PULLBACK ATTRACTORS FOR GLOBALLY MODIFIED NAVIER-STOKES EQUATIONS WITH INFINITE DELAYS

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Abstract. We establish the existence of pullback attractors for the dynamical system associated to a globally modified model of the Navier-Stokes equations containing delay operators with infinite delay in a suitable weighted space. Actually, we are able to prove the existence of attractors in different classes of universes, one is the classical of fixed bounded sets, and the other is given by a tempered condition. Relationship between these two kind of objects is also analyzed.

1. **Introduction.** Let $\Omega \subset \mathbb{R}^3$ be an open bounded set with regular boundary Γ, and let $N \in (0, +\infty)$ be fixed. Let us define $F_N : [0, +\infty) \to (0, 1]$ by

$$F_N(r) := \min\left\{1, \frac{N}{r}\right\}, \quad r \in [0, +\infty),$$

and consider the following system of globally modified Navier-Stokes equations (GMNSE for short) on Ω, with infinite delays and homogeneous Dirichlet boundary condition

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu \Delta u + F_N\left(\|u\|\right)\left[(u \cdot \nabla)u\right] + \nabla p = f(t) + g(t, u_t) & \text{in } (\tau, +\infty) \times \Omega, \\ \nabla \cdot u = 0 & \text{in } (\tau, +\infty) \times \Omega, \\ u = 0 \text{ on } (\tau, +\infty) \times \Gamma, \\ u(\tau + s, x) = \phi(s, x), \ s \in (-\infty, 0], \ x \in \Omega, \end{cases} \tag{1}$$

where $\nu > 0$ is the kinematic viscosity, u the velocity field of the fluid, p the pressure, $\tau \in \mathbb{R}$ an initial time, $f(t)$ a given external force field, g is another external force field containing some hereditary characteristic, ϕ is a given function defined in the interval $(-\infty, 0]$, and we denote by u_t the function defined on $(-\infty, 0]$ by the relation $u_t(s) = u(t + s), \ s \in (-\infty, 0]$.

The GMNSE (1), with or without delays, are indeed *global* modifications of the Navier-Stokes equations – the modifying factor $F_N(\|u\|)$ depends on the norm $\|u\| = \|\nabla u\|_{(L^2(\Omega))^{3\times 3}}$, which in turn depends on ∇u over the whole domain Ω and not just at or near the point $x \in \Omega$ under consideration. Essentially, it prevents large gradients dominating the dynamics and leading to explosions. It violates the basic laws of mechanics, but mathematically the GMNSE (1) are a well defined system of equations, just like the modified versions of the NSE of Leray and others with other mollifications of the nonlinear term, see the review paper [12]. It is worth mentioning that a global cut off function involving the $D(A^{1/4})$ norm for the two dimensional stochastic Navier-Stokes equations is used in [13], and a cut-off function similar to the one we will use here was considered in [32].

The globally modified Navier-Stokes equations, in the case without delays, were introduced and studied in [1] (see also [3, 2, 19, 20, 21, 7, 24] and the review paper [18]). This modified model of the three-dimensional Navier-Stokes equations has some good properties: global existence, uniqueness, regularity, contrary to the original Navier-Sokes model, where the analysis of the asymptotic behaviour of solutions needs to be carried out in some non-standard way, e.g. cf. [28] and the references therein. These results are interesting in their own right, but also GMNSE are useful in obtaining new results about the three-dimensional Navier-Stokes equations, e.g., they were used in [1] to establish the existence of bounded entire weak solutions for them. Also, in [21], GMNSE were used to show that the attainability set of the weak solutions of the three-dimensional Navier-Stokes equations satisfying an energy inequality are weakly compact and weakly connected. For convergence results of solutions of GMNSE to solutions of the three-dimensional Navier-Stokes equations, see [1, 24].

However, there are situations in which the model is better described if some terms containing delays appear in the equations. These delays may appear, for instance, when one wants to control the system by applying a force which takes into account not only the present state but the complete history of the solutions.

To our knowledge, the references [8, 9, 10] are the first papers devoted to consider existence of solutions for the Navier-Stokes equations with delays and to study their asymptotic behaviour (see also [15, 25] for the same task in some unbounded domains). However, all these papers deal with finite delays, while the case of infinite delays has been treated more recently for autonomous and non-autonomous dynamical systems (e.g. cf. [6, 27]).

In this paper we are interested in the case of a GMNSE model in which terms containing infinite delays appear (see [7] for the case with finite delays). The problem (1) was studied in [23], where existence and uniqueness of solution, and convergence to stationary solutions were obtained.

Our goal in this paper is to prove more general results on the asymptotic behaviour of problem (1) than those shown in [23]. Namely, we will establish for a suitable process related to problem (1) that we can assure the existence of minimal pullback attractors under less restrictive assumptions than those in Theorem 3 and Theorem 4 in [23]. In fact, we will obtain two minimal pullback attractors for the

process associated to problem (1). The first one is the minimal pullback attractor of fixed bounded sets of $C_\gamma(H)$, which is the most usual in the literature. The second one, is the pullback attractor in the framework of a universe of families of time dependent sets with a tempered growth condition, following the ideas of [4, 26].

The structure of the paper is the following. In Section 2 we recall the basic results on existence of solution for the GMNSE problem with infinite delays. Indeed, an improvement on the conditions imposed for the existence is done. In Section 3 we state some well-known results on the theory for the existence of minimal pullback attractors, in a unified approach for an abstract given universe. This will be applied to two cases, one the classical case of fixed bounded sets, and the other is a universe defined by a tempered condition. Finally, in Section 4 we apply the above results to problem (1) obtaining two different kind of families of minimal pullback attractors. The main key is an asymptotic compactness result, whose proof relies on an energy method that makes the most of the continuity properties of the solutions and the corresponding non-increasing energy functions. Relationship between these objects is also analyzed.

2. **Existence of solutions.** To set our problem in the abstract framework, we consider the following usual abstract spaces (e.g. cf. [22] and [30, 31]):

$$\mathcal{V} = \left\{ u \in (C_0^\infty(\Omega))^3 : \text{div } u = 0 \right\},$$

$H =$ the closure of $\mathcal{V}$ in $(L^2(\Omega))^3$ with inner product $(\cdot, \cdot)$ and associate norm $|\cdot|$, where for $u, v \in (L^2(\Omega))^3$,

$$(u, v) = \sum_{j=1}^{3} \int_\Omega u_j(x) v_j(x) \, dx,$$

$V =$ the closure of $\mathcal{V}$ in $(H_0^1(\Omega))^3$ with scalar product $((\cdot, \cdot))$ and associate norm $\|\cdot\|$, where for $u, v \in (H_0^1(\Omega))^3$,

$$((u, v)) = \sum_{i,j=1}^{3} \int_\Omega \frac{\partial u_j}{\partial x_i} \frac{\partial v_j}{\partial x_i} \, dx.$$

We will use $\|\cdot\|_*$ for the norm in V' and $\langle \cdot, \cdot \rangle$ for the duality pairing between V' and V. Finally, we will identify every $u \in H$ with the element $f_u \in V'$ given by

$$\langle f_u, v \rangle = (u, v) \quad \text{for all } v \in V.$$

It follows that $V \subset H \subset V'$, where the injections are dense and compact.

We consider the linear continuous operator $A : V \to V'$ defined by

$$\langle Au, v \rangle = ((u, v)) \quad \text{for all } u, v \in V.$$

Denoting $D(A) = \{u \in V : Au \in H\}$, with inner product $(u, v)_{D(A)} = (Au, Av)$, then, by the regularity of Γ, $D(A) = (H^2(\Omega))^3 \cap V$, and $Au = -P\Delta u$, for all $u \in D(A)$, is the Stokes operator (P is the ortho-projector from $(L^2(\Omega))^3$ onto H).

Let us denote

$$\lambda_1 = \inf_{v \in V \setminus \{0\}} \frac{\|v\|^2}{|v|^2} > 0,$$

the first eigenvalue of the Stokes operator.

Now we define

$$b(u, v, w) = \sum_{i,j=1}^{3} \int_{\Omega} u_i \frac{\partial v_j}{\partial x_i} w_j \, dx,$$

for all measurable functions u, v, w defined on Ω with values in $\mathbb{R}^3$ for which the integrals in the right-hand member of the above equality are finite.

In particular, b is a trilinear continuous form on $V \times V \times V$, i.e., there exists a constant $C_1 > 0$ only dependent on Ω (namely, $C_1 = (2\lambda_1^{1/4})^{-1}$) such that

$$|b(u, v, w)| \leq C_1 \|u\|\|v\|\|w\|, \quad \text{for all } u, v, w \in V,$$

and $b(u, v, v) = 0$, for all $u, v \in V$.

We denote

$$b_N(u, v, w) = F_N(\|v\|)b(u, v, w).$$

The form b_N is linear in u and w, but it is nonlinear in v.

By the definition of F_N, if we denote

$$\langle B_N(u, v), w \rangle = b_N(u, v, w), \quad \text{for all } u, v, w \in V,$$

we have

$$\|B_N(u, v)\|_* \leq NC_1 \|u\|, \quad \text{for all } u, v \in V. \tag{2}$$

We recall (cf. [30]) that there exists a constant $C_2 > 0$ depending only on Ω such that

$$|b(u, v, w)| \leq C_2\|u\|^{1/2}|Au|^{1/2}\|v\|\|w\|, \tag{3}$$

for all $u \in D(A), v \in V, w \in H$, and

$$|b(u, v, w)| \leq C_2\|u\|\|v\|\|w\|^{1/2}\|w\|^{1/2}, \tag{4}$$

for all $u, v, w \in V$. (See [29] for the proof of (4)).

Let $\gamma > 0$ be fixed. One possibility to deal with infinite delays, and which we will use here (cf. [27, 16, 17]), is to consider the space

$$C_\gamma(H) = \left\{ \varphi \in C((-\infty, 0]; H) \ : \ \exists \lim_{s \to -\infty} e^{\gamma s}\varphi(s) \in H \right\},$$

which is a Banach space with the norm

$$\|\varphi\|_\gamma := \sup_{s \in (-\infty, 0]} e^{\gamma s}|\varphi(s)|.$$

We will assume that $f \in L^2_{loc}(\mathbb{R}; (L^2(\Omega))^3)$. For the term g, in which the delay is present, we assume that $g : \mathbb{R} \times C_\gamma(H) \to (L^2(\Omega))^3$ satisfies

(g1) For any $\xi \in C_\gamma(H)$ the mapping $\mathbb{R} \ni t \mapsto g(t, \xi)$ is measurable,
(g2) $g(t, 0) = 0$ for all $t \in \mathbb{R}$,
(g3) there exists a constant $L_g > 0$ such that for any $t \in \mathbb{R}$ and all $\xi, \eta \in C_\gamma(H)$,

$$|g(t, \xi) - g(t, \eta)| \leq L_g\|\xi - \eta\|_\gamma.$$

An example of an operator satisfying assumptions (g1)-(g3) is given in [23].

Definition 1. A weak solution of (1) is a function $u \in C((-\infty, T]; H) \cap L^2(\tau, T; V)$ for all $T > \tau$, such that for all $v \in V$,

$$\frac{d}{dt}(u(t), v) + \nu((u(t), v)) + b_N(u(t), u(t), v) = (f(t), v) + (g(t, u_t), v),$$

in the sense of $\mathcal{D}'(\tau, +\infty)$, and $u_\tau = \phi$.

Remark 1. If u is a weak solution of (1), then u satisfies the energy equality,

$$|u(t)|^2 + 2\nu \int_s^t \|u(r)\|^2 dr$$

$$= |u(s)|^2 + 2 \int_s^t [(f(r), u(r)) + (g(r, u_r), u(r))] \, dr \quad \text{for all } s, t \in [\tau, +\infty).$$

We have the following existence and uniqueness result:

Theorem 1. *Suppose that $f \in L^2_{loc}(\mathbb{R}; (L^2(\Omega))^3)$, $\gamma > 0$, and $g : \mathbb{R} \times C_\gamma(H) \to (L^2(\Omega))^3$ satisfying the assumptions (g1)–(g3), are given. Then, for any $\tau \in \mathbb{R}$ and $\phi \in C_\gamma(H)$, there exists a unique weak solution $u = u(\cdot; \tau, \phi)$ of (1), which in fact is a strong solution in the sense that*

$$u \in C((\tau, T]; V) \cap L^2(\tau + \varepsilon, T; D(A)),$$

for all $\varepsilon > 0$ and any $T > \tau + \varepsilon$.
    Moreover, if $\phi(0) \in V$, then u satisfies

$$u \in C([\tau, T]; V) \cap L^2(\tau, T; D(A)),$$

for all $T > \tau$.

Proof. The proof can be seen in [23]. There, the additional assumption $2\gamma > \nu\lambda_1$ was made. That this assumption is unnecessary can be seen as follows.
    For the Galerkin approximations u^m defined by (8) on page 661 of [23], one has

$$\frac{d}{dt}|u^m(t)|^2 + 2\nu\|u^m(t)\|^2 = 2(f(t), u^m(t)) + 2(g(t, u_t^m), u^m(t))$$

$$\leq \nu\|u^m(t)\|^2 + \frac{1}{\nu\lambda_1}|f(t)|^2 + 2L_g\|u_t^m\|_\gamma^2,$$

and therefore,

$$|u^m(t)|^2 + \nu \int_\tau^t \|u^m(s)\|^2 ds \leq |u(\tau)|^2 + \int_\tau^t (|f(s)|^2/(\nu\lambda_1) + 2L_g\|u_s^m\|_\gamma^2)ds, \quad (5)$$

for all $t \geq \tau$.
    Using (5) instead of inequality (9) of [23], one obtains

$$\|u_t^m\|_\gamma^2 \leq \max \left\{ \sup_{\theta \in (-\infty, \tau-t]} e^{2\gamma\theta}|\phi(\theta + t - \tau)|^2, \right.$$

$$\sup_{\theta \in [\tau-t, 0]} \left(e^{2\gamma\theta}|u(\tau)|^2 + e^{2\gamma\theta} \int_\tau^{t+\theta} (|f(s)|^2/(\nu\lambda_1) + 2L_g\|u_s^m\|_\gamma^2)ds \right) \right\}$$

$$\leq \max \left\{ \sup_{\theta \in (-\infty, \tau-t]} e^{2\gamma\theta}|\phi(\theta + t - \tau)|^2, \right.$$

$$\left. |u(\tau)|^2 + \int_\tau^t (|f(s)|^2/(\nu\lambda_1) + 2L_g\|u_s^m\|_\gamma^2)ds \right\}, \quad (6)$$

and therefore, observing that

$$
\begin{aligned}
\sup_{\theta \in (-\infty, \tau-t]} e^{\gamma\theta} |\phi(\theta + t - \tau)| &= \sup_{\theta \leq 0} e^{\gamma(\theta - (t-\tau))} |\phi(\theta)| \\
&= e^{-\gamma(t-\tau)} \|\phi\|_\gamma \\
&\leq \|\phi\|_\gamma,
\end{aligned}
$$

and $|u(\tau)| = |\phi(0)| \leq \|\phi\|_\gamma$, we deduce from (6) that

$$
\|u_t^m\|_\gamma^2 \leq \|\phi\|_\gamma^2 + \int_\tau^t (|f(s)|^2/(\nu\lambda_1) + 2L_g\|u_s^m\|_\gamma^2)ds
$$

for all $t \geq \tau$.

Thus, by the Gronwall lemma, we have

$$
\|u_t^m\|_\gamma^2 \leq e^{2L_g(t-\tau)}\|\phi\|_\gamma^2 + (\nu\lambda_1)^{-1} \int_\tau^t e^{2L_g(t-s)}|f(s)|^2 ds,
$$

for all $t \geq \tau$.

Using this inequality and (5), one also obtains (10) and (12) in [23]. Now, the proof of the theorem follows as in that paper. □

The following continuous dependence result was also proved in [23, Prop.1].

Proposition 1 (Continuity of solutions with respect to initial data). *Under the assumptions of Theorem 1, for any $\tau \in \mathbb{R}$ the solutions obtained for (1) are continuous with respect to the initial condition ϕ, and more exactly, there exists a constant $C_3 > 0$, only dependent on ν and the constant C_2 appearing in (4), such that if $u^i = u^i(\cdot; \tau, \phi^i)$, for $i = 1, 2$, are the corresponding solutions to initial data $\phi^i \in C_\gamma(H)$, $i = 1, 2$, the following estimate holds:*

$$
\max_{r \in [\tau,t]} |u^1(r) - u^2(r)|^2 \leq \left(|\phi^1(0) - \phi^2(0)|^2 + \frac{L_g}{2\gamma}\|\phi^1 - \phi^2\|_\gamma^2 \right)
$$
$$
\times e^{(3L_g + 2C_3 N^4)(t-\tau)},
$$

for all $t \geq \tau$.

3. **Abstract results on attractors theory. Existence of minimal pullback attractors.** In this section we recall some abstract results on pullback attractors theory. We present a summary of some results on the existence of minimal pullback attractors obtained in [14] (see also [26, 4, 5]). In particular, we consider the process U being closed (see Definition 2 below).

Consider given a metric space (X, d_X), and let us denote $\mathbb{R}^2_d = \{(t, \tau) \in \mathbb{R}^2 : \tau \leq t\}$.

A process on X is a mapping U such that $\mathbb{R}^2_d \times X \ni (t, \tau, x) \mapsto U(t, \tau)x \in X$ with $U(\tau, \tau)x = x$ for any $(\tau, x) \in \mathbb{R} \times X$, and $U(t, r)(U(r, \tau)x) = U(t, \tau)x$ for any $\tau \leq r \leq t$ and all $x \in X$.

Definition 2. Let U be a process on X.

a) U is said to be continuous if for any pair $\tau \leq t$, the mapping $U(t, \tau): X \to X$ is continuous.

b) U is said to be closed if for any $\tau \leq t$, and any sequence $\{x_n\} \subset X$, if $x_n \to x \in X$ and $U(t, \tau)x_n \to y \in X$, then $U(t, \tau)x = y$.

Remark 2. It is clear that every continuous process is closed. More generally, every strong-weak continuous process (see [26] for the definition) is a closed process.

Let us denote $\mathcal{P}(X)$ the family of all nonempty subsets of X, and consider a family of nonempty sets $\widehat{D}_0 = \{D_0(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ [observe that we do not require any additional condition on these sets as compactness or boundedness].

Definition 3. We say that a process U on X is pullback $\widehat{D}_0$-asymptotically compact if for any $t \in \mathbb{R}$ and any sequences $\{\tau_n\} \subset (-\infty, t]$ and $\{x_n\} \subset X$ satisfying $\tau_n \to -\infty$ and $x_n \in D_0(\tau_n)$ for all n, the sequence $\{U(t, \tau_n)x_n\}$ is relatively compact in X.

Let be given $\mathcal{D}$ a nonempty class of families parameterized in time $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$. The class $\mathcal{D}$ will be called a universe in $\mathcal{P}(X)$.

Definition 4. It is said that $\widehat{D}_0 = \{D_0(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ is pullback $\mathcal{D}-$absorbing for the process U on X if for any $t \in \mathbb{R}$ and any $\widehat{D} \in \mathcal{D}$, there exists a $\tau_0(t, \widehat{D}) \leq t$ such that

$$U(t, \tau)D(\tau) \subset D_0(t) \quad \text{for all } \tau \leq \tau_0(t, \widehat{D}).$$

Observe that in the definition above $\widehat{D}_0$ does not belong necessarily to the class $\mathcal{D}$.

Definition 5. Given a family parameterized in time, $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$, it is said that a process U on X is $\widehat{D}-$asymptotically compact if for any $t \in \mathbb{R}$ and any sequences $\{\tau_n\} \subset (-\infty, t]$ and $\{x_n\} \subset X$ satisfying $\tau_n \to -\infty$ and $x_n \in D(\tau_n)$ for all n, the sequence $\{U(t, \tau_n)x_n\}$ is relatively compact in X.

Definition 6. A process U on X is said to be pullback $\mathcal{D}-$asymptotically compact if it is $\widehat{D}$-asymptotically compact for any $\widehat{D} \in \mathcal{D}$.

Denote

$$\Lambda(\widehat{D}_0, t) := \bigcap_{s \leq t} \overline{\bigcup_{\tau \leq s} U(t, \tau)D_0(\tau)}^X \quad \text{for all } t \in \mathbb{R},$$

where $\overline{\{\cdots\}}^X$ is the closure in X.

Finally, we denote by $\text{dist}_X(\mathcal{O}_1, \mathcal{O}_2)$ the Hausdorff semi-distance in X between two sets $\mathcal{O}_1$ and $\mathcal{O}_2$, defined as

$$\text{dist}_X(\mathcal{O}_1, \mathcal{O}_2) = \sup_{x \in \mathcal{O}_1} \inf_{y \in \mathcal{O}_2} d_X(x, y) \quad \text{for } \mathcal{O}_1, \mathcal{O}_2 \subset X.$$

We have the following result on existence of minimal pullback attractors (cf. [14]).

Theorem 2. *Consider a closed process $U : \mathbb{R}^2_d \times X \to X$, a universe $\mathcal{D}$ in $\mathcal{P}(X)$, and a family $\widehat{D}_0 = \{D_0(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ which is pullback $\mathcal{D}-$absorbing for U, and assume also that U is pullback $\widehat{D}_0-$asymptotically compact.*

Then, the family $\mathcal{A}_{\mathcal{D}} = \{\mathcal{A}_{\mathcal{D}}(t) : t \in \mathbb{R}\}$ defined by

$$\mathcal{A}_{\mathcal{D}}(t) = \overline{\bigcup_{\widehat{D} \in \mathcal{D}} \Lambda(\widehat{D}, t)}^X \quad t \in \mathbb{R},$$

has the following properties:

(a) *for any $t \in \mathbb{R}$, the set $\mathcal{A}_{\mathcal{D}}(t)$ is a nonempty compact subset of X, and*

$$\mathcal{A}_{\mathcal{D}}(t) \subset \Lambda(\widehat{D}_0, t),$$

(b) $\mathcal{A}_{\mathcal{D}}$ *is pullback $\mathcal{D}-$attracting, i.e.,*

$$\lim_{\tau \to -\infty} \operatorname{dist}_X(U(t, \tau)D(\tau), \mathcal{A}_{\mathcal{D}}(t)) = 0 \quad \text{for all } \widehat{D} \in \mathcal{D}, \quad t \in \mathbb{R},$$

(c) $\mathcal{A}_{\mathcal{D}}$ *is invariant, i.e., $U(t, \tau)\mathcal{A}_{\mathcal{D}}(\tau) = \mathcal{A}_{\mathcal{D}}(t)$ for all $\tau \leq t$,*

(d) *if $\widehat{D}_0 \in \mathcal{D}$, then $\mathcal{A}_{\mathcal{D}}(t) = \Lambda(\widehat{D}_0, t) \subset \overline{D_0(t)}^X$, for all $t \in \mathbb{R}$.*

The family $\mathcal{A}_{\mathcal{D}}$ is minimal in the sense that if $\widehat{C} = \{C(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ is a family of closed sets such that for any $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \in \mathcal{D}$,

$$\lim_{\tau \to -\infty} \operatorname{dist}_X(U(t, \tau)D(\tau), C(t)) = 0,$$

then $\mathcal{A}_{\mathcal{D}}(t) \subset C(t)$.

Remark 3. Under the assumptions of Theorem 2, the family $\mathcal{A}_{\mathcal{D}}$ is called the minimal pullback $\mathcal{D}-$attractor for the process U.

If $\mathcal{A}_{\mathcal{D}} \in \mathcal{D}$, then it is the unique family of closed subsets in $\mathcal{D}$ that satisfies (b)–(c).

A sufficient condition for $\mathcal{A}_{\mathcal{D}} \in \mathcal{D}$ is to have that $\widehat{D}_0 \in \mathcal{D}$, the set $D_0(t)$ is closed for all $t \in \mathbb{R}$, and the family $\mathcal{D}$ is inclusion-closed (i.e., if $\widehat{D} \in \mathcal{D}$, and $\widehat{D}' = \{D'(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ with $D'(t) \subset D(t)$ for all t, then $\widehat{D}' \in \mathcal{D}$).

We will denote by $\mathcal{D}_F(X)$ the universe of fixed nonempty bounded subsets of X, i.e., the class of all families $\widehat{D}$ of the form $\widehat{D} = \{D(t) = D : t \in \mathbb{R}\}$ with D a fixed nonempty bounded subset of X. In the particular case of the universe $\mathcal{D}_F(X)$, the corresponding minimal pullback $\mathcal{D}_F(X)-$attractor for the process U is the pullback attractor defined by Crauel, Debussche, and Flandoli, [11, Th.1.1, p.311], and will be denoted by $\mathcal{A}_{\mathcal{D}_F(X)}$.

Now, it is easy to conclude the following result.

Corollary 1. *[cf. [25, Cor.20]] Under the assumptions of Theorem 2, if the universe $\mathcal{D}$ contains the universe $\mathcal{D}_F(X)$, then both attractors, $\mathcal{A}_{\mathcal{D}_F(X)}$ and $\mathcal{A}_{\mathcal{D}}$, exist, and the following relation holds:*

$$\mathcal{A}_{\mathcal{D}_F(X)}(t) \subset \mathcal{A}_{\mathcal{D}}(t) \quad \text{for all } t \in \mathbb{R}.$$

Remark 4. It can be proved (see [26]) that, under the assumptions of the preceding corollary, if, moreover, $\widehat{D}_0 \in \mathcal{D}$, and for some $T \in \mathbb{R}$ the set $\cup_{t \leq T} D_0(t)$ is a bounded subset of X, then

$$\mathcal{A}_{\mathcal{D}_F(X)}(t) = \mathcal{A}_{\mathcal{D}}(t) \quad \text{for all } t \leq T.$$

4. **Existence of pullback attractors for the process associated to** (1)**.** Now, by the previous results, we are able to define correctly a process U on $C_\gamma(H)$ associated to (1), and to obtain the existence of minimal pullback attractors.

Proposition 2. *Assume that $f \in L^2_{loc}(\mathbb{R}; (L^2(\Omega))^3)$, $\gamma > 0$, and $g : \mathbb{R} \times C_\gamma(H) \to (L^2(\Omega))^3$ satisfying the assumptions (g1)–(g3), are given. Then, the bi-parametric family of maps $U(t, \tau) : C_\gamma(H) \to C_\gamma(H)$, with $\tau \leq t$, given by*

$$U(t, \tau)\phi = u_t, \tag{7}$$

where $u = u(\cdot; \tau, \phi)$ is the unique weak solution of (1), defines a continuous process on $C_\gamma(H)$.

Proof. It is a consequence of Theorem 1 and Proposition 1. □

Lemma 1. *Under the assumptions of Proposition 2, let μ be such that*

$$0 < \mu < \nu \quad and \quad (\nu - \mu)\lambda_1 \leq \gamma. \tag{8}$$

Then, the following estimates hold for the solution to (1) for all $t \geq \tau$:

$$\begin{aligned}
\|u_t\|_\gamma^2 &\leq e^{-2((\nu-\mu)\lambda_1 - L_g)(t-\tau)}\|\phi\|_\gamma^2 \\
&\quad + (\mu\lambda_1)^{-1}\int_\tau^t e^{-2((\nu-\mu)\lambda_1 - L_g)(t-s)}|f(s)|^2 ds,
\end{aligned} \tag{9}$$

$$\begin{aligned}
\mu\int_\tau^t \|u(s)\|^2 ds &\leq e^{2(\nu-\mu)\lambda_1(t-\tau)}|u(\tau)|^2 + e^{2L_g(t-\tau)}\|\phi\|_\gamma^2 \\
&\quad + (\mu\lambda_1)^{-1}e^{-2(\nu-\mu)\lambda_1\tau}\int_\tau^t e^{2(\nu-\mu)\lambda_1 s}|f(s)|^2 ds \\
&\quad + (\mu\lambda_1)^{-1}e^{2L_g t - 2(\nu-\mu)\lambda_1\tau}\int_\tau^t e^{2((\nu-\mu)\lambda_1 - L_g)s}|f(s)|^2 ds. \tag{10}
\end{aligned}$$

Proof. Take a μ such that $0 < \mu < \nu$. By the energy equality (see Remark 1), one has

$$\begin{aligned}
\frac{d}{dt}|u(t)|^2 + 2\nu\|u(t)\|^2 &= 2(f(t), u(t)) + 2(g(t, u_t), u(t)) \\
&\leq 2\lambda_1^{-1/2}|f(t)|\|u(t)\| + 2L_g\|u_t\|_\gamma|u(t)| \\
&\leq \mu\|u(t)\|^2 + \frac{1}{\mu\lambda_1}|f(t)|^2 + 2L_g\|u_t\|_\gamma^2, \quad \text{a.e. } t > \tau.
\end{aligned}$$

Thus,

$$\begin{aligned}
\frac{d}{dt}|u(t)|^2 + 2(\nu-\mu)\lambda_1|u(t)|^2 + \mu\|u(t)\|^2 &\leq \frac{d}{dt}|u(t)|^2 + 2(\nu-\mu)\|u(t)\|^2 + \mu\|u(t)\|^2 \\
&\leq \frac{1}{\mu\lambda_1}|f(t)|^2 + 2L_g\|u_t\|_\gamma^2, \quad \text{a.e. } t > \tau,
\end{aligned}$$

and therefore,

$$|u(t)|^2 + \mu\int_\tau^t e^{-2(\nu-\mu)\lambda_1(t-s)}\|u(s)\|^2 ds \tag{11}$$

$$\leq e^{-2(\nu-\mu)\lambda_1(t-\tau)}|u(\tau)|^2 + \int_\tau^t e^{-2(\nu-\mu)\lambda_1(t-s)}(|f(s)|^2/(\mu\lambda_1) + 2L_g\|u_s\|_\gamma^2)ds,$$

for all $t \geq \tau$.

Consequently

$$\begin{aligned}
\|u_t\|_\gamma^2 \leq \max\Big\{ &\sup_{\theta \in (-\infty, \tau-t]} e^{2\gamma\theta}|\phi(\theta + t - \tau)|^2, \\
&\sup_{\theta \in [\tau-t, 0]} (e^{2\gamma\theta - 2(\nu-\mu)\lambda_1(t-\tau+\theta)}|u(\tau)|^2 \\
&+ e^{2\gamma\theta}\int_\tau^{t+\theta} e^{-2(\nu-\mu)\lambda_1(t+\theta-s)}(|f(s)|^2/(\mu\lambda_1) + 2L_g\|u_s\|_\gamma^2)ds)\Big\}.
\end{aligned}$$

Since μ satisfies (8), we have that on the one hand

$$
\begin{aligned}
\sup_{\theta\in(-\infty,\tau-t]} e^{\gamma\theta}|\phi(\theta+t-\tau)| &= \sup_{\theta\le 0} e^{\gamma(\theta-(t-\tau))}|\phi(\theta)| \\
&= e^{-\gamma(t-\tau)}\|\phi\|_\gamma \\
&\le e^{-(\nu-\mu)\lambda_1(t-\tau)}\|\phi\|_\gamma.
\end{aligned}
$$

On the other hand,

$$
\sup_{\theta\in[\tau-t,0]} e^{2\gamma\theta-2(\nu-\mu)\lambda_1(t-\tau+\theta)}|u(\tau)|^2 \le e^{-2(\nu-\mu)\lambda_1(t-\tau)}|u(\tau)|^2
$$

and

$$
\sup_{\theta\in[\tau-t,0]} e^{2\gamma\theta}\int_\tau^{t+\theta} e^{-2(\nu-\mu)\lambda_1(t+\theta-s)}(|f(s)|^2/(\mu\lambda_1)+2L_g\|u_s\|_\gamma^2)ds
$$
$$
\le \int_\tau^t e^{-2(\nu-\mu)\lambda_1(t-s)}(|f(s)|^2/(\mu\lambda_1)+2L_g\|u_s\|_\gamma^2)ds.
$$

Collecting these inequalities we deduce

$$
\|u_t\|_\gamma^2 \le e^{-2(\nu-\mu)\lambda_1(t-\tau)}\|\phi\|_\gamma^2 + \int_\tau^t e^{-2(\nu-\mu)\lambda_1(t-s)}(|f(s)|^2/(\mu\lambda_1)+2L_g\|u_s\|_\gamma^2)ds,
$$

for all $t\ge\tau$.

Then, by the Gronwall lemma we conclude that (9) holds.

Now, from (11), (9), and Fubini's theorem, we conclude (10). $\square$

From now on we will assume that

$$
\text{there exists } 0 < \mu < \nu \text{ such that } L_g < (\nu-\mu)\lambda_1 \le \gamma, \tag{12}
$$

and

$$
\int_{-\infty}^0 e^{2((\nu-\mu)\lambda_1-L_g)s}|f(s)|^2 ds < +\infty. \tag{13}
$$

Remark 5. Condition (12) is equivalent to

$$
L_g < \min(\gamma, \nu\lambda_1). \tag{14}
$$

Indeed, it is clear that (12) implies (14).

Assume now that L_g satisfies (14). Then there are two possibilities:

If $\nu\lambda_1 \le \gamma$, then as $L_g < \nu\lambda_1$, it is evident that there exists a μ such that (12) is satisfied.

If $\nu\lambda_1 > \gamma$, let us take $\mu = \lambda_1^{-1}(\nu\lambda_1 - \gamma)$. Evidently, $0 < \mu < \nu$, and $L_g < \gamma = (\nu-\mu)\lambda_1$.

Remark 6. If we assume that $f \in L^2_{\text{loc}}(\mathbb{R};(L^2(\Omega))^3)$, assumption (13) is equivalent to

$$
\int_{-\infty}^t e^{-2((\nu-\mu)\lambda_1-L_g)(t-s)}|f(s)|^2 ds < +\infty, \quad \text{for all } t\in\mathbb{R}.
$$

From now on, for brevity we will denote

$$
\sigma = 2((\nu-\mu)\lambda_1 - L_g). \tag{15}
$$

Definition 7. We will denote by $\mathcal{D}_\sigma(C_\gamma(H))$ the class of all families of nonempty subsets $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{P}(C_\gamma(H))$ such that

$$\lim_{\tau \to -\infty} \left(e^{\sigma\tau} \sup_{v \in D(\tau)} \|v\|_\gamma^2 \right) = 0.$$

Accordingly to the notation introduced in the previous subsection, $\mathcal{D}_F(C_\gamma(H))$ will denote the class of families $\widehat{D} = \{D(t) = D : t \in \mathbb{R}\}$ with D a fixed nonempty bounded subset of $C_\gamma(H)$.

Remark 7. Observe that $\mathcal{D}_F(C_\gamma(H)) \subset \mathcal{D}_\sigma(C_\gamma(H))$, and that both are inclusion-closed.

Corollary 2. *Under the assumptions of Proposition 2, if moreover conditions (12) and (13) are satisfied, then the family $\widehat{D}_0 = \{D_0(t) : t \in \mathbb{R}\}$, with $D_0(t) = \overline{B}_{C_\gamma(H)}(0, \rho(t))$, the closed ball in $C_\gamma(H)$ of center zero and radius $\rho(t)$, where*

$$\rho^2(t) = 1 + (\mu\lambda_1)^{-1} \int_{-\infty}^t e^{-\sigma(t-s)}|f(s)|^2 ds, \tag{16}$$

is pullback $\mathcal{D}_\sigma(C_\gamma(H))$-absorbing for the process U defined by (7). Moreover, $\widehat{D}_0 \in \mathcal{D}_\sigma(C_\gamma(H))$.

Proof. This follows immediately from Lemma 1. $\qquad\qquad\square$

Proposition 3. *Under the assumptions of Corollary 2, the process U defined by (7) is $\widehat{D}_0$-asymptotically compact.*

Proof. Let us fix $t_0 \in \mathbb{R}$. Let $u^n = u^n(\cdot; \tau_n, \phi^n)$ be a sequence of weak solutions of (1), defined in their respective intervals $[\tau_n, +\infty)$, with initial data $\phi^n \in D_0(\tau_n) = \overline{B}_{C_\gamma(H)}(0, \rho(\tau_n))$, where $\tau_n \to -\infty$ as $n \to +\infty$. Without loss of generality, we may assume that $\tau_n < t_0$ for all n. Consider the sequence $\xi^n = u_{t_0}^n$. Then we will prove that this sequence is relatively compact in $C_\gamma(H)$. To do this, we will proceed in two steps.

Step 1: We will prove that from $\{\xi^n\}$ we may extract a subsequence, relabelled the same, and a continuous function $\psi : (-\infty, 0] \to H$, such that $\xi^n \to \psi$ in $C([-\overline{T}, 0]; H)$ for every $\overline{T} > 0$.

Consider two arbitrary values $0 < \overline{T} < T$.

It follows from (9) and (13) that there exists $n_0(t_0, T)$ such that $\tau_n < t_0 - T$ for $n \geq n_0(t_0, T)$, and

$$\|u_t^n\|_\gamma^2 \leq R(t_0, T) < +\infty \quad \text{for all } t \in [t_0 - T, t_0], \text{ and any } n \geq n_0(t_0, T), \tag{17}$$

where

$$R(t_0, T) = 1 + (\mu\lambda_1)^{-1} e^{-\sigma(t_0-T)} \int_{-\infty}^{t_0} e^{\sigma s}|f(s)|^2 ds,$$

so that, in particular,

$$|u^n(t)|^2 \leq R(t_0, T), \quad \text{for all } t \in [t_0 - T, t_0], \text{ and any } n \geq n_0(t_0, T). \tag{18}$$

Let

$$y^n(t) = u^n(t + t_0 - T), \quad \text{for all } t \in [0, T].$$

In particular, by (18), the sequence $\{y^n\}_{n \geq n_0(t_0, T)}$ is bounded in $L^\infty(0, T; H)$.

On the other hand, for each $n \geq n_0(t_0, T)$, the function y^n is a solution on $[0, T]$ of a problem similar to (1), namely with f and g replaced by

$$\tilde{f}(t) = f(t + t_0 - T) \text{ and } \tilde{g}(t, \cdot) = g(t + t_0 - T, \cdot), \text{ for all } t \in [0, T], \qquad (19)$$

respectively, and with $y_0^n = u_{t_0-T}^n$, $y_T^n = u_{t_0}^n = \xi^n$. By (17), $\|y_0^n\|_\gamma^2 \leq R(t_0, T)$ for all $n \geq n_0(t_0, T)$. From (10) we have

$$\|y^n\|_{L^2(0,T;V)}^2 \leq K(t_0, T), \quad \text{for all } n \geq n_0(t_0, T). \qquad (20)$$

Hence, the sequence $\{y^n\}_{n \geq n_0(t_0,T)}$ is also bounded in $L^2(0, T; V)$, and by (2), the sequence of time derivatives $\{(y^n)'\}_{n \geq n_0(t_0,T)}$ is bounded in $L^2(0, T; V')$. Thus, up to a subsequence (relabelled the same), for some function y we have that

$$\begin{aligned}
y^n &\overset{*}{\rightharpoonup} y & &\text{weakly star in } L^\infty(0, T; H), \\
y^n &\rightharpoonup y & &\text{weakly in } L^2(0, T; V), \\
(y^n)' &\rightharpoonup y' & &\text{weakly in } L^2(0, T; V'), \\
y^n &\to y & &\text{strongly in } L^2(0, T; H), \\
y^n(t) &\to y(t) & &\text{strongly in } H, \text{ a.e. } t \in (0, T).
\end{aligned}$$

Observe also that for every sequence $\{t_n\} \subset [0, T]$ with $t_n \to t^*$, one has

$$y^n(t_n) \rightharpoonup y(t^*) \text{ weakly in } H. \qquad (21)$$

This follows from the boundedness of the sequence $\{y^n\}_{n \geq n_0(t_0,T)}$ in $L^\infty(0, T; H)$, the boundedness of the sequence $\{(y^n)'\}_{n \geq n_0(t_0,T)}$ in $L^2(0, T; V')$, and the compactness of the injection of H into V' (see [23] for a similar argument).

In order to find the equation satisfied by y, we have the trouble that the weak convergence in $L^2(0, T; V)$ is not enough to ensure that

$$\|y^n(t)\| \to \|y(t)\|,$$

or at least

$$F_N(\|y^n(t)\|) \to F_N(\|y(t)\|) \quad \text{for a.a. } t,$$

which is needed to manage the nonlinear term $B_N(y^n, y^n)$.

Now, we are going to obtain a stronger estimate. From now on, we assume that $n \geq n_0(t_0, T)$. As y^n satisfies a problem similar to (1) on $[0, T]$, with $\tilde{f}$ and $\tilde{g}$ defined by (19), in place of f and g, taking the inner product with Ay^n, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|y^n(t)\|^2 + \nu|Ay^n(t)|^2 + b_N(y^n(t), y^n(t), Ay^n(t)) \\
&= (\tilde{f}(t), Ay^n(t)) + (\tilde{g}(t, y_t^n), Ay^n(t)).
\end{aligned} \qquad (22)$$

Obviously,

$$(\tilde{f}(t), Ay^n(t)) \leq |\tilde{f}(t)||Ay^n(t)| \leq \frac{\nu}{8}|Ay^n(t)| + \frac{2}{\nu}|\tilde{f}(t)|^2$$

and

$$|(\tilde{g}(t, y_t^n), Ay^n(t))| \leq \frac{\nu}{8}|Ay^n(t)|^2 + \frac{2}{\nu}|\tilde{g}(t, y_t^n)|^2.$$

From the definition of $F_N(\cdot)$, (3), and Young's inequality, it follows

$$
\begin{aligned}
|b_N(y^n(t), y^n(t), Ay^n(t))| &\leq \frac{N}{\|y^n(t)\|} C_2 \|y^n(t)\|^{3/2} |Ay^n(t)|^{3/2} \\
&= NC_2 \|y^n(t)\|^{1/2} |Ay^n(t)|^{3/2} \\
&\leq \frac{\nu}{4} |Ay^n(t)|^2 + C_N \|y^n(t)\|^2,
\end{aligned}
$$

with $C_N = \dfrac{27(NC_2)^4}{4\nu^3}$.

These inequalities combined with (22) lead to

$$
\frac{d}{dt} \|y^n(t)\|^2 + \nu |Ay^n(t)|^2 \leq \frac{4}{\nu} |\tilde{f}(t)|^2 + \frac{4}{\nu} |\tilde{g}(t, y_t^n)|^2 + 2C_N \|y^n(t)\|^2.
$$

Integrating between s and t with $0 \leq s \leq t \leq T$, we deduce that

$$
\begin{aligned}
\|y^n(t)\|^2 + \nu \int_s^t |Ay^n(r)|^2 dr &\leq \|y^n(s)\|^2 + \frac{4}{\nu} \int_0^T (|\tilde{f}(r)|^2 + |\tilde{g}(r, y_r^n)|^2) dr \\
&\quad + 2C_N \int_0^T \|y^n(r)\|^2 dr, \quad \text{for all } 0 \leq s \leq t \leq T \text{(23)}
\end{aligned}
$$

Now, integrating once more between 0 and t we obtain

$$
\begin{aligned}
t\|y^n(t)\|^2 &\leq \frac{4T}{\nu} \int_0^T (|\tilde{f}(r)|^2 + |\tilde{g}(r, y_r^n)|^2) dr \\
&\quad + (1 + 2C_N T) \int_0^T \|y^n(r)\|^2 dr, \quad \text{for all } 0 \leq s \leq t \leq T. \quad (24)
\end{aligned}
$$

By the assumptions on f and g, from (17), (20), (23) and (24) we deduce that the sequence $\{y^n\}_{n \geq n_0(t_0, T)}$ is bounded in $L^\infty(\varepsilon, T; V) \cap L^2(\varepsilon, T; D(A))$, for all $0 < \varepsilon < T$. Thus, as $D(A) \subset V$ with compact injection, by [22, Ch.1, Th.5.1], and using a sequence of positive values $\varepsilon_n \downarrow 0$ and a diagonal argument, eventually extracting a subsequence, in particular, we deduce that

$$
\|y^n(t)\| \to \|y(t)\| \quad \text{a.e. in } (0, T),
$$

and therefore

$$
F_N(\|y^n(t)\|) \to F(\|y(t)\|) \quad \text{a.e. in } (0, T).
$$

Also, by $(g3)$ and (17) we obtain

$$
\int_0^t |\tilde{g}(s, y_s^n)|^2 ds \leq Ct,
$$

where $C > 0$ does not depend neither on n nor $t \in [0, T]$. Thus, eventually extracting a subsequence, there exists $\xi \in L^2(0, T; (L^2(\Omega))^3)$ such that

$$
\tilde{g}(\cdot, y_\cdot^n) \rightharpoonup \xi \quad \text{weakly in } L^2(0, T; (L^2(\Omega))^3),
$$

and therefore

$$
\begin{aligned}
&\int_s^t |\tilde{g}(r, y_r^n)|^2 dr \leq C(t - s), \\
&\int_s^t |\xi(s)|^2 ds \leq \liminf_{n \to +\infty} \int_s^t |\tilde{g}(r, y_r^n)|^2 dr \leq C(t - s),
\end{aligned}
\quad (25)
$$

for all $0 \leq s \leq t \leq T$.

Then, in a standard way, one can prove that $y(\cdot)$ is the unique weak solution to the problem

$$
\begin{cases}
u_t - \nu\Delta u + F_N\left(\|u\|\right)\left[(u\cdot\nabla)u\right] = -\nabla p + \tilde{f}(t) + \xi(t), & \text{in } (0,T)\times\Omega, \\
\nabla\cdot u = 0 & \text{in } (0,T)\times\Omega, \\
u = 0 \text{ on } (0,T)\times\Gamma, \\
u(0,x) = y(0,x), x\in\Omega.
\end{cases}
$$

By the energy equality and (25), combined with Young's inequality, we obtain that

$$
|z(t)|^2 + \nu\int_s^t \|z(r)\|^2 dr \leq |z(s)|^2 + 2\int_s^t \langle\tilde{f}(r),z(r)\rangle dr + \frac{C}{\nu\lambda_1}(t-s),\ 0\leq s\leq t\leq T,
$$

where $z = y^n$ or $z = y$. Then the maps $J_n, J : [0,T]\to\mathbb{R}$ defined by

$$
\begin{aligned}
J(t) &= |y(t)|^2 - 2\int_0^t \langle\tilde{f}(r),y(r)\rangle dr - \frac{C}{\nu\lambda_1}t, \\
J_n(t) &= |y^n(t)|^2 - 2\int_0^t \langle\tilde{f}(r),y^n(r)\rangle dr - \frac{C}{\nu\lambda_1}t,
\end{aligned}
$$

are non-increasing and continuous, and satisfy

$$
J_n(t)\to J(t)\quad\text{a.e. } t\in[0,T]. \tag{26}
$$

We can use the functionals J_n and J to deduce that $y^n\to y$ in $C([\delta,T];H)$, for any $0 < \delta < T$. If this is not true, then there exist $0 < \delta^* < T$, $\varepsilon^* > 0$, and subsequences $\{y^m\}\subset\{y^n\}_{n\geq n_0(t_0,T)}$ and $\{t_m\}\subset[\delta^*,T]$, with $t_m\to t^*$, such that

$$
|y^m(t_m) - y(t^*)| \geq \varepsilon^*,\quad\text{for all } m. \tag{27}
$$

Let us fix $\varepsilon > 0$. Observe that $t^*\in[\delta^*,T]$, and therefore, by (26) and the continuity and non-increasing character of J, there exists $0\leq\hat{t}_\varepsilon < t^*$ such that

$$
\lim_{m\to+\infty} J_m(\hat{t}_\varepsilon) = J(\hat{t}_\varepsilon), \tag{28}
$$

and

$$
0\leq J(\hat{t}_\varepsilon) - J(t^*)\leq\varepsilon. \tag{29}
$$

As $t_m\to t^*$, there exists an m_ε such that $\hat{t}_\varepsilon < t_m$ for all $m\geq m_\varepsilon$. Then, by (29),

$$
\begin{aligned}
J_m(t_m) - J(t^*) &\leq J_m(\hat{t}_\varepsilon) - J(t^*) \\
&\leq |J_m(\hat{t}_\varepsilon) - J(\hat{t}_\varepsilon)| + |J(\hat{t}_\varepsilon) - J(t^*)| \\
&\leq |J_m(\hat{t}_\varepsilon) - J(\hat{t}_\varepsilon)| + \varepsilon
\end{aligned}
$$

for all $m\geq m_\varepsilon$, and consequently, by (28),

$$
\limsup_{m\to+\infty} J_m(t_m)\leq J(t^*) + \varepsilon.
$$

Thus, as $\varepsilon > 0$ is arbitrary, we deduce that

$$
\limsup_{m\to+\infty} J_m(t_m)\leq J(t^*). \tag{30}
$$

Taking into account that $t_m\to t^*$, and

$$
\int_0^{t_m}(\tilde{f}(r),y^m(r))dr\to\int_0^{t^*}(\tilde{f}(r),y(r))dr,
$$

from (30) we deduce that

$$\limsup_{m \to +\infty} |y^m(t_m)| \leq |y(t^*)|.$$

This last inequality and (21), imply that

$$y^m(t_m) \to y(t^*) \quad \text{strongly in } H,$$

which is in contradiction with (27).

We have thus proved that $y^n \to y$ in $C([\delta, T]; H)$, for any $0 < \delta < T$. As $T > \overline{T}$, we obtain in particular that $\xi^n|_{[-\overline{T},0]} \to \psi$ in $C([-\overline{T}, 0]; H)$, where $\psi(s) = y(s+T)$, for $s \in [-\overline{T}, 0]$.

Repeating the same procedure for $2\overline{T}$, $3\overline{T}$, etc. for a diagonal subsequence (relabelled the same) we can obtain a function $\psi \in C((-\infty, 0]; H)$ such that $\xi^n|_{[-\overline{T},0]} \to \psi$ in $C([-\overline{T}, 0]; H)$ on every interval $[-\overline{T}, 0]$.

Moreover, by the estimate (18), it is clear that we also have

$$|\psi(s)|^2 \leq 1 + Me^{\sigma T} \quad \text{for all } s \in [-T, 0], \text{ for any } T > 0, \tag{31}$$

where

$$M = (\mu\lambda_1)^{-1} e^{-\sigma t_0} \int_{-\infty}^{t_0} e^{\sigma s} |f(s)|^2 ds.$$

Step 2: We now prove that in fact ξ^n converges to ψ in $C_\gamma(H)$.

Indeed, we have to see that for every $\varepsilon > 0$ there exists n_ε such that

$$\sup_{s \in (-\infty, 0]} e^{2\gamma s} |\xi^n(s) - \psi(s)|^2 \leq \varepsilon \quad \text{for all } n \geq n_\varepsilon. \tag{32}$$

Fix $T_\varepsilon > 0$ such that $\max\{e^{-2\gamma T_\varepsilon}, Me^{\sigma} e^{(\sigma - 2\gamma)T_\varepsilon}\} \leq \varepsilon/8$, and take $n_\varepsilon \geq n_0(t_0, T_\varepsilon)$ such that $e^{2\gamma s} |\xi^n(s) - \psi(s)|^2 \leq \varepsilon$ for all $s \in [-T_\varepsilon, 0]$, and $\tau_n \leq t_0 - T_\varepsilon$, for all $n \geq n_\varepsilon$. This last choice is possible thanks to the Step 1.

So, in order to prove (32) we only have to check that

$$\sup_{s \in (-\infty, -T_\varepsilon]} |\xi^n(s) - \psi(s)|^2 e^{2\gamma s} \leq \varepsilon \quad \text{for all } n \geq n_\varepsilon.$$

By (31) and the choice of T_ε, and since $\sigma - 2\gamma \leq 0$, for all $k \geq 0$ we have that for all $s \in [-(T_\varepsilon + k + 1), -(T_\varepsilon + k)]$ it holds

$$\begin{aligned} e^{2\gamma s} |\psi(s)|^2 &\leq e^{-2\gamma(T_\varepsilon + k)}(1 + Me^{\sigma(T_\varepsilon + k + 1)}) \\ &= e^{-2\gamma T_\varepsilon} e^{-2\gamma k} + Me^{\sigma} e^{(\sigma - 2\gamma)T_\varepsilon} e^{k(\sigma - 2\gamma)} \\ &\leq \varepsilon/8 + \varepsilon/8 \\ &= \varepsilon/4. \end{aligned}$$

So, to finish, it suffices to prove that

$$\sup_{s \in (-\infty, -T_\varepsilon]} e^{2\gamma s} |\xi^n(s)|^2 \leq \varepsilon/4 \quad \text{for all } n \geq n_\varepsilon.$$

We remind that

$$\xi^n(s) = \begin{cases} \phi^n(s + t_0 - \tau_n), & \text{if } s \in (-\infty, \tau_n - t_0), \\ u^n(s + t_0), & \text{if } s \in [\tau_n - t_0, 0]. \end{cases}$$

Thus, the proof is finished if we prove that

$$\max\left\{ \sup_{s \in (-\infty, \tau_n - t_0)} e^{2\gamma s} |\phi^n(s + t_0 - \tau_n)|^2, \sup_{s \in [\tau_n - t_0, -T_\varepsilon]} e^{2\gamma s} |u^n(s + t_0)|^2 \right\} \leq \varepsilon/4.$$

But observe that

$$
\begin{aligned}
\sup_{s \le \tau_n - t_0} e^{2\gamma s} |\phi^n(s + t_0 - \tau_n)|^2 &= \sup_{s \le \tau_n - t_0} e^{2\gamma(s + t_0 - \tau_n)} e^{2\gamma(\tau_n - t_0)} |\phi^n(s + t_0 - \tau_n)|^2 \\
&= e^{2\gamma(\tau_n - t_0)} \|\phi^n\|_\gamma^2 \\
&\le e^{2\gamma(\tau_n - t_0)} \rho^2(\tau_n) \\
&\le e^{2\gamma(\tau_n - t_0)} + M e^{(2\gamma - \sigma)(\tau_n - t_0)} \le \varepsilon/4,
\end{aligned}
$$

thanks to the choice of n_ε.

Finally, by (17) with $T = T_\varepsilon$, we also have

$$
\begin{aligned}
\sup_{s \in [\tau_n - t_0, -T_\varepsilon]} e^{2\gamma s} |u^n(s + t_0)|^2 &= \sup_{\theta \in [\tau_n - t_0 + T_\varepsilon, 0]} e^{2\gamma(\theta - T_\varepsilon)} |u^n(t_0 - T_\varepsilon + \theta)|^2 \\
&\le e^{-2\gamma T_\varepsilon} \|u_{t_0 - T_\varepsilon}^n\|_\gamma^2 \\
&\le e^{-2\gamma T_\varepsilon} R(t_0, T_\varepsilon) \\
&= e^{-2\gamma T_\varepsilon} + M e^{(\sigma - 2\gamma) T_\varepsilon} \\
&\le \varepsilon/4.
\end{aligned}
$$

The proof is completed. $\qquad\square$

Joining all the above statements we obtain the existence of minimal pullback attractors for the process U on $C_\gamma(H)$ associated to problem (1).

Theorem 3. *Assume that $f \in L_{loc}^2(\mathbb{R}; (L^2(\Omega))^3)$, $\gamma > 0$, and $g : \mathbb{R} \times C_\gamma(H) \to (L^2(\Omega))^3$ satisfying the assumptions (g1)–(g3), (12) and (13), are given. Then, there exist the minimal pullback $\mathcal{D}_F(C_\gamma(H))$-attractor*

$$
\mathcal{A}_{\mathcal{D}_F(C_\gamma(H))} = \{\mathcal{A}_{\mathcal{D}_F(C_\gamma(H))}(t) : t \in \mathbb{R}\}
$$

and the minimal pullback $\mathcal{D}_\sigma((C_\gamma(H)))$-attractor

$$
\mathcal{A}_{\mathcal{D}_\sigma(C_\gamma(H))} = \{\mathcal{A}_{\mathcal{D}_\sigma(C_\gamma(H))}(t) : t \in \mathbb{R}\},
$$

for the process U defined by (7). The family $\mathcal{A}_{\mathcal{D}_\sigma(C_\gamma(H))}$ belongs to $\mathcal{D}_\sigma(C_\gamma(H))$, and the following relation holds:

$$
\mathcal{A}_{\mathcal{D}_F(C_\gamma(H))}(t) \subset \mathcal{A}_{\mathcal{D}_\sigma(C_\gamma(H))}(t) \subset \overline{B}_{C_\gamma(H)}(0, \rho(t)) \quad \text{for all } t \in \mathbb{R},
$$

where $\rho(t)$ is the expression given in (16).

Proof. The result is a direct consequence of Theorem 2, Remark 3, Corollary 1, Proposition 2, Corollary 2, and Proposition 3. $\qquad\square$

As a consequence of Theorem 3, we have the following result, which just discusses about the existence of a bigger tempered universe and bigger corresponding pullback attractor, if condition (12) is not optimized.

Corollary 3. *Under the assumptions of Theorem 3, if $\mu \in (0, \nu)$ satisfies $(\nu - \mu)\lambda_1 < \gamma$, then there exists $\tilde{\mu} \in (0, \mu)$ such that it still fulfills $(\nu - \tilde{\mu})\lambda_1 \le \gamma$, and therefore there exists the minimal pullback $\mathcal{D}_{\tilde{\sigma}}((C_\gamma(H)))$-attractor*

$$
\mathcal{A}_{\mathcal{D}_{\tilde{\sigma}}(C_\gamma(H))} = \{\mathcal{A}_{\mathcal{D}_{\tilde{\sigma}}(C_\gamma(H))}(t) : t \in \mathbb{R}\},
$$

for the process U defined by (7), where $\tilde{\sigma} = 2((\nu - \tilde{\mu})\lambda_1 - L_g)$, and the universe $\mathcal{D}_{\tilde{\sigma}}(C_\gamma(H))$ is defined analogously as in Definition 7, but with parameter $\tilde{\sigma}$.

The family $\mathcal{A}_{\mathcal{D}_{\tilde{\sigma}}(C_\gamma(H))}$ belongs to $\mathcal{D}_{\tilde{\sigma}}(C_\gamma(H))$, and the following relation holds:

$$
\mathcal{A}_{\mathcal{D}_\sigma(C_\gamma(H))}(t) \subset \mathcal{A}_{\mathcal{D}_{\tilde{\sigma}}(C_\gamma(H))}(t) \subset \overline{B}_{C_\gamma(H)}(0, \tilde{\rho}(t)) \quad \text{for all } t \in \mathbb{R}, \tag{33}
$$

where

$$\tilde{\rho}^2(t) = 1 + (\tilde{\mu}\lambda_1)^{-1} \int_{-\infty}^{t} e^{-\tilde{\sigma}(t-s)}|f(s)|^2 ds.$$

Proof. The result is a consequence of Theorem 3, provided that f and μ satisfying (12) and (13) imply that f and $\tilde{\mu}$ also satisfy the analogous conditions.

Moreover, observe that in this case any family of $\mathcal{D}_\sigma(C_\gamma(H))$ also belongs to $\mathcal{D}_{\tilde{\sigma}}(C_\gamma(H))$, whence the first inclusion in (33) follows. The second inclusion is again a consequence of Theorem 3 with parameter $\tilde{\mu}$. $\square$

Remark 8.
i) Under the assumptions of Corollary 3, i.e., with μ satisfying $(\nu - \mu)\lambda_1 < \gamma$, there are two possibilities.
   a) If $\nu\lambda_1 > \gamma$, the optimal value (cf. Remark 5) in order to obtain the maximal family of minimal pullback attractor in the biggest universe is $\tilde{\mu} = \nu - \gamma\lambda_1^{-1} \in (0, \nu)$.
   b) If $\nu\lambda_1 \le \gamma$, then for any arbitrarily small value $\tilde{\mu}$ the conditions (12) and (13) with such a value are satisfied. Accordingly to Corollary 3, this means that increasing families of minimal pullback attractors and of tempered universes exist when $\tilde{\mu} \downarrow 0^+$.
ii) Under the assumptions of Theorem 3, if additionally, we assume that

$$\sup_{r\le 0} \int_{-\infty}^{r} e^{-\sigma(r-s)}|f(s)|^2 ds < +\infty, \tag{34}$$

where σ is given by (15), then, taking into account Remark 4 and Corollary 2, we conclude that

$$\mathcal{A}_{\mathcal{D}_F(C_\gamma(H))}(t) = \mathcal{A}_{\mathcal{D}_\sigma(C_\gamma(H))}(t), \quad \text{for all } t \in \mathbb{R}. \tag{35}$$

In fact, under the assumptions of Corollary 3, then (34) also implies the analogous condition for parameter $\tilde{\sigma}$ associated to $\tilde{\mu}$, i.e.,

$$\sup_{r\le 0} \int_{-\infty}^{r} e^{-\tilde{\sigma}(r-s)}|f(s)|^2 ds < +\infty.$$

Thus, in this case we conclude that (35) also holds with σ replaced by $\tilde{\sigma}$.

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