Pullback attractors in V for non-autonomous 2D-Navier-Stokes equations and their tempered behaviour

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Abstract

In this paper the asymptotic behaviour of the solutions to a non-autonomous 2D-Navier-Stokes model is analyzed when the initial datum belongs to V, for two frameworks: the universe of fixed bounded sets, and also for another universe given by a tempered condition. The existence of pullback attractors in these different universes is established, and thanks to regularity properties, the relation between these several families of attractors and the corresponding in H is successfully studied. Finally, two results about the tempered behaviour in V and $(H^2(\Omega))^2$ of the pullback attractors, when time goes to $-\infty$, are obtained.

Key words: 2D-Navier-Stokes equations, pullback attractors, tempered behaviour.
Mathematics Subject Classifications (2010): 35B41, 35B65, 35Q30

1 Introduction

The Navier-Stokes equations govern the motion of usual fluids like water, air, oil, etc. These equations have been the object of numerous works since the first paper of Leray was published in 1933 (e.g. cf. [8,17,26,11,16], and the references therein).

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On the one hand, the theory of attractors was initiated to deal with some open problems as the understanding of turbulence. Actually, many related items have been developed in the last decades with partial or total success, as determining modes and nodes, simplification to finite-dimensional dynamics, and also applied to general problems in dynamical systems.

On the other hand, the appearance of more complex and realistic models that aimed to deal with terms depending non-trivially on time involved substantial changes. While a first (and natural) approach was that of uniform attractors (e.g. cf. [4,5] and the references therein), other different approaches appeared to allow unbounded time-depending terms and processes, as random or stochastic models.

In particular, the theory of pullback attractors has been extensively developed in the last years in a vast range of problems (e.g. cf. [7,15]). This approach studies under minimal requirements not only the future of the dynamical system but what are the current attracting sections when the initial data come from $-\infty$.

Namely, it has been applied in many different situations as for instance those coming from chemical, physical, and biological motivations, and also for several models related to the Navier-Stokes system (e.g. cf. [10,9,23,13,20,22]).

Recent advances in the theory of non-autonomous dynamical systems include the consideration of universes of initial data changing in time (usually in terms of a tempered condition of growth), accordingly to the intrinsically non-autonomous model (e.g. cf. [6,2]).

However, many questions remained open in this direction, as for instance a proper comparison between pullback attractors in the classical sense and the so-called pullback $\mathcal{D}-$attractors (this problem was addressed in [21]), and pointing out the usefulness of the last concept when dealing with non-compact but only asymptotically compact processes.

The goal of this paper is to continue the analysis of some of these questions, and indeed we aim to address them with a non-autonomous $2D-$Navier-Stokes model. Namely, we will present a study on the regularity of the different families of pullback attractors, the relation among them, and their tempered behaviour in different norms.

The structure of the paper is as follows. In Section 2 the statement of the problem is done, recalling some basic definitions and estimates that will be necessary bellow. Section 3 is devoted to present under minimal assumptions some abstract results on pullback attractors in different spaces and the relation among them. The existence of pullback $\mathcal{D}-$attractors in the H^1-norm in several universes is treated in Section 4 by using an energy method which relies on the continuity of the solutions (we deal with the two-dimensional case). Finally, under some suitable additional assumptions, some results about the tempered behaviour of these families are obtained in Section 5.

2 Statement of the problem

Consider an arbitrary value $\tau \in \mathbb{R}$, and the following Navier-Stokes problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla) u + \nabla p = f(t) \text{ in } (\tau, +\infty) \times \Omega, \\ \operatorname{div} u = 0 \text{ in } (\tau, +\infty) \times \Omega, \\ u = 0 \text{ on } (\tau, +\infty) \times \partial\Omega, \\ u(\tau, x) = u_\tau(x), \ x \in \Omega, \end{cases} \tag{1}$$

where the set $\Omega \subset \mathbb{R}^2$ is open and bounded with smooth enough boundary, $\nu > 0$ is the kinematic viscosity, u is the velocity field of the fluid, p is the pressure, u_τ is the initial velocity field, and f is the external force term depending on time.

To start, we consider the following usual function spaces:

$$\mathcal{V} = \left\{ u \in (C_0^\infty(\Omega))^2 : \operatorname{div} u \ = 0 \right\},$$

$H =$ the closure of $\mathcal{V}$ in $(L^2(\Omega))^2$ with the norm $|\cdot|$, and inner product $(\cdot, \cdot)$, where for $u, v \in (L^2(\Omega))^2$,

$$(u, v) = \sum_{j=1}^{2} \int_\Omega u_j(x) v_j(x) \mathrm{d}x,$$

$V =$ the closure of $\mathcal{V}$ in $(H_0^1(\Omega))^2$ with the norm $\|\cdot\|$ associated to the inner product $((\cdot, \cdot))$, where for $u, v \in (H_0^1(\Omega))^2$,

$$((u, v)) = \sum_{i,j=1}^{2} \int_\Omega \frac{\partial u_j}{\partial x_i} \frac{\partial v_j}{\partial x_i} \mathrm{d}x.$$

We will use $\|\cdot\|_*$ for the norm in V' and $\langle \cdot, \cdot \rangle$ for the duality $\langle V', V \rangle$. We consider every element $h \in H$ as an element of V', given by the equality $\langle h, v \rangle = (h, v)$ for all $v \in V$. It follows that $V \subset H \subset V'$, where the injections are dense and compact.

Define the operator $A : V \to V'$ as

$$\langle Au, v\rangle := ((u, v)) \quad \forall u, v \in V.$$

Denoting $D(A) = (H^2(\Omega))^2 \cap V$, then $Au = -P\Delta u$ for all $u \in D(A)$, is the Stokes operator (P is the ortho-projector from $(L^2(\Omega))^2$ onto H). On $D(A)$ we consider the norm $|\cdot|_{D(A)}$ defined by $|u|_{D(A)} = |Au|$. Observe that on $D(A)$ the norms $\|\cdot\|_{(H^2(\Omega))^2}$ and $|\cdot|_{D(A)}$ are equivalent, and $D(A)$ is compactly and densely injected in V.

Let us denote

$$b(u, v, w) = \sum_{i,j=1}^{2} \int_\Omega u_i \frac{\partial v_j}{\partial x_i} w_j \mathrm{d}x,$$

for every functions $u, v, w : \Omega \to \mathbb{R}^2$ for which the right-hand side is well defined.

In particular, b has sense for all $u, v, w \in V$, and is a continuous trilinear form on $V \times V \times V$.

Some useful properties concerning b that we will use in the next sections are the following (see [24] or [26]):

There exists a constant $C_1 > 0$, only dependent on Ω, such that (recall that we are in dimension two)

$$|b(u, v, w)| \le C_1 |u|^{1/2} |Au|^{1/2} \|v\| |w|, \quad \forall u \in D(A),\ v \in V,\ w \in H, \tag{2}$$

$$|b(u, v, w)| \le C_1 |Au| \|v\| |w|, \quad \forall u \in D(A),\ v \in V,\ w \in H, \tag{3}$$

and

$$|b(u, v, w)| \le C_1 |u|^{1/2} \|u\|^{1/2} \|v\| |w|^{1/2} \|w\|^{1/2}, \quad \forall\, u, v, w \in V. \tag{4}$$

Assume that $u_\tau \in H$ and $f \in L^2_{loc}(\mathbb{R}; V')$.

Definition 2.1 (Weak solution) *A weak solution of (1) is a function u that belongs to $L^2(\tau, T; V) \cap L^\infty(\tau, T; H)$ for all $T > \tau$, with $u(\tau) = u_\tau$, such that for all $v \in V$,*

$$\frac{d}{dt}(u(t), v) + \nu\langle Au(t), v\rangle + b(u(t), u(t), v) = \langle f(t), v\rangle, \tag{5}$$

where the equation must be understood in the sense of $\mathcal{D}'(\tau, +\infty)$.

Remark 2.2 *If u is a weak solution of (1), then from (5) we deduce that for any $T > \tau$, one has $u' \in L^2(\tau, T; V')$, and so $u \in C([\tau, +\infty); H)$, whence the initial datum has full sense. Moreover, in this case the following energy* *equality holds:*

$$|u(t)|^2 + 2\nu \int_s^t \langle Au(r), u(r)\rangle dr = |u(s)|^2 + 2\int_s^t \langle f(r), u(r)\rangle dr, \quad \forall \tau \le s \le t.$$

A notion of more regular solution is also suitable for problem (1).

Definition 2.3 (Strong solution) *A strong solution of (1) is a weak solution u of (1) such that $u \in L^2(\tau, T; D(A)) \cap L^\infty(\tau, T; V)$ for all $T > \tau$.*

Remark 2.4 *If $f \in L^2_{loc}(\mathbb{R}; H)$ and u is a strong solution of (1), then $u' \in L^2(\tau, T; H)$ for all $T > \tau$, and so $u \in C([\tau, +\infty); V)$. In this case the following energy equality holds:*

$$\begin{aligned} &\|u(t)\|^2 + 2\nu \int_s^t |Au(r)|^2\, dr + 2\int_s^t b(u(r), u(r), Au(r))\, dr \\ = &\|u(s)\|^2 + 2\int_s^t (f(r), Au(r))\, dr, \quad \forall \tau \le s \le t. \end{aligned} \tag{6}$$

3 Abstract results on attractors theory. Existence of minimal pullback attractors

The results in this section are a slight modification and generalization of those presented in [21] (see also [2] and [3]). In particular, we consider the process U being closed (cf. [18], see below Definition 3.1). The proofs are not difficult, but some of them are given explicitly for the sake of completeness.

Consider given a metric space (X, d_X), and let us denote $\mathbb{R}^2_d = \{(t, \tau) \in \mathbb{R}^2 : \tau \le t\}$.

A process on X is a mapping U such that $\mathbb{R}^2_d \times X \ni (t, \tau, x) \mapsto U(t, \tau)x \in X$ with $U(\tau, \tau)x = x$ for any $(\tau, x) \in \mathbb{R} \times X$, and $U(t, r)(U(r, \tau)x) = U(t, \tau)x$ for any $\tau \le r \le t$ and all $x \in X$.

Definition 3.1 *A process U on X is said to be closed if for any $\tau \le t$, and any sequence $\{x_n\} \subset X$ with $x_n \to x \in X$ and $U(t, \tau)x_n \to y \in X$, then $U(t, \tau)x = y$.*

Remark 3.2 *In [21] it was observed that the assumption of U being strong-weak (also known as norm-to weak) continuous is weaker than to ask to U being continuous (in the sense that for any pair $\tau \le t$, $U(t, \tau) : X \to X$ was continuous).*

Now we point out that to ask to U being closed is weaker than being strong-weak continuous. This more relaxed concept may be useful in some situations.

Let us denote $\mathcal{P}(X)$ the family of all nonempty subsets of X, and consider a family of nonempty sets $\widehat{D}_0 = \{D_0(t) : \ t \in \mathbb{R}\} \subset \mathcal{P}(X)$ [observe that we do not require any additional condition on these sets as compactness or boundedness].

Definition 3.3 *We say that a process U on X is pullback $\widehat{D}_0$-asymptotically compact if for any $t \in \mathbb{R}$ and any sequences $\{\tau_n\} \subset (-\infty, t]$ and $\{x_n\} \subset X$ satisfying $\tau_n \to -\infty$ and $x_n \in D_0(\tau_n)$ for all n, the sequence $\{U(t,\tau_n)x_n\}$ is relatively compact in X.*

Denote

$$\Lambda(\widehat{D}_0, t) := \bigcap_{s \le t} \overline{\bigcup_{\tau \le s} U(t,\tau) D_0(\tau)}^X \quad \forall\, t \in \mathbb{R}, \tag{7}$$

where $\overline{\{\cdots\}}^X$ is the closure in X.

We denote by $\mathrm{dist}_X(\mathcal{O}_1, \mathcal{O}_2)$ the Hausdorff semi-distance in X between two sets $\mathcal{O}_1$ and $\mathcal{O}_2$, defined as

$$\mathrm{dist}_X(\mathcal{O}_1, \mathcal{O}_2) = \sup_{x \in \mathcal{O}_1} \inf_{y \in \mathcal{O}_2} d_X(x,y) \quad \text{for } \mathcal{O}_1,\, \mathcal{O}_2 \subset X.$$

The following result is standard, and it does not use any continuity assumption on U (e.g. cf. [2,21]).

Proposition 3.4 *If the process U on X is pullback $\widehat{D}_0$-asymptotically compact, then, for all $t \in \mathbb{R}$, the set $\Lambda(\widehat{D}_0, t)$ given by (7) is a nonempty compact subset of X, and*

$$\lim_{\tau \to -\infty} \mathrm{dist}_X(U(t,\tau) D_0(\tau), \Lambda(\widehat{D}_0, t)) = 0. \tag{8}$$

Moreover, it is the minimal family of closed sets satisfying (8).

Assuming also that U is closed, we obtain the invariance of the family of sets $\{\Lambda(\widehat{D}_0, t) \ : \ t \in \mathbb{R}\}$.

Proposition 3.5 *Suppose that the process U on X is pullback $\widehat{D}_0$-asymptotically compact and closed, then the family of sets $\{\Lambda(\widehat{D}_0, t) \ : \ t \in \mathbb{R}\}$, defined by (7), is invariant for U, i.e.*

$$\Lambda(\widehat{D}_0, t) = U(t,\tau) \Lambda(\widehat{D}_0, \tau) \quad \forall\, \tau \le t.$$

Proof. Consider $\tau < t$ and $y \in \Lambda(\widehat{D}_0, \tau)$. Then, there exist sequences $\{\tau_n\} \subset$

$(-\infty, \tau]$ and $\{x_n\} \subset X$ satisfying $\lim_n \tau_n = -\infty$ and $x_n \in D_0(\tau_n)$ for all n, such that $U(\tau, \tau_n)x_n \to y$.

On the one hand, from the pullback $\widehat{D}_0-$asymptotic compactness we have that $\{U(t, \tau_n)x_n\}$ is relatively compact, so there exists a subsequence $U(t, \tau_{n'})x_{n'} \to z \in \Lambda(\widehat{D}_0, t)$. Since $U(t, \tau_n) = U(t, \tau)U(\tau, \tau_n)$ for all n, from the fact that U is closed, we deduce that $z = U(t, \tau)y$. The inclusion $U(t, \tau)\ \Lambda(\widehat{D}_0, \tau) \subset \Lambda(\widehat{D}_0, t)$ is thus proved.

On the other hand, consider $z \in \Lambda(\widehat{D}_0, t)$, and $\{\tau_n\} \subset (-\infty, \tau]$, with $\tau_n \to -\infty$ and $x_n \in D_0(\tau_n)$ for all n, such that $U(t, \tau_n)x_n \to z$. By using the concatenation property of the process, we have that $U(t, \tau_n) = U(t, \tau)U(\tau, \tau_n)$ for all n. Now, since the sequence $\{U(\tau, \tau_n)x_n\}$ is also relatively compact, for a subsequence we deduce that $U(\tau, \tau_{n'})x_{n'} \to y \in \Lambda(\widehat{D}_0, \tau)$. Again, since U is closed, we have that $z = U(t, \tau)y$. Thus we have proved the inclusion $U(t, \tau)\Lambda(\widehat{D}_0, \tau) \supset \Lambda(\widehat{D}_0, t)$. ■

Let be given $\mathcal{D}$ a nonempty class of families parameterized in time $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$. The class $\mathcal{D}$ will be called a universe in $\mathcal{P}(X)$.

Definition 3.6 *It is said that $\widehat{D}_0 = \{D_0(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ is pullback $\mathcal{D}-$absorbing for the process U on X if for any $t \in \mathbb{R}$ and any $\widehat{D} \in \mathcal{D}$, there exists a $\tau_0(t, \widehat{D}) \leq t$ such that*

$$U(t, \tau)D(\tau) \subset D_0(t) \quad \textit{for all } \tau \leq \tau_0(t, \widehat{D}).$$

Observe that in the definition above $\widehat{D}_0$ does not belong necessarily to the class $\mathcal{D}$.

Proposition 3.7 *[cf. [2, Prop.10]] If $\widehat{D}_0$ is pullback $\mathcal{D}-$absorbing for a process U, then*

$$\Lambda(\widehat{D}, t) \subset \Lambda(\widehat{D}_0, t) \quad \textit{for all } \widehat{D} \in \mathcal{D},\ t \in \mathbb{R}.$$

In addition, if $\widehat{D}_0 \in \mathcal{D}$, then

$$\Lambda(\widehat{D}_0, t) \subset \overline{D_0(t)} \quad \textit{for all } t \in \mathbb{R}.$$

Definition 3.8 *A process U on X is said to be pullback $\mathcal{D}-$asymptotically compact if it is $\widehat{D}$-asymptotically compact for any $\widehat{D} \in \mathcal{D}$, i.e. if for any $t \in \mathbb{R}$, any $\widehat{D} \in \mathcal{D}$, and any sequences $\{\tau_n\} \subset (-\infty, t]$ and $\{x_n\} \subset X$ satisfying $\tau_n \to -\infty$ and $x_n \in D(\tau_n)$ for all n, the sequence $\{U(t, \tau_n)x_n\}$ is relatively compact in X.*

As a consequence of Propositions 3.4 and 3.5, we have the following

Proposition 3.9 *Assume that the process U is closed and pullback $\mathcal{D}-$asymptotically compact. Then, for each $\widehat{D} \in \mathcal{D}$ and any $t \in \mathbb{R}$, the set $\Lambda(\widehat{D}, t)$ is a nonempty compact subset of X, invariant for U, that attracts $\widehat{D}$ in the pullback sense, i.e.*

$$\lim_{\tau \to -\infty} \operatorname{dist}_X(U(t,\tau)D(\tau), \Lambda(\widehat{D}, t)) = 0. \tag{9}$$

Moreover, it is the minimal family of closed sets satisfying (9).

Proposition 3.10 *Assume that $\widehat{D_0} = \{D_0(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ is pullback $\mathcal{D}-$absorbing for a process U on X, which is pullback $\widehat{D_0}$-asymptotically compact. Then, the process U is also pullback $\mathcal{D}-$asymptotically compact.*

Proof. Consider fixed $t \in \mathbb{R}$, $\widehat{D} \in \mathcal{D}$, and sequences $\{\tau_n\} \subset (-\infty, t]$ and $\{x_n\} \subset X$, with $\lim_n \tau_n = -\infty$, and $x_n \in D(\tau_n)$ for all n. We must prove that from the sequence $\{U(t,\tau_n)x_n\}$ we can extract a subsequence converging in X.

Observing that $\widehat{D_0}$ is pullback $\mathcal{D}-$absorbing for the process U, we deduce that for any integer $k \geq 1$ there exists a $\tau_{n_k} \in \{\tau_n\}$ such that $\tau_{n_k} \leq t - k$ and $y_{n_k} = U(t-k, \tau_{n_k})x_{n_k} \in D_0(t-k)$. As U is pullback $\widehat{D_0}$-asymptotically compact, from the sequence $\{U(t, t-k)y_{n_k}\}$ we can extract a subsequence $\{U(t, t-k')y_{n_{k'}}\}$ converging in X. But $U(t, t-k')y_{n_{k'}} = U(t, t-k')(U(t-k', \tau_{n_{k'}})x_{n_{k'}}) = U(t, \tau_{n_{k'}})x_{n_{k'}}$. This finishes the proof. ■

With the above definitions and results, we obtain the main result of this section.

Theorem 3.11 *Consider a closed process $U : \mathbb{R}^2_d \times X \to X$, a universe $\mathcal{D}$ in $\mathcal{P}(X)$, and a family $\widehat{D_0} = \{D_0(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ which is pullback $\mathcal{D}-$absorbing for U, and assume also that U is pullback $\widehat{D_0}-$asymptotically compact.*

Then, the family $\mathcal{A}_{\mathcal{D}} = \{\mathcal{A}_{\mathcal{D}}(t) : t \in \mathbb{R}\}$ defined by

$$\mathcal{A}_{\mathcal{D}}(t) = \overline{\bigcup_{\widehat{D} \in \mathcal{D}} \Lambda(\widehat{D}, t)}^X \quad t \in \mathbb{R},$$

has the following properties:

(a) for any $t \in \mathbb{R}$, the set $\mathcal{A}_{\mathcal{D}}(t)$ is a nonempty compact subset of X, and

$$\mathcal{A}_{\mathcal{D}}(t) \subset \Lambda(\widehat{D_0}, t),$$

(b) $\mathcal{A}_{\mathcal{D}}$ *is pullback* $\mathcal{D}-$*attracting, i.e.*

$$\lim_{\tau\to-\infty} \operatorname{dist}_X(U(t,\tau)D(\tau), \mathcal{A}_{\mathcal{D}}(t)) = 0 \quad \textit{for all } \widehat{D} \in \mathcal{D}, \quad t \in \mathbb{R},$$

(c) $\mathcal{A}_{\mathcal{D}}$ *is invariant, i.e.* $U(t,\tau)\mathcal{A}_{\mathcal{D}}(\tau) = \mathcal{A}_{\mathcal{D}}(t)$ *for all* $\tau \le t$,
(d) if $\widehat{D}_0 \in \mathcal{D}$, *then* $\mathcal{A}_{\mathcal{D}}(t) = \Lambda(\widehat{D}_0, t) \subset \overline{D_0(t)}^X$, *for all* $t \in \mathbb{R}$.

The family $\mathcal{A}_{\mathcal{D}}$ *is minimal in the sense that if* $\widehat{C} = \{C(t) \ : \ t \in \mathbb{R}\} \subset \mathcal{P}(X)$ *is a family of closed sets such that for any* $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \in \mathcal{D}$,

$$\lim_{\tau\to-\infty} \operatorname{dist}_X(U(t,\tau)D(\tau), C(t)) = 0,$$

then $\mathcal{A}_{\mathcal{D}}(t) \subset C(t)$.

Proof. As $\widehat{D}_0$ is pullback $\mathcal{D}-$absorbing for U, from Proposition 3.7 we know that $\Lambda(\widehat{D}, t) \subset \Lambda(\widehat{D}_0, t)$ for any $t \in \mathbb{R}$ and $\widehat{D} \in \mathcal{D}$, and if moreover $\widehat{D}_0 \in \mathcal{D}$, then $\Lambda(\widehat{D}_0, t) \subset \overline{D_0(t)}^X$ for all $t \in \mathbb{R}$.

As U is pullback $\widehat{D}_0-$asymptotically compact, by Proposition 3.4, the set $\Lambda(\widehat{D}_0, t)$ is nonempty and compact, for any $t \in \mathbb{R}$.

By Proposition 3.10, U is also pullback $\mathcal{D}-$asymptotically compact. Thus, again by Proposition 3.4 applied to $\widehat{D}$ instead of $\widehat{D}_0$, for any $t \in \mathbb{R}$ and $\widehat{D} \in \mathcal{D}$, the set $\Lambda(\widehat{D}, t)$ is nonempty and compact.

These considerations prove (a) and (d).

Moreover, as evidently

$$\operatorname{dist}_X(U(t,\tau)D(\tau), \mathcal{A}_{\mathcal{D}}(t)) \le \operatorname{dist}_X(U(t,\tau)D(\tau), \Lambda(\widehat{D}, t))$$

for any $\widehat{D} \in \mathcal{D}$, (b) is also a consequence of Proposition 3.4.

Now, in order to prove (c) we observe that by Proposition 3.5, we also have

$$U(t,\tau)\Lambda(\widehat{D}, \tau) = \Lambda(\widehat{D}, t) \quad \text{for all } \tau \le t \text{ and any } \widehat{D} \in \mathcal{D}. \tag{10}$$

If $y \in \mathcal{A}_{\mathcal{D}}(t)$, there exist two sequences $\{\widehat{D}_n\} \subset \mathcal{D}$ and $\{y_n\} \subset X$, such that $y_n \in \Lambda(\widehat{D}_n, t)$ and $y_n \to y$. But by (10), $y_n = U(t,\tau)x_n$, with $x_n \in \Lambda(\widehat{D}_n, \tau) \subset \mathcal{A}_{\mathcal{D}}(\tau)$. By the compactness of this last set, there exists a subsequence $\{x_{n'}\} \subset \{x_n\}$ such that $x_{n'} \to x \in \mathcal{A}_{\mathcal{D}}(\tau)$. But then, as U is closed, $y = U(t,\tau)x$, and this proves that $\mathcal{A}_{\mathcal{D}}(t) \subset U(t,\tau)\mathcal{A}_{\mathcal{D}}(\tau)$. The reverse inclusion can be proved analogously.

Finally, the minimality is also easy to obtain taking into account Proposition 3.9 and the definition of $\mathcal{A}_{\mathcal{D}}$. ■

Remark 3.12 *Under the assumptions of Theorem 3.11, the family* $\mathcal{A}_{\mathcal{D}}$ *is called the minimal pullback* $\mathcal{D}-$*attractor for the process* U.

If $\mathcal{A}_{\mathcal{D}} \in \mathcal{D}$, *then it is the unique family of closed subsets in* $\mathcal{D}$ *that satisfies (b)–(c).*

A sufficient condition for $\mathcal{A}_{\mathcal{D}} \in \mathcal{D}$ *is to have that* $\widehat{D}_0 \in \mathcal{D}$, *the set* $D_0(t)$ *is closed for all* $t \in \mathbb{R}$, *and the family* $\mathcal{D}$ *is inclusion-closed (i.e. if* $\widehat{D} \in \mathcal{D}$, *and* $\widehat{D'} = \{D'(t) \ : \ t \in \mathbb{R}\} \subset \mathcal{P}(X)$ *with* $D'(t) \subset D(t)$ *for all* t, *then* $\widehat{D'} \in \mathcal{D}$*).*

We will denote $\mathcal{D}_F^X$ the universe of fixed nonempty bounded subsets of X, i.e. the class of all families $\widehat{D}$ of the form $\widehat{D} = \{D(t) = D : t \in \mathbb{R}\}$ with D a fixed nonempty bounded subset of X. In the particular case of the universe $\mathcal{D}_F^X$, the corresponding minimal pullback $\mathcal{D}_F^X-$attractor for the process U is the pullback attractor defined by Crauel, Debussche, and Flandoli, [9, Th.1.1,p.311], and will be denoted $\mathcal{A}_{\mathcal{D}_F^X}$.

Now, it is easy to conclude the following result.

Corollary 3.13 *Under the assumptions of Theorem 3.11, if the universe* $\mathcal{D}$ *contains the universe* $\mathcal{D}_F^X$, *then both attractors,* $\mathcal{A}_{\mathcal{D}_F^X}$ *and* $\mathcal{A}_{\mathcal{D}}$, *exist, and the following relation holds:*

$$\mathcal{A}_{\mathcal{D}_F^X}(t) \subset \mathcal{A}_{\mathcal{D}}(t) \qquad \forall t \in \mathbb{R}.$$

Remark 3.14 *It can be proved (see [21]) that, under the assumptions of the preceding corollary, if for some* $T \in \mathbb{R}$ *the set* $\cup_{t \le T} D_0(t)$ *is a bounded subset of* X, *then*

$$\mathcal{A}_{\mathcal{D}_F^X}(t) = \mathcal{A}_{\mathcal{D}}(t) \qquad \forall t \le T.$$

Now, we establish an abstract result that allows to compare two attractors for a process under appropriate assumptions.

Theorem 3.15 *Let* $\{(X_i, d_{X_i})\}_{i=1,2}$ *be two metric spaces such that* $X_1 \subset X_2$ *with continuous injection, and for* $i = 1, 2$, *let* $\mathcal{D}_i$ *be a universe in* $\mathcal{P}(X_i)$, *with* $\mathcal{D}_1 \subset \mathcal{D}_2$. *Assume that we have a map* U *that acts as a process in both cases, i.e.* $U : \mathbb{R}_d^2 \times X_i \to X_i$ *for* $i = 1, 2$ *is a process.*

For each $t \in \mathbb{R}$, *let us denote*

$$\mathcal{A}_i(t) = \overline{\bigcup_{\widehat{D}_i \in \mathcal{D}_i} \Lambda_i(\widehat{D}_i, t)}^{X_i}, \quad i = 1, 2,$$

where the subscript i *in the symbol of the omega-limit set* Λ_i *is used to denote the dependence of the respective topology.*

Then,

$$\mathcal{A}_1(t) \subset \mathcal{A}_2(t) \quad \textit{for all } t \in \mathbb{R}.$$

Suppose moreover that the two following conditions are satisfied:

(i) $\mathcal{A}_1(t)$ *is a compact subset of* X_1 *for all* $t \in \mathbb{R}$*,*

(ii) for any $\widehat{D}_2 \in \mathcal{D}_2$ *and any* $t \in \mathbb{R}$*, there exist a family* $\widehat{D}_1 \in \mathcal{D}_1$ *and a* $t^*_{\widehat{D}_1} \leq t$ *(both possibly depending on* t *and* $\widehat{D}_2$*), such that* U *is pullback* $\widehat{D}_1$*-asymptotically compact, and for any* $s \leq t^*_{\widehat{D}_1}$ *there exists a* $\tau_s \leq s$ *such that*

$$U(s,\tau)D_2(\tau) \subset D_1(s) \quad \textit{for all } \tau \leq \tau_s.$$

Then, under all the conditions above,

$$\mathcal{A}_1(t) = \mathcal{A}_2(t) \quad \textit{for all } t \in \mathbb{R}.$$

Proof. Since the omega-limit set is characterized as

$$\Lambda_i(\widehat{D}_i, t) = \{x \in X_i : \exists \tau_n \to -\infty, x_n \in D_i(\tau_n), x = X_i - \lim_n U(t,\tau_n)x_n\},$$

by the continuous injection of X_1 into X_2 we have that $\Lambda_1(\widehat{D}_1, t) \subset \Lambda_2(\widehat{D}_1, t)$, for all $\widehat{D}_1 \in \mathcal{D}_1$ and any $t \in \mathbb{R}$. This implies that

$$\bigcup_{\widehat{D}_1 \in \mathcal{D}_1} \Lambda_1(\widehat{D}_1, t) \subset \bigcup_{\widehat{D}_1 \in \mathcal{D}_1} \Lambda_2(\widehat{D}_1, t) \subset \bigcup_{\widehat{D}_2 \in \mathcal{D}_2} \Lambda_2(\widehat{D}_2, t).$$

Again from the continuous injection of X_1 into X_2, we obtain one inclusion:

$$\mathcal{A}_1(t) = \overline{\bigcup_{\widehat{D}_1 \in \mathcal{D}_1} \Lambda_1(\widehat{D}_1, t)}^{X_1} \subset \overline{\bigcup_{\widehat{D}_2 \in \mathcal{D}_2} \Lambda_2(\widehat{D}_2, t)}^{X_2} = \mathcal{A}_2(t).$$

For the opposite inclusion, assuming (i) and (ii), consider $\widehat{D}_2 \in \mathcal{D}_2$ and $t \in \mathbb{R}$ given. For any $x \in \Lambda_2(\widehat{D}_2, t)$ there exist two sequences $\{\tau_n\}$ and $\{x_n\}$ with $\tau_n \leq t$ for all n, satisfying $\lim_n \tau_n = -\infty$, $x_n \in D_2(\tau_n)$, and $x = X_2 - \lim_n U(t,\tau_n)x_n$. By assumption (ii), there exist a $\widehat{D}_1 \in \mathcal{D}_1$ and an integer $k_{\widehat{D}_1} \geq 1$ such that U is pullback $\widehat{D}_1$-asymptotically compact, and for any $k \geq k_{\widehat{D}_1}$ there exist $x_{n_k} \in \{x_n\}$ and $\tau_{n_k} \leq t - k$ such that

$$y_{n_k} = U(t-k, \tau_{n_k})x_{n_k} \in D_1(t-k).$$

As U is pullback $\widehat{D}_1-$asymptotically compact, there exists a subsequence of the sequence $\{x_{n_k}\}$ (relabelled the same) such that

$$X_1 - \lim_k U(t, t-k)y_{n_k} = z \in \Lambda_1(\widehat{D}_1, t).$$

But taking into account that $U(t,t-k)y_{n_k} = U(t,\tau_{n_k})x_{n_k}$, by the continuous injection of X_1 into X_2, we deduce that $z = x$. Thus, $x \in \Lambda_1(\widehat{D}_1, t)$.

Consequently,

$$\bigcup_{\widehat{D}_2 \in \mathcal{D}_2} \Lambda_2(\widehat{D}_2, t) \subset \bigcup_{\widehat{D}_1 \in \mathcal{D}_1} \Lambda_1(\widehat{D}_1, t) \subset \mathcal{A}_1(t).$$

As $\mathcal{A}_1(t)$ is compact in X_1, from the continuous injection, it is also compact in X_2, and in particular, closed. Taking closure in X_2 in the above inclusion, we conclude that $\mathcal{A}_2(t) \subset \mathcal{A}_1(t)$. The proof is finished. ■

Remark 3.16 *In the preceding theorem, if instead of assumption (ii) we consider the following condition:*

(ii') for any $\widehat{D}_2 \in \mathcal{D}_2$ and any sequence $\tau_n \to -\infty$ there exist another family $\widehat{D}_1 \in \mathcal{D}_1$ and another sequence $\tau'_n \to -\infty$ with $\tau'_n \geq \tau_n$ for all n, such that U is pullback $\widehat{D}_1$-asymptotically compact, and

$$U(\tau'_n, \tau_n) D_2(\tau_n) \subset D_1(\tau'_n), \quad \textit{for all } n, \tag{11}$$

then, with a similar proof, one can obtain that the equality $\mathcal{A}_2(t) = \mathcal{A}_1(t)$ for all $t \in \mathbb{R}$, also holds.

Observe that a sufficient condition for (11) is that there exists $T > 0$ such that for any $\widehat{D}_2 \in \mathcal{D}_2$, there exists a $\widehat{D}_1 \in \mathcal{D}_1$ satisfying $U(\tau + T, \tau) D_2(\tau) \subset D_1(\tau + T)$, for all $\tau \in \mathbb{R}$.

4 Pullback attractors for the non-autonomous 2D-Navier-Stokes model

4.1 Pullback attractors in H

The following results concerning existence and uniqueness of solution for (1), and continuity with respect to initial datum, are well-known (e.g. cf. [17,26,24]). We present them summarized.

Theorem 4.1 (Weak and strong solutions) *Assume that $f \in L^2_{\rm loc}(\mathbb{R}; V')$ and $u_\tau \in H$. Then, problem (1) possesses a unique weak solution, which will be denoted $u(\cdot) = u(\cdot; \tau, u_\tau)$.*

Moreover, if $f \in L^2_{\rm loc}(\mathbb{R}; H)$, this solution u satisfies that $u \in C((\tau, T]; V) \cap L^2(\tau + \varepsilon, T; (H^2(\Omega))^2)$ for every $\varepsilon > 0$ and $T > \tau + \varepsilon$. In fact, if $u_\tau \in V$, *then $u \in C([\tau,T];V) \cap L^2(\tau,T;(H^2(\Omega))^2)$ for every $T > \tau$, i.e. u is a strong solution.*

Therefore, when $f \in L^2_{\rm loc}(\mathbb{R};V')$, we can define a process $U : \mathbb{R}^2_d \times H \to H$ as

$$U(t,\tau)u_\tau = u(t;\tau,u_\tau) \quad \forall u_\tau \in H, \quad \forall \tau \le t, \tag{12}$$

and if $f \in L^2_{\rm loc}(\mathbb{R};H)$, the restriction of this process to $\mathbb{R}^2_d \times V$ is a process in V.

Proposition 4.2 (Continuity of the process) *If $f \in L^2_{\rm loc}(\mathbb{R};V')$, for any pair $(t,\tau) \in \mathbb{R}^2_d$, the map $U(t,\tau)$ is continuous from H into H. Moreover, if $f \in L^2_{\rm loc}(\mathbb{R};H)$, then $U(t,\tau)$ is also continuous from V into V.*

The asymptotic behaviour in H is also well-known, and again we only summarize the main facts (e.g. cf. [2,3]). Actually, the results in this case can be obtained in a way analogous, but simpler, to that which we will use later for the asymptotic behaviour in V.

We will denote $\lambda_1 > 0$ the first eigenvalue of the Stokes operator A.

Lemma 4.3 *Assume that $f \in L^2_{\rm loc}(\mathbb{R};V')$ and $u_\tau \in H$. Consider any $\mu \in (0, 2\nu\lambda_1)$ fixed. Then, the solution u to (1) satisfies for all $t \ge \tau$:*

$$|u(t)|^2 \le e^{-\mu(t-\tau)}|u_\tau|^2 + \frac{e^{-\mu t}}{2\nu - \mu\lambda_1^{-1}} \int_\tau^t e^{\mu s}\|f(s)\|_*^2 ds.$$

Once the above estimate has been established, we define the following universe.

Definition 4.4 (Universe) *We will denote by $\mathcal{D}^H_\mu$ the class of all families of nonempty subsets $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{P}(H)$ such that*

$$\lim_{\tau \to -\infty} \left(e^{\mu\tau} \sup_{v \in D(\tau)} |v|^2 \right) = 0.$$

Accordingly to the notation introduced in the previous section, $\mathcal{D}^H_F$ will denote the class of families $\widehat{D} = \{D(t) = D : t \in \mathbb{R}\}$ with D a fixed nonempty bounded subset of H.

Remark 4.5 *Observe that $\mathcal{D}^H_F \subset \mathcal{D}^H_\mu$ and that both are inclusion-closed.*

Corollary 4.6 ($\mathcal{D}^H_\mu-$absorbing family) *Assume that $f \in L^2_{\rm loc}(\mathbb{R};V')$ sat-*

isfies that there exists some $\mu \in (0, 2\nu\lambda_1)$ such that

$$\int_{-\infty}^{0} e^{\mu s}\|f(s)\|_*^2 ds < +\infty. \tag{13}$$

Then, the family $\widehat{D}_0 = \{D_0(t) : t \in \mathbb{R}\}$ defined by $D_0(t) = \overline{B}_H(0, R_H^{1/2}(t))$, the closed ball in H of center zero and radius $R_H^{1/2}(t)$, where

$$R_H(t) = 1 + \frac{e^{-\mu t}}{2\nu - \mu\lambda_1^{-1}} \int_{-\infty}^{t} e^{\mu s}\|f(s)\|_*^2 ds,$$

is pullback $\mathcal{D}_\mu^H-$absorbing for the process $U : \mathbb{R}_d^2 \times H \to H$ given by (12) (and therefore $\mathcal{D}_F^H-$absorbing too), and $\widehat{D}_0 \in \mathcal{D}_\mu^H$.

Indeed, we also have

Lemma 4.7 ($\mathcal{D}_\mu^H-$asymptotic compactness) *Under the assumptions of Lemma 4.3, the process U defined by (12) is pullback $\mathcal{D}_\mu^H-$asymptotically compact, i.e. for any $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \in \mathcal{D}_\mu^H$, any $t \in \mathbb{R}$, and any sequences $\{\tau_n\} \subset (-\infty, t]$ and $\{u_{\tau_n}\} \subset H$ satisfying $\tau_n \to -\infty$ and $u_{\tau_n} \in D(\tau_n)$ for all n, the sequence $\{U(t, \tau_n)u_{\tau_n}\}$ is relatively compact in H.*

As a consequence of above, we obtain the existence of minimal pullback attractors for the process $U : \mathbb{R}_d^2 \times H \to H$ defined by (12).

Theorem 4.8 *Assume that $f \in L^2_{\mathrm{loc}}(\mathbb{R}; V')$ satisfies for some $\mu \in (0, 2\nu\lambda_1)$ the condition (13). Then, there exist the minimal pullback $\mathcal{D}_F^H$-attractor*

$$\mathcal{A}_{\mathcal{D}_F^H} = \{\mathcal{A}_{\mathcal{D}_F^H}(t) : t \in \mathbb{R}\}$$

and the minimal pullback $\mathcal{D}_\mu^H$-attractor

$$\mathcal{A}_{\mathcal{D}_\mu^H} = \{\mathcal{A}_{\mathcal{D}_\mu^H}(t) : t \in \mathbb{R}\},$$

for the process U defined by (12). The family $\mathcal{A}_{\mathcal{D}_\mu^H}$ belongs to $\mathcal{D}_\mu^H$, and the following relation holds:

$$\mathcal{A}_{\mathcal{D}_F^H}(t) \subset \mathcal{A}_{\mathcal{D}_\mu^H}(t) \subset \overline{B}_H(0, R_H^{1/2}(t)) \quad \forall t \in \mathbb{R}.$$

4.2 Pullback attractors in V

The goal of this section is to prove analogous results to those given above, but concerning to the map U defined as a process in V.

First, we recall a lemma (see [24]) which we will use in the proof of some of our results.

Lemma 4.9 *Let X, Y be Banach spaces such that X is reflexive, and the inclusion $X \subset Y$ is continuous. Assume that $\{u_n\}$ is a bounded sequence in $L^\infty(t_0, T; X)$ such that $u_n \rightharpoonup u$ weakly in $L^q(t_0, T; X)$ for some $q \in [1, +\infty)$ and $u \in C([t_0, T]; Y)$.*

Then, $u(t) \in X$ and $\|u(t)\|_X \le \liminf_{n\to+\infty} \|u_n\|_{L^\infty(t_0,T;X)}$, for all $t \in [t_0, T]$.

From now on we assume that $f \in L^2_{\rm loc}(\mathbb{R}; H)$, and satisfies

$$\int_{-\infty}^{0} e^{\mu s}|f(s)|^2\, ds < +\infty, \quad \text{for some } \mu \in (0, 2\nu\lambda_1). \tag{14}$$

We have the following result, which is proved analogously to [12, Cor.2.3 and Cor.2.5].

Lemma 4.10 *Suppose that $f \in L^2_{\rm loc}(\mathbb{R}; H)$ satisfies the condition (14). Then, for any $t \in \mathbb{R}$ and $\widehat{D} \in \mathcal{D}^H_\mu$, there exists $\tau_1(\widehat{D}, t) < t - 3$, such that for any $\tau \le \tau_1(\widehat{D}, t)$ and any $u_\tau \in D(\tau)$, it holds*

$$\begin{cases} |u(r;\tau,u_\tau)|^2 \le \rho_1(t) \text{ for all } r \in [t-3, t], \\ \|u(r;\tau,u_\tau)\|^2 \le \rho_2(t) \text{ for all } r \in [t-2, t], \\ \displaystyle\int_{r-1}^{r} |Au(\theta;\tau,u_\tau)|^2 d\theta \le \rho_3(t) \text{ for all } r \in [t-1, t], \\ \displaystyle\int_{r-1}^{r} |u'(\theta;\tau,u_\tau)|^2 d\theta \le \rho_4(t) \text{ for all } r \in [t-1, t], \end{cases} \tag{15}$$

where

$$\rho_1(t) = 1 + \frac{e^{\mu(3-t)}}{2\nu\lambda_1 - \mu}\int_{-\infty}^{t} e^{\mu\theta}\,|f(\theta)|^2\, d\theta, \tag{16}$$

$$\begin{aligned} \rho_2(t) = \max_{r\in[t-2,t]} \Bigg\{ &\left(\frac{1}{\nu}\rho_1(r) + \left(\frac{1}{\nu^2\lambda_1} + \frac{2}{\nu}\right)\int_{r-1}^{r} |f(\theta)|^2\, d\theta\right) \\ &\times \exp\left[2C^{(\nu)}\rho_1(r)\left(\frac{1}{\nu}\rho_1(r) + \frac{1}{\nu^2\lambda_1}\int_{r-1}^{r}|f(\theta)|^2\, d\theta\right)\right]\Bigg\}, \end{aligned} \tag{17}$$

$$\rho_3(t) = \frac{1}{\nu}\left(\rho_2(t) + \frac{2}{\nu}\int_{t-2}^{t}|f(\theta)|^2\, d\theta + 2C^{(\nu)}\rho_1(t)\rho_2^2(t)\right), \tag{18}$$

$$\rho_4(t) = \nu\rho_2(t) + 2\int_{t-2}^{t}|f(\theta)|^2\, d\theta + 2C_1^2\rho_2(t)\rho_3(t), \tag{19}$$

and $C^{(\nu)} = 27C_1^4(4\nu^3)^{-1}$.

Proof. In order to obtain all the estimates in (15), we will proceed with the Galerkin approximations and then passing to the limit using Lemma 4.9.

For each integer $n \geq 1$, we denote by $u_n(s) = u_n(s;\tau,u_\tau)$ the Galerkin approximation of the solution $u(s;\tau,u_\tau)$ of (1), which is given by

$$u_n(s) = \sum_{j=1}^{n} \gamma_{nj}(s) w_j,$$

and is the solution of

$$\begin{cases} \dfrac{d}{ds}\left(u_n(s), w_j\right) + \nu((u_n(s), w_j)) + b(u_n(s), u_n(s), w_j) = \left(f(s), w_j\right), \\ \\ (u_n(\tau), w_j) = (u_\tau, w_j) \qquad j = 1, \ldots, n, \end{cases} \tag{20}$$

where $\{w_j : \ j \geq 1\} \subset V$ is the Hilbert basis of H formed by the eigenfunctions of the Stokes operator A. Observe that by the regularity of Ω, all w_j belong to $(H^2(\Omega))^2$.

Multiplying by $\gamma_{nj}(s)$ in (20), and summing from $j = 1$ to n, we obtain

$$\frac{d}{d\theta}\left|u_n(\theta)\right|^2 + 2\nu\left\|u_n(\theta)\right\|^2 = 2\left(f(\theta), u_n(\theta)\right), \quad \text{a.e. } \theta > \tau, \tag{21}$$

and therefore,

$$\frac{d}{d\theta}\left(e^{\mu\theta}\left|u_n(\theta)\right|^2\right) + 2\nu e^{\mu\theta}\left\|u_n(\theta)\right\|^2 = \mu e^{\mu\theta}\left|u_n(\theta)\right|^2 + 2e^{\mu\theta}\left(f(\theta), u_n(\theta)\right), \tag{22}$$

a.e. $\theta > \tau$.

Observing that $\lambda_1 |u_n(\theta)|^2 \leq \|u_n(\theta)\|^2$, and

$$2\left|\left(f(\theta), u_n(\theta)\right)\right| \leq \frac{1}{2\nu\lambda_1 - \mu}\left|f(\theta)\right|^2 + \left(2\nu\lambda_1 - \mu\right)\left|u_n(\theta)\right|^2,$$

from (22) we deduce

$$\frac{d}{d\theta}\left(e^{\mu\theta}\left|u_n(\theta)\right|^2\right) \leq \frac{e^{\mu\theta}}{2\nu\lambda_1 - \mu}\left|f(\theta)\right|^2 \quad \text{a.e. } \theta > \tau,$$

and therefore

$$e^{\mu r}\left|u_n(r)\right|^2 \leq e^{\mu\tau}\left|u_\tau\right|^2 + \frac{1}{2\nu\lambda_1 - \mu}\int_{-\infty}^{r} e^{\mu\theta}\left|f(\theta)\right|^2 d\theta \quad \forall\, r \geq \tau. \tag{23}$$

From (23) we deduce that for each $t \in \mathbb{R}$ and $\widehat{D} \in \mathcal{D}_\mu^H$, there exists a $\tau_1(\widehat{D}, t) < t - 3$ such that for any $n \geq 1$,

$$|u_n(r;\tau,u_\tau)|^2 \leq \rho_1(t) \quad \text{for all } r \in [t-3, t],\ \tau \leq \tau_1(\widehat{D}, t),\ u_\tau \in D(\tau), \tag{24}$$

where $\rho_1(t)$ is given by (16).

Now, multiplying in (20) by $\lambda_j \gamma_{nj}(s)$, where λ_j is the eigenvalue associated to the eigenfunction w_j, and summing from $j = 1$ to n, we obtain

$$\frac{1}{2}\frac{d}{d\theta}\left\|u_n(\theta)\right\|^2 + \nu\left|Au_n(\theta)\right|^2 + b(u_n(\theta), u_n(\theta), Au_n(\theta)) = (f(\theta), Au_n(\theta)), \quad (25)$$

a.e. $\theta > \tau$.

Observe that

$$|(f(\theta), Au_n(\theta))| \leq \frac{1}{\nu}\left|f(\theta)\right|^2 + \frac{\nu}{4}\left|Au_n(\theta)\right|^2,$$

and by (2) and Young's inequality,

$$\begin{aligned} |b(u_n(\theta), u_n(\theta), Au_n(\theta))| &\leq C_1|u_n(\theta)|^{1/2}\|u_n(\theta)\||Au_n(\theta)|^{3/2} \\ &\leq \frac{\nu}{4}\left|Au_n(\theta)\right|^2 + C^{(\nu)}|u_n(\theta)|^2\|u_n(\theta)\|^4. \end{aligned} \quad (26)$$

Thus, from (25) we deduce

$$\begin{aligned} &\frac{d}{d\theta}\left\|u_n(\theta)\right\|^2 + \nu\left|Au_n(\theta)\right|^2 \\ &\leq \frac{2}{\nu}\left|f(\theta)\right|^2 + 2C^{(\nu)}|u_n(\theta)|^2\|u_n(\theta)\|^4, \quad \text{a.e. } \theta > \tau. \end{aligned} \quad (27)$$

From this inequality, in particular we obtain

$$\begin{aligned} \|u_n(r)\|^2 \leq\, & \|u_n(s)\|^2 + \frac{2}{\nu}\int_{r-1}^{r}\left|f(\theta)\right|^2 d\theta \\ & +2C^{(\nu)}\int_s^r |u_n(\theta)|^2\|u_n(\theta)\|^4\, d\theta \end{aligned}$$

for all $\tau \leq r - 1 \leq s \leq r$, and therefore, by Gronwall's lemma,

$$\begin{aligned} \|u_n(r)\|^2 \leq\, & \left(\|u_n(s)\|^2 + \frac{2}{\nu}\int_{r-1}^{r}\left|f(\theta)\right|^2 d\theta\right) \\ & \times \exp\left(2C^{(\nu)}\int_{r-1}^{r}|u_n(\theta)|^2\|u_n(\theta)\|^2\, d\theta\right) \end{aligned}$$

for all $\tau \leq r - 1 \leq s \leq r$.

Integrating this last inequality for s between $r - 1$ and r, we obtain

$$
\begin{aligned}
\|u_n(r)\|^2 \leq & \left(\int_{r-1}^r \|u_n(s)\|^2\, ds + \frac{2}{\nu}\int_{r-1}^r |f(\theta)|^2\ d\theta\right) \\
& \times \exp\left(2C^{(\nu)}\int_{r-1}^r |u_n(\theta)|^2\|u_n(\theta)\|^2\, d\theta\right) \qquad (28)
\end{aligned}
$$

for all $\tau \leq r-1$.

Observe that by (21),

$$
\nu\int_{r-1}^r \|u_n(\theta)\|^2\, d\theta \leq |u_n(r-1)|^2 + \frac{1}{\nu\lambda_1}\int_{r-1}^r |f(\theta)|^2\, d\theta,
$$

and therefore, from (24) and (28) we deduce that for any $n \geq 1$,

$$
\|u_n(r;\tau,u_\tau)\|^2 \leq \rho_2(t) \quad \text{for all } r \in [t-2,t],\ \tau \leq \tau_1(\widehat{D},t),\ u_\tau \in D(\tau), \qquad (29)
$$

where $\rho_2(t)$ is given by (17).

Now, by (27),

$$
\begin{aligned}
\nu\int_{r-1}^r |Au_n(\theta)|^2\ d\theta \leq & \|u_n(r-1)\|^2 + \frac{2}{\nu}\int_{r-1}^r |f(\theta)|^2\ d\theta \\
& + 2C^{(\nu)}\int_{r-1}^r |u_n(\theta)|^2\|u_n(\theta)\|^4\, d\theta, \quad \text{for all } \tau \leq r-1,
\end{aligned}
$$

and therefore, by (24) and (29), for every $n \geq 1$,

$$
\int_{r-1}^r |Au_n(\theta;\tau,u_\tau)|^2\ d\theta \leq \rho_3(t) \qquad (30)
$$

for all $r \in [t-1,t]$, $\tau \leq \tau_1(\widehat{D},t)$, $u_\tau \in D(\tau)$, where $\rho_3(t)$ is given by (18).

On the other hand, multiplying by the derivative $\gamma'_{nj}(s)$ in (20), and summing from $j=1$ till n, we obtain

$$
|u_n'(\theta)|^2 + \frac{\nu}{2}\frac{d}{d\theta}\|u_n(\theta)\|^2 + b(u_n(\theta),u_n(\theta),u_n'(\theta)) = (f(\theta),u_n'(\theta)), \qquad (31)
$$

a.e. $\theta > \tau$.

Observing that

$$
|\,(f(\theta),u_n'(\theta))\,| \leq \frac{1}{4}|u_n'(\theta)|^2 + |f(\theta)|^2,
$$

and by (3)

$$
\begin{aligned}
|b(u_n(\theta),u_n(\theta),u_n'(\theta))| \leq & C_1|Au_n(\theta)|\|u_n(\theta)\||u_n'(\theta)| \\
\leq & \frac{1}{4}|u_n'(\theta)|^2 + C_1^2|Au_n(\theta)|^2\|u_n(\theta)\|^2,
\end{aligned}
$$

we obtain from (31)

$$|u_n'(\theta)|^2 + \nu \frac{d}{d\theta} \|u_n(\theta)\|^2 \le 2|f(\theta)|^2 + 2C_1^2 |Au_n(\theta)|^2 \|u_n(\theta)\|^2.$$

Integrating this last inequality, we deduce that

$$\int_{r-1}^{r} |u_n'(\theta)|^2 \, d\theta \le \nu \|u_n(r-1)\|^2 + 2\int_{r-1}^{r} |f(\theta)|^2 \, d\theta \\ +2C_1^2 \sup_{\theta\in[r-1,r]} \|u_n(\theta)\|^2 \int_{r-1}^{r} |Au_n(\theta)|^2 \, d\theta,$$

and therefore, by (24), (29) and (30), we obtain

$$\int_{r-1}^{r} |u_n'(\theta)|^2 \, d\theta \le \rho_4(t) \text{ for all } r \in [t-1,t],\ \tau \le \tau_1(\widehat{D},t),\ u_\tau \in D(\tau), \quad (32)$$

where $\rho_4(t)$ is defined by (19).

By Lemma 4.9, and the well-known facts that u_n converges to $u(\cdot;\tau,u_\tau)$ weakly in $L^2(t-3,t;D(A))$, u_n' converges to $u'(\cdot;\tau,u_\tau)$ weakly in $L^2(t-3,t;H)$, and $u(\cdot;\tau,u_\tau) \in C([t-3,t];V)$, we can pass to the limit when $n \to +\infty$ in (24), (29), (30), and (32), and it turns out that (15) holds. ■

Remark 4.11 *It is clear that under the assumptions of Lemma 4.10,*

$$\lim_{t\to-\infty} e^{\mu t}\rho_1(t) = 0.$$

In other words, the family $\{\overline{B}_H(0,\rho_1^{1/2}(t)) : t \in \mathbb{R}\}$, where $\overline{B}_H(0,\rho_1^{1/2}(t))$ is the closed ball in H of center zero and radius $\rho_1^{1/2}(t)$, with $\rho_1(t)$ given by (16), belongs to $\mathcal{D}_\mu^H$.

We will denote by $\mathcal{D}_\mu^{H,V}$ the class of all families $\widehat{D}_V$ of elements of $\mathcal{P}(V)$ of the form $\widehat{D}_V = \{D(t)\cap V : t \in \mathbb{R}\}$, where $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \in \mathcal{D}_\mu^H$.

Again, accordingly to the notation in the previous section, we denote $\mathcal{D}_F^V$ the universe of families (parameterized in time but constant for all $t \in \mathbb{R}$) of nonempty fixed bounded subsets of V.

Both classes, $\mathcal{D}_\mu^{H,V}$ and $\mathcal{D}_F^V$, are (inclusion-closed) universes in $\mathcal{P}(V)$, and evidently $\mathcal{D}_F^V \subset \mathcal{D}_\mu^{H,V}$.

Now, the following result is immediate.

Corollary 4.12 *Under the assumptions of Lemma 4.10, the family*

$$\widehat{D}_{0,V} = \{\overline{B}_H(0,\rho_1^{1/2}(t)) \cap V : t \in \mathbb{R}\}$$

belongs to $\mathcal{D}_\mu^{H,V}$ and satisfies that for any $t \in \mathbb{R}$ and any $\widehat{D} \in \mathcal{D}_\mu^H$, there exists a $\tau(\widehat{D}, t) < t$ such that

$$U(t,\tau)D(\tau) \subset D_{0,V}(t) \quad \textit{for all } \tau \leq \tau(\widehat{D}, t).$$

In particular, the family $\widehat{D}_{0,V}$ is pullback $\mathcal{D}_\mu^{H,V}-$absorbing for the process $U : \mathbb{R}_d^2 \times V \to V$.

Now we apply an energy method with continuous functions (e.g. cf. [14,19,22]) in order to obtain the pullback asymptotic compactness in V for the universe $\mathcal{D}_\mu^{H,V}$.

Lemma 4.13 *Suppose that $f \in L^2_{\text{loc}}(\mathbb{R}; H)$ satisfies the condition (14). Then, the process $U : \mathbb{R}_d^2 \times V \to V$ is pullback $\mathcal{D}_\mu^{H,V}-$ asymptotically compact.*

Proof. Let us fix $t \in \mathbb{R}$, a family $\widehat{D}_V \in \mathcal{D}_\mu^{H,V}$, a sequence $\{\tau_n\} \subset (-\infty, t]$ with $\tau_n \to -\infty$, and a sequence $\{u_{\tau_n}\} \subset V$, with $u_{\tau_n} \in D_V(\tau_n)$, for all n. We must prove that the sequence $\{u(t; \tau_n, u_{\tau_n})\}$ is relatively compact in V. For short, let us denote $u^n(s) = u(s; \tau_n, u_{\tau_n})$.

From Lemma 4.10 we know that there exists a $\tau_1(\widehat{D}_V, t) < t-3$, such that the subsequence $\{u^n : \tau_n \leq \tau_1(\widehat{D}_V, t)\} \subset \{u^n\}$ is uniformly bounded in $L^\infty(t-2, t; V) \cap L^2(t-2, t; D(A))$, with $\{(u^n)'\}$ also uniformly bounded in $L^2(t-2, t; H)$. Then, using in particular the Aubin-Lions compactness lemma (see [1], [17] or [25]) there exists an element $u \in L^\infty(t-2, t; V) \cap L^2(t-2, t; D(A))$ with $u' \in L^2(t-2, t; H)$, such that for a subsequence (relabelled the same) the following convergences hold:

$$\begin{cases} u^n \overset{*}{\rightharpoonup} u & \text{weak-star in } L^\infty(t-2, t; V), \\ u^n \rightharpoonup u & \text{weakly in } L^2(t-2, t; D(A)), \\ (u^n)' \rightharpoonup u' & \text{weakly in } L^2(t-2, t; H), \\ u^n \to u & \text{strongly in } L^2(t-2, t; V), \\ u^n(s) \to u(s) & \text{strongly in } V, \ \textit{a.e. } s \in (t-2, t). \end{cases} \tag{33}$$

Observe that $u \in C([t-2, t]; V)$, and due to (33), u satisfies the equation (5) in the interval $(t-2, t)$.

From (33) we also deduce that $\{u^n\}$ is equi-continuous in H, on $[t-2, t]$. Thus, taking into account that the sequence $\{u^n\}$ is uniformly bounded in $C([t-2, t]; V)$, by the compactness of the injection of V into H, and the Ascoli-Arzelà Theorem, we obtain that

$$u^n \to u \quad \text{strongly in} \quad C([t-2, t]; H). \tag{34}$$

Again by the uniform boundedness of $\{u^n\}$ in $C([t-2,t];V)$, we have that for all sequence $\{s_n\} \subset [t-2,t]$ with $s_n \to s_*$, it holds that

$$u^n(s_n) \rightharpoonup u(s_*) \quad \text{weakly in } V, \tag{35}$$

where we have used (34) to identify the weak limit.

Actually, we claim that

$$u^n \to u \quad \text{strongly in } C([t-1,t];V), \tag{36}$$

which in particular will imply the relative compactness.

Indeed, if (36) does not hold, there exist $\varepsilon > 0$, a sequence $\{t_n\} \subset [t-1,t]$, without loss of generality converging to some t_* and such that

$$\|u^n(t_n) - u(t_*)\| \geq \varepsilon \quad \forall n \geq 1. \tag{37}$$

From (35) we already have that

$$\|u(t_*)\| \leq \liminf_{n\to\infty} \|u^n(t_n)\|. \tag{38}$$

On the other hand, using the energy equality (6) for u and all u^n, and reasoning as for the obtention of (27), we have that for all $t-2 \leq s_1 \leq s_2 \leq t$,

$$\begin{aligned} &\|u^n(s_2)\|^2 + \nu \int_{s_1}^{s_2} |Au^n(r)|^2 dr \\ &\leq \|u^n(s_1)\|^2 + 2C^{(\nu)} \int_{s_1}^{s_2} |u^n(r)|^2 \|u^n(r)\|^4 dr + \frac{2}{\nu} \int_{s_1}^{s_2} |f(r)|^2 dr, \end{aligned} \tag{39}$$

and

$$\begin{aligned} &\|u(s_2)\|^2 + \nu \int_{s_1}^{s_2} |Au(r)|^2 dr \\ &\leq \|u(s_1)\|^2 + 2C^{(\nu)} \int_{s_1}^{s_2} |u(r)|^2 \|u(r)\|^4 dr + \frac{2}{\nu} \int_{s_1}^{s_2} |f(r)|^2 dr. \end{aligned} \tag{40}$$

In particular we can define the functions

$$\begin{aligned} J_n(s) &= \|u^n(s)\|^2 - 2C^{(\nu)} \int_{t-2}^{s} |u^n(r)|^2 \|u^n(r)\|^4 dr - \frac{2}{\nu} \int_{t-2}^{s} |f(r)|^2 dr, \\ J(s) &= \|u(s)\|^2 - 2C^{(\nu)} \int_{t-2}^{s} |u(r)|^2 \|u(r)\|^4 dr - \frac{2}{\nu} \int_{t-2}^{s} |f(r)|^2 dr. \end{aligned}$$

It is clear from the regularity of u and all u^n that these functions are continuous on $[t-2,t]$. Moreover, from the definition of J_n and (39), we have

$$
\begin{aligned}
&J_n(s_2) - J_n(s_1)\\
&= \|u^n(s_2)\|^2 - 2C^{(\nu)} \int_{t-2}^{s_2} |u^n(r)|^2 \|u^n(r)\|^4 dr - \frac{2}{\nu} \int_{t-2}^{s_2} |f(r)|^2 dr\\
&\quad - \|u^n(s_1)\|^2 + 2C^{(\nu)} \int_{t-2}^{s_1} |u^n(r)|^2 \|u^n(r)\|^4 dr + \frac{2}{\nu} \int_{t-2}^{s_1} |f(r)|^2 dr\\
&= \|u^n(s_2)\|^2 - \|u^n(s_1)\|^2 - 2C^{(\nu)} \int_{s_1}^{s_2} |u^n(r)|^2 \|u^n(r)\|^4 dr - \frac{2}{\nu} \int_{s_1}^{s_2} |f(r)|^2 dr\\
&\le -\nu \int_{s_1}^{s_2} |Au^n(r)|^2 dr\\
&\le 0 \quad \text{for all } t-2 \le s_1 \le s_2 \le t,
\end{aligned}
$$

and therefore all J_n are non-increasing functions in $[t-2, t]$. Analogously, using (40) and the definition of J, one deduces that J is also a non-increasing function in $[t-2, t]$.

Observe now that by the last convergence in (33), and (34), $\|u^n(s)\| \to \|u(s)\|$ and $|u^n(s)|^2 \|u^n(s)\|^4 \to |u(s)|^2 \|u(s)\|^4$, a.e. $s \in (t-2, t)$. Moreover, as the sequence $\{u^n\}$ is bounded in $L^\infty(t-2, t; V) \subset L^\infty(t-2, t; H)$, we have that the sequence $\{|u^n(s)|^2 \|u^n(s)\|^4\}$ is bounded in $L^\infty(t-2, t)$. Therefore, from the Lebesgue's dominated convergence theorem we deduce that

$$
\int_{t-2}^{s} |u^n(r)|^2 \|u^n(r)\|^4 dr \to \int_{t-2}^{s} |u(r)|^2 \|u(r)\|^4 dr \quad \text{for all } s \in [t-2, t].
$$

Thus,

$$
J_n(s) \to J(s) \quad a.e.\ s \in (t-2, t).
$$

Hence, there exists a sequence $\{\tilde{t}_k\} \subset (t-2, t_*)$ such that $\tilde{t}_k \to t_*$, when $k \to +\infty$, and

$$
\lim_{n \to +\infty} J_n(\tilde{t}_k) = J(\tilde{t}_k) \qquad \text{for all } k.
$$

Fix an arbitrary value $\delta > 0$. From the continuity of J, there exists k_δ such that

$$
|J(\tilde{t}_k) - J(t_*)| < \delta/2 \quad \forall k \ge k_\delta.
$$

Now consider $n(k_\delta)$ such that for all $n \ge n(k_\delta)$ it holds

$$
t_n \ge \tilde{t}_{k_\delta} \quad \text{and} \quad |J_n(\tilde{t}_{k_\delta}) - J(\tilde{t}_{k_\delta})| < \delta/2.
$$

Then, since all J_n are non-increasing, we deduce that for all $n \ge n(k_\delta)$

$$
\begin{aligned}
J_n(t_n) - J(t_*) &\le J_n(\tilde{t}_{k_\delta}) - J(t_*)\\
&\le |J_n(\tilde{t}_{k_\delta}) - J(t_*)|\\
&\le |J_n(\tilde{t}_{k_\delta}) - J(\tilde{t}_{k_\delta})| + |J(\tilde{t}_{k_\delta}) - J(t_*)| < \delta.
\end{aligned}
$$

This yields that

$$
\limsup_{n \to \infty} J_n(t_n) \le J(t_*),
$$

and therefore, by (33),

$$\limsup_{n\to\infty} \|u^n(t_n)\| \leq \|u(t_*)\|,$$

which joined to (38) and (35) implies that $u^n(t_n) \to u(t_*)$ strongly in V, in contradiction with (37). Thus, (36) holds and the relatively compactness of $\{u(t;\tau_n, u_{\tau_n})\}$ in V is proved. ■

As a consequence of the previous results, we obtain the following Theorem.

Theorem 4.14 *Suppose that $f \in L^2_{\mathrm{loc}}(\mathbb{R}; H)$ satisfies the condition (14). Then, there exist the minimal pullback $\mathcal{D}_F^V$-attractor*

$$\mathcal{A}_{\mathcal{D}_F^V} = \{\mathcal{A}_{\mathcal{D}_F^V}(t) : t \in \mathbb{R}\},$$

and the minimal pullback $\mathcal{D}_\mu^{H,V}$-attractor

$$\mathcal{A}_{\mathcal{D}_\mu^{H,V}} = \{\mathcal{A}_{\mathcal{D}_\mu^{H,V}}(t) : t \in \mathbb{R}\}$$

for the process $U : \mathbb{R}^2_d \times V \to V$ defined by (12), and the following relation holds:

$$\mathcal{A}_{\mathcal{D}_F^V}(t) \subset \mathcal{A}_{\mathcal{D}_F^H}(t) \subset \mathcal{A}_{\mathcal{D}_\mu^H}(t) = \mathcal{A}_{\mathcal{D}_\mu^{H,V}}(t) \quad \text{for all } t \in \mathbb{R}, \tag{41}$$

where $\mathcal{A}_{\mathcal{D}_F^H}$ and $\mathcal{A}_{\mathcal{D}_\mu^H}$ are respectively the minimal pullback $\mathcal{D}_F^H$-attractor and the minimal pullback $\mathcal{D}_\mu^H$-attractor for the process $U : \mathbb{R}^2_d \times H \to H$, whose existence is guaranteed by Theorem 4.8. In particular, the following pullback attraction result in V holds:

$$\lim_{\tau\to-\infty} \mathrm{dist}_V(U(t,\tau)D(\tau), \mathcal{A}_{\mathcal{D}_\mu^H}(t)) = 0 \quad \text{for all } t \in \mathbb{R} \text{ and any } \widehat{D} \in \mathcal{D}_\mu^H. \tag{42}$$

Finally, if moreover f satisfies

$$\sup_{s\leq 0} \left(e^{-\mu s} \int_{-\infty}^{s} e^{\mu\theta} |f(\theta)|^2 \, d\theta \right) < +\infty, \tag{43}$$

then

$$\mathcal{A}_{\mathcal{D}_F^V}(t) = \mathcal{A}_{\mathcal{D}_F^H}(t) = \mathcal{A}_{\mathcal{D}_\mu^H}(t) = \mathcal{A}_{\mathcal{D}_\mu^{H,V}}(t) \quad \text{for all } t \in \mathbb{R}, \tag{44}$$

and for any bounded subset B of H

$$\lim_{\tau\to-\infty} \mathrm{dist}_V(U(t,\tau)B, \mathcal{A}_{\mathcal{D}_F^H}(t)) = 0 \quad \text{for all } t \in \mathbb{R}.$$

Proof. The existence of $\mathcal{A}_{\mathcal{D}_F^V}$ and $\mathcal{A}_{\mathcal{D}_\mu^{H,V}}$ is a direct consequence of Theorem 3.11, Corollary 3.13, Proposition 4.2, Corollary 4.12, and Lemma 4.13.

The inclusions and equality in (41) are a consequence of Corollary 3.13, Theorem 3.15, and Corollary 4.12. Then, (42) is evident.

If moreover f satisfies (43), then, taking into account (16), the equality $\mathcal{A}_{\mathcal{D}_F^H}(t) = \mathcal{A}_{\mathcal{D}_\mu^H}(t)$ is a consequence of Remark 3.14, and the equality $\mathcal{A}_{\mathcal{D}_F^V}(t) = \mathcal{A}_{\mathcal{D}_F^H}(t)$, is a consequence of Theorem 3.15. ■

Remark 4.15 *(a) Observe that if $f \in L^2_{\mathrm{loc}}(\mathbb{R}; H)$ satisfies the condition (14), then it also satisfies*

$$\int_{-\infty}^0 e^{\sigma s}|f(s)|^2\,ds < +\infty, \quad \text{for all } \sigma \in (\mu, 2\nu\lambda_1).$$

Thus, for any $\sigma \in (\mu, 2\nu\lambda_1)$ there exists the corresponding minimal pullback $\mathcal{D}_\sigma^H$-attractor, $\mathcal{A}_{\mathcal{D}_\sigma^H}$.

By Theorem 3.15, since $\mathcal{D}_\mu^H \subset \mathcal{D}_\sigma^H$, it is evident that, for any $t \in \mathbb{R}$,

$$\mathcal{A}_{\mathcal{D}_\mu^H}(t) \subset \mathcal{A}_{\mathcal{D}_\sigma^H}(t) \quad \textit{for all } \sigma \in (\mu, 2\nu\lambda_1).$$

Moreover, if f satisfies (43), then, by (44),

$$\mathcal{A}_{\mathcal{D}_F^H}(t) = \mathcal{A}_{\mathcal{D}_\mu^H}(t) = \mathcal{A}_{\mathcal{D}_\sigma^H}(t) \quad \textit{for all } t \in \mathbb{R}, \textit{ and any } \sigma \in (\mu, 2\nu\lambda_1).$$

(b) In the above results, Theorem 3.15 can also be used with (ii) replaced by (ii') from Remark 3.16.

5 Tempered behaviour of the pullback attractors

The tempered behaviour in H of the pullback attractor $\mathcal{A}_{\mathcal{D}_\mu^H}$ is given by Theorem 4.8. Indeed, under the assumptions of that result, $\mathcal{A}_{\mathcal{D}_\mu^H} \in \mathcal{D}_\mu^H$, i.e. one has that

$$\lim_{t\to-\infty}\left(e^{\mu t}\sup_{v\in\mathcal{A}_{\mathcal{D}_\mu^H}(t)}|v|^2\right) = 0.$$

In this section we obtain two results about the tempered behaviour of $\mathcal{A}_{\mathcal{D}_\mu^H}(t)$, in V and $(H^2(\Omega))^2$, when time goes to $-\infty$. In fact, we will obtain the tempered behaviour for any invariant family belonging to $\mathcal{D}_\mu^H$. For related results, see [12].

Proposition 5.1 *Suppose that $f \in L^2_{loc}(\mathbb{R}; H)$ satisfies the assumption (43) in Theorem 4.14, and let $\widehat{D} \in \mathcal{D}_\mu^H$ be invariant with respect to the process U*

defined by (12) (i.e. such that $D(t) = U(t,\tau)D(\tau)$ for all $\tau \leq t$). Then,

$$\lim_{t\to-\infty}\left(e^{\mu t}\sup_{v\in D(t)}\|v\|^2\right) = 0.$$

Proof. The result is a consequence of the invariance of $\widehat{D}$, the second estimate in (15) in Lemma 4.10, and the tempered character of the expression (17), since for $f \in L^2_{loc}(\mathbb{R}; H)$, the condition (43) is equivalent to

$$\sup_{s\leq t}\int_{s-1}^{s}|f(\theta)|^2\,d\theta < +\infty, \quad \text{for all } t \in \mathbb{R}. \tag{45}$$

■

Assuming now that $f' \in L^2_{loc}(\mathbb{R}; H)$, we can obtain the tempered behaviour in $(H^2(\Omega))^2$, for any invariant family belonging to $\mathcal{D}^H_\mu$. We first prove the following result, which completes the estimates obtained in Lemma 4.10.

Proposition 5.2 *If $f \in W^{1,2}_{loc}(\mathbb{R}; H)$ and satisfies (14), then for each $t \in \mathbb{R}$ and $\widehat{D} \in \mathcal{D}^H_\mu$ there exists $\tau_1(\widehat{D}, t) < t - 3$ such that*

$$|AU(r,\tau)u_\tau|^2 \leq \rho_6(t) \quad \text{for all } r \in [t-1,t],\ \tau \leq \tau_1(\widehat{D},t),\ u_\tau \in D(\tau),$$

where

$$\rho_6(t) = \frac{4}{\nu^2}(\rho_5(t) + \max_{r\in[t-1,t]}|f(r)|^2) + \frac{2C^{(\nu)}}{\nu}\rho_1(t)\rho_2(t)^2, \tag{46}$$

with $\rho_5(t)$ defined by

$$\rho_5(t) = \left(\rho_4(t) + \frac{1}{\nu\lambda_1}\int_{t-2}^{t}|f'(\theta)|^2\,d\theta\right)\exp\left(\frac{C_1^2}{\nu}\rho_2(t)\right), \tag{47}$$

and where the $\rho_i(t)$, $i = 1, 2, 4$, are given by (16), (17) and (19).

Proof. We consider the Galerkin approximations used in the proof of Lemma 4.10.

As we are assuming that $f \in W^{1,2}_{loc}(\mathbb{R}; H)$, we can differentiate with respect to time in (20), and then, multiplying by $\gamma'_{nj}(s)$, and summing from $j = 1$ to n, we obtain

$$\frac{1}{2}\frac{d}{d\theta}|u'_n(\theta)|^2 + \nu\|u'_n(\theta)\|^2 + b(u'_n(\theta), u_n(\theta), u'_n(\theta)) = (f'(\theta), u'_n(\theta)),$$

a.e. $\theta > \tau$.

From this inequality, taking into account that

$$|(f'(\theta), u'_n(\theta))| \leq \frac{\nu}{2}\|u'_n(\theta)\|^2 + \frac{1}{2\nu\lambda_1}|f'(\theta)|^2,$$

and that by (4)

$$
\begin{aligned}
|b(u_n'(\theta), u_n(\theta), u_n'(\theta))| &\leq C_1 |u_n'(\theta)| \|u_n'(\theta)\| \|u_n(\theta)\| \\
&\leq \frac{\nu}{2} \|u_n'(\theta)\|^2 + \frac{C_1^2}{2\nu} |u_n'(\theta)|^2 \|u_n(\theta)\|^2,
\end{aligned}
$$

we deduce

$$
\frac{d}{d\theta} |u_n'(\theta)|^2 \leq \frac{1}{\nu\lambda_1} |f'(\theta)|^2 + \frac{C_1^2}{\nu} |u_n'(\theta)|^2 \|u_n(\theta)\|^2, \quad \text{a.e. } \theta > \tau.
$$

Integrating in the last inequality,

$$
|u_n'(r)|^2 \leq |u_n'(s)|^2 + \frac{1}{\nu\lambda_1} \int_{r-1}^{r} |f'(\theta)|^2 \, d\theta + \frac{C_1^2}{\nu} \int_s^r |u_n'(\theta)|^2 \|u_n(\theta)\|^2 \, d\theta,
$$

for all $\tau \leq r - 1 \leq s \leq r$.

Thus, by Gronwall's inequality,

$$
|u_n'(r)|^2 \leq \left(|u_n'(s)|^2 + \frac{1}{\nu\lambda_1} \int_{r-1}^{r} |f'(\theta)|^2 \, d\theta \right) \exp\left(\frac{C_1^2}{\nu} \int_{r-1}^{r} \|u_n(\theta)\|^2 \, d\theta \right),
$$

for all $\tau \leq r - 1 \leq s \leq r$.

Now, integrating this inequality with respect to s between $r - 1$ and r, we obtain

$$
\begin{aligned}
|u_n'(r)|^2 \leq & \left(\int_{r-1}^{r} |u_n'(s)|^2 \, ds + \frac{1}{\nu\lambda_1} \int_{r-1}^{r} |f'(\theta)|^2 \, d\theta \right) \\
& \times \exp\left(\frac{C_1^2}{\nu} \int_{r-1}^{r} \|u_n(\theta)\|^2 \, d\theta \right),
\end{aligned}
$$

for all $\tau \leq r - 1$ and any $n \geq 1$, and therefore, by (29) and (32) we deduce that for any $n \geq 1$,

$$
|u_n'(r; \tau, u_\tau)|^2 \leq \rho_5(t) \quad \text{for all } r \in [t-1, t],\ \tau \leq \tau_1(\widehat{D}, t),\ u_\tau \in D(\tau), \tag{48}
$$

where $\rho_5(t)$ is given by (47).

Finally, multiplying again in (20) by $\lambda_j \gamma_{nj}(s)$, and summing once more from $j = 1$ to n, we obtain

$$
(u_n'(r), Au_n(r)) + \nu |Au_n(r)|^2 + b(u_n(r), u_n(r), Au_n(r)) = (f(r), Au_n(r)), \tag{49}
$$

a.e. $r \geq \tau$.

But
$$|\,(u_n'(r), Au_n(r))\,| \leq \frac{2}{\nu}\,|u_n'(r)|^2 + \frac{\nu}{8}\,|Au_n(r)|^2\,,$$
and
$$|\,(f(r), Au_n(r))\,| \leq \frac{2}{\nu}\,|f(r)|^2 + \frac{\nu}{8}\,|Au_n(r)|^2\,.$$
Therefore, taking into account (26), we deduce from (49) that

$$\frac{\nu}{2}\,|Au_n(r)|^2 \leq \frac{2}{\nu}(|u_n'(r)|^2 + |f(r)|^2) + C^{(\nu)}|u_n(r)|^2\|u_n(r)\|^4,$$

for all $r \geq \tau$.

Thus, since in particular $f \in C(\mathbb{R}; H)$, from (24), (29) and (48) we deduce that for any $n \geq 1$,

$$|Au_n(r;\tau,u_\tau)|^2 \leq \rho_6(t) \quad \text{for all } r \in [t-1,t],\ \tau \leq \tau_1(\widehat{D},t),\ u_\tau \in D(\tau), \quad (50)$$

where $\rho_6(t)$ is given by (46).

The result now is a consequence of Lemma 4.9 and (50), taking into account the well known facts that $u_n(\cdot;\tau,u_\tau)$ converges weakly to $u(\cdot;\tau,u_\tau)$ in $L^2(t-1,t;V)$, and $u(\cdot;\tau,u_\tau) \in C([t-1,t];V)$. ■

Now, we may conclude a result about tempered behaviour in $(H^2(\Omega))^2$.

Proposition 5.3 *Suppose that $f \in W^{1,2}_{loc}(\mathbb{R}; H)$ satisfies the assumption (43) in Theorem 4.14, and moreover*

$$\lim_{t\to-\infty}\left(e^{\mu t}\int_{t-1}^{t}|f'(\theta)|^2\,d\theta\right) = 0, \quad (51)$$

and
$$\lim_{t\to-\infty}\left(e^{\mu t}|f(t)|^2\right) = 0. \quad (52)$$

Then, for every family $\widehat{D} \in \mathcal{D}^H_\mu$ invariant with respect to the process U defined by (12), one has
$$\lim_{t\to-\infty}\left(e^{\mu t}\sup_{v\in D(t)}\|v\|^2_{(H^2(\Omega))^2}\right) = 0.$$

Proof. Observe that

$$|f(r)| \leq |f(t-1)| + \left(\int_{t-1}^{t}|f'(\theta)|^2\,d\theta\right)^{1/2} \quad \text{for all } r \in [t-1,t].$$

Thus, taking into account (51) and (52), the result follows from the invariance of $\widehat{D}$, Proposition 5.2, (16), (17), (19), and the fact that, as we observed in the proof of Proposition 5.1, the condition (43) is equivalent to (45). ■

References

[1] J. P. Aubin, Un théorème de compacité, *C. R. Acad. Sci. Paris* **256** (1963), 5042–5044.

[2] T. Caraballo, G. Łukaszewicz & J. Real, Pullback attractors for asymptotically compact non-autonomous dynamical systems, *Nonlinear Anal.* **64** (2006), 484–498.

[3] T. Caraballo, G. Łukaszewicz & J. Real, Pullback attractors for non-autonomous 2D-Navier-Stokes equations in some unbounded domains, *C. R. Acad. Sci. Paris* **342** (2006), 263–268.

[4] V. V. Chepyzhov and M. I. Vishik, Attractors of non-autonomous dynamical systems and their dimension , *J. Math. Pures Appl.* **73** (1994), 279–333.

[5] V. V. Chepyzhov and M. I. Vishik, *Attractors for Equations of Mathematical Physics*, Colloquium Publications **49**, Providence, American Mathematical Society, 2002.

[6] I. D. Chueshov, *Monotone Random Systems Theory and Applications*, *Lecture Notes in Mathematics* **1779** Berlin Heidelberg: Springer-Verlag, 2002.

[7] I. Chueshov, T. Caraballo, P. Marín-Rubio, and J. Real, Existence and asymptotic behaviour for stochastic heat equations with multiplicative noise in materials with memory, *Discrete Contin. Dyn. Syst.* **18** (2007), 253–270.

[8] P. Constantin and C. Foias, *Navier Stokes Equations*, The University of Chicago Press, Chicago, 1988.

[9] H. Crauel, A. Debussche, and F. Flandoli, Random attractors, *J. Dynam. Differential Equations* **9** (1997), 307–341.

[10] H. Crauel and F. Flandoli, Attractors for random dynamical systems, *Probab. Theory Relat. Fields* **100** (1994), 365–393.

[11] C. Foias, O. Manley, R. Rosa, and R. Temam, *Navier-Stokes Equations and Turbulence,* Encyclopedia of Mathematics and its Applications, 83. Cambridge University Press, Cambridge, 2001.

[12] J. García-Luengo, P. Marín-Rubio, and J. Real, H^2-boundedness of the pullback attractors for non-autonomous 2D Navier-Stokes equations in bounded domains, *Nonlinear Anal.* **74** (2011), 4882–4887.

[13] M. J. Garrido-Atienza and P. Marín-Rubio, Navier-Stokes equations with delays on unbounded domains, *Nonlinear Anal.* **64** (2006), 1100–1118.

[14] A. V. Kapustyan, V. S. Melnik, and J. Valero, Attractos fo multivalued dynamical processes generated by phase-field equations, *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* **13** (2003), 1969–1983.

[15] P. Kloeden, P. Marín-Rubio, and J. Real, Pullback attractors for a semilinear heat equation in a non-cylindrical domain, *J. Differential Equations* **244** (2008), 2062–2090.

[16] P. Kloeden, P. Marín-Rubio, and J. Real, Equivalence of invariant measures and stationary statistical solutions for the autonomous globally modified Navier-Stokes equations, *Commun. Pure Appl. Anal.* **8** (2009), 785–802.

[17] J. L. Lions, *Quelques Méthodes de Résolution des Problèmes aux Limites Non Linéaires,* Dunod, Paris, 1969.

[18] Y. Lu, Uniform attractors for the closed process and applications to the reaction-diffusion equation with dynamical boundary condition, *Nonlinear Anal.* **71** (2009), 4012–4025.

[19] P. Marín-Rubio, G. Planas, and J. Real, Asymptotic behaviour of a phase-field model with three coupled equations without uniqueness, *J. Differential Equations* **246** (2009), 4632–4652.

[20] P. Marín-Rubio and J. Real, Attractors for 2D-Navier-Stokes equations with delays on some unbounded domains, *Nonlinear Anal.* **67** (2007), 2784–2799.

[21] P. Marín-Rubio and J. Real, On the relation between two different concepts of pullback attractors for non-autonomous dynamical systems, *Nonlinear Anal.* **71** (2009), 3956–3963.

[22] P. Marín-Rubio and J. Real, Pullback attractors for $2D-$Navier-Stokes equations with delays in continuous and sub-linear operators, *Discrete Contin. Dyn. Syst.* **26** (2010), 989–1006.

[23] P. Marín-Rubio and J. Robinson, Attractors for the three-dimensional stochastic Navier-Stokes equations, *Stoch. Dyn.* **3** (2003), 279–297.

[24] J. C. Robinson, *Infinite-dimensional dynamical systems,* Cambridge University Press, 2001.

[25] J. Simon, Compact sets in the space $L^p(0,T;B)$, *Ann. Mat. Pura Appl.(4)* **146** (1987), 65–96.

[26] R. Temam, *Infinite Dimensional Dynamical Systems in Mechanics and Physics,* Springer, New York, 1988.