Some results on stochastic differential equations with reflecting boundary conditions

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Abstract

Some results related to stochastic differential equations with reflecting boundary conditions are obtained. Existence and uniqueness of strong solution is ensured under the relaxation on the drift coefficient (instead of the Lipschitz character, a monotonicity condition is supposed).

KEY WORDS: Skorokhod problem; Reflected Stochastic Differential Equations; Monotonicity condition; Strong solution

1 Introduction

In this paper we extend some results of Tanaka[(6)] and Lions and Sznitman[(4)] on existence and uniqueness of strong solutions for stochastic differential equations with reflecting boundary conditions (SDER) to the case in which the drift coefficient b satisfies the monotonicity condition

$$(x - x', b(t, x) - b(t, x')) \leq L_{b_x} |x - x'|^2,$$

instead of the classical Lipschitz condition. As far as we know, for this type of drift coefficient there is not in the literature a general result of existence of strong solutions for SDER (an exception is the 1-dimensional case, cf. Zhang[(7)] and Matoussi[(5)]). In the case of stochastic differential equations without reflection the same kind of problem has been previously solved, for instance, in Jacod[(3)] and Gyöngy and Krylov[(2)].

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In section 2 we give the framework, definitions and claim the main result. Section 3 is devoted to prove a previous and similar result on the deterministic Skorokhod problem. Finally, the stochastic version is treated in section 4.

2 Statement of the problem and main result

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, $\{\mathcal{F}_t\}_{t\geq 0}$ an increasing and right continuous family of sub-σ-algebras of $\mathcal{F}$ such that $\mathcal{F}_0$ contains all the P-null sets of $\mathcal{F}$, and $\{W_t;\ t \geq 0\}$ an m-dimensional standard $\{\mathcal{F}_t\}$-Wiener process.

Let $\mathcal{O}$ be an open connected bounded subset of $\mathbb{R}^d$ given by $\mathcal{O} = \{\phi > 0\}$, with $\phi \in C^2(\mathbb{R}^d)$, and such that $\partial\mathcal{O} = \{\phi = 0\}$, with $|\nabla\phi(x)| = 1$ for all $x \in \partial\mathcal{O}$. Observe that in particular ϕ, $\nabla\phi$ and $D^2\phi$ are bounded in $\bar{\mathcal{O}}$. Also, observe that $n(x)$, the unit outward normal to $\partial\mathcal{O}$ at x, coincides with $-\nabla\phi(x)$, and that we can assert that there exists a constant $C_0 > 0$ such that

$$2(x' - x, \nabla\phi(x)) + C_0|x' - x|^2 \geq 0, \qquad \forall\, x \in \partial\mathcal{O},\ \forall\, x' \in \bar{\mathcal{O}}. \tag{2.1}$$

We are also given a final time $T > 0$, and two random functions:

$$b : \Omega \times [0, T] \times \bar{\mathcal{O}} \to \mathbb{R}^d,$$

$$\sigma : \Omega \times [0, T] \times \bar{\mathcal{O}} \to \mathbb{R}^{d\times m},$$

such that

(i) b and σ are uniformly bounded;

(ii) for all $x \in \bar{\mathcal{O}}$ the processes $b(\cdot, \cdot, x)$ and $\sigma(\cdot, \cdot, x)$ are $\{\mathcal{F}_t\}$-progressively measurable;

(iii) for all $t \in [0, T]$ and a.s. ω, the function $b(\omega, t, \cdot)$ is continuous on $\bar{\mathcal{O}}$;

(iv) there exist two constants $L_{b_x} \in \mathbb{R}$ and $L_{\sigma_x} \geq 0$ such that for all $t \in [0, T]$ and all $x, x' \in \bar{\mathcal{O}}$,

$$(x - x', b(\omega, t, x) - b(\omega, t, x')) \leq L_{b_x}|x - x'|^2, \quad a.s.,$$

$$\|\sigma(\omega, t, x) - \sigma(\omega, t, x')\| \leq L_{\sigma_x}|x - x'|, \quad a.s.,$$

where $|\cdot|$ and $\|\cdot\|$ denote the usual Euclidean and trace norm for vectors and matrices respectively.

From now on, in general we will omit the explicit dependence of the processes on ω.

Remark 1. Observe that if b satisfies the conditions (i)-(iii) above, then, reasoning as in the proof of Tietze's Extension Theorem, and using the theorems 8.1.4 and 8.2.9 in Aubin and Frankowska[1], one can see that there exists an extension of b,

$$\tilde{b} : \Omega \times [0,T] \times \mathbb{R}^d \to \mathbb{R}^d,$$

such that $\tilde{b}$ is also uniformly bounded and satisfies (ii) and (iii) in $\mathbb{R}^d$ instead of $\bar{\mathcal{O}}$.

We seek strong solutions for the problem:

$$X_t = x_0 + \int_0^t b(s, X_s)\, ds + \int_0^t \sigma(s, X_s)\, dW_s - k_t, \tag{2.2}$$

$$k_t = -\int_0^t \nabla\phi(X_s)\, d|k|_s, \quad |k|_t = \int_0^t 1_{\{X_s \in \partial\mathcal{O}\}}\, d|k|_s, \quad t \in [0,T], \tag{2.3}$$

where $x_0 \in \bar{\mathcal{O}}$ is given, and $|k|_t$ stands for the total variation of k on $[0,t]$.

Definition 1. *A strong solution to the above problem is a pair of $\{\mathcal{F}_t\}$-adapted and continuous processes (X,k) defined on $\Omega \times [0,T]$, the first one with values in $\bar{\mathcal{O}}$, the second one with values in $\mathbb{R}^d$ and paths of bounded variation in $[0,T]$, satisfying the equations* (2.2)-(2.3) *a.s. for all $t \in [0,T]$.*

We now state our main result, which generalizes that given in Lions and Sznitman[4] when b is Lipschitz.

Theorem 1. *Under the assumptions* (i)-(iv)*, for each $x_0 \in \bar{\mathcal{O}}$ given there exists a unique pair (X,k), strong solution of* (2.2)-(2.3)*.*

To prove this theorem, we will analyze a deterministic problem which generalizes the Skorokhod problem studied in Lions and Sznitman[4].

3 A generalization of the Skorokhod problem

In this section, we consider the open set $\mathcal{O}$ given in section 2 but we assume that the coefficient b is independent of ω.

We suppose given $x_0 \in \bar{\mathcal{O}}$ and a function $g \in C([0,T];\mathbb{R}^d)$ such that $g_0 = 0$. We want to solve the deterministic problem

$$x_t = x_0 + \int_0^t b(s, x_s)\, ds + g_t - k_t, \tag{3.1}$$

$$k_t = -\int_0^t \nabla\phi(x_s)\, d|k|_s, \quad |k|_t = \int_0^t 1_{\{x_s \in \partial\mathcal{O}\}}\, d|k|_s, \quad t \in [0,T]. \tag{3.2}$$

Definition 2. *A solution of the problem* (3.1)-(3.2) *is a pair* (x, k) *of continuous functions defined on* $[0, T]$ *with values in* $\mathbb{R}^d$*, such that* $x_t \in \bar{\mathcal{O}}$ *for all* $t \in [0, T]$*,* k *is of bounded variation on* $[0, T]$*, and the equations* (3.1)-(3.2) *are satisfied for all* $t \in [0, T]$*.*

From Theorem 2.1 and Remark 2.1 in Lions and Sznitman[(4)] we can assert the following result:

Theorem 2. *Let suppose* $b \equiv 0$ *and* $x_0 \in \bar{\mathcal{O}}$ *given. Then, for any function* $g \in C([0, T]; \mathbb{R}^d)$ *such that* $g_0 = 0$ *there exists a unique pair* (x, k) *solution of the problem* (3.1)-(3.2)*. Moreover, the mapping* $g \mapsto x$ *is Hölder continuous of order* $1/2$ *on compact sets of* $C([0, T]; \mathbb{R}^d)$*.*

Remark 2. Observe that, as a direct consequence of the Hölder continuity of order $1/2$ on compact sets of $C([0, T]; \mathbb{R}^d)$ of the the mapping $g \mapsto x$, we can assert that under the conditions of Theorem 2, if $\{g^n\} \subset C([0, T]; \mathbb{R}^d)$ is a sequence of functions such that $g_0^n = 0$ and $g^n \to g$ in $C([0, T]; \mathbb{R}^d)$, then, if we denote by (x^n, k^n) (resp. (x, k)) the pair solution of (3.1)-(3.2) corresponding to $b \equiv 0$ and g^n (resp. g), we have that $x^n \to x$ in $C([0, T]; \bar{\mathcal{O}})$.

We will see now that we can extend Theorem 2 to the case in which $b \not\equiv 0$. First at all, we have the following result:

Theorem 3. *Let be* $\mathcal{O}$ *satisfying the conditions in section* 2*. Consider given a measurable and bounded function* $b : [0, T] \times \bar{\mathcal{O}} \to \mathbb{R}^d$*, such that for all* $t \in [0, T]$ *the function* $b(t, \cdot)$ *is continuous on* $\bar{\mathcal{O}}$*. Then, for each* $x_0 \in \bar{\mathcal{O}}$ *and* $g \in C([0, T]; \mathbb{R}^d)$ *such that* $g_0 = 0$ *given, there exists at least one solution* (x, k) *of the problem* (3.1)-(3.2)*.*

Proof. We will proceed in two steps.

Step 1 Let also suppose that $g \in C^1([0, T]; \mathbb{R}^d)$ and b is Lipschitz in x on $\bar{\mathcal{O}}$, i.e. there exists $L > 0$ such that

$$|b(t, x) - b(t, x')| \leq L|x - x'|$$

for all $t \in [0, T]$ and all $x, x' \in \bar{\mathcal{O}}$.

In this case, the existence and uniqueness of solution to (3.1)-(3.2) can be deduced from the stochastic results in Lions and Sznitman[(4)]. However, for more clarity, we give a completely deterministic proof.

Denote by f the derivative of g. For each $y \in C([0, T]; \mathbb{R}^d)$ given, consider the problem

$$x_t = x_0 + \int_0^t (b(s, y_s) + f_s)\, ds - k_t, \tag{3.3}$$

$$k_t = -\int_0^t \nabla\phi(x_s)\, d|k|_s, \quad |k|_t = \int_0^t 1_{\{x_s\in\partial\mathcal{O}\}}\, d|k|_s, \quad t\in[0,T]. \tag{3.4}$$

Obviously, the function

$$\tilde{g}_t = \int_0^t (b(s,y_s)+f_s)\, ds$$

is continuous on $[0,T]$, with $\tilde{g}_0 = 0$, thus by Theorem 2, there exists a unique solution (x,k) of (3.3)-(3.4). It is enough to prove that there exists a unique fixed point for the mapping

$$F : y \in C([0,T];\bar{\mathcal{O}}) \mapsto x \in C([0,T];\bar{\mathcal{O}})$$

defined by (3.3)-(3.4).

Let $x = Fy$ and $x' = Fy'$. Using (2.1), it is easy to see that

$$\begin{aligned}
&\exp\{-C_0(\phi(x_t)+\phi(x'_t))\}|x_t - x'_t|^2 \\
\le\ & -C_0\int_0^t \exp\{-C_0(\phi(x_s)+\phi(x'_s))\}(\nabla\phi(x_s), b(s,y_s)+f_s)|x_s-x'_s|^2\, ds \\
& -C_0\int_0^t \exp\{-C_0(\phi(x_s)+\phi(x'_s))\}(\nabla\phi(x'_s), b(s,y'_s)+f_s)|x_s-x'_s|^2\, ds \\
& +2\int_0^t \exp\{-C_0(\phi(x_s)+\phi(x'_s))\}(x_s-x'_s, b(s,y_s)-b(s,y'_s))\, ds. \quad (3.5)
\end{aligned}$$

As x and x' take values in $\bar{\mathcal{O}}$,

$$\exp\{-2C_0 \max_{\bar{\mathcal{O}}}\phi\} \le \exp\{-C_0(\phi(x_t)+\phi(x'_t))\} \le 1$$

for all $t\in[0,T]$.

Moreover, $\nabla\phi$, b and f are uniformly bounded, and so, using that b is Lipschitz, it is easy to obtain from (3.5) the existence of a constant $C>0$, independent of y, y' and t, such that

$$|x_t - x'_t|^2 \le C\int_0^t \left(|x_s-x'_s|^2 + |y_s-y'_s|^2\right)\, ds,$$

for all $t\in[0,T]$.

Therefore,

$$\sup_{r\in[0,t]}|x_r - x'_r|^2 \le C\int_0^t \left(\sup_{r\in[0,s]}|x_r-x'_r|^2 + \sup_{r\in[0,s]}|y_r-y'_r|^2\right) ds,$$

for all $t \in [0,T]$, and from Gronwall's lemma we have

$$\sup_{r\in[0,t]} |x_r - x'_r|^2 \le C \exp(CT) \int_0^t \sup_{r\in[0,s]} |y_r - y'_r|^2 \, ds, \tag{3.6}$$

for all $t \in [0,T]$.

It is known that (3.6) implies that a power of F is a contraction in $C([0,T];\mathbb{R}^d)$, and so there exists a unique fixed point for F.

Step 2 Suppose now that we are in the conditions of the theorem.

In this case, we can approach g by a sequence of functions $g^n \in C^1([0,T];\mathbb{R}^d)$ such that $g_0^n = 0$, converging to g in $C([0,T];\mathbb{R}^d)$.

Furthermore, if we fix a regularizing sequence $\{\rho_n\} \subset \mathcal{D}(\mathbb{R}^d)$, i.e.

$$\rho_n : \mathbb{R}^d \to \mathbb{R}, \quad \rho_n \ge 0, \quad \operatorname{supp}(\rho_n) \subset \overline{B_{\mathbb{R}^d}(0,1/n)}, \quad \int_{\mathbb{R}^d} \rho_n dx = 1$$

and define

$$b_n(t,x) = \int_{\mathbb{R}^d} \rho_n(y) \tilde{b}(t, x-y)\, dy, \qquad (t,x) \in [0,T] \times \mathbb{R}^d, \tag{3.7}$$

with $\tilde{b}$ a measurable and uniformly bounded extension of b to $[0,T] \times \mathbb{R}^d$, such that $\tilde{b}(t,\cdot)$ is continuous in $\mathbb{R}^d$, we obtain a sequence of measurable functions $b_n : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ such that, in particular, for all $t \in [0,T]$ the function $b_n(t,\cdot)$ is continuous on $\mathbb{R}^d$,

$$|b_n(t,x)| \le \sup_{y\in\mathbb{R}^d} |\tilde{b}(t,y)|, \quad \forall\, x \in \mathbb{R}^d, \tag{3.8}$$

$$|b_n(t,x) - b_n(t,x')| \le L_n |x - x'|, \quad \forall\, x, x' \in \mathbb{R}^d, \tag{3.9}$$

$$b_n(t,\cdot) \to b(t,\cdot) \quad \text{uniformly in } \bar{\mathcal{O}}. \tag{3.10}$$

By Step 1, for each n we have a unique solution (x^n, k^n) of the problem:

$$\begin{aligned} x_t^n &= x_0 + \int_0^t b_n(s, x_s^n)\, ds + g_t^n - k_t^n, \\ k_t^n &= -\int_0^t \nabla\phi(x_s^n)\, d|k^n|_s, \quad |k^n|_t = \int_0^t 1_{\{x_s^n \in \partial\mathcal{O}\}}\, d|k^n|_s, \quad t \in [0,T]. \end{aligned}$$

It is obvious that $\{b_n(\cdot, x_\cdot^n)\}$ is bounded in $L^2(0,T;\mathbb{R}^d)$, and thus the sequence $\{\int_0^\cdot b_n(s, x_s^n)\, ds\}$ is bounded in $C([0,T];\mathbb{R}^d)$ and equicontinuous. Therefore, it is easy to see that there exist a subsequence $\{x^\mu\} \subset \{x^n\}$ and an element $\mathcal{B} \in L^2(0,T;\mathbb{R}^d)$, such that

$$b_\mu(\cdot, x_\cdot^\mu) \rightharpoonup \mathcal{B} \quad \text{in } L^2(0,T;\mathbb{R}^d) \quad \text{and}$$

$$\int_0^\cdot b_\mu(s,x_s^\mu)\,ds \to \int_0^\cdot \mathcal{B}_s\,ds \quad \text{in } C([0,T];\mathbb{R}^d).$$

Then, according to Theorem 2, $x^\mu \to x$ in $C([0,T];\mathbb{R}^d)$, with (x,k) the solution of

$$\begin{aligned} x_t &= x_0 + \int_0^t \mathcal{B}_s\,ds + g_t - k_t, \\ k_t &= -\int_0^t \nabla\phi(x_s)\,d|k|_s, \quad |k|_t = \int_0^t 1_{\{x_s\in\partial\mathcal{O}\}}\,d|k|_s, \quad t\in[0,T]. \end{aligned}$$

But, as $x^\mu \to x$ in $C([0,T];\mathbb{R}^d)$, it is easy to obtain from (3.8), (3.10), and the continuity of $b(t,\cdot)$, that $b_\mu(\cdot,x^\mu_\cdot) \to b(\cdot,x_\cdot)$ in $L^2(0,T;\mathbb{R}^d)$. Thus, $\mathcal{B} = b(\cdot,x_\cdot)$, and (x,k) is a solution of (3.1)-(3.2). □

In the proof of Theorem 3 we have seen that, under the conditions of the theorem, if b is also Lipschitz in x, then the solution of (3.1)-(3.2) is unique. In fact, we have the following result

Theorem 4. *Under the conditions of Theorem* 3*, suppose that there exists* $L_{b_x}\in\mathbb{R}$ *such that for all* $t\in[0,T]$ *and all* $x,x'\in\bar{\mathcal{O}}$*,*

$$(x-x', b(t,x)-b(t,x')) \le L_{b_x}|x-x'|^2.$$

Then, for each $x_0\in\bar{\mathcal{O}}$ *and* $g\in C([0,T];\mathbb{R}^d)$ *given such that* $g_0=0$*, there exists a unique solution* (x,k) *of the problem* (3.1)-(3.2)*.*

Proof. Because of Theorem 3, we only have to check uniqueness. Let (x,k) and (x',k') two solutions of (3.1)-(3.2) corresponding to the same x_0 and g. Then, for all $t\in[0,T]$

$$x_t - x'_t = \int_0^t (b(s,x_s)-b(s,x'_s))\,ds - k_t + k'_t,$$

and consequently,

$$\begin{aligned} & \exp\{-C_0(|k|_t+|k'|_t)\}|x_t-x'_t|^2 \\ = \quad & -C_0\int_0^t \exp\{-C_0(|k|_s+|k'|_s)\}|x_s-x'_s|^2\,(d|k|_s+d|k'|_s) \\ & +2\int_0^t \exp\{-C_0(|k|_s+|k'|_s)\}(x_s-x'_s, b(s,x_s)-b(s,x'_s))\,ds \\ & +2\int_0^t \exp\{-C_0(|k|_s+|k'|_s)\}(x_s-x'_s, \nabla\phi(x_s))\,d|k|_s \\ & -2\int_0^t \exp\{-C_0(|k|_s+|k'|_s)\}(x_s-x'_s, \nabla\phi(x'_s))\,d|k'|_s. \end{aligned} \tag{3.11}$$

It is easy to see that, by (2.1), (3.2), and the hypotheses on b, we obtain from (3.11)

$$\exp\{-C_0(|k|_t+|k'|_t)\}|x_t-x'_t|^2 \leq 2|L_{b_x}|\int_0^t \exp\{-C_0(|k|_s+|k'|_s)\}|x_s-x'_s|^2\,ds$$

for all $t \in [0,T]$, and thus, from Gronwall's lemma, we obtain the claimed result. $\square$

Remark 3. Consider the hypotheses of Theorem 4, and the sequence b_n given by (3.7). Denote by (x^n, k^n) the unique solution of the problem

$$\begin{aligned} x_t^n &= x_0 + \int_0^t b_n(s, x_s^n)\,ds + g_t - k_t^n, \\ k_t^n &= -\int_0^t \nabla\phi(x_s^n)\,d|k^n|_s, \quad |k^n|_t = \int_0^t 1_{\{x_s^n \in \partial\mathcal{O}\}}\,d|k^n|_s, \quad t \in [0,T]. \end{aligned}$$

Then, reasoning as in Step 2 of the proof of Theorem 3, and by the uniqueness of the solution (x,k) of (3.1)-(3.2), we can assert that all the sequence x^n converges to x in $C([0,T];\mathbb{R}^d)$.

4 Proof of Theorem 1

For the proof, we will proceed in two steps.

Step 1 Let σ be independent of x, i.e. $\sigma(\omega,t,x) = \sigma(\omega,t)$, a.s. for all $(t,x) \in [0,T] \times \bar{\mathcal{O}}$.

In this case, denote

$$M_t = \int_0^t \sigma(s)\,dW_s,$$

and observe that a pair (X,k) of $\{\mathcal{F}_t\}$-progressively measurable processes with values in $\mathbb{R}^d$ is a solution of (2.2)-(2.3) if and only if, a.s. $\omega \in \Omega$, $(X(\omega), k(\omega))$ is a solution of the problem

$$X_t(\omega) = x_0 + \int_0^t b(\omega, s, X_s(\omega))\,ds + M_t(\omega) - k_t(\omega), \tag{4.1}$$

$$k_t(\omega) = -\int_0^t \nabla\phi(X_s(\omega))d|k(\omega)|_s, \quad |k(\omega)|_t = \int_0^t 1_{\{X_s(\omega)\in\partial\mathcal{O}\}}d|k(\omega)|_s, \tag{4.2}$$

for all $t \in [0,T]$.

But, according to Theorem 4, for each $\omega \in \Omega$ there exists a unique solution $(X(\omega), k(\omega))$ of (4.1)-(4.2). Thus, in order to prove that the random pair (X,k) defined by (4.1)-(4.2) is the unique strong solution of (2.2)-(2.3), we must only see that X (and so k) is $\{\mathcal{F}_t\}$-progressively measurable. To this end, observe that, by Theorem 3.1 and Remark 3.3 in Lions and Sznitman[(4)], the existence of strong solution to (2.2)-(2.3) is guaranteed if b is also Lipschitz in x. Consequently, if we fix a regularizing sequence $\{\rho_n\} \subset \mathcal{D}(\mathbb{R}^d)$ and define for $(t,x) \in [0,T] \times \mathbb{R}^d$,

$$b_n(\omega,t,x) = \int_{\mathbb{R}^d} \rho_n(y)\tilde{b}(\omega,t,x-y)\,dy, \qquad a.s.,$$

with $\tilde{b}$ the extension of b whose existence is observed in Remark 1, we obtain a sequence (X^n,k^n) of $\{\mathcal{F}_t\}$-progressively measurable processes such that a.s. they are solutions of

$$X_t^n(\omega) = x_0 + \int_0^t b_n(\omega,s,X_s^n(\omega))\,ds + M_t(\omega) - k_t^n(\omega),$$

$$k_t^n(\omega) = -\int_0^t \nabla\phi(X_s^n(\omega))\,d|k^n(\omega)|_s, \quad |k^n(\omega)|_t = \int_0^t 1_{\{X_s^n(\omega)\in\partial\mathcal{O}\}}\,d|k^n(\omega)|_s,$$

for all $t \in [0,T]$.

Moreover, by Remark 3, a.s. $X^n(\omega)$ converges to $X(\omega)$ in $C([0,T];\mathbb{R}^d)$. Thus, in particular, X (and therefore, k) is $\{\mathcal{F}_t\}$-progressively measurable.

Step 2 In the conditions of Theorem 1.

We proceed in a similar way to the proof of Lemma 3.1 in Lions and Sznitman[(4)]. Denote by $L^4_{\mathcal{F}_t}(\Omega;C([0,T];\mathbb{R}^d))$ the space of the elements of $L^4(\Omega;C([0,T];\mathbb{R}^d))$ that are $\{\mathcal{F}_t\}$-progressively measurable. Then, the space $L^4_{\mathcal{F}_t}(\Omega;C([0,T];\mathbb{R}^d))$ is a Banach subspace of $L^4(\Omega;C([0,T];\mathbb{R}^d))$.

Consider the mapping

$$\hat{F} : L^4_{\mathcal{F}_t}(\Omega;C([0,T];\mathbb{R}^d)) \to L^4_{\mathcal{F}_t}(\Omega;C([0,T];\mathbb{R}^d))$$

that to each $Y \in L^4_{\mathcal{F}_t}(\Omega;C([0,T];\mathbb{R}^d))$ associates $\hat{F}(Y) = X$, with (X,k) the strong solution of

$$X_t = x_0 + \int_0^t b(s,X_s)\,ds + \int_0^t \sigma(s,Y_s)\,dW_s - k_t, \tag{4.3}$$

$$k_t = -\int_0^t \nabla\phi(X_s)\,d|k|_s, \quad |k|_t = \int_0^t 1_{\{X_s\in\partial\mathcal{O}\}}\,d|k|_s, \quad t \in [0,T], \tag{4.4}$$

whose existence and uniqueness is guaranteed by Step 1. Observe that, as $X_t \in \bar{\mathcal{O}}$, and $\mathcal{O}$ is bounded, we have that, of course, $X \in L^4_{\mathcal{F}_t}(\Omega;C([0,T];\mathbb{R}^d))$.

It is easy to see that (X,k) is a strong solution of (2.2)-(2.3) if and only if $\hat{F}(X)=X$. Consequently, to finish the proof, it is enough to prove that $\hat{F}$ has a unique fixed point.

Let Y and Y' be two processes in $L^4_{\mathcal{F}_t}(\Omega;C([0,T];\mathbb{R}^d))$, and denote $\hat{F}(Y)=X$, $\hat{F}(Y')=X'$. Then, applying Itô's formula to

$$\exp\{-C_0(\phi(X_t)+\phi(X'_t))\}|X_t-X'_t|^2,$$

we get a.s. for all $t\in[0,T]$:

$$\begin{aligned}
&\exp\{-C_0(\phi(X_t)+\phi(X'_t))\}|X_t-X'_t|^2\\
= \ & 2\int_0^t \exp\{-C_0(\phi(X_s)+\phi(X'_s))\}\big[(X_s-X'_s)^*(\sigma(s,Y_s)-\sigma(s,Y'_s))\,dW_s\\
&\quad +(X_s-X'_s,b(s,X_s)-b(s,X'_s))\,ds\\
&\quad +(X_s-X'_s,\nabla\phi(X_s))\,d|k|_s-(X_s-X'_s,\nabla\phi(X'_s))\,d|k'|_s\big]\\
+ \ & \int_0^t \exp\{-C_0(\phi(X_s)+\phi(X'_s))\}\|\sigma(s,Y_s)-\sigma(s,Y'_s)\|^2\,ds\\
- \ & C_0\int_0^t \exp\{-C_0(\phi(X_s)+\phi(X'_s))\}|X_s-X'_s|^2\\
&\qquad \times\big\{((\nabla\phi(X_s))^*\sigma(s,Y_s)+(\nabla\phi(X'_s))^*\sigma(s,Y'_s))\,dW_s\\
&\qquad +\frac{1}{2}\mathrm{tr}(D^2\phi(X_s)(\sigma\sigma^*)(s,Y_s)+D^2\phi(X'_s)(\sigma\sigma^*)(s,Y'_s))\,ds\\
&\qquad +[(\nabla\phi(X_s),b(s,X_s))+(\nabla\phi(X'_s),b(s,X'_s))]\,ds\\
&\qquad +|\nabla\phi(X_s)|^2\,d|k|_s+|\nabla\phi(X'_s)|^2\,d|k'|_s\big\}\\
+ \ & \frac{C_0^2}{2}\int_0^t \exp\{-C_0(\phi(X_s)+\phi(X'_s))\}|X_s-X'_s|^2\\
&\qquad \times|\sigma^*(s,Y_s)\nabla\phi(X_s)+\sigma^*(s,Y'_s)\nabla\phi(X'_s)|^2\,ds\\
- \ & 2C_0\int_0^t \exp\{-C_0(\phi(X_s)+\phi(X'_s))\}(X_s-X'_s)^*(\sigma(s,Y_s)-\sigma(s,Y'_s))\\
&\qquad \times(\sigma^*(s,Y_s)\nabla\phi(X_s)+\sigma^*(s,Y'_s)\nabla\phi(X'_s))\,ds. \qquad (4.5)
\end{aligned}$$

Since $|\nabla\phi(x)|=1$ for $x\in\partial\mathcal{O}$, by (2.1) and (4.4) we have, a.s. for all $t\in[0,T]$,

$$\begin{aligned}
&2\int_0^t \exp\{-C_0(\phi(X_s)+\phi(X'_s))\}(X_s-X'_s,\nabla\phi(X_s))\,d|k|_s\\
&-C_0\int_0^t \exp\{-C_0(\phi(X_s)+\phi(X'_s))\}|X_s-X'_s|^2|\nabla\phi(X_s)|^2\,d|k|_s\le 0,(4.6)
\end{aligned}$$

and, analogously,

$$-2\int_0^t \exp\{-C_0(\phi(X_s)+\phi(X'_s))\}(X_s - X'_s, \nabla\phi(X'_s))\, d|k'|_s$$
$$-C_0\int_0^t \exp\{-C_0(\phi(X_s)+\phi(X'_s))\}|X_s - X'_s|^2|\nabla\phi(X'_s)|^2\, d|k'|_s \le 0. \quad (4.7)$$

Using inequalities (4.6) and (4.7) in (4.5), reasoning as in the proof of Lemma 3.1 in Lions and Sznitman[(4)], and in particular using Doob's inequality, the boundeness of the exponential term, b, σ, ϕ, $\nabla\phi$ and $D^2\phi$, and the condition (iv) on b and σ, it is not difficult to see that there exists a constant $C > 0$, depending only on C_0, $\mathcal{O}$, b, σ and ϕ, such that for all $t \in [0, T]$,

$$E(\sup_{0\le s\le t}|X_s - X'_s|^4) \le C\left(\int_0^t E(|X_s - X'_s|^4)\, ds + \int_0^t E(|Y_s - Y'_s|^4) ds + \int_0^t E(|X_s - X'_s|^2|Y_s - Y'_s|^2) ds\right).$$

Now, by Young's inequality, introducing sup in the integrals, and using Gronwall's lemma, it is easy to see that

$$E(\sup_{0\le s\le t}|X_s - X'_s|^4) \le 2C\exp(2CT)\int_0^t E(\sup_{0\le r\le s}|Y_r - Y'_r|^4)\, ds, \qquad (4.8)$$

for all $t \in [0, T]$.

It is a standard matter to prove that, by (4.8), a power of $\hat{F}$ is a contraction in $L^4_{\mathcal{F}_t}(\Omega; C([0, T]; \mathbb{R}^d))$, and, thus, there exists a unique fixed point of $\hat{F}$.

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