left[\frac{1}{2}x, 2x\right], \qquad x \in \mathbb{R}.$$

Here the set $\{0\}$ is strongly invariant and hence weakly invariant. It is weakly attracting, but is not strongly attracting. In fact, any set of the form $B = [a,b]$ with finite $a < 0 < b$ is weakly positively invariant and weakly absorbing. The weak maximal attractor A relative to B is the set B itself. If we take two sets $B_1 \subset B_2$ of this form, then we have two maximal weak attractors relative to these absorbing sets, namely $A_1 = B_1 \subset A_2 = B_2$. The weak attractor $\{0\}$ is unique in that it can only be approach asymptotically from outside, i.e, it is not its own absorbing set.

Our fourth example is properly nonautonomous with pullback attraction but not forward attraction. We define

$$F_t(x) := \begin{cases} [0,1] & \text{if } x = 0 \\ \{2^t x\} & \text{otherwise} \end{cases}$$

for each $t \in \mathbb{Z}$. The setvalued difference inclusion process here is given by

$$\Phi(t, t_0, 0) := \begin{cases} [0,1] & \text{if} \quad t = t_0 + 1 \\ \left[0, \max\left\{1, 2^{(t+t_0)(t-t_0-1)/2}\right\}\right] & \text{if} \quad t \geq t_0 + 2 \end{cases}$$

with $\Phi(t, t-k, x) = 2^{k(2t-k-1)/2} x$ for $x \neq 0$ and $k \in \mathbb{Z}^+$ with $\Phi(t_0, t_0, x) = \{x\}$. In particular,

$$\Phi(t, t-k, x) = 2^{k(2t-k-1)/2} x \longrightarrow 0 \qquad \text{for} \quad k \to \infty,$$

so the family $\mathcal{A} = \{A_t,\ t \in \mathbb{Z}\}$ with $A_t \equiv \{0\}$ for all $t \in \mathbb{Z}$ is a weak pullback attractor. It is weakly but not strongly invariant and weakly but not strongly pullback attracting.

For our final example we consider the nonautonomous difference inclusion (1) with

$$F_t(x) = 2^t x + [-1, +1],$$

for which the difference inclusion process is given by

$$\Phi(t_0, t_0, x) = \{x\}, \qquad \Phi(t, t-k, x) = 2^{k(2t-k-1)/2} x + [-D_{t,k}, D_{t,k}],$$

where $D_{t,k} = 1 + \sum_{j=1}^{k-2} 2^{j(2t-j-1)/2}$, which is finite for each $k \in \mathbb{Z}$. Since $D_{t,k} \to D_t$ $= 1 + \sum_{j=1}^{\infty} 2^{j(2t-j-1)/2} < \infty$ as $k \to \infty$, this process has the strong pullback (but not forward) attractor $\mathcal{A} = \{A_t,\ t \in \mathbb{Z}\}$ with $A_t = [-D_t, D_t]$.

In addition, the family $\mathcal{A}^* = \{A_t^*,\ t \in \mathbb{Z}\}$ with $A_t^* \equiv [-1, 1]$ for all $t \in \mathbb{Z}$ is a weak (but not strong) pullback attractor. In particular, it is only weakly positively invariant. In fact, any family $\mathcal{A}^\alpha$ of subintervals $A_t^\alpha \equiv [-\alpha, \alpha]$ for all $t \in \mathbb{Z}$ with α $\in [0, 1]$ is also a weak pullback attractor. Such weak pullback attractors may thus be useful in investigating the internal structure of a strong pullback attractor.

8 Proofs

We will need the following lemmata in the proof of Theorem 6.1

Lemma 8.1 *Suppose that a setvalued difference process Φ has a weak pullback absorbing family $\mathcal{B} = \{B_t, t \in \mathbb{Z}\}$ and a weak pullback attractor $\mathcal{A} = \{A_t, t \in \mathbb{Z}\}$. Then an entire trajectory ϕ of Φ satisfies $\phi(t) \in B_t$ for all $t \in \mathbb{Z}$ if and only if $\phi(t)$ $\in A_t$ for all $t \in \mathbb{Z}$.*

Proof: Suppose that ϕ is an entire trajectory with $\phi(t) \in B_t$ for each $t \in \mathbb{Z}$. Fix $t_0 \in \mathbb{Z}$. Then there is a sequence of trajectories $\phi_n : [t_0 - n, t_0] \cap \mathbb{Z} \to \mathbb{R}^d$, namely $\phi_n \equiv \phi$, with $\phi_n(t) = \phi(t) \in B_t$ for each $t \in [t_0 - n, t_0] \cap \mathbb{Z}$. In particular, $\phi_n(t_0) \equiv \phi(t_0) \to \phi(t_0)$ as $n \to \infty$. By the definition, $\phi(t_0) \in A_{t_0}$. Since t_0 was otherwise arbitrary, we thus have $\phi(t) \in A_t$ for all $t \in \mathbb{Z}$. The converse follows from the fact that $A_t \subset B_t$ for all $t \in \mathbb{Z}$.

Lemma 8.2 *Suppose that $H^*(B_n, B) \to 0$ as $n \to \infty$ for nonempty compact subsets B, B_1 , B_2, ... Then for any sequence $b_n \in B_n$, $n \in \mathbb{Z}^+$, there exists a convergent subsequence $b_{n_j} \to b^* \in B$ as $n_j \to \infty$.*

Proof: Clearly $\text{dist}(b_n, B) \leq H^*(B_n, B)$ for all $n \in \mathbb{Z}^+$ and since B is compact, there exist $b_n^* \in B$ such that $\text{dist}(b_n, B) = \|b_n - b_n^*\|$. By the compactness of B again, there exists a convergent subsequence $b_{n_j}^* \to b^* \in B$ as $n_j \to \infty$. Then $b_{n_j} \to b^*$ too as $n_j \to \infty$ since

$$\|b_{n_j} - b^*\| \leq \|b_{n_j} - b_{n_j}^*\| + \|b_{n_j}^* - b^*\| = \text{dist}\left(b_{n_j}, B\right) + \|b_{n_j}^* - b^*\|$$

for all $n_j \in \mathbb{Z}^+$.

Lemma 8.3 *Suppose F and $F^\epsilon \in USC(\mathbb{R}^d, \mathcal{H}(\mathbb{R}^d))$ with $\epsilon > 0$ are such that $F^\epsilon(x) \subset N_\epsilon(F(x))$ for all $x \in \mathbb{R}^d$. Then*

$$H^*\left(F^{\epsilon_n}(x_n), F(x^*)\right) \longrightarrow 0 \qquad \textit{as} \quad n \to \infty$$

for any convergent sequences $x_n \to x^*$ in $\mathbb{R}^d$ and $\epsilon_n \to 0$ as $n \to \infty$.

Proof: For every $\nu > 0$ there exists an $K_\nu \in \mathbb{Z}$ such that $\|x_n - x^*\| < \nu/2$ and $0 < \epsilon_n < \nu/2$ for all $n \geq K_\nu$. Thus

$$x_n \in N_{\nu/2}(x^*) \quad \text{and} \quad F^{\epsilon_n}(x_n) \subset N_{\epsilon_n}(F(x_n)) \subset N_{\nu/2}(F(x_n))$$

for all $n \geq K_\nu$. Since $F \in USC(\mathbb{R}^d, \mathcal{H}(\mathbb{R}^d))$ there exists a $\delta(\nu/2, x^*) > 0$ such that $F(x_n) \subset N_{\nu/2}(F(x^*))$ for all x_n with $\|x_n - x^*\| < \delta(\nu/2, x^*)$. Thus we have

$$F^{\epsilon_n}(x_n) \subset N_{\nu/2}(F(x_n)) \subset N_{\nu/2}\left(N_{\nu/2}(F(x^*))\right) = N_\nu(F(x^*))$$

for all $n \geq \max\left\{K_{\nu/2}, K_{\delta(\nu/2,x^*)}\right\}$.

8.1 Proof of Theorem 5.2

We divide the proof into three parts.

8.1.1 Existence and compactness

Fix $t_0 \in \mathbb{Z}$. By the weak positive invariance of $\mathcal{B} = \{B_t, t \in \mathbb{Z}\}$, there exist trajectories $\phi_n : [t_0 - n, t_0] \cap \mathbb{Z} \to \mathbb{R}^d$ with $\phi_n(t) \in B_t$ for each $t \in [t_0 - n, t_0] \cap \mathbb{Z}$ and all $n \in \mathbb{Z}^+$. In particular, $\phi_n(t_0) \in B_{t_0}$ for each $n \in \mathbb{Z}^+$. Since B_{t_0} is compact, there exists a convergent subsequence $\phi_{n_j}(t_0) \to a_0 \in B_{t_0}$. Taking this subsequence to be the original sequence in the definition (10) of A_{t_0}, we have $a_0 \in A_{t_0}$, which proves that A_{t_0} is nonempty.

To show that A_{t_0} is compact, we need only to show that it is closed because A_{t_0} is a subset of the compact set B_{t_0}. Suppose that $a_k \in A_{t_0}$ and $a_k \to a^*$ as $k \to \infty$. Then for each $k \in \mathbb{Z}^+$ there exist subsequences $t_{k,n} \to \infty$ as $n \to \infty$ and trajectories $\phi_{k,n} : [t_0 - t_{k,n}, t_0] \cap \mathbb{Z} \to \mathbb{R}^d$ with $\phi_{k,n}(t) \in B_t$ for each $t \in [t_0 - t_{k,n}, t_0] \cap \mathbb{Z}$ and $n \in \mathbb{Z}^+$ for which $\lim_{k\to\infty} \phi_{k,n}(t_0) = a_k$. Pick n_k so that

$$\|\phi_{k,n_k}(t_0) - a_k\| \leq \frac{1}{k} \qquad \text{and} \qquad t_{k+1,n_{k+1}} \geq t_{k,n_k} + 1$$

for each $k \in \mathbb{Z}^+$. Then

$$\|\phi_{k,n_k}(t_0) - a^*\| \leq \|\phi_{k,n_k}(t_0) - a_k\| + \|a_k - a^*\| \leq \frac{1}{k} + \|a_k - a^*\| \to 0$$

as $k \to \infty$. Write $\bar{\phi}_k \equiv \phi_{k,n_k}$ and $\bar{t}_k \equiv t_{k,n_k}$. Then $\bar{\phi}_k : [t_0 - \bar{t}_k, t_0] \cap \mathbb{Z} \to \mathbb{R}^d$ with $\bar{\phi}_k(t) \in B_t$ for each $t \in [t_0 - \bar{t}_k, t_0] \cap \mathbb{Z}$ and $k \in \mathbb{Z}^+$. Moreover, $\bar{t}_k \to \infty$ as $k \to \infty$ with $\bar{\phi}_k(t_0) \to a^*$ as $k \to \infty$. Thus $a^* \in A_{t_0}$, so A_{t_0} is closed and hence compact.

8.1.2 Weak invariance

Let us first prove that the family $\mathcal{A} = \{A_t,\ t \in \mathbb{Z}\}$ is weakly positively invariant. Fix $t_0 \in \mathbb{Z}$ and take $x_0 \in A_{t_0}$. Then, there exists $t_n \to +\infty$ and trajectories $\phi_n : [t_0 - t_n, t_0] \cap \mathbb{Z} \to \mathbb{R}^d$ with $\phi_n(t) \in B_t$ for each $t \in [t_0 - t_n, t_0] \cap \mathbb{Z}$ and $n \in \mathbb{Z}^+$ for which $\lim_{n\to\infty} \phi_n(t_0) = x_0$. Since $\mathcal{B}$ is weakly positively invariant, each trajectory ϕ_n can be extended to $[t_0 - t_n, \infty) \cap \mathbb{Z}$ so that $\phi_n(t) \in B_t$ for all $t \geq t_0$. By the compactness of each B_t, we can find a (diagonal) subsequence $n'_k \to \infty$ as $k \to \infty$ such that $\phi_{n'_k}(t) \to \bar{\phi}(t) \in B_t$ for each $t \geq t_0$. Obviously $\bar{\phi}(t_0) = x_0 \in A_{t_0}$ since the original subsequence $\phi_{n_k}(t_0) \to x_0$. By the construction, $\bar{\phi}(t) \in A_t$ for all $t \geq t_0$.

The mapping $\bar{\phi} : [t_0, \infty) \cap \mathbb{Z} \to \mathbb{R}^d$ is a trajectory of the setvalued mapping Φ since $\text{dist}\left(\bar{\phi}(t+1), F_t(\bar{\phi}(t))\right) = 0$, i.e., $\bar{\phi}(t+1) \in F_t(\bar{\phi}(t))$ for all $t \geq t_0$. This follows from

$$\text{dist}\left(\bar{\phi}(t+1), F_t(\bar{\phi}(t))\right) \quad \leq \quad \left\|\bar{\phi}(t+1) - \phi_{n'_k}(t+1)\right\| + \text{dist}\left(\phi_{n'_k}(t+1), F_t(\phi_{n'_k}(t))\right)$$

$$
\begin{aligned}
& +H^*\left(F_t(\phi_{n'_k}(t)), F_t(\bar{\phi}(t))\right) \\
= & \left\|\bar{\phi}(t+1) - \phi_{n'_k}(t+1)\right\| + H^*\left(F_t(\phi_{n'_k}(t)), F_t(\bar{\phi}(t))\right) \\
\longrightarrow & \ 0 \ \text{ as } \ n'_k \to \infty,
\end{aligned}
$$

for each $t \geq t_0$, since $\phi_{n'_k}(t+1) \in F_t(\phi_{n'_k}(t))$ for the trajectories $\phi_{n'_k}$.

Now $t_0 \in \mathbb{Z}$ and $x_0 \in A_{t_0}$were arbitrary, so $\mathcal{A} = \{A_t, t \in \mathbb{Z}\}$ is weakly positively invariant.

A similar argument holds with a little more care for all $t \leq t_0$. This will show the weak invariance. Fix an $N \in \mathbb{Z}^+$ and take k large enough so that $n_k \geq N$ in the above subsequence of trajectories $\phi_{n_k} : [t_0 - n_k, t_0] \cap \mathbb{Z} \to \mathbb{R}^d$ with $\phi_{n_k}(t) \in B_t$ for $t \in [t_0 - n_k, t_0] \cap \mathbb{Z}$ and $\phi_{n_k}(t_0) \to x_0$. We now restrict these trajectories to the common definition interval $[t_0 - N, t_0] \cap \mathbb{Z} \subset [t_0 - n_k, t_0] \cap \mathbb{Z}$. Since each B_t is compact, there is a convergent subsequence with $\phi_{n'_k}(t) \to \bar{\phi}(t) \in B_t$ for each $t \in [t_0 - N, t_0] \cap \mathbb{Z}$. Obviously $\bar{\phi}(t_0) = x_0$. By a diagonal subsequence argument we have a (diagonal) subsequence such that $\phi_{n'_k}(t) \to \bar{\phi}(t) \in B_t$ for all $t \leq t_0$. It then follows as above that $\bar{\phi}$ is a trajectory of the setvalued difference process Φ with $\bar{\phi}(t) \in A_t$ for all $t \leq t_0$. Concatenating the two parts of $\bar{\phi}$ to all of $\mathbb{Z}$ gives us an entire trajectory $\bar{\phi}$ of the setvalued difference process Φ with $\bar{\phi}(t) \in A_t$ for all $t \in \mathbb{Z}$. Thus $\mathcal{A} = \{A_t, t \in \mathbb{Z}\}$ is weakly invariant.

8.1.3 Weak pullback attraction

Fix $t_0 \in \mathbb{Z}$ and a bounded subset D of $\mathbb{R}^d$. Since $\mathcal{B} = \{B_t, t \in \mathbb{Z}\}$ is a weakly pullback absorbing family for the setvalued difference process Φ on $\mathbb{R}^d$, for every $n \in \mathbb{Z}^+$ there is an integer $N_{t_0-n,D} \in \mathbb{Z}^+$ such that for each $k \geq N_{t_0-n,D}$ and $d_n \in D$ there exists a trajectory $\phi_{k,n}$ of Φ on $[t_0 - k - n, t_0 - n] \cap \mathbb{Z}$ with $\phi_{k,n}(t_0 - k - n) = d_n$ and $b_{k,n} = \phi_{k,n}(t_0 - n) \in B_{t_0-n}$ for all $k \geq N_{t_0-n,D}$ and $n \in \mathbb{Z}^+$. Since $\mathcal{B}$ is weakly positively invariant, each $\phi_{k,n}$ can be extended indefinitely so that $\phi_{k,n}(t) \in B_t$ for all $t \geq t_0 - n$. In particular, $\phi_{k,n}(t_0) \in B_{t_0}$ and B_{t_0} is compact, so there is a subsequence $k_n < k_{n+1} \to \infty$ as $n \to \infty$ with $k_n \geq N_{t_0-n,D}$ and $k_{n+1} \geq N_{t_0-n-1,D}$ such that $\phi_{k_n,n}(t_0) \to a^* \in B_{t_0}$ as $n \to \infty$.

Write $\bar{\phi}_n \equiv \phi_{k_n,n}$ and $t_n \equiv n + k_n$. Then $\bar{\phi}_n$ is defined on $[t_0 - t_n, \infty) \cap \mathbb{Z}$ with $\bar{\phi}_n(t_0 - t_n) = d_{k_n} \in D$ and $\bar{\phi}_n(t_0) \to a^*$ as $k \to \infty$. By the construction $a^* \in A_{t_0}$, so $\lim_{n\to\infty} \operatorname{dist}\left(\bar{\phi}_n(t_0), A_{t_0}\right) = 0$. Thus property (9) holds and $\mathcal{A} = \{A_t, t \in \mathbb{Z}\}$ is weakly pullback attracting.

8.2 Proof of Theorem 6.1

Let $\mathcal{A} = \{A_t, t \in \mathbb{Z}\}$ be the maximal weak pullback attractor in $\mathcal{B} = \{B_t, t \in \mathbb{Z}\}$ of the unperturbed setvalued difference process Φ and let $\mathcal{A}^\epsilon = \{A_t^\epsilon, t \in \mathbb{Z}\}$ be the maximal weak pullback attractor in $\mathcal{B}^\epsilon = \{B_t^\epsilon, t \in \mathbb{Z}\}$ of the perturbed setvalued difference process Φ^ϵ. Suppose for some $t_0 \in \mathbb{Z}$ that

$$\lim_{\epsilon \to 0} H^*\left(A_{t_0}^\epsilon, A_{t_0}\right) \neq 0.$$

Then there exists an $\eta_0 > 0$ and a subsequence $\epsilon_j \to 0$ as $j \to \infty$ such that

$$H^*\left(A_{t_0}^{\epsilon_j}, A_{t_0}\right) \geq \eta_0 \tag{16}$$

for all $j \in \mathbb{Z}^+$. We will show that this leads to a contradiction.

Let $a^{\epsilon_j} \in A_{t_0}^\epsilon$ be such that $\operatorname{dist}(a^{\epsilon_j}, A_{t_0}) = H^*\left(A_{t_0}^{\epsilon_j}, A_{t_0}\right)$, so $\operatorname{dist}(a^{\epsilon_j}, A_{t_0}) \geq \eta_0$ for $j \in \mathbb{Z}^+$, which is possible since $A_{t_0}^\epsilon$ is compact. By Lemma 8.1 there is an entire trajectory ϕ^{ϵ_j} of the perturbed setvalued difference process Φ^{ϵ_j} such that $\phi^{\epsilon_j}(t) \in A_t^{\epsilon_j} \subset B_t^{\epsilon_j}$ for each $t \in \mathbb{Z}$ with $\phi^{\epsilon_j}(t_0) = a^{\epsilon_j}$. Since for each t, the $B_t^{\epsilon_j}$ and B_t are compact with $H^*\left(B_t^{\epsilon_j}, B_t\right) \to 0$ as $\epsilon_j \to 0$, by Lemma 8.2 there exists a convergent (diagonal) subsequence $\phi^{\epsilon_j'}(t) \to \bar{\phi}(t) \in B_t$ as $\epsilon_j' \to 0$ for each $t \in \mathbb{Z}$. Obviously a^{ϵ_j} $= \phi^{\epsilon_j'}(t_0) \to \bar{\phi}(t_0)$, so from (16) we have

$$\operatorname{dist}\left(\bar{\phi}(t_0), A_{t_0}\right) \geq \eta_0/2. \tag{17}$$

We will show that $\bar{\phi}$ is a trajectory of the unperturbed setvalued difference process Φ. We have

$$\begin{aligned}
\operatorname{dist}\left(\bar{\phi}(t+1), F_t(\bar{\phi}(t))\right) &\leq \left\|\bar{\phi}(t+1) - \phi^{\epsilon_j'}(t+1)\right\| + \operatorname{dist}\left(\phi^{\epsilon_j'}(t+1), F_t^{\epsilon_j'}(\phi^{\epsilon_j'}(t))\right) \\
&\qquad + H^*\left(F_t^{\epsilon_j'}(\phi^{\epsilon_j'}(t)), F_t(\bar{\phi}(t))\right) \\
&= \left\|\bar{\phi}(t+1) - \phi^{\epsilon_j'}(t+1)\right\| + H^*\left(F_t^{\epsilon_j'}(\phi^{\epsilon_j'}(t)), F_t(\bar{\phi}(t))\right)
\end{aligned}$$

for each $t \geq t_0$, since $\phi^{\epsilon_j'}(t+1) \in F_t^{\epsilon_j'}(\phi^{\epsilon_j}(t))$ for the trajectories $\phi^{\epsilon_j'}$ of $F^{\epsilon_j'}$.

¿From above

$$\phi^{\epsilon_j'}(t+1) \to \bar{\phi}(t+1), \qquad \phi^{\epsilon_j'}(t) \to \bar{\phi}(t) \quad \text{as} \quad \epsilon_j' \to 0.$$

Since the setvalued mappings $F_t^{\epsilon_j'}$ and F_t are upper semi continuous and the $F_t^{\epsilon_j'}$ converge upper semi continuously to F_t, it follows by Lemma 8.3 that

$$H^*\left(F_t^{\epsilon_j'}(\phi^{\epsilon_j'}(t)), F_t(\bar{\phi}(t))\right) \longrightarrow 0 \quad \text{as} \quad \epsilon_j' \to 0.$$

Thus $\operatorname{dist}\big(\bar{\phi}(t+1), F_t(\bar{\phi}(t))\big) = 0$ for all $t \in \mathbb{Z}$, i.e., $\bar{\phi}(t+1) \in F_t(\bar{\phi}(t))$ for all $t \in \mathbb{Z}$, which means that $\bar{\phi}$ is an entire trajectory of the unperturbed setvalued difference process Φ with $\bar{\phi}(t) \in B_t$ for each $t \in \mathbb{Z}$. By Lemma 8.1 it follows that $\bar{\phi}(t) \in A_t$ for each $t \in \mathbb{Z}$. However, this contradicts (17) and hence (16). This contradiction means that the A_t^ϵ converge upper semi continuously to A_t for each $t \in \mathbb{Z}$.

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