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PROBABILISTIC REPRESENTATION OF SOLUTIONS FOR QUASI-LINEAR PARABOLIC PDE VIA FBSDE WITH REFLECTING BOUNDARY CONDITIONS

PEDRO MARÍN-RUBIO AND JOSÉ REAL

Dpto. Ecuaciones Diferenciales y Análisis Numérico,
Universidad de Sevilla
Apdo. de Correos 1160, 41080 Sevilla (SPAIN)

pmr@us.es jreal@us.es

Abstract

A probabilistic representation of the solution (in the viscosity sense) of a quasi-linear parabolic PDE system with non-lipschitz terms and a Neumann boundary condition is given via a fully coupled forward-backward stochastic differential equation with a reflecting term in the forward equation. The extension of previous results consists on the relaxation on the Lipschitz assumption on the drift coefficient of the forward equation, using a previous result of the authors.

Key words: *Probabilistic formulae for PDE, Forward backward stochastic differential equations, Skorokhod problem, Reflected Stochastic Differential Equations.*
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Introduction

Deeper relations between stochastic differential equations and systems of PDE have been established since [4] developed the theory of backward stochastic differential equations. Roughly speaking, combining a forward stochastic differential equation with a BSDE, the Feyman-Kac formula can be extended to nonlinear PDE, and not only in a classical sense, but also via viscosity solutions.

Usually, the deterministic problems treated in this way are posed in the whole domain $\mathbb{R}^d$, or in a bounded domain of $\mathbb{R}^d$ with Dirichlet boundary condition. With a Neumann boundary condition, the problem was studied by Y. Hu using local time around the boundary of the domain. This technique is closely related to a stochastic version of the Skorokhod problem (see e.g. [6], for a direct application in this sense). We extend these studies and relations to the case of fully coupled systems of FBSDER in which the open set is not necessarily convex but still smooth (this restriction is for commodity and may be removed), and the drift coefficient of the forward equation is monotone in x, instead of Lipschitz. In this way, we generalize some results from [5] and [1].

In this paper we give a probabilistic representation of the solution of a quasi-linear PDE system extending some results of those given in [5] and [1] on a system of a fully coupled forward-backward stochastic differential equations with a reflecting term in the forward equation (FBSDER) and its relation with a system of quasi-linear partial differential equations, in short PDE. Preceding works on this line were due to Y. Hu and to E. Pardoux and S. Zhang (cf. [6]). In our case, the drift satisfies the monotonicity condition introduced before, and the domain $\mathcal{O}$ is not necessarily convex. Existence of solution under such conditions was proved in a precedent paper by the authors (cf. [3]).

In Section 1 we start giving the suitable framework for the reflected problem and recall a previous result which will be used later on. In Section 2, we state the general framework for the study of a fully coupled FBSDER, and provide a probabilistic interpretation for a system of quasi-linear PDE with homogeneous Neumann boundary condition.

1 Statement of the "reflected" problem

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, $\{\mathcal{F}_t\}_{t\geq 0}$ an increasing and right continuous family of sub-σ-algebras of $\mathcal{F}$ such that $\mathcal{F}_0$ contains all the P-null sets of $\mathcal{F}$, and $\{W_t;\ t\geq 0\}$ an m-dimensional standard $\{\mathcal{F}_t\}$-Wiener process.

Let $\mathcal{O}$ be an open connected bounded subset of $\mathbb{R}^d$ given by $\mathcal{O} = \{\phi > 0\}$, with $\phi \in C^2(\mathbb{R}^d)$, and such that $\partial\mathcal{O} = \{\phi = 0\}$, with $|\nabla\phi(x)| = 1$ for all $x \in \partial\mathcal{O}$. Observe that in particular ϕ, $\nabla\phi$ and $D^2\phi$ are bounded in $\bar{\mathcal{O}}$. Then there exists a constant $C_0 > 0$ such that

$$2(x'-x, \nabla\phi(x)) + C_0|x'-x|^2 \geq 0, \qquad \forall\, x \in \partial\mathcal{O},\ \forall\, x' \in \bar{\mathcal{O}}. \tag{1}$$

We are also given a final time $T > 0$, and two random functions:

$$b : \Omega \times [0,T] \times \bar{\mathcal{O}} \to \mathbb{R}^d, \qquad \sigma : \Omega \times [0,T] \times \bar{\mathcal{O}} \to \mathbb{R}^{d\times m},$$

such that

(i) b and σ are uniformly bounded;

(ii) for all $x \in \bar{\mathcal{O}}$ the processes $b(\cdot,\cdot,x)$ and $\sigma(\cdot,\cdot,x)$ are $\{\mathcal{F}_t\}$-progressively measurable;

(iii) for all $t \in [0,T]$ and a.s. ω, the function $b(\omega,t,\cdot)$ is continuous on $\bar{\mathcal{O}}$;

(iv) there exist two constants $L_{b_x} \in \mathbb{R}$ and $L_{\sigma_x} \geq 0$ such that for all $t \in [0,T]$ and all $x, x' \in \bar{\mathcal{O}}$,

$$(x-x', b(\omega,t,x) - b(\omega,t,x')) \leq L_{b_x}|x-x'|^2, \quad a.s.,$$

$$\|\sigma(\omega,t,x) - \sigma(\omega,t,x')\| \leq L_{\sigma_x}|x-x'|, \quad a.s.,$$

where $|\cdot|$ and $\|\cdot\|$ denote the usual Euclidean and trace norm for vectors and matrices respectively.

From now on, we will omit the explicit dependence of the processes on ω.

Consider the following problem:

$$X_t = x_0 + \int_0^t b(s, X_s)\, ds + \int_0^t \sigma(s, X_s)\, dW_s - k_t, \tag{2}$$

$$k_t = -\int_0^t \nabla\phi(X_s)\, d|k|_s, \quad |k|_t = \int_0^t 1_{\{X_s \in \partial\mathcal{O}\}}\, d|k|_s, \quad t \in [0,T], \tag{3}$$

where $x_0 \in \bar{\mathcal{O}}$ is given, and $|k|_t$ stands for the total variation of k on $[0,t]$.

Definition 1 *A strong solution to the above problem is a pair of $\{\mathcal{F}_t\}$-adapted and continuous processes (X,k) defined on $\Omega \times [0,T]$, the first one with values in $\bar{\mathcal{O}}$, the second one with values in $\mathbb{R}^d$ and paths of bounded variation in $[0,T]$, satisfying the equations (2)-(3) a.s. for all $t \in [0,T]$.*

Main result stated in [3], which generalizes a result by Lions and Sznitman when b is Lipschitz, is the following:

Theorem 1 *Under the assumptions* (i)-(iv)*, for each $x_0 \in \bar{\mathcal{O}}$ given there exists a unique pair (X,k), strong solution of (2)-(3).*

2 Forward-Backward Stochastic Differential Equations with Reflection and representation of a PDE system

We continue considering the complete probability space $(\Omega, \mathcal{F}, P)$, and the m-dimensional standard $\{\mathcal{F}_t\}$-Wiener process $\{W_t;\ t \geq 0\}$ given in Section 1, but now we suppose that, for each $t \geq 0$, $\mathcal{F}_t$ coincides with the σ-algebra $\sigma(W_s;\, 0 \leq s \leq t)$ augmented with all the P-null sets of $\mathcal{F}$.

Let $T > 0$ be fixed, and consider the open set $\mathcal{O}$ introduced in Section 1.

For each integer $l \geq 1$, we shall denote by $M^2_{\mathcal{F}_t}(0,T;\mathbb{R}^l)$ the Hilbert subspace of $L^2(\Omega \times (0,T);\mathbb{R}^l)$ formed by those elements that are $\{\mathcal{F}_t\}$-progressively measurable, and we will write $L^2_{\mathcal{F}_t}(\Omega; C([0,T];\mathbb{R}^l))$ to denote the space of the elements of $L^2(\Omega; C([0,T];\mathbb{R}^l))$ that are $\{\mathcal{F}_t\}$-progressively measurable. Thus, $L^2_{\mathcal{F}_t}(\Omega; C([0,T];\mathbb{R}^l))$ is a Banach subspace of $L^2(\Omega; C([0,T];\mathbb{R}^l))$.

Similarly, we denote by $M^2_{\mathcal{F}_t}(0,T;\bar{\mathcal{O}})$ the complete metric subspace of the space $M^2_{\mathcal{F}_t}(0,T;\mathbb{R}^d)$ constituted by the elements $X \in M^2_{\mathcal{F}_t}(0,T;\mathbb{R}^d)$ such that a.e. $t \in (0,T)$, $X_t \in \bar{\mathcal{O}}$ a.s.; we shall also use $L^2_{\mathcal{F}_t}(\Omega; C([0,T];\bar{\mathcal{O}}))$ to denote the complete metric subspace of $L^2_{\mathcal{F}_t}(\Omega; C([0,T];\mathbb{R}^l))$ formed by those elements X in the last space such that a.s. $X_t \in \bar{\mathcal{O}}$ for all $t \in [0,T]$. Finally, we shall denote by $L^2(\Omega, \mathcal{F}_T; \bar{\mathcal{O}})$ the complete metric subspace of $L^2(\Omega, \mathcal{F}_T; \mathbb{R}^d)$ formed by the $\mathcal{F}_T$-measurable random variables $\xi \in L^2(\Omega;\mathbb{R}^d)$ such that a.s. $\xi \in \bar{\mathcal{O}}$.

We are given four random functions:

$$b : \Omega \times [0,T] \times \bar{\mathcal{O}} \times \mathbb{R}^n \times \mathbb{R}^{n\times m} \to \mathbb{R}^d,\ f : \Omega \times [0,T] \times \bar{\mathcal{O}} \times \mathbb{R}^n \times \mathbb{R}^{n\times m} \to \mathbb{R}^n,$$

$$\sigma : \Omega \times [0,T] \times \bar{\mathcal{O}} \times \mathbb{R}^n \times \mathbb{R}^{n\times m} \to \mathbb{R}^{d\times m},\ h : \Omega \times \bar{\mathcal{O}} \to \mathbb{R}^n,$$

such that

(i') b and σ are uniformly bounded;

(ii') for all $(x, y, z) \in \bar{\mathcal{O}} \times \mathbb{R}^n \times \mathbb{R}^{n\times m}$ the processes $b(\cdot, x, y, z)$, $f(\cdot, x, y, z)$ and $\sigma(\cdot, x, y, z)$ are $\{\mathcal{F}_t\}$-progressively measurable, and the random variable $h(\cdot, x)$ is $\mathcal{F}_T$-measurable;

(iii') for all $(t, x, y, z) \in [0, T] \times \bar{\mathcal{O}} \times \mathbb{R}^n \times \mathbb{R}^{n\times m}$ the functions $b(t, \cdot, y, z)$ and $f(t, x, \cdot, z)$ are a.s. continuous on $\bar{\mathcal{O}}$ and $\mathbb{R}^n$ respectively;

(iv') there exist real constants L_{b_x} and L_{f_y}, and nonnegative constants L_{b_y}, L_{b_z}, L_{f_x}, L_{f_z}, L_{σ_x}, L_{σ_y}, L_{σ_z}, L_h and l_0 such that for all $t \in [0, T]$, all $x, x' \in \bar{\mathcal{O}}$, all $y, y' \in \mathbb{R}^n$, all $z, z' \in \mathbb{R}^{n\times m}$, and a.s.,

$$(x - x', b(t, x, y, z) - b(t, x', y, z)) \le L_{b_x}|x - x'|^2,$$

$$|b(t, x, y, z) - b(t, x, y', z')| \le L_{b_y}|y - y'| + L_{b_z}\|z - z'\|,$$

$$\|\sigma(t, x, y, z) - \sigma(t, x', y', z')\|^2 \le L^2_{\sigma_x}|x - x'|^2 + L^2_{\sigma_y}|y - y'|^2 + L^2_{\sigma_z}\|z - z'\|^2,$$

$$(y - y', f(t, x, y, z) - f(t, x, y', z)) \le L_{f_y}|y - y'|^2,$$

$$|f(t, x, y, z) - f(t, x', y, z')| \le L_{f_x}|x - x'| + L_{f_z}\|z - z'\|,$$

$$|f(t, x, y, z)| \le |f(t, x, 0, z)| + l_0(1 + |y|),$$

$$|h(x) - h(x')| \le L_h|x - x'|;$$

(v') $\quad E\int_0^T |f(t, 0, 0, 0)|^2\, dt + E|h(0)|^2 < \infty.$

We want to study the following problem:

$$X_t = x_0 + \int_0^t b(s, X_s, Y_s, Z_s)\, ds + \int_0^t \sigma(s, X_s, Y_s, Z_s)\, dW_s - k_t, \quad (4)$$

$$Y_t = h(X_T) + \int_t^T f(s, X_s, Y_s, Z_s)\, ds - \int_t^T Z_s\, dW_s, \quad (5)$$

$$k_t = -\int_0^t \nabla\phi(X_s)\, d|k|_s, \quad |k|_t = \int_0^t 1_{\{X_s\in\partial\mathcal{O}\}}\, d|k|_s, \quad t \in [0, T], \quad (6)$$

where $x_0 \in \bar{\mathcal{O}}$ is given.

Definition 2 *A solution to the problem (4)-(6) is a set (X, Y, Z, k) of four $\{\mathcal{F}_t\}$-progressively measurable processes defined on $\Omega \times [0, T]$, such that X is continuous with values in $\bar{\mathcal{O}}$, k is continuous with values in $\mathbb{R}^d$ and paths of bounded variation in $[0, T]$, $(Y, Z) \in M^2_{\mathcal{F}_t}(0, T; \mathbb{R}^n) \times M^2_{\mathcal{F}_t}(0, T; \mathbb{R}^{n\times m})$, and the equations (4)-(6) are satisfied a.s. for all $t \in [0, T]$.*

For the resolution of the above fully coupled FBSDER, we will use the following result, that is a direct consequence of Theorem 1:

Corollary 2 *Under the assumptions* (i')-(iv')*, if $(Y, Z) \in M^2_{\mathcal{F}_t}(0, T; \mathbb{R}^n) \times M^2_{\mathcal{F}_t}(0, T; \mathbb{R}^{n\times m})$ is fixed, there exists a unique pair (X, k) of $\{\mathcal{F}_t\}$-progressively measurable processes defined on $\Omega\times[0, T]$, such that X is continuous with values* *in $\bar{\mathcal{O}}$, k is continuous with values in $\mathbb{R}^d$ and paths of bounded variation in $[0,T]$, and they satisfy a.s. for all $t \in [0,T]$ that*

$$X_t = x_0 + \int_0^t b(s, X_s, Y_s, Z_s)\, ds + \int_0^t \sigma(s, X_s, Y_s, Z_s)\, dW_s - k_t, \quad (7)$$

$$k_t = -\int_0^t \nabla\phi(X_s)\, d|k|_s, \quad |k|_t = \int_0^t 1_{\{X_s \in \partial\mathcal{O}\}}\, d|k|_s. \quad (8)$$

We will also need the following well-known result (see for instance Pardoux's notes at Geilo, 1996) for the backward equation:

Theorem 3 *Under the assumptions* (ii')-(v')*, let be given $X \in M^2_{\mathcal{F}_t}(0,T;\bar{\mathcal{O}})$ and $\xi \in L^2(\Omega, \mathcal{F}_T; \bar{\mathcal{O}})$. Then, there exists a unique pair $(Y,Z) \in M^2_{\mathcal{F}_t}(0,T;\mathbb{R}^n) \times M^2_{\mathcal{F}_t}(0,T;\mathbb{R}^{n\times m})$ such that*

$$Y_t = h(\xi) + \int_t^T f(s, X_s, Y_s, Z_s)\, ds - \int_t^T Z_s\, dW_s, \quad (9)$$

a.s. for all $t \in [0,T]$. Moreover, we have that $Y \in L^2_{\mathcal{F}_t}(\Omega; C([0,T];\mathbb{R}^n))$.

Using the two results above, it is not difficult to prove existence and uniqueness of solution of problem (4)-(6) if T is small enough. More exactly, we have the following result, whose proof we will omit for the sake of brevity:

Theorem 4 *Suppose the assumptions* (i')-(v')*, and that moreover σ does not depend on z. Then, there exists a $T_* > 0$ such that if $T \le T_*$, the application Φ defined from*

$$L^2_{\mathcal{F}_t}(\Omega; C([0,T];\bar{\mathcal{O}})) \times L^2_{\mathcal{F}_t}(\Omega; C([0,T];\mathbb{R}^n)) \times M^2_{\mathcal{F}_t}(0,T;\mathbb{R}^{n\times m})$$

on itself by $\Phi(X,Y,Z) = (\bar{X}, \bar{Y}, \bar{Z})$, with $(\bar{X}, \bar{Y}, \bar{Z})$ the unique solution of

$$\bar{X}_t = x_0 + \int_0^t b(s, \bar{X}_s, Y_s, Z_s)\, ds + \int_0^t \sigma(s, \bar{X}_s, Y_s)\, dW_s - \bar{k}_t,$$

$$\bar{k}_t = -\int_0^t \nabla\phi(\bar{X}_s)\, d|\bar{k}|_s, \quad |\bar{k}|_t = \int_0^t 1_{\{\bar{X}_s \in \partial\mathcal{O}\}}\, d|\bar{k}|_s,$$

$$\bar{Y}_t = h(\bar{X}_T) + \int_t^T f(s, \bar{X}_s, \bar{Y}_s, \bar{Z}_s)\, ds - \int_t^T \bar{Z}_s\, dW_s,$$

a.s. for all $t \in [0,T]$, is a contraction. So, if $T \le T_*$, the problem (4)-(6) has a unique solution.

For the resolution of the above fully coupled FBSDER for any $T > 0$, we follow [5] and [1].

We shall denote by Γ_1 the mapping

$$\Gamma_1 : M^2_{\mathcal{F}_t}(0,T;\mathbb{R}^n) \times M^2_{\mathcal{F}_t}(0,T;\mathbb{R}^{n\times m}) \to M^2_{\mathcal{F}_t}(0,T;\mathbb{R}^n) \times M^2_{\mathcal{F}_t}(0,T;\mathbb{R}^{n\times m}),$$

defined by $\Gamma_1(Y,Z) = (\bar{Y},\bar{Z})$, with $(\bar{X},\bar{Y},\bar{Z},\bar{k})$ the unique solution of

$$\begin{aligned}
\bar{X}_t &= x_0 + \int_0^t b(s,\bar{X}_s,Y_s,Z_s)\,ds + \int_0^t \sigma(s,\bar{X}_s,Y_s,Z_s)\,dW_s - \bar{k}_t,\\
\bar{k}_t &= -\int_0^t \nabla\phi(\bar{X}_s)\,d|\bar{k}|_s, \quad |\bar{k}|_t = \int_0^t 1_{\{\bar{X}_s\in\partial\mathcal{O}\}}\,d|\bar{k}|_s,\\
\bar{Y}_t &= h(\bar{X}_T) + \int_t^T f(s,\bar{X}_s,\bar{Y}_s,\bar{Z}_s)\,ds - \int_t^T \bar{Z}_s\,dW_s,
\end{aligned}$$

a.s. for all $t\in[0,T]$.

We will denote by Γ_2 the mapping

$$\Gamma_2 : M^2_{\mathcal{F}_t}(0,T;\bar{\mathcal{O}})\times L^2(\Omega,\mathcal{F}_T;\bar{\mathcal{O}}) \to M^2_{\mathcal{F}_t}(0,T;\bar{\mathcal{O}})\times L^2(\Omega,\mathcal{F}_T;\bar{\mathcal{O}}),$$

defined by $\Gamma_2(X,\xi) = (\bar{X},\bar{X}_T)$, with $\bar{X}$ such that $(\bar{X},\bar{Y},\bar{Z},\bar{k})$ is the unique solution of

$$\begin{aligned}
\bar{Y}_t &= h(\xi) + \int_t^T f(s,X_s,\bar{Y}_s,\bar{Z}_s)\,ds - \int_t^T \bar{Z}_s\,dW_s,\\
\bar{X}_t &= x_0 + \int_0^t b(s,\bar{X}_s,\bar{Y}_s,\bar{Z}_s)\,ds + \int_0^t \sigma(s,\bar{X}_s,\bar{Y}_s,\bar{Z}_s)\,dW_s - \bar{k}_t,\\
\bar{k}_t &= -\int_0^t \nabla\phi(\bar{X}_s)\,d|\bar{k}|_s, \quad |\bar{k}|_t = \int_0^t 1_{\{\bar{X}_s\in\partial\mathcal{O}\}}\,d|\bar{k}|_s,
\end{aligned}$$

a.s. for all $t\in[0,T]$.

By Corollary 2 and Theorem 3, under the conditions (i')-(v') the maps Γ_1 and Γ_2 are well defined. Also, it is clear that to solve the problem (4)-(6) is equivalent to finding a fixed point for Γ_1 or Γ_2. Thus, in order to prove existence and uniqueness of solution to problem (4)-(6), it is enough to find a Hilbert norm in $M^2_{\mathcal{F}_t}(0,T;\mathbb{R}^n)\times M^2_{\mathcal{F}_t}(0,T;\mathbb{R}^{n\times m})$, such that Γ_1 is a contraction for this norm. Analogously, it is enough to find a complete metric in $M^2_{\mathcal{F}_t}(0,T;\bar{\mathcal{O}})\times L^2(\Omega,\mathcal{F}_T;\bar{\mathcal{O}})$, for which the map Γ_2 is a contraction.

From now on, for $l\geq 1$ integer, and $\lambda\in\mathbb{R}$, we will denote by $\|\cdot\|_\lambda$ the norm on $M^2_{\mathcal{F}_t}(0,T;\mathbb{R}^l)$, equivalent to the usual one, given by

$$\|\zeta\|^2_\lambda = E\int_0^T e^{-\lambda s}|\zeta|^2 ds.$$

For the sake of brevity on these notes we omit here the estimates on the difference of two solutions (X,k) and (X',k') associated respectively to processes (Y,Z) and (Y',Z'), or the inverse. If we combine these estimates in the two possible orders, to obtain estimations for Γ_1 and Γ_2, we have two possibilities.

On the one hand, one can search for a $\lambda\in\mathbb{R}$ such that, with the norm on $M^2_{\mathcal{F}_t}(0,T;\mathbb{R}^n)\times M^2_{\mathcal{F}_t}(0,T;\mathbb{R}^{n\times m})$ defined by

$$\|(Y,Z)\|^2_\lambda = \|Y\|^2_\lambda + \|Z\|^2_\lambda,$$

the mapping Γ_1 is a contraction.

On the other hand, one can search for a λ such that, with the metric on $M^2_{\mathcal{F}_t}(0,T;\bar{\mathcal{O}}) \times L^2(\Omega,\mathcal{F}_T;\bar{\mathcal{O}})$ induced by the norm on $M^2_{\mathcal{F}_t}(0,T;\mathbb{R}^d) \times L^2(\Omega,\mathcal{F}_T;\mathbb{R}^d)$ defined by

$$\|(X,\xi)\|^2_\lambda = \exp(-\lambda T)E|\xi|^2 + \lambda_1\|X\|^2_\lambda,$$

the mapping Γ_2 is a contraction.

Then, one obtains existence and uniqueness for (4)-(6) that generalize to b monotone and $\mathcal{O}$ not necessarily convex some of the results in [5] and [1].

For example, existence and uniqueness of solution for (4)-(6) hold when its coupling is weak, that is, when dependence of b and σ respect to their variables y and z is small, or, analogously for the backward equation, when the dependence of f and h with respect to x is small. More exactly, we have:

Theorem 5 *Let conditions* (i')-(v') *hold. Then there exists an* $\varepsilon_0 > 0$ *depending on* L_{σ_x}, L_{b_x}, L_{f_x}, L_{f_y}, L_{f_z}, L_h *and* T *such that if* L_{b_y}, L_{b_z}, L_{σ_y}, $L_{\sigma_z} \in [0,\varepsilon_0)$, *then there exists* λ *such that* Γ_1 *is a contraction, and thus there exists a unique solution to (4)-(6). On the other hand, the same thesis holds for* Γ_2, *changing roles of* L_{b_y}, L_{b_z}, L_{σ_y}, *and* L_{σ_z}, *with* L_h *and* L_{f_x}.

Also, using Γ_2, and reasoning as in [1] or [2], one can prove

Theorem 6 *Let conditions* (i')-(v') *hold, and suppose one of the following two conditions:*
a) *If h is independent of x, there exists* $\alpha \in (0,1)$ *such that* $\mu(\alpha,T)L_{f_x}C_3 < \lambda_1$.
b) *If h does depend on x, there exists* $\alpha \in (k_1 L^2_{\sigma_z} L^2_h, 1)$ *such that* $\mu(\alpha,T)L^2_h < 1$.
Then, there exists a unique solution for (4)-(6).

Remark 1 *Reasoning as in [2], one can make some (technical) improvements. Namely, it is possible to consider that* σ *can depend on* z, *but introducing compatibility conditions. On other hand, if* L_{f_y} *is negative enough, then (4)-(6) has a unique solution for all final time* $T > 0$.

Finally, as in [5], and in [1], with the previous results on the problem (4)-(6), one can prove existence of viscosity solution to a homogeneous Neumann problem for an associated system of quasi-linear parabolic PDE. We briefly recall here how this can be done.

For each $(t,x) \in [0,T] \times \bar{\mathcal{O}}$, consider the problem

$$\begin{aligned}
X^{t,x}_s &= x + \int_t^s b(r,X^{t,x}_r,Y^{t,x}_r,Z^{t,x}_r)dr + \int_t^s \sigma(r,X^{t,x}_r,Y^{t,x}_r,Z^{t,x}_r)dW_r - k^{t,x}_s,\\
Y^{t,x}_s &= h(X^{t,x}_T) + \int_s^T f(r,X^{t,x}_r,Y^{t,x}_r,Z^{t,x}_r)dr - \int_s^T Z^{t,x}_r\,dW_r,\\
k^{t,x}_s &= -\int_t^s \nabla\phi(X^{t,x}_r)\,d|k^{t,x}|_r, \quad |k^{t,x}|_s = \int_t^s 1_{\{X^{t,x}_r \in \partial\mathcal{O}\}}\,d|k^{t,x}|_r, \quad s \in [t,T].
\end{aligned}$$

It is immediate to extend to this family of problems the previous theorems on existence and uniqueness of solution for problem (4)-(6).

To establish the relation with PDE, we assume now that b, σ, f and h are deterministic, moreover, we suppose that σ does not depend on z. Also, for simplicity, we consider $n = 1$. For short, we introduce the following notation:

$$(L\varphi)(s,x,y,z) = \frac{1}{2}\sum_{i,j=1}^{d}(\sigma\sigma^*)_{ij}(s,x,y)\frac{\partial^2\varphi}{\partial x_i \partial x_j}(s,x) + (b(s,x,y,z), \nabla\varphi(s,x)),$$

and consider the homogeneous Neumann problem

$$\begin{aligned}
&\frac{\partial u}{\partial t}(t,x) + (Lu)(t,x,u(t,x),(\nabla u(t,x))^*\sigma(t,x,u(t,x))) \\
&+f(t,x,u(t,x),(\nabla u(t,x))^*\sigma(t,x,u(t,x))) = 0, \quad (t,x) \in (0,T)\times\mathcal{O}, \\
&\frac{\partial u}{\partial n}(t,x) = 0, \quad (t,x) \in (0,T)\times\partial\mathcal{O}, \\
&u(T,x) = h(x), \quad x \in \mathcal{O}.
\end{aligned} \tag{10}$$

Then, we have, for example, the following result, that can be proved as Theorem 3.8 in [1], and actually can also be adapted to deal with a system.

Theorem 7 *Under the assumptions of Theorem* 6*, suppose, moreover,* $n = 1$*. Suppose also that* b*,* σ*,* f *and* h *are deterministic, continuous in all its variables, and* σ *does not depend on* z*. Then, the function* u *defined by* $u(t,x) = Y_t^{x,t}$*,* $(t,x) \in [0,T]\times\bar{\mathcal{O}}$*, is a viscosity solution of (10).*

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