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PULLBACK ATTRACTORS FOR THE NON-AUTONOMOUS 2D NAVIER–STOKES EQUATIONS FOR MINIMALLY REGULAR FORCING

Julia García-Luengo, Pedro Marín-Rubio, and José Real

Dpto. de Ecuaciones Diferenciales y Análisis Numérico, Universidad de Sevilla
Apdo. de Correos 1160, 41080-Sevilla (Spain)

James C. Robinson

Mathematics Institute, University of Warwick, Coventry CV4 7AL, UK

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Abstract. This paper treats the existence of pullback attractors for the non-autonomous 2D Navier–Stokes equations in two different spaces, namely L^2 and H^1. The non-autonomous forcing term is taken in $L^2_{\rm loc}(\mathbb{R};H^{-1})$ and $L^2_{\rm loc}(\mathbb{R};L^2)$ respectively for these two results: even in the autonomous case it is not straightforward to show the required asymptotic compactness of the flow with this regularity of the forcing term. Here we prove the asymptotic compactness of the corresponding processes by verifying the flattening property – also known as "condition (C)". We also show, using the semigroup method, that a little additional regularity – $f \in L^p_{\rm loc}(\mathbb{R};H^{-1})$ or $f \in L^p_{\rm loc}(\mathbb{R};L^2)$ for some $p > 2$ – is enough to ensure the existence of a compact pullback absorbing family (not only asymptotic compactness). Even in the autonomous case the existence of a compact absorbing set for this model is new when f has such limited regularity.

1. **Introduction.** In this paper we consider the existence of attractors in H (essentially L^2) and in V (essentially H^1) for the incompressible two-dimensional Navier–Stokes equations

$$u_t - \nu\Delta u + (u\cdot\nabla)u + \nabla p = f(t) \qquad \nabla\cdot u = 0$$

when the forcing term has the minimal regularity required to obtain solutions that evolve continuously in these phase spaces. Similar problems have previously been considered by Rosa [24] in the autonomous case (in the phase space H with $f \in V'$), and by García-Luengo et al. [13] in the non-autonomous case (in the phase space V with f satisfying the same conditions as in this paper). In all these cases, the existence of attractors has been shown by proving some sort of asymptotic compactness of the corresponding flow (or process in the non-autonomous case).

To verify asymptotic compactness one can either proceed directly, or make use of a splitting of the solutions into high and low components. Such a splitting is a very common technique in the study of the qualitative behaviour of solutions for

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Joseé Real died on January 27th, 2012. J.G.-L., P.M.-R., and J.C.R. would like to dedicate this paper to his memory.

PDE problems, in particular when considering the long-time behaviour of dynamics, as in the construction of invariant manifolds [6, 15] and inertial manifolds [10, 7], the squeezing property [9, 26], the notion of 'determining modes' [8, 16], and the theory of attractors [20]. In the context of proofs of the existence of attractors it was formalised by Ma, Wang, and Zhong [20] as their celebrated 'condition (C)'. A more descriptive terminology, 'the flattening property', was coined by Kloeden and Langa [17], and we adopt this terminology here. However, it is worth making the observation that this is not so much a 'property' as a (powerful) technique for obtaining the asymptotic compactness of a flow, be it autonomous or non-autonomous. We return to this point of view later in the paper.

Here we consider attractors in both H and V. We show that when $f \in L^2_{\rm loc}(\mathbb{R}; V')$ – which is the minimum regularity of f consistent with weak solutions that have $u \in L^2_{\rm loc}(\mathbb{R}; V)$ and $u_t \in L^2_{\rm loc}(\mathbb{R}; V')$ – the process is pullback asymptotically compact. We do this using the method developed in [13], and also show as a consequence that the process satisfies 'Condition (C)'. With only a little more regularity of f, namely $f \in L^p_{\rm loc}(\mathbb{R}; V')$ for some $p > 2$, we are able to show, using the semigroup approach of Fujita and Kato [11] and ideas from the ϵ-regularity theory developed by Arrieta and Carvalho [1], that in fact there is a compact pullback absorbing family in H. In particular, in the autonomous case it follows that for $f \in V'$ there is a compact absorbing set. All the proofs depend in a crucial way on Lemma 4.5, which (essentially) guarantees the asymptotic compactness in H.

To treat attractors in V when $f \in L^2_{\rm loc}(\mathbb{R}; H)$ is significantly more straightforward. One can seek to prove asymptotic compactness directly, as in [13], but this is little easier than the analysis we present here for the phase space H. In fact in this case use of the Fourier splitting technique ('the flattening property') makes the analysis significantly simpler, and the argument is much shorter than that in [13]. With a little extra regularity (again, $f \in L^p_{\rm loc}(\mathbb{R}; H)$ for some $p > 2$) the semigroup approach yields - in this case very quickly - the existence of a compact pullback absorbing family in V.

Our goal in this paper is to obtain the flattening property for a non-autonomous 2D Navier–Stokes model in different norms, namely in L^2 and H^1, when the forcing has the minimal regularity for generating weak and strong solutions, respectively.

While in the case of L^2 a direct proof of asymptotic compactness is no harder than a proof of the flattening property (indeed, we will obtain both from Lemma 4.5, the asymptotic compactness following almost immediately), in H^1 a proof of asymptotic compactness via the flattening property is significantly shorter than proofs in previous papers, see [9] and [22, 13], which were based on more involved inequalities and on the energy method used by Rosa [24], respectively. This is due to the fact that there are stronger estimates available for the nonlinear term in H^1 than in L^2:

$$|b(u,u,q)| \le c|u|\|u\|\|q\|, \qquad |b(u,u,Aq)| \le c\|u\||u|^{1/2}\|\nabla u\|^{1/2}\|\nabla q\|,$$

where $\|\cdot\|$ denotes the norm in H^1, see properties (4) and (5), below.

The structure of the paper is as follows. Section 2 contains some preliminaries, including the functional setting of the problem. Section 3 is devoted to recalling standard results from the theory of pullback attractors (within the framework of time-dependent universes of sets), such as existence and comparison, and the flattening property in a Banach space.

The analysis in the space L^2 for $u_\tau \in L^2$ and $f \in L^2_{\rm loc}(\mathbb{R}; V')$ is carried out in Section 4, where we conclude the existence of minimal pullback attractors for a universe not only of fixed bounded sets but also for a set of tempered universes, that depends on integrability conditions of the force in the problem under a suitable weight, namely

$$\int_{-\infty}^{0} e^{\mu t}\|f(t)\|_*^2\, dt < \infty \tag{1}$$

for some $0 < \mu < 2\nu\lambda_1$, where λ_1 is the first eigenvalue of the Stokes operator. As announced before, we obtain the existence of a compact pullback absorbing family in H if we strengthen the regularity of f to $f \in L^p_{\rm loc}(\mathbb{R}; V')$ for some $p > 2$.

We then establish additional regularity results in Section 5, under the assumption that $f \in L^2_{\rm loc}(\mathbb{R}; H)$ and an integrability condition similar to (1); we obtain the flattening property in the H^1 norm, which implies the asymptotic compactness of the corresponding process in this norm, to finish with the existence of minimal pullback attractors and comparison among them under suitable additional assumptions. Again, we are able to show the existence of a compact pullback absorbing family in V if we strengthen the regularity of f to $f \in L^p_{\rm loc}(\mathbb{R}; H)$ for some $p > 2$.

2. **Statement of the problem.** Let $\Omega \subset \mathbb{R}^2$ be an open bounded set with smooth enough boundary $\partial\Omega$, and consider an arbitrary initial time $\tau \in \mathbb{R}$, and the following Navier-Stokes problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu\Delta u + (u\cdot\nabla)u + \nabla p = f(t) \quad \text{in} \;\; \Omega\times(\tau,\infty), \\ \operatorname{div} u = 0 \quad \text{in} \;\; \Omega\times(\tau,\infty), \\ u = 0 \quad \text{on} \;\; \partial\Omega\times(\tau,\infty), \\ u(x,\tau) = u_\tau(x), \quad x \in \Omega, \end{cases} \tag{2}$$

where we assume that $\nu > 0$ is the kinematic viscosity, $u = (u_1, u_2)$ is the velocity field of the fluid, p is the pressure, u_τ is the initial velocity field, and f is an external force term depending on time.

To start, we consider the usual spaces in the variational theory of Navier-Stokes equations:

$$\mathcal{V} = \{u \in (C_0^\infty(\Omega))^2 : \operatorname{div} u = 0\},$$

$H =$ the closure of $\mathcal{V}$ in $(L^2(\Omega))^2$ with the norm $|\cdot|$, and inner product $(\cdot,\cdot)$, where for $u, v \in (L^2(\Omega))^2$,

$$(u,v) = \sum_{j=1}^{2}\int_\Omega u_j(x)v_j(x)\, dx,$$

$V =$ the closure of $\mathcal{V}$ in $(H_0^1(\Omega))^2$ with the norm $\|\cdot\|$ associated to the inner product $((\cdot,\cdot))$, where for $u, v \in (H_0^1(\Omega))^2$,

$$((u,v)) = \sum_{i,j=1}^{2}\int_\Omega \frac{\partial u_j}{\partial x_i}\frac{\partial v_j}{\partial x_i}\, dx.$$

We will use $\|\cdot\|_*$ for the norm in V' and $\langle\cdot,\cdot\rangle$ for the duality product between V' and V. We consider every element $h \in H$ as an element of V', given by the equality $\langle h, v\rangle = (h,v)$ for all $v \in V$. Then, it follows that $V \subset H \subset V'$, where the injections are dense and compact.

Define the operator $A : V \to V'$ as $\langle Au, v\rangle = ((u, v))$ for all $u,\ v \in V$. Let us denote by $D(A) = \{u \in V : Au \in H\}$. By the regularity of $\partial\Omega$, one has $D(A) = (H^2(\Omega))^2 \cap V$, and $Au = -P\Delta u$ for all $u \in D(A)$ is the Stokes operator (P is the ortho-projector from $(L^2(\Omega))^2$ onto H). On $D(A)$ we consider the norm $|\cdot|_{D(A)}$ defined by $|u|_{D(A)} = |Au|$. Observe that on $D(A)$ the norms $\|\cdot\|_{(H^2(\Omega))^2}$ and $|\cdot|_{D(A)}$ are equivalent (see [5] or [25]), and $D(A)$ is compactly and densely injected in V.

Let us define

$$b(u, v, w) = \sum_{i,j=1}^{2} \int_\Omega u_i \frac{\partial v_j}{\partial x_i} w_j \, dx,$$

for all functions $u, v, w : \Omega \to \mathbb{R}^2$ for which the right-hand side is well defined.

In particular, b has sense for all $u, v, w \in V$, and is a continuous trilinear form on $V \times V \times V$.

Some useful properties concerning b that we will use throughout the paper are the following (see [26] or [23]):

$$b(u, v, w) = -b(u, w, v) \quad \forall\, u, v, w \in V, \tag{3}$$

$$|b(u, v, w)| \le 2^{-1/2}|u|^{1/2}\|u\|^{1/2}\|v\||w|^{1/2}\|w\|^{1/2} \quad \forall\, u, v, w \in V, \tag{4}$$

and there exists a constant $C_1 > 0$, depending only on Ω, such that

$$|b(u, v, w)| \le C_1|u|^{1/2}|Au|^{1/2}\|v\||w| \quad \forall\, u \in D(A),\ v \in V,\ w \in H. \tag{5}$$

For any $u,\ v \in V$, we will also denote by $B(u, v)$ the operator of V' given by

$$\langle B(u, v), w\rangle = b(u, v, w) \quad \forall\, w \in V$$

and $B(u) = B(u, u)$.

Assume that $u_\tau \in H$ and $f \in L^2_{\mathrm{loc}}(\mathbb{R}; V')$.

Definition 2.1. A weak solution to (2) is a function u that belongs to $L^2(\tau, T; V) \cap L^\infty(\tau, T; H)$ for all $T > \tau$, with $u(\tau) = u_\tau$, and such that for all $v \in V$,

$$\frac{d}{dt}(u(t), v) + \nu\langle Au(t), v\rangle + b(u(t), u(t), v) = \langle f(t), v\rangle, \tag{6}$$

where the equation must be understood in the sense of $\mathcal{D}'(\tau, \infty)$.

Note that for the right-hand side to be defined we certainly require $f(t) \in V'$ for almost every $t > \tau$; we choose $f \in L^2_{\mathrm{loc}}(\mathbb{R}; V')$ so that we can interpret the initial condition and obtain an energy equality for solutions. Indeed, if u is a weak solution to (2) and $f \in L^2_{\mathrm{loc}}(\mathbb{R}; V')$ then from (6) we deduce that for any $T > \tau$, one has $u' \in L^2(\tau, T; V')$, and so $u \in C([\tau, \infty); H)$, whence the initial datum has full sense. Moreover, in this case the following energy equality holds:

$$|u(t)|^2 + 2\nu \int_s^t \|u(r)\|^2 \, dr = |u(s)|^2 + 2\int_s^t \langle f(r), u(r)\rangle \, dr \quad \forall\, \tau \le s \le t.$$

In Section 4 we will prove the existence of pullback attractors in H with this (minimal) regularity requirement on f, coupled with the boundedness condition

$$\int_{-\infty}^0 e^{\mu s}\|f(s)\|_*^2 \, ds < \infty \tag{7}$$

for some $\mu \in (0, 2\nu\lambda_1)$.

A notion of more regular solution is also suitable for problem (2).

Definition 2.2. A strong solution to problem (2) is a weak solution u to (2) such that u belongs to $L^2(\tau,T;D(A))\cap L^\infty(\tau,T;V)$ for all $T>\tau$.

If $f\in L^2_{\rm loc}(\mathbb{R};H)$ and u is a strong solution to (2), then $u'\in L^2(\tau,T;H)$ for all $T>\tau$, and so $u\in C([\tau,\infty);V)$. In this case the following energy equality holds:

$$\begin{aligned}&\|u(t)\|^2+2\nu\int_s^t|Au(r)|^2\,dr+2\int_s^t b(u(r),u(r),Au(r))\,dr\\ =\;&\|u(s)\|^2+2\int_s^t(f(r),Au(r))\,dr\quad\forall\,\tau\le s\le t.\end{aligned}\tag{8}$$

We study pullback attractors in the space V in Section 5, again taking the minimal regularity requirement on f for the existence of such solutions, along with a condition parallel to (7), namely

$$\int_{-\infty}^0 e^{\mu s}|f(s)|^2\,ds<\infty$$

for some $\mu\in(0,2\nu\lambda_1)$.

3. Abstract results on minimal pullback attractors. Pullback $\widehat{D}_0$-flattening property.

In this section we recall some results from [13] about the existence of minimal pullback attractors (see also [2, 3, 21]).

Let (X,d_X) be a metric space, and define $\mathbb{R}^2_d=\{(t,\tau)\in\mathbb{R}^2:\tau\le t\}$.

A process U on X is a mapping $\mathbb{R}^2_d\times X\ni(t,\tau,x)\mapsto U(t,\tau)x\in X$ such that $U(\tau,\tau)x=x$ for any $(\tau,x)\in\mathbb{R}\times X$, and $U(t,r)(U(r,\tau)x)=U(t,\tau)x$ for any $\tau\le r\le t$ and all $x\in X$.

Definition 3.1. Let U be a process on X.

(a) U is said to be continuous if for any pair $\tau\le t$, the mapping $U(t,\tau):X\to X$ is continuous.

(b) U is said to be closed if for any $\tau\le t$, and any sequence $\{x_n\}\subset X$, if $x_n\to x\in X$ and $U(t,\tau)x_n\to y\in X$, then $U(t,\tau)x=y$.

It is clear that every continuous process is closed.

Let us denote by $\mathcal{P}(X)$ the family of all nonempty subsets of X, and consider a family of nonempty sets $\widehat{D}_0=\{D_0(t):t\in\mathbb{R}\}\subset\mathcal{P}(X)$.

Definition 3.2. We say that a process U on X is pullback $\widehat{D}_0$-asymptotically compact if for any $t\in\mathbb{R}$ and any sequences $\{\tau_n\}\subset(-\infty,t]$ and $\{x_n\}\subset X$ satisfying $\tau_n\to-\infty$ and $x_n\in D_0(\tau_n)$ for all n, the sequence $\{U(t,\tau_n)x_n\}$ is relatively compact in X.

Define

$$\Lambda(\widehat{D}_0,t)=\bigcap_{s\le t}\overline{\bigcup_{\tau\le s}U(t,\tau)D_0(\tau)}^X\qquad\forall\,t\in\mathbb{R},$$

where $\overline{\{\cdots\}}^X$ is the closure in X.

Given two subsets of X, $\mathcal{O}_1$ and $\mathcal{O}_2$, we denote by $\mathrm{dist}_X(\mathcal{O}_1,\mathcal{O}_2)$ the Hausdorff semi-distance in X between them, defined as

$$\mathrm{dist}_X(\mathcal{O}_1,\mathcal{O}_2)=\sup_{x\in\mathcal{O}_1}\inf_{y\in\mathcal{O}_2}d_X(x,y).$$

Let $\mathcal{D}$ be a nonempty class of families parameterized in time $\widehat{D}=\{D(t):t\in\mathbb{R}\}\subset\mathcal{P}(X)$. The class $\mathcal{D}$ will be called a universe in $\mathcal{P}(X)$.

Definition 3.3. A process U on X is said to be pullback $\mathcal{D}$-asymptotically compact if it is pullback $\widehat{D}$-asymptotically compact for any $\widehat{D} \in \mathcal{D}$.

We say that $\widehat{D}_0 = \{D_0(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ is pullback $\mathcal{D}$-absorbing for the process U on X if for any $t \in \mathbb{R}$ and any $\widehat{D} \in \mathcal{D}$, there exists a $\tau_0(\widehat{D}, t) \leq t$ such that

$$U(t,\tau)D(\tau) \subset D_0(t) \quad \forall \tau \leq \tau_0(\widehat{D}, t).$$

We have the following result (see [13, Theorem 3.11]).

Theorem 3.4. *Consider a closed process $U : \mathbb{R}^2_d \times X \to X$, a universe $\mathcal{D}$ in $\mathcal{P}(X)$, and a family $\widehat{D}_0 = \{D_0(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ that is pullback $\mathcal{D}$-absorbing for U, and assume also that U is pullback $\widehat{D}_0$-asymptotically compact.*

Then, the family $\mathcal{A}_\mathcal{D} = \{\mathcal{A}_\mathcal{D}(t) : t \in \mathbb{R}\}$ defined by $\mathcal{A}_\mathcal{D}(t) = \overline{\bigcup_{\widehat{D} \in \mathcal{D}} \Lambda(\widehat{D}, t)}^X$, has the following properties:

- *(a) for any $t \in \mathbb{R}$, the set $\mathcal{A}_\mathcal{D}(t)$ is a nonempty compact subset of X, and $\mathcal{A}_\mathcal{D}(t) \subset \Lambda(\widehat{D}_0, t)$,*
- *(b) $\mathcal{A}_\mathcal{D}$ is pullback $\mathcal{D}$-attracting, i.e. $\lim_{\tau \to -\infty} \mathrm{dist}_X(U(t,\tau)D(\tau), \mathcal{A}_\mathcal{D}(t)) = 0$ for all $\widehat{D} \in \mathcal{D}$, and any $t \in \mathbb{R}$,*
- *(c) $\mathcal{A}_\mathcal{D}$ is invariant, i.e. $U(t,\tau)\mathcal{A}_\mathcal{D}(\tau) = \mathcal{A}_\mathcal{D}(t)$ for all $(t,\tau) \in \mathbb{R}^2_d$,*
- *(d) if $\widehat{D}_0 \in \mathcal{D}$, then $\mathcal{A}_\mathcal{D}(t) = \Lambda(\widehat{D}_0, t) \subset \overline{D_0(t)}^X$ for all $t \in \mathbb{R}$.*

The family $\mathcal{A}_\mathcal{D}$ is minimal in the sense that if $\widehat{C} = \{C(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ is a family of closed sets such that for any $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \in \mathcal{D}$, $\lim_{\tau \to -\infty} \mathrm{dist}_X(U(t,\tau)D(\tau), C(t)) = 0$, then $\mathcal{A}_\mathcal{D}(t) \subset C(t)$.

A family $\mathcal{A}_\mathcal{D}$ that satisfies properties (a)–(c) in Theorem 3.4 is called a minimal pullback $\mathcal{D}$-attractor for the process U. If $\mathcal{A}_\mathcal{D} \in \mathcal{D}$ then it is the unique family of closed subsets in $\mathcal{D}$ that satisfies (b) and (c).

Sufficient conditions that $\mathcal{A}_\mathcal{D} \in \mathcal{D}$ are

- (i) $\widehat{D}_0 \in \mathcal{D}$,
- (ii) the set $D_0(t)$ is closed for all $t \in \mathbb{R}$, and
- (iii) the universe $\mathcal{D}$ is *inclusion-closed*, i.e. if $\widehat{D} \in \mathcal{D}$, and $\widehat{D}' = \{D'(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ with $D'(t) \subset D(t)$ for all t, then $\widehat{D}' \in \mathcal{D}$.

We will denote by $\mathcal{D}^X_F$ the universe of fixed nonempty bounded subsets of X, i.e., the class of all families $\widehat{D}$ of the form $\widehat{D} = \{D(t) = D : t \in \mathbb{R}\}$ with D a fixed nonempty bounded subset of X.

Now, it is easy to conclude the following result [21].

Corollary 1. *Under the assumptions of Theorem 3.4, if the universe $\mathcal{D}$ contains the universe $\mathcal{D}^X_F$, then both attractors, $\mathcal{A}_{\mathcal{D}^X_F}$ and $\mathcal{A}_\mathcal{D}$, exist, and $\mathcal{A}_{\mathcal{D}^X_F}(t) \subset \mathcal{A}_\mathcal{D}(t)$ for all $t \in \mathbb{R}$.*

Moreover, if for some $T \in \mathbb{R}$, the set $\cup_{t \leq T} D_0(t)$ is a bounded subset of X, then $\mathcal{A}_{\mathcal{D}^X_F}(t) = \mathcal{A}_\mathcal{D}(t)$ for all $t \leq T$.

Now, we introduce a notion which is a slight modification of Ma, Wang, and Zhong's "Condition (C)" [20] (re-christened the "flattening property" by Kloeden and Langa [17]), after Definition XXX in the book by Carvalho, Langa, and Robinson [4], where P_ε need not be a projection operator.

Definition 3.5. Assume that X is a Banach space with norm $\|\cdot\|_X$, and $\widehat{D}_0 = \{D_0(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ is a given family. We will say that the process U on X satisfies the pullback $\widehat{D}_0$-flattening property if for any $t \in \mathbb{R}$, and $\varepsilon > 0$, there exist $\tau_\varepsilon < t$, a finite dimensional subspace X_ε of X, and a mapping $P_\varepsilon : X \to X_\varepsilon$, all depending on $\widehat{D}_0$, t and ε, such that

$$\{P_\varepsilon U(t,\tau)u_\tau : \tau \le \tau_\varepsilon,\ u_\tau \in D_0(\tau)\} \text{ is bounded in } X$$

and

$$\|(I - P_\varepsilon)U(t,\tau)u_\tau\|_X < \varepsilon \quad \text{for any } \tau \le \tau_\varepsilon,\ u_\tau \in D_0(\tau).$$

Similarly to the results in [20] and [17] (see also [4]) we will see that to show that a process U is pullback $\widehat{D}_0$-asymptotically compact, it is enough to verify the pullback $\widehat{D}_0$-flattening property given in the definition above.

Proposition 1. *Assume that X is a Banach space and $\widehat{D}_0 = \{D_0(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ is a given family such that the process U on X satisfies the pullback $\widehat{D}_0$-flattening property. Then the process U is pullback $\widehat{D}_0$-asymptotically compact.*

Proof. Let $t \in \mathbb{R}$, a sequence $\{\tau_n\} \subset (-\infty, t]$ such that $\tau_n \to -\infty$, and a sequence $\{x_n\} \subset X$ such that $x_n \in D_0(\tau_n)$ for all n, be fixed.

For a fixed integer $k \ge 1$, by the pullback $\widehat{D}_0$-flattening property, there exist $N_k \ge 1$, a finite dimensional subspace X_k of X, and a mapping $P_k : X \to X_k$, such that $\{P_k U(t,\tau_n)x_n : n \ge N_k\}$ is a bounded subset of X_k, and therefore a relatively compact subset of X, and $\|(I - P_k)U(t,\tau_n)x_n\|_X \le 1/(2k)$ for all $n \ge N_k$. Thus, $\{U(t,\tau_n)x_n : n \ge 1\}$ can be covered by a finite number of balls in X of radius $1/k$. As k is arbitrary, it is not difficult to check that $\{U(t,\tau_n)x_n : n \ge 1\}$ possesses a Cauchy subsequence in X. Since X is complete, this subsequence is convergent, whence $\{U(t,\tau_n)x_n : n \ge 1\}$ is relatively compact in X. □

Remark 1. It can be proved (see [4]) that, reciprocally, when X is a uniformly convex Banach space, if the process U is pullback $\widehat{D}_0$-asymptotically compact, then it satisfies the pullback $\widehat{D}_0$-flattening property.

Finally, we recall an abstract result that allows us to compare two attractors for a process under appropriate assumptions (cf. [13, Theorem 3.15]).

Theorem 3.6. *Let $\{(X_i, d_{X_i})\}_{i=1,2}$ be two metric spaces such that $X_1 \subset X_2$ with continuous injection, and for $i = 1, 2$, let $\mathcal{D}_i$ be a universe in $\mathcal{P}(X_i)$, with $\mathcal{D}_1 \subset \mathcal{D}_2$. Assume that we have a map U that acts as a process in both cases, i.e., $U : \mathbb{R}^2_d \times X_i \to X_i$ for $i = 1, 2$ is a process.*

For each $t \in \mathbb{R}$, let us denote

$$\mathcal{A}_i(t) = \overline{\bigcup_{\widehat{D}_i \in \mathcal{D}_i} \Lambda_i(\widehat{D}_i, t)}^{X_i} \qquad i = 1, 2,$$

where the subscript i in the symbol of the omega-limit set Λ_i is used to denote the dependence of the respective topology.

Then, $\mathcal{A}_1(t) \subset \mathcal{A}_2(t)$ for all $t \in \mathbb{R}$.

If in addition

(i) $\mathcal{A}_1(t)$ is a compact subset of X_1 for all $t \in \mathbb{R}$, and

(ii) *for any* $\widehat{D}_2 \in \mathcal{D}_2$ *and any* $t \in \mathbb{R}$*, there exist a family* $\widehat{D}_1 \in \mathcal{D}_1$ *and a* $t^*_{\widehat{D}_1} \leq t$ *(both possibly depending on* t *and* $\widehat{D}_2$*), such that* U *is pullback* $\widehat{D}_1$*-asymptotically compact, and for any* $s \leq t^*_{\widehat{D}_1}$ *there exists a* $\tau_s \leq s$ *such that* $U(s,\tau)D_2(\tau) \subset D_1(s)$ *for all* $\tau \leq \tau_s$*,*

then $\mathcal{A}_1(t) = \mathcal{A}_2(t)$ *for all* $t \in \mathbb{R}$*.*

4. **Existence of minimal pullback attractors in H norm.** Now, we define a suitable process U on H associated to problem (2), and, by the previous results, we are able to obtain the existence of minimal pullback attractors by using the pullback flattening property. As pointed out in the Introduction, the proofs of the flattening property and the asymptotic compactness of this process are in fact very similar.

Results concerning existence and uniqueness of weak solutions for problem (2), and continuity with respect to the initial condition, summarized in the following theorem and proposition, are well known (see [19, 23, 26], for example).

Theorem 4.1. *Let* $f \in L^2_{\rm loc}(\mathbb{R}; V')$ *be given. Then, for each* $\tau \in \mathbb{R}$ *and* $u_\tau \in H$*, there exists a unique weak solution* $u(\cdot) = u(\cdot;\tau,u_\tau)$ *of (2).*

Moreover, if $f \in L^2_{\rm loc}(\mathbb{R}; H)$*, then*

(a) $u \in C([\tau+\varepsilon, T]; V) \cap L^2(\tau+\varepsilon, T; D(A))$ *for all* $T > \tau+\varepsilon > \tau$*.*

(b) If $u_\tau \in V$*, in fact* u *is a strong solution of (2).*

Therefore, when $f \in L^2_{\rm loc}(\mathbb{R}; V')$, we can define a process $U : \mathbb{R}^2_d \times H \to H$ as

$$U(t,\tau)u_\tau = u(t;\tau,u_\tau) \quad \forall\, (t,\tau) \in \mathbb{R}^2_d,\ u_\tau \in H, \tag{9}$$

and if $f \in L^2_{\rm loc}(\mathbb{R}; H)$, the restriction of this process to $\mathbb{R}^2_d \times V$ is a process on V.

Proposition 2. *If* $f \in L^2_{\rm loc}(\mathbb{R}; V')$*, for any pair* $(t,\tau) \in \mathbb{R}^2_d$*, the map* $U(t,\tau)$ *is continuous from* H *into* H*. Moreover, if* $f \in L^2_{\rm loc}(\mathbb{R}; H)$*, then* $U(t,\tau)$ *is also continuous from* V *into* V*.*

The following result guarantees the existence of a pullback absorbing family for the process U on H.

Lemma 4.2. *Let* $f \in L^2_{\rm loc}(\mathbb{R}; V')$ *be given and consider any fixed* $\mu \in (0, 2\nu\lambda_1)$*. Then, for any* $\tau \in \mathbb{R}$*, and* $u_\tau \in H$*, the solution* $u(\cdot) = u(\cdot;\tau,u_\tau)$ *to (2) satisfies*

$$|u(t)|^2 \leq e^{-\mu(t-\tau)}|u_\tau|^2 + \frac{e^{-\mu t}}{2\nu - \mu\lambda_1^{-1}} \int_\tau^t e^{\mu\theta}\|f(\theta)\|_*^2\, d\theta \quad \forall\, t \geq \tau. \tag{10}$$

Proof. By the energy equality we have

$$\frac{d}{d\theta}|u(\theta)|^2 + 2\nu\|u(\theta)\|^2 = 2\langle f(\theta), u(\theta)\rangle, \quad \text{a.e. } \theta > \tau,$$

and therefore,

$$\frac{d}{d\theta}(e^{\mu\theta}|u(\theta)|^2) + 2\nu e^{\mu\theta}\|u(\theta)\|^2 = \mu e^{\mu\theta}|u(\theta)|^2 + 2e^{\mu\theta}\langle f(\theta), u(\theta)\rangle, \quad \text{a.e. } \theta > \tau.$$

Observing that by Young's inequality,

$$2|\langle f(\theta), u(\theta)\rangle| \leq \frac{1}{2\nu - \mu\lambda_1^{-1}}\|f(\theta)\|_*^2 + (2\nu - \mu\lambda_1^{-1})\|u(\theta)\|^2,$$

from above we deduce

$$\frac{d}{d\theta}(e^{\mu\theta}|u(\theta)|^2) \leq \frac{e^{\mu\theta}}{2\nu - \mu\lambda_1^{-1}}\|f(\theta)\|_*^2, \quad \text{a.e. } \theta > \tau,$$

and thus, integrating in time,

$$e^{\mu t}|u(t)|^2 \leq e^{\mu\tau}|u_\tau|^2 + \frac{1}{2\nu - \mu\lambda_1^{-1}}\int_\tau^t e^{\mu\theta}\|f(\theta)\|_*^2\, d\theta \quad \forall\, t \geq \tau.$$

So, from this last inequality we obtain (10). □

Once the above estimate has been established, we introduce the following universe in $\mathcal{P}(H)$.

Definition 4.3. For any $\mu > 0$, we will denote by $\mathcal{D}_\mu^H$ the class of all families of nonempty subsets $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{P}(H)$ such that

$$\lim_{\tau\to-\infty}\left(e^{\mu\tau}\sup_{v\in D(\tau)}|v|^2\right) = 0.$$

Accordingly to the notation introduced in the previous section, $\mathcal{D}_F^H$ will denote the class of families $\widehat{D} = \{D(t) = D : t \in \mathbb{R}\}$ with D a fixed nonempty bounded subset of H.

Observe that for any $\mu > 0$, $\mathcal{D}_F^H \subset \mathcal{D}_\mu^H$ and that the universe $\mathcal{D}_\mu^H$ is inclusion-closed.

Corollary 2. *Suppose that $f \in L^2_{\mathrm{loc}}(\mathbb{R}; V')$ satisfies*

$$\int_{-\infty}^0 e^{\mu s}\|f(s)\|_*^2\, ds < \infty \quad \textit{for some } \mu \in (0, 2\nu\lambda_1). \tag{11}$$

Then, the family $\widehat{D}_0 = \{D_0(t) : t \in \mathbb{R}\}$ defined by $D_0(t) = \overline{B}_H(0, R_H(t))$, the closed ball in H of center zero and radius $R_H(t)$, where

$$R_H^2(t) = 1 + \frac{e^{-\mu t}}{2\nu - \mu\lambda_1^{-1}}\int_{-\infty}^t e^{\mu s}\|f(s)\|_*^2\, ds,$$

is pullback $\mathcal{D}_\mu^H$-absorbing for the process U on H given by (9) (and thus pullback $\mathcal{D}_F^H$-absorbing too), and $\widehat{D}_0 \in \mathcal{D}_\mu^H$.

Now, we establish several estimates for the process U in finite intervals of time when the initial time is sufficiently shifted in a pullback sense.

Lemma 4.4. *Assume that $f \in L^2_{\mathrm{loc}}(\mathbb{R}; V')$ satisfies (11). Then, for any $t \in \mathbb{R}$ and $\widehat{D} \in \mathcal{D}_\mu^H$, there exists $\tau_1(\widehat{D}, t) < t - 2$, such that for any $\tau \leq \tau_1(\widehat{D}, t)$ and any $u_\tau \in D(\tau)$,*

$$|u(r;\tau,u_\tau)|^2 \leq \rho_1^2(t) \quad \forall\, r \in [t-2, t], \tag{12}$$

$$\nu\int_{r-1}^r \|u(\theta;\tau,u_\tau)\|^2\, d\theta \leq \rho_2^2(t) \quad \forall\, r \in [t-1, t], \tag{13}$$

$$\int_{r-1}^r \|u'(\theta;\tau,u_\tau)\|_*^2\, d\theta \leq \rho_3^2(t) \quad \forall\, r \in [t-1, t], \tag{14}$$

where

$$\begin{aligned}\rho_1^2(t) &= 1 + e^{-\mu(t-2)}(2\nu - \mu\lambda_1^{-1})^{-1}\int_{-\infty}^t e^{\mu\theta}\|f(\theta)\|_*^2\,d\theta,\\ \rho_2^2(t) &= \rho_1^2(t) + \nu^{-1}\int_{t-2}^t \|f(\theta)\|_*^2\,d\theta,\\ \rho_3^2(t) &= 3\nu\rho_2^2(t) + \frac{3}{2}\rho_1^2(t)\frac{\rho_2^2(t)}{\nu} + 3\int_{t-2}^t \|f(\theta)\|_*^2\,d\theta.\end{aligned}$$

Proof. Let $\tau_1(\widehat{D}, t) < t-2$ be such that

$$e^{-\mu(t-2)}e^{\mu\tau}|u_\tau|^2 \le 1 \quad \forall\, \tau \le \tau_1(\widehat{D}, t),\ u_\tau \in D(\tau).$$

Consider fixed $\tau \le \tau_1(\widehat{D}, t)$ and $u_\tau \in D(\tau)$.

The estimate (12) follows directly from (10), using the increasing character of the exponential.

Now, observing that

$$\frac{d}{d\theta}|u(\theta)|^2 + 2\nu\|u(\theta)\|^2 \le \nu^{-1}\|f(\theta)\|_*^2 + \nu\|u(\theta)\|^2, \quad \text{a.e. } \theta > \tau, \tag{15}$$

and using (12), we obtain (13).

Finally, from (3), (4), (6), and the fact that A is an isometric isomorphism, we have

$$\|u'(\theta)\|_* \le \nu\|u(\theta)\| + 2^{-1/2}|u(\theta)|\|u(\theta)\| + \|f(\theta)\|_*, \quad \text{a.e. } \theta > \tau,$$

and therefore,

$$\|u'(\theta)\|_*^2 \le 3\nu^2\|u(\theta)\|^2 + \frac{3}{2}|u(\theta)|^2\|u(\theta)\|^2 + 3\|f(\theta)\|_*^2, \quad \text{a.e. } \theta > \tau,$$

whence, using (12) and (13), the estimate (14) follows. □

In order to prove the pullback $\widehat{D}_0$-flattening property for the process U on H, we need the following auxiliary result.

Lemma 4.5. *Under the assumptions of Lemma 4.4, for any $t \in \mathbb{R}$, $\widehat{D} \in \mathcal{D}_\mu^H$, and sequences $\{\tau_n\} \subset (-\infty, t-1]$ and $\{u_{\tau_n}\} \subset H$ such that $\tau_n \to -\infty$ and $u_{\tau_n} \in D(\tau_n)$ for all n, the sequence $\{u(\cdot;\tau_n, u_{\tau_n})\}$ is relatively compact in $C([t-1,t];H)$.*

Proof. Consider fixed $t \in \mathbb{R}$, a family $\widehat{D} \in \mathcal{D}_\mu^H$, and sequences $\{\tau_n\} \subset (-\infty, t-1]$ with $\tau_n \to -\infty$, and $\{u_{\tau_n}\}$ with $u_{\tau_n} \in D(\tau_n)$ for all n. For simplicity of notation we write $u^n(\cdot) = u(\cdot;\tau_n, u_{\tau_n})$.

From Lemma 4.4 and compactness arguments, there exist a value $\tau_1(\widehat{D}, t) < t-2$ and a function $u \in C([t-2,t];H) \cap L^2(t-2,t;V)$ with $u' \in L^2(t-2,t;V')$, such that for a subsequence of $\{u^n : \tau_n \le \tau_1(\widehat{D}, t)\} \subset \{u^n\}$, which we relabel the same, it holds that $u^n \overset{*}{\rightharpoonup} u$ weakly-star in $L^\infty(t-2,t;H)$, $u^n \rightharpoonup u$ weakly in $L^2(t-2,t;V)$, and $(u^n)' \rightharpoonup u'$ weakly in $L^2(t-2,t;V')$. Therefore, again up to a subsequence (relabelled the same), $u^n \to u$ strongly in $L^2(t-2,t;H)$, and $u^n(s) \to u(s)$ strongly in H a.e. $s \in (t-2,t)$.

From these convergences, the function u satisfies (6) in the interval $(t-2,t)$.

By the Ascoli-Arzelà Theorem, we deduce that $u^n \to u$ in $C([t-2,t];V')$, and so, for any sequence $\{s_n\} \subset [t-2,t]$ with $s_n \to s_*$, we have

$$u^n(s_n) \rightharpoonup u(s_*) \ \text{ weakly in } H. \tag{16}$$

We claim that

$$u^n \to u \ \text{ in } C([t-1,t];H), \tag{17}$$

which in particular will imply the relative compactness. If this were not true, there would exist a subsequence $\{u^n\}$ (relabelled the same), $\epsilon > 0$, and $\{t_n\} \subset [t-1,t]$ with $t_n \to t_*$ such that

$$|u^n(t_n) - u(t_*)| \geq \epsilon \quad \forall\, n \geq 1. \tag{18}$$

Recall that by (16) we already have

$$|u(t_*)| \leq \liminf_{n\to\infty} |u^n(t_n)|. \tag{19}$$

On the other hand, applying the energy equality to $z = u^n$ and $z = u$, and reasoning as in (15), we obtain in particular that

$$|z(s_2)|^2 \leq |z(s_1)|^2 + \nu^{-1} \int_{s_1}^{s_2} \|f(\theta)\|_*^2 \, d\theta \quad \forall\, t-2 \leq s_1 \leq s_2 \leq t.$$

We may now define the functions

$$\begin{aligned} J_n(s) &= |u^n(s)|^2 - \nu^{-1} \int_{t-2}^{s} \|f(\theta)\|_*^2 \, d\theta, \\ J(s) &= |u(s)|^2 - \nu^{-1} \int_{t-2}^{s} \|f(\theta)\|_*^2 \, d\theta. \end{aligned}$$

Observe that J and all J_n are continuous functions on $[t-2,t]$, non-increasing, and $J_n(s) \to J(s)$ a.e. $s \in (t-2,t)$.

Take now $\{\tilde{t}_k\} \subset (t-2, t_*)$ such that $\tilde{t}_k \uparrow t_*$ and $\lim_{n\to\infty} J_n(\tilde{t}_k) = J(\tilde{t}_k)$ for all $k \geq 1$.

Fix an arbitrary value $\eta > 0$. There exists k_η such that $|J(\tilde{t}_k) - J(t_*)| < \eta/2$ for all $k \geq k_\eta$. Now consider $n(k_\eta)$ such that for any $n \geq n(k_\eta)$ it holds that

$$t_n \geq \tilde{t}_{k_\eta} \quad \text{and} \quad |J_n(\tilde{t}_{k_\eta}) - J(\tilde{t}_{k_\eta})| < \eta/2.$$

Then, since all J_n are non-increasing, we deduce that for all $n \geq n(k_\eta)$,

$$\begin{aligned} J_n(t_n) - J(t_*) &\leq J_n(\tilde{t}_{k_\eta}) - J(t_*) \\ &\leq |J_n(\tilde{t}_{k_\eta}) - J(t_*)| \\ &\leq |J_n(\tilde{t}_{k_\eta}) - J(\tilde{t}_{k_\eta})| + |J(\tilde{t}_{k_\eta}) - J(t_*)| < \eta. \end{aligned}$$

Thus, we conclude that $\limsup_{n\to\infty} |u^n(t_n)| \leq |u(t_*)|$, with joined to (16) and (19), proves that (18) is absurd, and so claim (17) is true. This finishes the proof. $\square$

Note that the asymptotic compactness of U is an immediate corollary of this result.

Corollary 3. *Under the assumptions of Lemma 4.4, the process U on H defined by (9) is pullback $\mathcal{D}_\mu^H$-asymptotically compact.*

However, in order to prove the pullback flattening property directly (it is known that it is equivalent to asymptotic compactness in any uniformly convex Banach space, see [4]) we need to do a little more, beginning with the next corollary of Lemma 4.5.

Corollary 4. *Under the assumptions of Lemma 4.4, for any $\varepsilon > 0$, $t \in \mathbb{R}$, and $\widehat{D} \in \mathcal{D}_\mu^H$, there exists $\delta = \delta(\varepsilon, t, \widehat{D}) \in (0,1)$, such that*

$$\nu^{-1}\left||u(t;\tau,u_\tau)|^2 - |u(t-s;\tau,u_\tau)|^2\right| < \varepsilon/2 \quad \forall\, s \in [0,\delta],\ \tau \le \tau_1(\widehat{D},t),\ u_\tau \in D(\tau), \tag{20}$$

where $\tau_1(\widehat{D},t)$ is given in Lemma 4.4.

In particular,

$$\int_{t-\delta}^{t} \|u(\theta;\tau,u_\tau)\|^2\, d\theta < \varepsilon \quad \forall\, \tau \le \tau_1(\widehat{D},t),\ u_\tau \in D(\tau). \tag{21}$$

Proof. First at all, observe that if we consider $t \in \mathbb{R}$ and $\widehat{D} \in \mathcal{D}_\mu^H$, for any $\delta \in (0,1)$ and $\tau \le t-1$, an integration in (15) with any $u_\tau \in D(\tau)$ yields

$$\nu \int_{t-\delta}^{t} \|u(\theta;\tau,u_\tau)\|^2\, d\theta \le |u(t-\delta;\tau,u_\tau)|^2 - |u(t;\tau,u_\tau)|^2 + \nu^{-1}\int_{t-\delta}^{t} \|f(\theta)\|_*^2\, d\theta.$$

Therefore, since $f \in L^2_{\rm loc}(\mathbb{R};V')$, (21) is a consequence of (20).

We prove now (20) by a contradiction argument. If (20) were not true, there would exist $\varepsilon > 0$, $t \in \mathbb{R}$, a family $\widehat{D} \in \mathcal{D}_\mu^H$, and sequences $\{\tau_n\} \subset (-\infty, t-1]$ with $\tau_n \to -\infty$, $\{s_n\}$ with $0 \le s_n \le 1/n$, and $\{u_{\tau_n}\}$ with $u_{\tau_n} \in D(\tau_n)$ for all n, such that

$$\nu^{-1}\left||u(t;\tau_n,u_{\tau_n})|^2 - |u(t-s_n;\tau_n,u_{\tau_n})|^2\right| \ge \varepsilon/2 \quad \forall\, n \ge 1,$$

which is absurd, since from (17) we know that for a subsequence (which we relabel the same) it holds that $u(t;\tau_n,u_{\tau_n})$ and $u(t-s_n;\tau_n,u_{\tau_n})$ converge to $u(t)$. □

We will also use the following result, whose proof is analogous to that of [18, Lemma 12].

Lemma 4.6. *If $f \in L^2_{\rm loc}(\mathbb{R};V')$ satisfies the condition (11), then, for any $t \in \mathbb{R}$,*

$$\lim_{\rho\to\infty} e^{-\rho t}\int_{-\infty}^{t} e^{\rho s}\|f(s)\|_*^2\, ds = 0.$$

Now, we are able to prove the pullback $\widehat{D}_0$-flattening property for the process U on H defined by (9). Actually, we will prove that U satisfies the pullback $\widehat{D}$-flattening property for any $\widehat{D} \in \mathcal{D}_\mu^H$.

Proposition 3. *Under the assumptions of Lemma 4.4, for any $\varepsilon > 0$, $t \in \mathbb{R}$, and $\widehat{D} \in \mathcal{D}_\mu^H$, there exists $m = m(\varepsilon, t, \widehat{D}) \in \mathbb{N}$, such that the projection $P_m : H \to H_m :=$span$[w_1,\ldots,w_m]$ (where $\{w_j\}_{j\ge1}$ is the Hilbert basis of H formed by all the ortho-normalized eigenfunctions of the Stokes operator A) satisfies the following properties:*

$$\{P_m U(t,\tau)D(\tau) : \tau \le \tau_1(\widehat{D},t)\} \text{ is bounded in } H, \tag{22}$$

$$|(I-P_m)U(t,\tau)u_\tau| < \varepsilon \quad \text{for any } \tau \le \tau_1(\widehat{D},t),\ u_\tau \in D(\tau), \tag{23}$$

where $\tau_1(\widehat{D},t)$ is given in Lemma 4.4.

In particular, the process U on H satisfies the pullback $\widehat{D}$-flattening property for any $\widehat{D} \in \mathcal{D}_\mu^H$.

Proof. Let $\varepsilon > 0$, $t \in \mathbb{R}$, and $\widehat{D} \in \mathcal{D}_\mu^H$ be fixed.

Since P_m is non-expansive and taking into account (12), property (22) is automatically satisfied for any $m \in \mathbb{N}$. Therefore, we concentrate on proving (23).

Consider fixed $\tau \le \tau_1(\widehat{D},t)$, $u_\tau \in D(\tau)$, and let us define $u(r) = U(r,\tau)u_\tau$ and $q_m(r) = u(r) - P_m u(r)$.

Then, using the energy equality, for each $m \ge 1$ we have

$$\frac{1}{2}\frac{d}{dr}|q_m(r)|^2 + \nu\|q_m(r)\|^2 + b(u(r),u(r),q_m(r)) = \langle f(r), q_m(r)\rangle, \quad \text{a.e. } r > \tau.$$

Observing that by (3), (4), and Young's inequality,

$$\begin{aligned} |b(u(r),u(r),q_m(r))| &\le 2^{-1/2}|u(r)|\|u(r)\|\|q_m(r)\| \\ &\le \frac{\nu}{4}\|q_m(r)\|^2 + \frac{1}{2\nu}|u(r)|^2\|u(r)\|^2, \end{aligned}$$

we obtain

$$\frac{d}{dr}|q_m(r)|^2 + \nu\|q_m(r)\|^2 \le \nu^{-1}|u(r)|^2\|u(r)\|^2 + 2\nu^{-1}\|f(r)\|_*^2, \quad \text{a.e. } r > \tau.$$

Consequently, as $\|q_m(r)\|^2 \ge \lambda_{m+1}|q_m(r)|^2$, where λ_{m+1} is the eigenvalue associated to the eigenfunction w_{m+1}, we deduce that

$$\frac{d}{dr}|q_m(r)|^2 + \nu\lambda_{m+1}|q_m(r)|^2 \le \nu^{-1}|u(r)|^2\|u(r)\|^2 + 2\nu^{-1}\|f(r)\|_*^2, \quad \text{a.e. } r > \tau.$$

Thus, multiplying this last inequality by $e^{\nu\lambda_{m+1}r}$, integrating in $[t-1,t]$, and again taking into account (12), we obtain

$$\begin{aligned} e^{\nu\lambda_{m+1}t}|q_m(t)|^2 \le\ & e^{\nu\lambda_{m+1}(t-1)}|q_m(t-1)|^2 + \nu^{-1}\rho_1^2(t)\int_{t-1}^t e^{\nu\lambda_{m+1}r}\|u(r)\|^2\,dr \\ & +2\nu^{-1}\int_{t-1}^t e^{\nu\lambda_{m+1}r}\|f(r)\|_*^2\,dr. \end{aligned}$$

Therefore, from Lemma 4.6, and since $|q_m(t-1)|^2 \le |u(t-1)|^2 \le \rho_1^2(t)$ and $\lambda_m \to \infty$ as $m \to \infty$, in order to have (23), it suffices to check that for the previously fixed $\varepsilon > 0$, $t \in \mathbb{R}$, and $\widehat{D} \in \mathcal{D}_\mu^H$, there exists $m = m(\varepsilon,t,\widehat{D}) \in \mathbb{N}$, such that for any $\tau \le \tau_1(\widehat{D},t)$ and $u_\tau \in D(\tau)$,

$$e^{-\nu\lambda_{m+1}t}\int_{t-1}^t e^{\nu\lambda_{m+1}r}\|u(r;\tau,u_\tau)\|^2\,dr < \frac{\varepsilon\nu}{3\rho_1^2(t)}. \tag{24}$$

Take $\delta = \delta\big(\frac{\varepsilon\nu}{6\rho_1^2(t)}, t, \widehat{D}\big) \in (0,1)$ as in Corollary 4. Then, using (13), for each $m \ge 1$ we have

$$\begin{aligned} & e^{-\nu\lambda_{m+1}t}\int_{t-1}^t e^{\nu\lambda_{m+1}r}\|u(r)\|^2\,dr \\ =\ & e^{-\nu\lambda_{m+1}t}\int_{t-1}^{t-\delta} e^{\nu\lambda_{m+1}r}\|u(r)\|^2\,dr + e^{-\nu\lambda_{m+1}t}\int_{t-\delta}^t e^{\nu\lambda_{m+1}r}\|u(r)\|^2\,dr \\ \le\ & e^{-\nu\lambda_{m+1}\delta}\int_{t-1}^{t-\delta}\|u(r)\|^2\,dr + \int_{t-\delta}^t\|u(r)\|^2\,dr \\ \le\ & e^{-\nu\lambda_{m+1}\delta}\nu^{-1}\rho_2^2(t) + \int_{t-\delta}^t\|u(r)\|^2\,dr. \end{aligned}$$

By taking now $m = m(\varepsilon,t,\widehat{D})$ such that $e^{-\nu\lambda_{m+1}\delta}\nu^{-1}\rho_2^2(t) < \frac{\varepsilon\nu}{6\rho_1^2(t)}$, jointly with (21) in Corollary 4, we conclude (24). □

Combining all the above statements, we obtain the existence of minimal pullback attractors for the process $U : \mathbb{R}_d^2 \times H \to H$.

Theorem 4.7. *Suppose that $f \in L^2_{\mathrm{loc}}(\mathbb{R};V')$ satisfies the condition (11). Then, there exist the minimal pullback $\mathcal{D}^H_F$-attractor $\mathcal{A}_{\mathcal{D}^H_F}$ and the minimal pullback $\mathcal{D}^H_\mu$-attractor $\mathcal{A}_{\mathcal{D}^H_\mu}$ for the process U on H given by (9). The family $\mathcal{A}_{\mathcal{D}^H_\mu}$ belongs to $\mathcal{D}^H_\mu$, and it holds that*

$$\mathcal{A}_{\mathcal{D}^H_F}(t) \subset \mathcal{A}_{\mathcal{D}^H_\mu}(t) \subset \overline{B}_H(0, R_H(t)) \quad \forall\, t \in \mathbb{R}. \tag{25}$$

Proof. The existence of $\mathcal{A}_{\mathcal{D}^H_\mu}$ and $\mathcal{A}_{\mathcal{D}^H_F}$ is a direct consequence of the abstract results given in Theorem 3.4 and Corollary 1 respectively, since all the assumptions, closed process (continuous in fact, by Proposition 2), pullback absorbing family (Corollary 2) and pullback asymptotic compactness (Corollary 3), are satisfied.

Then, the claim that $\mathcal{A}_{\mathcal{D}^H_\mu}$ belongs to $\mathcal{D}^H_\mu$ follows from Theorem 3.4 and the remarks following Theorem 3.4, since the universe $\mathcal{D}^H_\mu$ is inclusion-closed, the family $\widehat{D}_0$ belongs to $\mathcal{D}^H_\mu$, and the set $D_0(t)$ is closed in H for all $t \in \mathbb{R}$.

Finally, the first inclusion in (25) is a consequence of Corollary 1, since $\mathcal{D}^H_F \subset \mathcal{D}^H_\mu$. The last inclusion in (25) follows again from Theorem 3.4. □

4.1. **A compact pullback absorbing family using semigroup theory.** With only a slightly more stringent requirement on the forcing function f, namely that

$$f \in L^p_{\mathrm{loc}}(\mathbb{R};V') \qquad \text{for some} \quad p > 2$$

we can in fact use the semigroup approach of Fujita and Kato [11] (see also [15]) to show, using Corollary 4 again (and hence Lemma 4.5 once more) that in fact there is a compact pullback absorbing family in H. Our analysis is inspired by the paper by Arrieta and Carvalho [1] on the ϵ-regularity method for proving existence and uniqueness of semilinear problems.

In order to state the result precisely we need to define the fractional power spaces $D(A^\alpha)$ as the domains of the operators A^α, where

$$A^\alpha u := \sum_{j=1}^\infty \lambda_j^\alpha (u, w_j) w_j,$$

where λ_j and w_j are the eigenvalues and eigenfunctions of the Stokes operator as defined in Section 2. We recall the key estimate

$$|A^\gamma e^{-At} x| \le c_\gamma t^{-\gamma} |x| \tag{26}$$

for any $0 \le \gamma < 1$ (e.g., see Henry [15]). In the proof we write

$$\|u\|_s = |A^s u|,$$

and in particular we have $\|\cdot\|_0 = |\cdot|$, $\|\cdot\|_{1/2} = \|\cdot\|$, and $\|\cdot\|_{-1/2} = \|\cdot\|_*$.

Theorem 4.8. *Suppose that $f \in L^p_{\mathrm{loc}}(\mathbb{R};V')$ for some $p > 2$ and that*

$$\int_{-\infty}^0 e^{\mu s} \|f(s)\|_*^2 \, ds < \infty \quad \textit{for some } \mu \in (0, 2\nu\lambda_1).$$

Choose $\epsilon < \frac{1}{2} - \frac{1}{p}$. Then there exists a function $\rho_\epsilon : \mathbb{R} \to \mathbb{R}$ such that for any $t \in \mathbb{R}$ and $\widehat{D} \in \mathcal{D}^H_\mu$,

$$|A^\epsilon u(t;\tau,u_\tau)| \le \rho_\epsilon(t) \qquad \textit{for all} \quad u_\tau \in D(\tau),\ \tau \le \tau_1(\widehat{D}, t),$$

where $\tau_1(\widehat{D}, t)$ is the same as in Lemma 4.4; hence there is a compact pullback absorbing family in H. In particular if $f \in L^\infty_{\mathrm{loc}}(\mathbb{R};V')$ then we obtain a bounded *absorbing family in $D(A^\epsilon)$ for any $\epsilon < 1/2$; this holds therefore in the autonomous case when $f \in V'$.*

Proof. Given the improved regularity assumption on f, any weak solution u to (2) satisfies the variation of constants formula

$$u(t) = e^{-A(t-s)}u(s) + \int_s^t e^{-A(t-r)}[B(u(r)) + f(r)]\,dr$$

for all $\tau \le s \le t$ (see [15]). We use this formulation to find estimates on u in $D(A^\epsilon)$. The key observation (after [1]) is that $B : D(A^\epsilon) \to D(A^{-(1-2\epsilon)})$, which can be seen by taking $w \in D(A^{1-2\epsilon})$ and using Hölder's inequality to obtain

$$\begin{aligned} |\langle B(u), w\rangle| &= |\langle B(u,w), u\rangle| \\ &\le \|u\|^2_{L^{2/(1-2\epsilon)}} \|\nabla w\|_{L^{1/(2\epsilon)}} \\ &\le \tilde{c}_\epsilon \|u\|_\epsilon^2 \|w\|_{1-2\epsilon}, \end{aligned}$$

from which it follows that

$$\|B(u)\|_{-(1-2\epsilon)} \le \tilde{c}_\epsilon \|u\|_\epsilon^2. \tag{27}$$

Lemma 4.4 guarantees that for any $\widehat{D} \in \mathcal{D}_\mu^H$ and any $t \in \mathbb{R}$ there exists a $\tau_1(\widehat{D}, t) < t - 2$ such that for any $r \in [t-2, t]$,

$$|u(r; \tau, u_\tau)| \le \rho_1(t) \qquad \forall\, u_\tau \in D(\tau),\ \tau \le \tau_1(\widehat{D}, t). \tag{28}$$

Fix $t \in \mathbb{R}$, $\widehat{D} \in \mathcal{D}_\mu^H$, $\tau \le \tau_1(\widehat{D}, t)$, and $u_\tau \in D(\tau)$ and write

$$u_\sigma(s) = u(\sigma + s; \tau, u_\tau) \qquad \text{and} \qquad f_\sigma(s) = f(\sigma + s).$$

We can now rewrite the variation of constants formula in the notationally convenient form (for $\sigma \ge \tau$)

$$u_\sigma(s) = e^{-As}u_\sigma(0) + \int_0^s e^{-A(s-r)}[B(u_\sigma(r)) + f_\sigma(r)]\,dr \qquad \forall\, s \in [0, t-\sigma], \tag{29}$$

noting from (28) that

$$|u_\sigma(s)| \le \rho_1(t) \qquad \forall\, \sigma \in [t-1, t],\ s \in [0, t-\sigma]. \tag{30}$$

Pick $\sigma \in [t-1, t]$, let $s \le t - \sigma$, then take the norm of (29) in $D(A^\epsilon)$ and multiply by s^ϵ to obtain

$$\begin{aligned} s^\epsilon \|u_\sigma(s)\|_\epsilon &\le s^\epsilon \|e^{-As}u_\sigma(0)\|_\epsilon + s^\epsilon \int_0^s \left\| e^{-A(s-r)}[B(u_\sigma(r)) + f_\sigma(r)] \right\|_\epsilon dr \\ &\le c_\epsilon |u_\sigma(0)| + c_{1-\epsilon} s^\epsilon \int_0^s (s-r)^{-(1-\epsilon)} \|B(u_\sigma(r))\|_{-(1-2\epsilon)}\,dr \\ &\qquad + c_{1/2+\epsilon} s^\epsilon \int_0^s (s-r)^{-1/2-\epsilon} \|f_\sigma(r)\|_*\,dr \\ &\le c_\epsilon \rho_1(t) + \tilde{c}_\epsilon c_{1-\epsilon} s^\epsilon \int_0^s (s-r)^{-(1-\epsilon)} \|u_\sigma(r)\|_\epsilon^2\,dr \\ &\qquad + c_{1/2+\epsilon} s^\epsilon \int_0^s (s-r)^{-1/2-\epsilon} \|f_\sigma(r)\|_*\,dr, \end{aligned}$$

using (26), (27), and (30). The second term on the right-hand side can be bounded by

$$\tilde{c}_\epsilon c_{1-\epsilon} s^\epsilon \int_0^s (s-r)^{-(1-\epsilon)} \|u_\sigma(r)\|_\epsilon^{3/2} |u_\sigma(r)|^{1/2-\epsilon} \|u_\sigma(r)\|^\epsilon \, dr$$
$$\leq R(s) s^\epsilon \left(\int_0^s (s-r)^{-(1-\epsilon)/(1-\epsilon/2)} \|u_\sigma(r)\|_\epsilon^{3/(2-\epsilon)} \, dr \right)^{1-\epsilon/2},$$

where

$$R(s) = \tilde{c}_\epsilon c_{1-\epsilon} \left(\sup_{0\leq r\leq s} |u_\sigma(r)| \right)^{1/2-\epsilon} \left(\int_0^s \|u_\sigma(r)\|^2 \, dr \right)^{\epsilon/2}$$
$$\leq \tilde{c}_\epsilon c_{1-\epsilon} \rho_1(t)^{1/2-\epsilon} \left(\int_\sigma^t \|u(r)\|^2 \, dr \right)^{\epsilon/2} =: P(\sigma, t).$$

Setting $X(s) = s^\epsilon \|u_\sigma(s)\|_\epsilon$ we obtain an integral inequality for $X(s)$:

$$X(s) \leq \delta(s) + P(\sigma,t) s^\epsilon \left(\int_0^s (s-r)^{-(1-\epsilon)/(1-\epsilon/2)} r^{-3\epsilon/(2-\epsilon)} X(r)^{3/(2-\epsilon)} \, dr \right)^{1-\epsilon/2}, \tag{31}$$

where[1]

$$\delta(s) = c_\epsilon \rho_1(t) + c_{1/2+\epsilon} s^\epsilon \int_0^s (s-r)^{-1/2-\epsilon} \|f_\sigma(r)\|_* \, dr$$
$$\leq c_\epsilon \rho_1(t) + c_{1/2+\epsilon} s^\epsilon \left(\int_0^s (s-r)^{-p(1/2+\epsilon)/(p-1)} \, dr \right)^{1-(1/p)} \left(\int_0^s \|f_\sigma(r)\|_*^p \, dr \right)^{1/p}$$
$$\leq c_\epsilon \rho_1(t) + C_{\epsilon,p} \left(\int_{t-1}^t \|f(r)\|_*^p \, dr \right)^{1/p} =: \Phi(t),$$

using Hölder's inequality and the choice of ϵ which ensures that $p(1/2+\epsilon)/(p-1) < 1$.

In order to find an upper bound on $X(s)$ it suffices to find a continuous function $Y(t)$ with $X(0) < Y(0)$ that is a supersolution of (31), i.e. that satisfies

$$Y(s) \geq \Phi(t) + P(\sigma,t) s^\epsilon \left(\int_0^s (s-r)^{-(1-\epsilon)/(1-\epsilon/2)} r^{-3\epsilon/(2-\epsilon)} Y(r)^{3/(2-\epsilon)} \, dr \right)^{1-\epsilon/2} \tag{32}$$

for all $s \in [0, t-\sigma]$, to conclude that $X(s) \leq Y(s)$ for all $s \in [0, t-\sigma]$. Indeed, if it were not true, there would exists $\hat{s} \in (0, t-\sigma]$ with $Y(\hat{s}) < X(\hat{s})$ and so we may define $0 < \tilde{s} = \sup\{s \in [0, t-\sigma] : X(r) - Y(r) < 0 \ \forall r \in [0, s)\}$. Then, we would

[1] Note that in fact $\delta(s)$ is simply a bound on $s^\epsilon U(s+\sigma, \sigma)$ in $D(A^\epsilon)$, where $U(s,\sigma)$ is a solution of the linear equation $u_t + Au = f(t)$ with initial data $u(\sigma) = u_\sigma(0)$; this where we require the additional regularity for f, so it has nothing to do with the nature of the nonlinear term. We return briefly to this issue in the Conclusion.

have $X(s) < Y(s)$ for all $s \in [0, \tilde{s})$ and $X(\tilde{s}) = Y(\tilde{s})$ (by continuity), but therefore

$$\begin{aligned} Y(\tilde{s}) &\geq \Phi(t) + P(\sigma,t)\tilde{s}^{\epsilon}\left(\int_0^{\tilde{s}} (\tilde{s}-r)^{-(1-\epsilon)/(1-\epsilon/2)} r^{-3\epsilon/(2-\epsilon)} Y(r)^{3/(2-\epsilon)}\,dr\right)^{1-\epsilon/2} \\ &> \Phi(t) + P(\sigma,t)\tilde{s}^{\epsilon}\left(\int_0^{\tilde{s}} (\tilde{s}-r)^{-(1-\epsilon)/(1-\epsilon/2)} r^{-3\epsilon/(2-\epsilon)} X(r)^{3/(2-\epsilon)}\,dr\right)^{1-\epsilon/2} \\ &\geq X(\tilde{s}), \end{aligned}$$

i.e. $Y(\tilde{s}) > X(\tilde{s})$, a contradiction.

Now, we will prove that $Y(s) = 2\Phi(t)$ satisfies (32) for $s \in [0, t-\sigma]$, i.e. we need to ensure that

$$2\Phi(t) \geq \Phi(t) + 2\sqrt{2}P(\sigma,t)\Phi^{3/2}(t)s^{\epsilon}\left(\int_0^s (s-r)^{-(1-\epsilon)/(1-\epsilon/2)} r^{-3\epsilon/(2-\epsilon)}\,dr\right)^{1-\epsilon/2}$$

for all $s \in [0, t-\sigma]$. This holds if

$$2\sqrt{2}P(\sigma,t)\Phi^{1/2}(t)s^{\epsilon}\left(\int_0^s (s-r)^{-(1-\epsilon)/(1-\epsilon/2)} r^{-3\epsilon/(2-\epsilon)}\,dr\right)^{1-\epsilon/2} \leq 1\; \forall\, s \in [0, t-\sigma].$$

By substituting $r = s\theta$ one can see that

$$\begin{aligned} & s^{\epsilon}\left(\int_0^s (s-r)^{-(1-\epsilon)/(1-\epsilon/2)} r^{-3\epsilon/(2-\epsilon)}\,dr\right)^{1-\epsilon/2} \\ &\quad = s^{\epsilon}\left(\int_0^1 s^{-(1-\epsilon)/(1-\epsilon/2)}(1-\theta)^{-(1-\epsilon)/(1-\epsilon/2)} s^{-3\epsilon/(2-\epsilon)}\theta^{-3\epsilon/(2-\epsilon)} s\,d\theta\right)^{1-\epsilon/2} \\ &\quad = \left(\int_0^1 (1-\theta)^{-(1-\epsilon)/(1-\epsilon/2)}\theta^{-3\epsilon/(2-\epsilon)}\,d\theta\right)^{1-\epsilon/2} =: C_{\epsilon}. \end{aligned}$$

So in order to guarantee that $X(s) \leq 2\Phi(t)$ for $s \in [0, t-\sigma]$ it suffices to ensure that

$$2\sqrt{2}P(\sigma,t)\Phi^{1/2}(t) \leq C_{\epsilon}^{-1}. \tag{33}$$

Now, recall that

$$P(\sigma,t) = \tilde{c}_{\epsilon} c_{1-\epsilon}\rho_1(t)^{1/2-\epsilon}\left(\int_{\sigma}^t \|u(r)\|^2\,dr\right)^{\epsilon/2},$$

and that it follows from Corollary 4 that for any $\varepsilon > 0$, $t \in \mathbb{R}$, and $\widehat{D} \in \mathcal{D}_{\mu}^H$, there exists a $\sigma = \sigma(\varepsilon, t, \widehat{D}) \in (t-1, t)$, such that

$$\int_{\sigma}^t \|u(\theta;\tau,u_{\tau})\|^2\,d\theta < \varepsilon \quad \forall\, \tau \leq \tau_1(\widehat{D}, t),\, u_{\tau} \in D(\tau)$$

(this was (21)). We can therefore satisfy the condition (33), and deduce that

$$|A^{\epsilon}u(t;\tau,u_{\tau})| = (t-\sigma)^{-\epsilon}X(t-\sigma) \leq 2(t-\sigma)^{-\epsilon}\Phi(t) =: \rho_{\epsilon}(t)\; \forall\, u_{\tau} \in D(\tau),\ \tau \leq \tau_1(\widehat{D}, t).$$

Since $D(A^{\epsilon})$ is compactly embedded in H, the existence of a compact pullback absorbing family in H follows immediately. □

5. **Regularity of pullback attractors and attraction in V norm.** The goal of this section is to prove analogous results to those given above, but considering the map U defined as a process on V. Actually, our aim is to show a sharper conclusion than above. Firstly, we will prove the flattening property for the process U defined on V. Then, as a consequence of the pullback flattening property, we will recover some results concerning the existence of pullback attractors in V, which were already proved (by using an energy method) in [13], but here with a shorter proof.

From now on we assume that $f \in L^2_{\rm loc}(\mathbb{R}; H)$, and satisfies

$$\int_{-\infty}^{0} e^{\mu s}|f(s)|^2\, ds < \infty \quad \text{for some } \mu \in (0, 2\nu\lambda_1). \tag{34}$$

We have the following result, which is similar to [13, Lemma 4.10] and [14, Lemma 5.2] (see also [12] for close results).

Lemma 5.1. *Suppose that $f \in L^2_{\rm loc}(\mathbb{R}; H)$ satisfies (34). Then, for any $t \in \mathbb{R}$ and $\widehat{D} \in \mathcal{D}^H_\mu$, there exists $\tau_1(\widehat{D}, t) < t - 2$ (the one given in Lemma 4.4), such that for any $\tau \le \tau_1(\widehat{D}, t)$ and any $u_\tau \in D(\tau)$,*

$$|u(r;\tau,u_\tau)|^2 \le R_1^2(t) \quad \forall\, r \in [t-2, t], \tag{35}$$

$$\|u(r;\tau,u_\tau)\|^2 \le R_2^2(t) \quad \forall\, r \in [t-1, t], \tag{36}$$

$$\nu \int_{t-1}^{t} |Au(\theta;\tau,u_\tau)|^2\, d\theta \le R_3^2(t), \tag{37}$$

where

$$\begin{aligned}
R_1^2(t) &= 1 + e^{-\mu(t-2)}(2\nu\lambda_1 - \mu)^{-1}\int_{-\infty}^{t} e^{\mu\theta}|f(\theta)|^2\, d\theta,\\
R_2^2(t) &= \nu^{-1}\left(R_1^2(t) + (\nu^{-1}\lambda_1^{-1} + 2)\int_{t-2}^{t}|f(\theta)|^2\, d\theta\right)\\
&\quad \times \exp\left[2\nu^{-1}C^{(\nu)}R_1^2(t)\left(R_1^2(t) + \nu^{-1}\lambda_1^{-1}\int_{t-2}^{t}|f(\theta)|^2\, d\theta\right)\right],\\
R_3^2(t) &= R_2^2(t) + 2\nu^{-1}\int_{t-1}^{t}|f(\theta)|^2\, d\theta + 2C^{(\nu)}R_1^2(t)R_2^4(t),
\end{aligned}$$

with $C^{(\nu)} = 27C_1^4(4\nu^3)^{-1}$.

Proof. The first estimate (35) follows immediately from (12).

Now, consider fixed $\tau \le \tau_1(\widehat{D}, t)$ and $u_\tau \in D(\tau)$. Let us denote, for short, $u(r) = u(r;\tau,u_\tau)$.

In order to obtain (36), we first observe that by the energy equality and Young's inequality,

$$\frac{d}{d\theta}|u(\theta)|^2 + 2\nu\|u(\theta)\|^2 \le (\nu\lambda_1)^{-1}|f(\theta)|^2 + \nu\lambda_1|u(\theta)|^2, \quad \text{a.e. } \theta > \tau,$$

and therefore,

$$\frac{d}{d\theta}|u(\theta)|^2 + \nu\|u(\theta)\|^2 \le (\nu\lambda_1)^{-1}|f(\theta)|^2, \quad \text{a.e. } \theta > \tau.$$

Thus, in particular,

$$\nu \int_{r-1}^{r} \|u(\theta)\|^2 \, d\theta \leq |u(r-1)|^2 + (\nu\lambda_1)^{-1} \int_{r-1}^{r} |f(\theta)|^2 \, d\theta \quad \forall \tau \leq r-1. \tag{38}$$

On the other hand, by (8) and the regularity result (a) in Theorem 4.1, we have

$$\frac{1}{2}\frac{d}{d\theta}\|u(\theta)\|^2 + \nu|Au(\theta)|^2 + b(u(\theta), u(\theta), Au(\theta)) = (f(\theta), Au(\theta)), \quad \text{a.e. } \theta > \tau.$$

Therefore, observing that

$$|(f(\theta), Au(\theta))| \leq \frac{1}{\nu}|f(\theta)|^2 + \frac{\nu}{4}|Au(\theta)|^2,$$

and by (5) and Young's inequality,

$$\begin{aligned} |b(u(\theta), u(\theta), Au(\theta))| &\leq C_1|u(\theta)|^{1/2}\|u(\theta)\||Au(\theta)|^{3/2} \\ &\leq \frac{\nu}{4}|Au(\theta)|^2 + C^{(\nu)}|u(\theta)|^2\|u(\theta)\|^4, \end{aligned}$$

from above we deduce that

$$\frac{d}{d\theta}\|u(\theta)\|^2 + \nu|Au(\theta)|^2 \leq 2\nu^{-1}|f(\theta)|^2 + 2C^{(\nu)}|u(\theta)|^2\|u(\theta)\|^4, \quad \text{a.e. } \theta > \tau. \tag{39}$$

From (39), in particular we obtain

$$\|u(r)\|^2 \leq \|u(s)\|^2 + 2\nu^{-1}\int_{r-1}^{r} |f(\theta)|^2 \, d\theta + 2C^{(\nu)} \int_{s}^{r} |u(\theta)|^2\|u(\theta)\|^4 \, d\theta$$

for all $\tau < r-1 \leq s \leq r$, and therefore, by Gronwall's lemma,

$$\|u(r)\|^2 \leq \left(\|u(s)\|^2 + 2\nu^{-1}\int_{r-1}^{r} |f(\theta)|^2 \, d\theta\right) \exp\left(2C^{(\nu)}\int_{r-1}^{r} |u(\theta)|^2\|u(\theta)\|^2 \, d\theta\right)$$

for all $\tau < r-1 \leq s \leq r$.

Integrating this last inequality for s between $r-1$ and r, we obtain

$$\|u(r)\|^2 \leq \left(\int_{r-1}^{r}\|u(s)\|^2 \, ds + 2\nu^{-1}\int_{r-1}^{r}|f(\theta)|^2 \, d\theta\right) \exp\left(2C^{(\nu)}\int_{r-1}^{r}|u(\theta)|^2\|u(\theta)\|^2 \, d\theta\right)$$

for all $\tau < r-1$.

Thus, from this inequality and (38), using estimate (35), we deduce (36).

Finally, we observe that by (39),

$$\nu\int_{t-1}^{t} |Au(\theta)|^2 \, d\theta \leq \|u(t-1)\|^2 + 2\nu^{-1}\int_{t-1}^{t} |f(\theta)|^2 \, d\theta + 2C^{(\nu)}\int_{t-1}^{t} |u(\theta)|^2\|u(\theta)\|^4 \, d\theta,$$

and therefore, taking into account estimates (35) and (36), we conclude (37). □

Remark 2. It is clear that under the assumptions of Lemma 5.1, $\lim_{t\to-\infty} e^{\mu t}R_1^2(t) = 0$.

Now, we introduce the following universe in $\mathcal{P}(V)$.

Definition 5.2. For any $\mu > 0$, we will denote by $\mathcal{D}_\mu^{H,V}$ the class of all families $\widehat{D}_V$ of elements of $\mathcal{P}(V)$ of the form $\widehat{D}_V = \{D(t) \cap V : t \in \mathbb{R}\}$, where $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \in \mathcal{D}_\mu^H$.

Again, accordingly to the notation in the previous section, we denote by $\mathcal{D}_F^V$ the universe of families (parameterized in time but constant for all $t \in \mathbb{R}$) of nonempty fixed bounded subsets of V.

Remark 3. For any $\mu > 0$, $\mathcal{D}_F^V \subset \mathcal{D}_\mu^{H,V} \subset \mathcal{D}_\mu^H$. It must also be pointed out that the universe $\mathcal{D}_\mu^{H,V}$ is inclusion-closed.

The following result is immediate.

Corollary 5. *Under the assumptions of Lemma 5.1, the family*

$$\widehat{D}_{0,V} = \{\overline{B}_H(0, R_1(t)) \cap V : t \in \mathbb{R}\}$$

belongs to $\mathcal{D}_\mu^{H,V}$ *and satisfies that for any* $t \in \mathbb{R}$ *and any* $\widehat{D} \in \mathcal{D}_\mu^H$*, there exists* $\tau(\widehat{D}, t) < t$ *such that*

$$U(t,\tau)D(\tau) \subset D_{0,V}(t) \quad \forall \tau \leq \tau(\widehat{D}, t).$$

In particular, the family $\widehat{D}_{0,V}$ *is pullback* $\mathcal{D}_\mu^{H,V}$*-absorbing for the process* $U : \mathbb{R}_d^2 \times V \to V$.

Now, we will prove that the process $U : \mathbb{R}_d^2 \times V \to V$ satisfies the pullback $\widehat{D}_{0,V}$-flattening property. In fact, we will prove that U satisfies the pullback $\widehat{D}$-flattening property for any $\widehat{D} \in \mathcal{D}_\mu^H$.

Analogously to Lemma 4.6, we have the following result.

Lemma 5.3. *If* $f \in L^2_{\mathrm{loc}}(\mathbb{R}; H)$ *satisfies* (34)*, then, for any* $t \in \mathbb{R}$,

$$\lim_{\rho \to \infty} e^{-\rho t} \int_{-\infty}^t e^{\rho s} |f(s)|^2 \, ds = 0.$$

Proposition 4. *Under the assumptions of Lemma 5.1, for any* $\varepsilon > 0$ *and* $t \in \mathbb{R}$*, there exists* $m = m(\varepsilon, t) \in \mathbb{N}$ *such that for any* $\widehat{D} \in \mathcal{D}_\mu^H$*, the projection* $P_m : V \to V_m := \mathrm{span}[w_1, \ldots, w_m]$ *satisfies the following properties:*

$$\{P_m U(t,\tau) D(\tau) : \tau \leq \tau_1(\widehat{D}, t)\} \text{ is bounded in } V,$$

and

$$\|(I - P_m) U(t,\tau) u_\tau\| < \varepsilon \quad \text{for any } \tau \leq \tau_1(\widehat{D}, t),\ u_\tau \in D(\tau),$$

where $\tau_1(\widehat{D}, t)$ *is given in Lemma 4.4.*

In particular, the process U *on* V *satisfies the pullback* $\widehat{D}$*-flattening property for any* $\widehat{D} \in \mathcal{D}_\mu^H$.

Proof. Let $\varepsilon > 0$, $t \in \mathbb{R}$, and $\widehat{D} \in \mathcal{D}_\mu^H$ be fixed.

Since $\{w_j\}_{j \geq 1}$ is a special basis, P_m is non-expansive in V. From this and (36), we deduce the boundedness in V of the set $\{P_m U(t,\tau) D(\tau) : \tau \leq \tau_1(\widehat{D}, t)\}$, for all $m \geq 1$.

On the other hand, let us fix $\tau \leq \tau_1(\widehat{D}, t)$, $u_\tau \in D(\tau)$, and let us define again $u(r) = U(r,\tau) u_\tau$ and $q_m(r) = u(r) - P_m u(r)$.

Then, by (5) and Lemma 5.1, for each $m \geq 1$ one has

$$\begin{aligned} \frac{1}{2}\frac{d}{dr}\|q_m(r)\|^2 + \nu |A q_m(r)|^2 &= -b(u(r), u(r), A q_m(r)) + (f(r), A q_m(r)) \\ &\leq \frac{\nu}{2}|A q_m(r)|^2 + \frac{1}{\nu}|f(r)|^2 + \frac{C_1^2}{\nu} R_1(t) R_2^2(t) |A u(r)| \end{aligned}$$

a.e. $t - 1 < r < t$.

Consequently, as $|A q_m(r)|^2 \geq \lambda_{m+1} \|q_m(r)\|^2$, from above we deduce that

$$\frac{d}{dr}\|q_m(r)\|^2 + \nu \lambda_{m+1} \|q_m(r)\|^2 \leq 2\nu^{-1} |f(r)|^2 + 2 C_1^2 \nu^{-1} R_1(t) R_2^2(t) |A u(r)|$$

a.e. $t-1<r<t$.

Thus, multiplying this last inequality by $e^{\nu\lambda_{m+1}r}$, integrating from $t-1$ to t, and taking into account Lemma 5.1, we obtain

$$\begin{aligned}
e^{\nu\lambda_{m+1}t}\|q_m(t)\|^2 &\le e^{\nu\lambda_{m+1}(t-1)}\|q_m(t-1)\|^2 + 2\nu^{-1}\int_{t-1}^t e^{\nu\lambda_{m+1}r}|f(r)|^2\,dr \\
&\quad +2C_1^2\nu^{-1}R_1(t)R_2^2(t)\int_{t-1}^t e^{\nu\lambda_{m+1}r}|Au(r)|\,dr \\
&\le e^{\nu\lambda_{m+1}(t-1)}\|u(t-1)\|^2 + 2\nu^{-1}\int_{t-1}^t e^{\nu\lambda_{m+1}r}|f(r)|^2\,dr \\
&\quad +2C_1^2\nu^{-1}R_1(t)R_2^2(t)\left(\int_{t-1}^t e^{2\nu\lambda_{m+1}r}\,dr\right)^{1/2}\left(\int_{t-1}^t |Au(r)|^2\,dr\right)^{1/2} \\
&\le e^{\nu\lambda_{m+1}(t-1)}R_2^2(t) + 2\nu^{-1}\int_{t-1}^t e^{\nu\lambda_{m+1}r}|f(r)|^2\,dr \\
&\quad +2C_1^2\nu^{-3/2}R_1(t)R_2^2(t)R_3(t)(2\nu\lambda_{m+1})^{-1/2}e^{\nu\lambda_{m+1}t}.
\end{aligned}$$

Therefore, from Lemma 5.3 and since $\lambda_m\to\infty$ as $m\to\infty$, we conclude that there exists $m=m(\varepsilon,t)\in\mathbb{N}$ such that $\|(I-P_m)U(t,\tau)u_\tau\|<\varepsilon$ for all $\tau\le\tau_1(\widehat{D},t)$, $u_\tau\in D(\tau)$. □

As a consequence of the above result and Proposition 1, we obtain the asymptotic compactness in the V norm. It is worth to point out that in this way the proof is much shorter than [13, Lemma 4.13].

Lemma 5.4. *Under the assumptions of Lemma 5.1, the process U on V is pullback $\mathcal{D}_\mu^{H,V}$-asymptotically compact.*

From the previous results, we obtain the existence of minimal pullback attractors for the process U on V (see [13, Theorem 4.14]).

Theorem 5.5. *Suppose that $f\in L^2_{\mathrm{loc}}(\mathbb{R};H)$ satisfies the condition (34). Then, there exist the minimal pullback $\mathcal{D}_F^V$-attractor $\mathcal{A}_{\mathcal{D}_F^V}$ and the minimal pullback $\mathcal{D}_\mu^{H,V}$-attractor $\mathcal{A}_{\mathcal{D}_\mu^{H,V}}$ for the process U on V defined by (9), and the following relation holds:*

$$\mathcal{A}_{\mathcal{D}_F^V}(t)\subset\mathcal{A}_{\mathcal{D}_F^H}(t)\subset\mathcal{A}_{\mathcal{D}_\mu^H}(t)=\mathcal{A}_{\mathcal{D}_\mu^{H,V}}(t)\quad\forall\, t\in\mathbb{R}.$$

In particular, for any family $\widehat{D}\in\mathcal{D}_\mu^H$, the following pullback attraction result in V holds:

$$\lim_{\tau\to-\infty}\mathrm{dist}_V(U(t,\tau)D(\tau),\mathcal{A}_{\mathcal{D}_\mu^H}(t))=0\quad\forall\, t\in\mathbb{R}.$$

Finally, if moreover f satisfies

$$\sup_{s\le 0}\left(e^{-\mu s}\int_{-\infty}^s e^{\mu\theta}|f(\theta)|^2\,d\theta\right)<\infty, \tag{40}$$

then

$$\mathcal{A}_{\mathcal{D}_F^V}(t)=\mathcal{A}_{\mathcal{D}_F^H}(t)=\mathcal{A}_{\mathcal{D}_\mu^H}(t)=\mathcal{A}_{\mathcal{D}_\mu^{H,V}}(t)\quad\forall\, t\in\mathbb{R}, \tag{41}$$

and for any bounded subset B of H,

$$\lim_{\tau\to-\infty}\mathrm{dist}_V(U(t,\tau)B,\mathcal{A}_{\mathcal{D}_F^H}(t))=0\quad\forall\, t\in\mathbb{R}.$$

Observe that if $f \in L^2_{\rm loc}(\mathbb{R}; H)$ satisfies the condition (34), then it also satisfies

$$\int_{-\infty}^0 e^{\sigma s}|f(s)|^2\,ds < \infty \quad \forall\, \sigma \in (\mu, 2\nu\lambda_1).$$

Thus, for any $\sigma \in (\mu, 2\nu\lambda_1)$ there exists the corresponding minimal pullback $\mathcal{D}^H_\sigma$-attractor $\mathcal{A}_{\mathcal{D}^H_\sigma}$.

By Theorem 3.6, since $\mathcal{D}^H_\mu \subset \mathcal{D}^H_\sigma$, it is evident that, for any $t \in \mathbb{R}$,

$$\mathcal{A}_{\mathcal{D}^H_\mu}(t) \subset \mathcal{A}_{\mathcal{D}^H_\sigma}(t) \quad \forall\, \sigma \in (\mu, 2\nu\lambda_1).$$

Moreover, if f satisfies (40), then, by (41),

$$\mathcal{A}_{\mathcal{D}^H_F}(t) = \mathcal{A}_{\mathcal{D}^H_\mu}(t) = \mathcal{A}_{\mathcal{D}^H_\sigma}(t) \quad \forall\, t \in \mathbb{R},\ \sigma \in (\mu, 2\nu\lambda_1).$$

In light of the fact that our analysis in H required a similar amount of work to obtain asymptotic compactness or the flattening property, one might ask if one could 'simplify' the direct proof of asymptotic compactness in V from [13] by using some ideas from the above 'flattening' analysis. It then becomes apparent that the idea of 'direct' proof in this case simply means trying to prove asymptotic compactness with resorting to a splitting of the solution into high and low modes; this serves to emphasise that the 'flattening property' can more rightly be thought of as a technique (splitting) that is always available should we require it.

5.1. **Compactness of the process in V via semigroups.** Finally we show that in V, too, a little more regularity of f yields the existence of a compact pullback absorbing family. To this end we assume that

$$f \in L^p_{\rm loc}(\mathbb{R}; H) \qquad \text{for some } p > 2.$$

With this assumption we show that there is a bounded absorbing family in $D(A^{1/2+\delta})$ for an appropriately chosen $\delta > 0$.

Theorem 5.6. *Suppose that $f \in L^p_{\rm loc}(\mathbb{R}; H)$ for some $p > 2$ and that*

$$\int_{-\infty}^0 e^{\mu s}|f(s)|^2\,ds < \infty \quad \text{for some } \mu \in (0, 2\nu\lambda_1).$$

Fix $\delta < \frac{1}{2} - \frac{1}{p}$. Then, for any $t \in \mathbb{R}$ and $\widehat{D} \in \mathcal{D}^H_\mu$, there exists $\tau_1(\widehat{D}, t)$ (the one given in Lemma 4.4), such that for any $\tau \le \tau_1(\widehat{D}, t)$ and any $u_\tau \in D(\tau)$,

$$|A^{1/2+\delta}u(t; \tau, u_\tau)| \le R_\delta(t).$$

Proof. The analysis in V is significantly simpler than in H. Indeed, for any $\epsilon > 0$ the nonlinear term maps V into $D(A^{-\epsilon})$: taking the inner product of $B(u)$ with $w \in D(A^\epsilon)$ we obtain

$$\begin{aligned}|\langle B(u), w\rangle| &\le \|u\|_{L^{1/\epsilon}}\|\nabla u\|_{L^2}\|w\|_{L^{2/(1-2\epsilon)}}\\ &\le \tilde{c}_\epsilon\|u\|^2\|w\|_{L^{2/(1-2\epsilon)}}\\ &\le \tilde{c}_\epsilon\|u\|^2\|w\|_\epsilon,\end{aligned}$$

since $D(A^s) \subset L^{2/(1-2s)}$. Thus

$$\|B(u)\|_{-\epsilon} \le \tilde{c}_\epsilon\|u\|^2.$$

Given a solution $u(t) = u(t; \tau, u_\tau)$ we write

$$u(t) = e^{-A}u(t-1) + \int_{t-1}^t e^{-A(t-s)}(B(u(s)) + f(s))\,ds.$$

Take the norm in $D(A^{1/2+\delta})$, using (26) and choosing ϵ so that $\delta+\epsilon<1/2$, we obtain

$$\begin{aligned}|A^{1/2+\delta}u(t)| &\le c_\delta\|u(t-1)\| + \tilde{c}_\epsilon c_{1/2+\delta+\epsilon}\int_{t-1}^t (t-s)^{-(1/2+\delta+\epsilon)}\|u(s)\|^2\,ds \\ &\quad + c_{1/2+\delta}\int_{t-1}^t (t-s)^{-(1/2+\delta)}|f(s)|\,ds \\ &\le c_\delta R_2(t) + \frac{\tilde{c}_\epsilon c_{1/2+\delta+\epsilon}}{1/2-\delta-\epsilon}R_2^2(t) + C_{p,\delta}\left(\int_{t-1}^t |f(s)|^p\,ds\right)^{1/p} =: R_\delta(t);\end{aligned}$$

the first term is bounded using Lemma 5.1; the second since $(t-s)^{-(1/2+\delta+\epsilon)}$ is integrable and $\|u(s)\|^2 \le R_2^2(t)$ uniformly for $s\in[t-1,t]$ (Lemma 5.1 again); and for the third term we can argue as in the proof of Theorem 4.8 using Hölder's inequality since $\delta < \frac{1}{2}-\frac{1}{p}$. □

We note in particular that in the autonomous case this gives a very quick method of proving the existence of a compact absorbing set in V when we assume only $f\in H$. As in the more complicated case in H, the higher regularity of f is the same as would be required to obtain a similar result for the linear problem $u_t+Au=f(t)$.

Conclusion. We have shown the existence of pullback attractors in H and V under minimal regularity assumptions on the forcing f, proving asymptotic compactness of the dynamical process via the Fourier splitting method, i.e. a proof of 'Condition (C)'/'the flattening property'. With a little additional regularity we have been able to use the semigroup approach to prove the existence of a compact pullback absorbing family in both cases.

It is interesting that in order to obtain the compact pullback absorbing family we require the same regularity of f as we would in the purely linear problem. One can see that this is to be expected if we consider solutions given by the variation of constants formula

$$u(t) = e^{-A(t-s)}u(s) + \int_s^t e^{-A(t-r)}[B(u(r))+f(r)]\,dr,$$

noting that the expression

$$U(t,s) := e^{-A(t-s)}u(s) + \int_s^t e^{-A(t-r)}f(r)\,dr$$

is simply the solution of the linear equation

$$v_t + Av = f(t), \qquad v(s) = u(s)$$

at time t. So we could write the following variation on the variation-of-constants formula,

$$u(t) = U(t,s) + \int_s^t e^{-A(t-r)}B(u(r))\,dr.$$

For the analysis in H, the key step was the estimate (31), which we can write as

$$\begin{aligned}X(s) \quad &\le \quad s^\epsilon|A^\epsilon U(s+\sigma,\sigma)| \\ &\quad +P(\sigma,t)s^\epsilon\left(\int_0^s (s-r)^{-(1-\epsilon)/(1-\epsilon/2)}r^{-3\epsilon/(2-\epsilon)}X(r)^{3/(2-\epsilon)}\,dr\right)^{1-\epsilon/2}.\end{aligned}$$

Conclusion of the argument requires a bound on $|A^\epsilon U(s+\sigma,\sigma)|$ uniform for $\sigma\in[t-1,t]$, $s\in[0,t-\sigma]$, i.e. relies on solutions of the linear equation.

Similarly, if we estimate u in $D(A^{1/2+\delta})$ as in Section 5.1 then we obtain

$$\begin{aligned}|A^{1/2+\delta}u(t)| &\le |A^{1/2+\delta}U(t,t-1)| + \tilde{c}_\epsilon c_{1/2+\delta+\epsilon}\int_{t-1}^{t}(t-s)^{-(1/2+\delta+\epsilon)}\|u(s)\|^2\,ds \\ &\le |A^{1/2+\delta}v(t)| + \frac{\tilde{c}_\epsilon c_{1/2+\delta+\epsilon}}{1/2-\delta-\epsilon}R_2^2(t),\end{aligned}$$

and the key point is again an estimate on U, i.e. smoothness for the linear equation.

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E-mail address: `luengo@us.es`
E-mail address: `pmr@us.es`
E-mail address: `j.c.robinson@warwick.ac.uk`