Solutions of the 3D Navier–Stokes equations for initial data in $\dot{H}^{1/2}$: robustness of regularity and numerical verification of regularity for bounded sets of initial data in $\dot{H}^1$

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Abstract

We consider the three-dimensional Navier–Stokes equations on a periodic domain. We give a simple proof of the local existence of solutions in $\dot{H}^{1/2}$, and show that the existence of a regular solution on a bounded time interval $[0,T]$ is stable with respect to perturbations of the initial data in $\dot{H}^{1/2}$ and perturbations of the forcing function in $L^2(0,T;H^{-1/2})$. This forms the key ingredient in a proof that the *assumption* of regularity for all initial conditions in any given ball in $\dot{H}^1$ can be verified computationally in a finite time, strengthening a previous result of Robinson & Sadowski (*Asymptotic Analysis* **59** (2008) 39–50).

Keywords: Navier–Stokes equations, critical spaces, regularity

1. Introduction

The existence and uniqueness of regular solutions for the three-dimensional incompressible Navier–Stokes equations

$$u_t - \Delta u + (u \cdot \nabla)u + \nabla p = 0, \qquad \nabla \cdot u = 0 \tag{1}$$

is a long-standing and well known open problem.

Much research recently has focused on the question of existence of solutions in critical spaces, i.e. those in which the norm is invariant under the rescaling

$$u(x,t) \mapsto u_\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t),$$

since for the equation on the whole of $\mathbb{R}^3$, $u_\lambda(x,t)$ still solves (1) whenever $u(x,t)$ does. The Lebesgue space $L^3(\mathbb{R}^3)$ is a critical space, as is $\dot{H}^{1/2}(\mathbb{R}^3)$ [the space whose norm is given by $\int_{\mathbb{R}^3} |k||\hat{u}(k)|^2\,\mathrm{d}k$]. Generally, one has global existence for small data in such spaces (L^3 in [18]; $H^{1/2}$ in [5, 12]; BMO^{-1} in [16]) and in some cases local existence for all data (L^3 in [18]; $H^{1/2}$ in [6, 12]); the book by Lemarié-Rieusset [17] treats this subject in some detail. But there are some negative results, showing that arbitrarily small initial data can produce arbitrarily large solutions in an arbitrarily short time in the spaces $\dot{B}^{-1,\infty}_{\infty}$ [3] and in $B^{-1}_{\infty,\infty} \cap \dot{H}^{\alpha}$ for any $\alpha < 1/2$ [8].

In this paper we consider various problems connected with the local existence of solutions for (possibly large) data in $\dot{H}^{1/2}(Q)$, where $Q = [0, 2\pi]^3$ is a periodic domain in $\mathbb{R}^3$ and the dot denotes zero average over Q. To begin with we present a simplified version of the local existence argument from [6], which forms the basis of our subsequent analysis. We then show that the property of a solution belonging to $L^\infty(0,T;\dot{H}^{1/2}) \cap L^2(0,T;\dot{H}^{3/2})$ is 'stable' with respect to perturbations of the initial condition in $\dot{H}^{1/2}$, and of the forcing function in $L^2(0,T;H^{-1/2})$.

We then use this stability result as part of a proof that it is possible to 'verify numerically' the statement that every initial condition in $\overline{B}_{\dot{H}^1}(0,R)$ gives rise to a strong solution that exists for all $t \ge 0$: we give an explicit algorithm that will verify this statement, if true, in a finite time. This generalises and clarifies an earlier result of Robinson & Sadowski [20].

Throughout the paper by a 'solution' we mean a 'Leray–Hopf solution', i.e. a weak solution that satisfies the energy inequality. By a result of Serrin [22] (see also Galdi [14]), if u is a weak solution with $u \in L^r(0,T;L^s)$ with $3/s + 2/r = 1$ and $3 \le s \le \infty$ then it is unique in the class of Leray–Hopf solutions. In particular, if $u \in L^4(0,T;L^6)$ then u is unique; this will imply that the solutions we obtain in Theorem 1 are unique.

2. Preliminaries

We study solutions of (1) using periodic boundary conditions on the cubic domain $Q = [0, 2\pi]^3$, and enforce zero total momentum, i.e. $\int_Q u = 0$.

We write $\dot{\mathbb{Z}}^3 = \mathbb{Z}^3 \setminus \{0,0,0\}$, let $\dot{H}^s$ be the subspace of the Sobolev space H^s consisting of divergence-free, zero-average, periodic real functions,

$$\dot{H}^s = \left\{ u = \sum_{k \in \dot{\mathbb{Z}}^3} \hat{u}_k \mathrm{e}^{\mathrm{i}k\cdot x} : \ \hat{u}_k = \overline{\hat{u}_{-k}},\ \sum |k|^{2s} |\hat{u}_k|^2 < \infty,\ k \cdot \hat{u}_k = 0 \right\},$$

and equip $\dot{H}^s$ with the norm

$$\|u\|_s^2 = \sum |k|^{2s} |\hat{u}_k|^2.$$

We write $\dot{L}^2$ for $\dot{H}^0$, and $\|u\|$ for $\|u\|_0$.

Let Π be the orthogonal projection from L^2 onto $\dot{L}^2$ (divergence-free L^2), and denote by A the Stokes operator on Q, that is

$$A = -\Pi\Delta.$$

In the periodic case $Au = -\Delta\Pi u$, so $Au = -\Delta u$ for $u \in \dot{H}^s$. We make continual use of the equivalence of the norms $\|u\|_s = \|u\|_{\dot{H}^s}$ and $\|A^{s/2}u\|$ for $u \in \dot{H}^s = D(A^{s/2})$, and denote by $H^{-1/2}$ the dual space of $\dot{H}^{1/2}$. We denote by $\{\lambda_j\}_{j=1}^\infty$ the (positive) eigenvalues of A, arranged with $\lambda_{n+1} \geq \lambda_n$; these correspond to orthonormal eigenfunctions $\{w_j\}_{j=1}^\infty$, and we denote by P_n the orthogonal projection onto the span of the first n of these eigenfunctions,

$$P_n u := \sum_{j=1}^{n} (u, w_j) w_j. \tag{2}$$

We denote by $B(u,u)$ the bilinear form defined by

$$B(u,u) = \Pi[(u \cdot \nabla)u].$$

The Navier–Stokes equations can then be written as

$$u_t + Au + B(u,u) = 0.$$

For further details see [9], or Chapter 2 of [6].

From here on, c denotes an absolute constant which may change from line to line.

3. Local existence for initial data in $\dot{H}^{1/2}$

In this section we follow the argument in [6], giving a simplified version of their proof of the local existence of strong solutions for initial data in $\dot{H}^{1/2}$ (their Theorem 3.4). While the result guaranteeing global existence for small data is well known [5, 12], this proof of the local existence result based on relatively simple energy estimates appears to be much less familiar. It is striking that the local existence time depends only on properties of solutions of the heat equation (see (4)); such results hold in more general critical spaces, see for example the nice review article by Cannone [4]. (While the operator appearing in (3) is the Stokes operator A, note that the initial condition u_0 is assumed to be divergence free; as remarked in the previous Section, $Au = -\Delta u$ when $u \in \dot{H}^s$, so (3) is indeed the heat equation.)

Theorem 1. *There exists an absolute constant $\varepsilon > 0$ such that if $u_0 \in \dot{H}^{1/2}$ and $v(t)$ is the solution of the heat equation*

$$v_t + Av = 0 \qquad v(0) = u_0, \tag{3}$$

whenever

$$\int_0^{T^*} \|v(s)\|_1^4 \,\mathrm{d}s < \varepsilon \tag{4}$$

the equation

$$u_t + Au + B(u,u) = 0 \qquad u(0) = u_0$$

has a (unique) solution $u \in L^\infty(0,T^*;\dot H^{1/2}) \cap L^2(0,T^*;\dot H^{3/2})$.

Proof. Consider the Galerkin approximants of u, i.e. the solutions u_N of

$$(u_N)_t + Au_N + P_N B(u_N,u_N) = 0.$$

Standard arguments (see [9], [19]) guarantee that a subsequence of the u_N converges to a function $u \in L^\infty(0,T;L^2) \cap L^2(0,T;\dot H^1)$ that is a weak solution of the Navier–Stokes equations. To show that when $u_0 \in \dot H^{1/2}$ we in fact have $u \in L^\infty(0,T^*;\dot H^{1/2}) \cap L^2(0,T^*;\dot H^{3/2})$ we make some further estimates on the Galerkin approximants in order to show that u_N (for N sufficiently large) is uniformly bounded in the appropriate spaces for some $T^* > 0$.

Decompose u_N as $v + w_N$, where v and w_N are the solutions of

$$v_t + Av = 0, \qquad v(0) = u_0,$$

and

$$(w_N)_t + Aw_N + P_N B(u_N,u_N) = 0, \qquad w_N(0) = 0.$$

Then, taking the inner product of the v equation with $A^{1/2}v$ we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v\|_{1/2}^2 + \|v\|_{3/2}^2 \le 0,$$

whence

$$\frac{1}{2}\|v(t)\|_{1/2}^2 + \int_0^t \|v(s)\|_{3/2}^2\,\mathrm{d}s \le \frac{1}{2}\|u_0\|_{1/2}^2,$$

and so in particular $\|v(\cdot)\|_{3/2}^2$ is integrable.

From the equation for w_N, taking the inner product with $A^{1/2}w_N$ and using the inequality

$$|(B(u,v),A^{1/2}w)| \le c\|u\|_{L^6}\|Dv\|_{L^2}\|A^{1/2}w\|_{L^3} \le c\|u\|_1\|v\|_1\|w\|_{3/2} \qquad (5)$$

yields

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w_N\|_{1/2}^2 + \|w_N\|_{3/2}^2 &\le c\|u_N\|_1^2\|w_N\|_{3/2}\\ &\le c(\|v\|_1^2 + \|w_N\|_1^2)\|w_N\|_{3/2}.\end{aligned}$$

Interpolate $\|w_N\|_1^2 \le \|w_N\|_{1/2}\|w_N\|_{3/2}$ and use Young's inequality to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|w_N\|_{1/2}^2 + \|w_N\|_{3/2}^2 \le c\|w_N\|_{1/2}\|w_N\|_{3/2}^2 + c\|v\|_1^4.$$

Now integrate:

$$\begin{aligned}&\|w_N(t)\|_{1/2}^2 + \int_0^t \|w_N(s)\|_{3/2}^2\,\mathrm{d}s\\ &\qquad \le c\left\{\int_0^t \|w_N(s)\|_{3/2}^2\,\mathrm{d}s\right\}\left\{\sup_{0\le s\le t}\|w_N(s)\|_{1/2}\right\} + c\int_0^t \|v(s)\|_1^4\,\mathrm{d}s.\end{aligned}$$

And so, after using Young's inequality,

$$\sup_{0\le s\le t}\|w_N(s)\|_{1/2}^2+\int_0^t\|w_N(s)\|_{3/2}^2\,\mathrm{d}s\le\frac{1}{2}\sup_{0\le s\le t}\|w_N(s)\|_{1/2}^2 +\frac{k}{2}\left(\int_0^t\|w_N(s)\|_{3/2}^2\,\mathrm{d}s\right)^2+c\int_0^t\|v(s)\|_1^4\,\mathrm{d}s,$$

whence

$$\frac{1}{2}\sup_{0\le s\le t}\|w_N(s)\|_{1/2}^2+\int_0^t\|w_N(s)\|_{3/2}^2\,\mathrm{d}s \le\frac{k}{2}\left(\int_0^t\|w_N(s)\|_{3/2}^2\,\mathrm{d}s\right)^2+c\int_0^t\|v(s)\|_1^4\,\mathrm{d}s,\tag{6}$$

so

$$\sup_{0\le s\le t}\|w_N(s)\|_{1/2}^2+2\int_0^t\|w_N(s)\|_{3/2}^2\,\mathrm{d}s \le k\left(\int_0^t\|w_N(s)\|_{3/2}^2\,\mathrm{d}s\right)^2+2c\int_0^t\|v(s)\|_1^4\,\mathrm{d}s.\tag{7}$$

Now for each N sufficiently large let us set

$$T_N=\sup\left\{T\ge 0:\ \int_0^T\|w_N(s)\|_{3/2}^2\,\mathrm{d}s\le\frac{1}{k}\right\},$$

so that for all $t\in[0,T_N]$

$$\int_0^t\|w_N(s)\|_{3/2}^2\,\mathrm{d}s\le 2c\int_0^t\|v(s)\|_1^4\,\mathrm{d}s.$$

Choose T^* sufficiently small that

$$2c\int_0^{T^*}\|v(s)\|_1^4\,\mathrm{d}s\le\frac{1}{2k},\tag{8}$$

which is possible since $v\in L^2(0,T;\dot H^{3/2})$; it follows that $T_N\ge T^*$ for all N.

We therefore obtain a uniform bound on w_N, and hence on u_N, in the space $L^2(0,T^*;\dot H^{3/2})$. This in turn (via (7)) provides a uniform bound on u_N in $L^\infty(0,T^*;\dot H^{1/2})$. These limits are preserved as we let $N\to\infty$, so that the limit satisfies $u\in L^\infty(0,T^*;\dot H^{1/2})\cap L^2(0,T^*;\dot H^{3/2})$.

To prove uniqueness of solutions, we use the interpolation

$$\|u\|_1^4\le\|u\|_{1/2}^2\|u\|_{3/2}^2,\tag{9}$$

which implies that $u\in L^4(0,T^*;\dot H^1)$. Since it follows that $u\in L^4(0,T;L^6)$ we can use the uniqueness criterion of Serrin [22] to deduce that u is unique (in the class of Leray–Hopf weak solutions). □

We now show that solutions are more regular if the initial condition and forcing allow.

Theorem 2. *Let u be the solution of the Navier–Stokes equations with an initial condition $u_0 \in \dot{H}^1$ and the external forcing $f \in L^2(0,T;L^2)$. If*

$$u \in L^\infty(0,T;\dot{H}^{1/2}) \cap L^2(0,T;\dot{H}^{3/2})$$

then u is the strong solution of the Navier–Stokes equations:

$$u \in L^\infty(0,T;\dot{H}^1) \cap L^2(0,T;\dot{H}^2).$$

Proof. Take the inner product of the Navier–Stokes equations with Au and estimate

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_1^2 + \|u\|_2^2 &= -(B(u,u),Au) + (f,Au)\\ &\le \|u\|_{L^6}\|\nabla u\|_{L^3}\|Au\|_{L^2} + \|f\|\|Au\|\\ &\le c\|u\|_1\|\nabla u\|_{L^2}^{1/2}\|\nabla u\|_{L^6}^{1/2}\|u\|_2 + \|f\|\|Au\|\\ &\le c\|u\|_1^{3/2}\|u\|_2^{3/2} + \|f\|\|Au\|,\end{aligned}$$

whence (using Young's inequality)

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_1^2 + \|u\|_2^2 \le c\|u\|_1^6 + \|f\|^2. \tag{10}$$

Dropping the $\|u\|_2^2$ term and integrating yields

$$\|u(t)\|_1^2 \le \left[\|u_0\|_1^2 + \int_0^t \|f\|^2\right]\exp\left(\int_0^t c\|u(s)\|_1^4\,\mathrm{d}s\right),$$

From the fact that $u \in L^\infty(0,T;\dot{H}^1) \cap L^2(0,T;\dot{H}^2)$ it follows by interpolation (as in (9)) that $u \in L^4(0,T;\dot{H}^1)$ and hence $u \in L^\infty(0,T;\dot{H}^1)$. The bound in $L^2(0,T;\dot{H}^2)$ follows by integrating (10) a second time, retaining the $\|u\|_2^2$. □

4. Stability of local existence in $\dot{H}^{1/2}$

We now show that the property of local existence is stable under perturbations to the initial data in $\dot{H}^{1/2}$ (cf. [13]). This is a simplified version of Theorem 3.6 in [6] for a finite time interval, and in a form particularly suited to the application in the following section. For more general stability results in larger critical spaces see [15]. Note that the constant c in the theorem depends only on the constant c in (5), which in turn depends only on the constants in various Sobolev embedding results, and which can be determined explicitly.

Theorem 3. *There exists a constant $c > 0$ such that the following holds. Suppose that for a given $u_0 \in \dot{H}^{1/2}$ and $f \in L^2(0,T;H^{-1/2})$ there exists a solution $u \in L^\infty(0,T;\dot{H}^{1/2}) \cap L^2(0,T;\dot{H}^{3/2})$ of*

$$u_t + Au + B(u,u) = f(t), \qquad u(0) = u_0.$$

Then for every $v_0 \in \dot{H}^{1/2}$ and $g \in L^2(0,T;H^{-1/2})$ with

$$\|u_0 - v_0\|_{1/2} + \int_0^T \|f(t) - g(t)\|_{-1/2}^2 \, \mathrm{d}t < c \exp\left\{-c \int_0^T \|u(s)\|_1^4 \, \mathrm{d}s\right\}, \tag{11}$$

the solution v of

$$v_t + Av + B(v,v) = g(t), \qquad v(0) = v_0,$$

exists on $[0,T]$ with $v \in L^\infty(0,T;\dot{H}^{1/2}) \cap L^2(0,T;\dot{H}^{3/2})$ and satisfies

$$\sup_{0\le t\le T} \|u(t) - v(t)\|_{1/2}^2 + \int_0^T \|u(s) - v(s)\|_{3/2}^2 \le c. \tag{12}$$

Proof. If $w = u - v$ then

$$\frac{\mathrm{d}w}{\mathrm{d}t} + Aw + B(u,w) + B(w,u) - B(w,w) = f - g.$$

Take the inner product with $A^{1/2}w$:

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w\|_{1/2}^2 + \|w\|_{3/2}^2 \le |(B(u,w),A^{1/2}w)| + |(B(w,u),A^{1/2}w)| \\ + |(B(w,w),A^{1/2}w)| + \|f-g\|_{-1/2}\|w\|_{3/2}\end{aligned}$$

and use (5) again to obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w\|_{1/2}^2 + \|w\|_{3/2}^2 \le 2c\|u\|_1\|w\|_1\|w\|_{3/2} + c\|w\|_1^2\|w\|_{3/2} + \|f-g\|_{-1/2}\|w\|_{3/2}.$$

Therefore

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|w\|_{1/2}^2 + \|w\|_{3/2}^2 &\le \|f-g\|_{-1/2}^2 + c\|u\|_1^2\|w\|_1^2 + c\|w\|_1^4 \\ &\le \|f-g\|_{-1/2}^2 + c\|u\|_1^2\|w\|_{1/2}\|w\|_{3/2} + c\|w\|_{1/2}^2\|w\|_{3/2}^2,\end{aligned}$$

and so finally

$$\frac{\mathrm{d}}{\mathrm{d}t}\|w\|_{1/2}^2 + \frac{1}{2}\|w\|_{3/2}^2 \le \|f-g\|_{-1/2}^2 + c\|u\|_1^4\|w\|_{1/2}^2 + c\|w\|_{1/2}^2\|w\|_{3/2}^2.$$

Set

$$E(t) = \exp\left\{-c\int_0^t \|u(s)\|_1^4 \, \mathrm{d}s\right\};$$

then

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{E(t)\|w(t)\|_{1/2}^2\right\} + \frac{1}{2}\|w\|_{3/2}^2 E(t) \le \|f-g\|_{-1/2}^2 E(t) + c\|w\|_{1/2}^2\|w\|_{3/2}^2 E(t),$$

and so

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{E(t)\|w(t)\|_{1/2}^2\right\} + \|w\|_{3/2}^2 E(t)\left\{\frac{1}{2} - c\|w(t)\|_{1/2}^2\right\} \le \|f-g\|_{-1/2}^2 E(t).$$

Thus while $E(t)\|w\|_{1/2}^2 \leq cE(t)/4$ (i.e. while $\|w(t)\|_{1/2}^2 \leq c/4$) we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{E(t)\|w\|_{1/2}^2\right\} + \frac{1}{4}E(t)\|w(t)\|_{3/2}^2 \leq \|f-g\|_{-1/2}^2 E(t),$$

and so while $\|w\|_{1/2}^2 \leq c/4$,

$$E(t)\|w(t)\|_{1/2}^2 + \frac{1}{4}\int_0^t \|w(s)\|_{3/2}^2 E(s)\,\mathrm{d}s \leq \|w(0)\|_{1/2}^2 + \int_0^t \|f-g\|_{-1/2}^2 E(s)\,\mathrm{d}s,$$

which yields

$$\|w(t)\|_{1/2}^2 + \frac{1}{4}\int_0^t \|w(s)\|_{3/2}^2\,\mathrm{d}s \leq \frac{1}{E(t)}\|w(0)\|_{1/2}^2 + \frac{1}{E(t)}\int_0^t \|f-g\|_{-1/2}^2\,\mathrm{d}s.$$

On the time interval $[0,T]$ where $u \in L^\infty(0,T;\dot{H}^{1/2}) \cap L^2(0,T;\dot{H}^{3/2})$,

$$\frac{1}{E(t)} \leq \frac{1}{E(T)} < \infty.$$

Now choose $\|w_0\|_{1/2}$ and $\int_0^T \|f-g\|_{-1/2}^2\,\mathrm{d}s$ sufficiently small to guarantee that

$$\|w(t)\|_{1/2}^2 + \frac{1}{4}\int_0^t \|w(s)\|_{3/2}^2\,\mathrm{d}s \leq c/4$$

for all $t \in [0,T]$, i.e.

$$\|w_0\|_{1/2}^2 + \int_0^T \|f(s)-g(s)\|_{-1/2}^2\,\mathrm{d}s < \frac{c}{4}E(T) = \frac{c}{4}\exp\left\{-c\int_0^T \|u(s)\|_1^4\,\mathrm{d}s\right\}.$$

Then w remains bounded in $L^\infty(0,T;\dot{H}^{1/2}) \cap L^2(0,T;\dot{H}^{3/2})$, i.e. v is bounded in the same space and the theorem follows. □

5. Numerical verification of regularity

Our aim in this section is to show that one can 'verify numerically' the existence of global regular solutions of the unforced Navier–Stokes equations

$$u_t + Au + B(u,u) = 0 \qquad u(0) = u_0 \tag{13}$$

for every initial condition $u_0 \in \overline{B}_1(0,R)$. (We use the notation $B_s(0,R)$ to denote the open ball of radius R, centred at 0, in the space $\dot{H}^s$; $\overline{B}_s(0,R)$ denotes its closure.)

Definition 4. *We will say that a property Q can be* verified numerically *if, assuming that Q holds, there is an explicit numerical algorithm that will verify the veracity of Q in a finite time.*

Note that the definition is clearly one of theory rather than practice. Of course, if one could 'numerically verify' both Q and 'not Q' then this would be very powerful (even in theory).

5.1. Numerical verification for a single initial condition

Given a single, fixed, initial condition $u_0 \in \dot{H}^1$, one can ask if it is possible to 'verify numerically' the fact that u_0 gives rise to a strong solution on $[0,T]$. To answer this question we will use an idea due to Chernyshenko (see [7], where one can find a similar result in $\dot{H}^s$ for $s \geq 3$; a result in $\dot{H}^1$ can be found in [11]) and consider Galerkin approximations of the nth order:

$$\frac{\mathrm{d}u_n}{\mathrm{d}t} + Au_n + P_n B(u_n, u_n) = 0 \qquad u_n(0) = P_n u_0,$$

where P_n is the projection operator defined in (2). The above equation can also be written in the form

$$\frac{\mathrm{d}u_n}{\mathrm{d}t} + Au_n + B(u_n, u_n) = Q_n B(u_n, u_n) \qquad u_n(0) = P_n u_0,$$

where $Q_n = I - P_n$. Hence u_n is an exact solution of the Navier–Stokes equations with the force $g = Q_n B(u_n, u_n)$. Therefore the following *a posteriori* test for regularity is an easy consequence of the robustness result of Theorem 3 and Theorem 2. It is important to notice that the absolute constant c in (5) depends only on the domain of the flow and on the constants from the Sobolev embedding theorem, and can be computed explicitly for any domain $Q = [0, L]^3$.

Theorem 5. *Let u_n be a Galerkin approximation of the solution u arising from an initial condition $u_0 \in \dot{H}^1$ and let c be a constant from Theorem 3. If*

$$\begin{aligned} \|u_0 - u_n(0)\|_{1/2} + \int_0^T \|Q_n B(u_n, u_n)\|^2_{-1/2}\, \mathrm{d}s & \\ < \varrho(v) := c \exp\left(-c \int_0^T \|u_n(s)\|_1^4\, \mathrm{d}s \right) & \end{aligned} \tag{14}$$

then the solution of

$$u_t + Au + B(u, u) = 0 \tag{15}$$

exists on $[0,T]$ with $u \in L^\infty(0,T;\dot{H}^1) \cap L^2(0,T;\dot{H}^2)$.

To make sure that the above condition can really be used to check regularity of the solution u we need to check that both terms on the left-hand side converge to zero as n tends to infinity and that the right-hand side is bounded below by the same constant for all n.

The lower bound on the right-hand side is a consequence of the following theorem of Dashti & Robinson [11] (again after [7]) which ensures the convergence of Galerkin approximations to strong solutions (i.e. if one assumes regularity then these approximations converge).

Theorem 6. *Suppose that $u_0 \in \dot{H}^1$ gives rise to a strong solution u of (15), i.e. $u \in L^\infty(0,T;\dot{H}^1) \cap L^2(0,T;\dot{H}^2)$. Then the solutions u_n of the Galerkin approximations converge strongly to u in both $L^\infty(0,T;\dot{H}^1)$ and in $L^2(0,T;\dot{H}^2)$.*

We now consider the left-hand side. The convergence of $\|u_0 - u_n(0)\|_{1/2}$ to zero is obvious, but the convergence of $\int_0^T \|Q_n B(u_n, u_n)\|_{-1/2}^2 \, \mathrm{d}s$ requires more argumentation, and is a consequence of the following (somewhat stronger) result.

Lemma 7. *Under the assumptions of Theorem 6 we have*

$$Q_n B(u_n, u_n) \to 0 \quad \textit{in} \quad L^2(0, T; L^2) \tag{16}$$

and in consequence

$$Q_n B(u_n, u_n) \to 0 \quad \textit{in} \quad L^2(0, T; H^{-1/2}). \tag{17}$$

Proof. First we show that $B(u_n, u_n) \to B(u, u)$ in $L^2(0, T; L^2)$. Indeed, setting $b_n = B(u_n, u_n)$ and $b = B(u, u)$ we have

$$\begin{aligned}
\int_0^T \|b_n - b\|^2 \leq & \int_0^T \int_Q |(u \cdot \nabla)u - (u_n \cdot \nabla)u_n|^2 \\
\leq & c \int_0^T \int_Q |u - u_n|^2 |Du|^2 + |u_n|^2 |Du_n - Du|^2 \\
\leq & c \int_0^T \|u - u_n\|_{L^\infty(Q)}^2 \|Du_n\|^2 + c \int_0^T \|u_n\|_{L^\infty(Q)}^2 \|Du_n - Du\|^2 \\
\leq & c \int_0^T \|Au - Au_n\|^2 \|Du_n\|^2 + c \int_0^T \|Au_n\|^2 \|Du_n - Du\|^2 \\
\leq & c \|u_n\|_{L^\infty(0,T;\dot{H}^1)}^2 \|u - u_n\|_{L^2(0,T;\dot{H}^2)}^2 \\
& \quad + c \|u_n - u\|_{L^\infty(0,T;\dot{H}^1)}^2 \|u_n\|_{L^2(0,T;\dot{H}^2)}^2.
\end{aligned}$$

From Theorem 6 follows that $B_n \to B$ in $L^2(0, T; L^2)$. Now observe that

$$\int_0^T \|Q_n b_n\|^2 = \int_0^T \|Q_n(b_n - b) + Q_n b\|^2 \leq 2 \int_0^T \|Q_n(b - b_n)\|^2 + 2 \int_0^T \|Q_n b\|^2.$$

We have

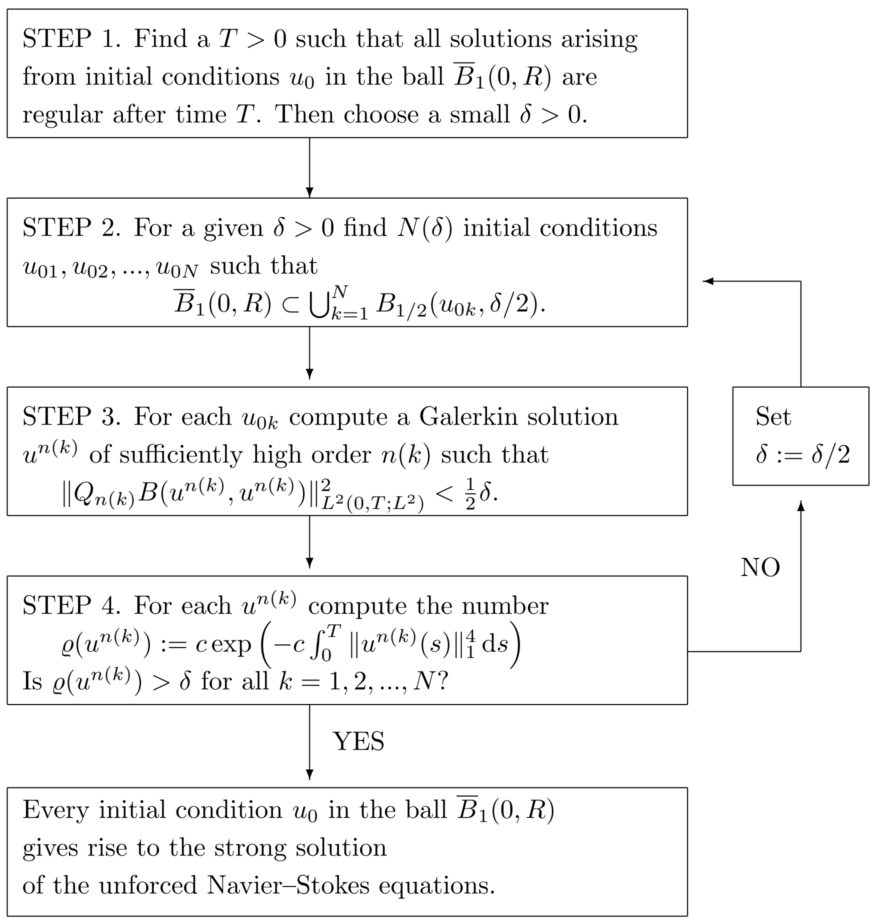
$$\int_0^T \|Q_n(b - b_n)\|^2 \leq \int_0^T \|(b - b_n)\|^2 \to 0.$$

Moreover, for all $s \in [0, T]$ we have $\|Q_n b(s)\|^2 \leq \|b(s)\|^2$ and since $b(s) \in L^2(Q)$ for almost all s we have $\|Q^n b(s)\| \to 0$ for almost all s. From the Lebesgue Dominated Convergence Theorem it follows that $\int_0^T \|Q_n b\|^2 \to 0$ and hence $\int_0^T \|Q_n b_n\|^2 \to 0$ as required. □

It is now clear that we can numerically verify regularity of the solution arising from a single initial condition in $\dot{H}^1$.

5.2. Numerical verification for a set of initial conditions

To prove that the 'numerical verification of regularity' for all initial data in a given ball in $\dot{H}^1$ is possible, it is enough to show that the following algorithm stops in a final time.



Step 1 can be easily done as was shown in Section 4 in [20].

Step 2 is also easy as the following lemma shows (the proof of this result follows exactly that of Lemma 5.5 in [20]).

Lemma 8. *Given $\delta > 0$ there exist N_δ and M_δ such that every $u_0 \in \overline{B}_1(0,R)$ can be approximated to within δ in the $\dot{H}^{1/2}$-norm by elements of the set*

$$U(\delta) := \{v_0 = \sum_{j=1}^{N_\delta} \alpha_j w_j, \ \textit{with } \alpha_j = a_j 2^{-M_\delta}, \ a_j \in \mathbb{Z}; \ \|v_0\|_1 \le R\},$$

where the $\{w_j\}$ are the eigenfunctions of the Stokes operator, arranged in 'increasing order' ($Aw_j = \lambda_j w_j$, $\lambda_{j+1} \geq \lambda_j$).

The fact that Step 3 can be done follows from Lemma 7.

Finally we need to show that the loop in the algorithm is not infinite and that - under the assumption that the hypothesis we verify is true - we must end up with sufficiently small δ for which the answer to the question in Step 4 is 'YES'. The arguments are quite delicate; the following result, after [10] and Theorem 12.10 in [23] (cf. Proposition 5.3) is crucial (the result in this form, along with a sketch of the proof, can be found in [20]).

Lemma 9. *Suppose that for every $u_0 \in B_1(0,R)$ there exists a strong solution $u \in L^\infty(0,T;H^1) \cap L^2(0,T;H^2)$ of the unforced Navier–Stokes equations. Then there exists an $M = M(R)$ such that for any such solution,*

$$\|u\|_{L^\infty(0,T;\dot H^1)} + \|u\|_{L^2(0,T;\dot H^2)} \leq M. \tag{18}$$

The following corollary is immediate, but the key deduction is the uniform lower bound on $\varrho(u)$ in (20).

Corollary 10. *Under the assumptions of Lemma 9 we also have*

$$\|u\|_{L^\infty(0,T;\dot H^{1/2})} \leq M, \quad \|u\|_{L^2(0,T;\dot H^{3/2})} \leq M, \tag{19}$$

and hence

$$\varrho(u) > c\mathrm{e}^{-cTM^4} := C_1 \tag{20}$$

Now we need to obtain a lower bound similar to (20) for all solutions v of the Navier–Stokes equations with initial condition $v_0 \in \bar B_1(0,R)$ and an additional non-zero force that is small in $L^2(0,T;L^2)$.

Lemma 11. *Let C_1 be the constant from (20) and assume that all initial conditions in the ball $\overline{B}_1(0,R)$ give rise to strong solutions of the unforced Navier–Stokes equations. Then for any $f \in L^2(0,T;L^2)$ with*

$$\int_0^T \|f(t)\|^2 \,\mathrm{d}t \leq C_1$$

and for any initial condition $v_0 \in \overline{B}(0,R_1)$ the Leray–Hopf solution of the Navier–Stokes equations with initial condition v_0 and the forcing f is strong: $v \in L^\infty(0,T;\dot H^1) \cap L^2(0,T;\dot H^2)$. Moreover, there exists a constant $C_2 > 0$ such that for all such solutions v we have

$$\varrho(v) > C_2. \tag{21}$$

Proof. From Lemma 9 it follows that for any solution u of the unforced Navier–Stokes equations arising from an initial condition $u_0 \in \overline{B}_1(0,R)$ we have

$$\varrho(u) := c\exp\left(-c\int_0^T \|u(s)\|_1^4 \,\mathrm{d}s\right) > C_1.$$

Then if $\|f\|^2_{L^2(0,T;L^2)} < C_1$ and $u_0 \in \overline{B}_1(0,R)$ we have

$$\|u_0 - v_0\|_{H^{1/2}} + \int_0^T \|f-g\|^2_{H^{-1/2}} < C_1 < \varrho(u)$$

for $v_0 = u_0$ and the force $g \equiv 0$. Thus the condition (11) is satisfied and so choosing $u_0 = v_0$ gives rise to a solution v of the Navier–Stokes equations with the forcing f such that $v \in L^\infty(0,T;\dot{H}^{1/2}) \cap L^2(0,T;\dot{H}^{3/2})$. Since $f \in L^2(0,T;L^2)$ it follows from Theorem 2 that we also have $v \in L^\infty(0,T;\dot{H}^1) \cap L^2(0,T;\dot{H}^2)$.

We will now prove the lower bound (21). From Corollary 10 it follows that for any solution u of the unforced Navier–Stokes equations

$$u_t + Au + B(u,u) = 0, \quad u(0) = u_0$$

where $u_0 \in \overline{B}_1(0,R)$ we have

$$\|u\|_{L^\infty(0,T;\dot{H}^{1/2})} \le M \quad \text{and} \quad \|u\|_{L^2(0,T;\dot{H}^{3/2})} \le M.$$

Let now v be the solution of

$$v_t + Av + B(v,v) = f, \quad v(0) = u_0$$

From the estimate (12) of Theorem 3 it follows that

$$\|v-u\|_{L^\infty(0,T;\dot{H}^{1/2})} + \|v-u\|_{L^2(0,T;\dot{H}^{3/2})} \le c.$$

Hence there exists a constant C_3 such that

$$\|v\|_{L^\infty(0,T;\dot{H}^{1/2})} + \|v\|_{L^2(0,T;\dot{H}^{3/2})} < C_3.$$

for all such solutions v. The lower bound (21) follows easily from interpolation of the norm in $\dot{H}^1$ between the norms $\dot{H}^{1/2}$ and $\dot{H}^{3/2}$. □

We emphasise that although neither the constant C_1 nor C_2 can be computed explicitly, we can still guarantee that our algorithm must terminate. Indeed, as $\delta \to 0$ we must have at some point $2\delta < \min(C_1, C_2)$. Once this happens we will obtain

$$\varrho(u^{n(k)}) > C_2 > \delta, \quad \text{for all } k = 1,2,...,N.$$

In consequence the answer to the question in Step 4 must be 'YES'. Therefore we proved the following theorem.

Theorem 12. *For any fixed $R > 0$, the following statement can be verified numerically: every initial condition $u_0 \in \overline{B}_1(0,R)$ gives rise to a unique solution u of (13) that for any $T > 0$ satisfies $u \in L^\infty(0,T;\dot{H}^1) \cap L^2(0,T;\dot{H}^2)$.*

Conclusion

We have presented a simple proof of local existence of solutions for initial data in $\dot{H}^{1/2}$, using the splitting of solutions into two parts, one satisfying the linear heat equation, and simple energy estimates. Such a splitting technique is standard, but is usually used in conjunction with the semigroup approach, as in [12] or [17]; for a proof of local existence in L^3 using splitting and energy estimates see [21].

The robustness of regularity for such solution has enabled us to give an explicit algorithm to verify the regularity of the equations for any ball of initial data in $\dot{H}^1$. Numerical implementation of this scheme for the Navier–Stokes system may well prove too expensive, but there are simpler systems which share many of the mathematical difficulties of the Navier–Stokes equations for which such methods may be more feasible: Blömker & Nolde are currently applying these ideas to a scalar model of surface growth (see [1, 2] for analytical studies) with encouraging preliminary results.

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