Negatively invariant sets and entire solutions *

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Abstract

Negatively invariant compact sets of autonomous and nonautonomous dynamical systems on a metric space, the latter formulated in terms of processes, are shown to contain a strictly invariant set and hence entire solutions. For completeness the positively invariant case is also considered. Both discrete and continuous time systems are considered. In the nonautonomous case, the various types of invariant sets are in fact families of subsets of the state space that are mapped onto each other by the process. A simple example shows the usefulness of the result for showing the occurrence of a bifurcation in a nonautonomous system.

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Dedicated to Russell Johnson on the occasion of his sixtieth birthday

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1 Introduction

Let φ be an autonomous semi-dynamical system on a metric space (X, d_X) and let A be a nonempty compact subset of X which is φ-invariant, i.e., $\varphi(t, A) = A$ for all $t \in \mathbb{R}^+$. It is well known that there exists at least one entire trajectory through each point $a \in A$ that is contained in A, i.e., there exists a mapping $\chi : \mathbb{R} \to X$ such that $\chi(t + s) = \varphi(s, \chi(t))$ for all $t \in \mathbb{R}$ and $s \in \mathbb{R}^+$, with $\chi(0) = a$ and $\chi(t) \in A$ for all $t \in \mathbb{R}$. Positively invariant sets are often encountered from absorbing sets, which is a first step to prove the existence of an attractor. Negatively invariant sets are not discussed directly so often in the literature, e.g., [13], but are present in many unstable situations such as following the loss of stability in a bifurcation or on an unstable manifold about an equilibrium point as well as in discretization and persistency problems, e.g., [2, 3, 4, 6, 12, 17, 18, 19].

Our aim in this paper is to establish the existence of invariant sets or families when positively invariant and negatively invariant compact subsets are assumed to exist. We will do this for both autonomous and nonautonomous systems with both discrete and continuous time sets. In the nonautonomous case, the various types of invariant sets are in fact families of subsets of the state space that are mapped onto each other by the nonautonomous process. We start with the autonomous discrete time case since everything is straightforward in that case, but the details form part of later constructions. In the positively invariant case this is the same as the construction of a global attractor once one is inside a compact positively invariant absorbing set. The main technical difficulty arises in the case of negatively invariant subsets for continuous time systems, as a trajectory joining two points in the set may leave it at intermediate times. To overcome this a limiting argument is used that involves the systems frozen at discrete dyadic times.

We will use $\mathrm{dist}_X(\cdot, \cdot)$ to denote the Hausdorff semi-distance between nonempty compact subsets of X, i.e.,

$$\mathrm{dist}_X(A, B) = \sup_{a \in A} \inf_{b \in B} d_X(a, b),$$

while we will denote by $H_X(\cdot, \cdot)$ the Hausdorff distance, i.e.,

$$H_X(A, B) = \max\{\mathrm{dist}_X(A, B), \mathrm{dist}_X(B, A)\}.$$

All through the paper we will only use the symbol $\subset$ for inclusion of one set into another one (equality of the two sets is allowed), but not the symbol $\subseteq$ (analogous comment with $\supset$ and $\supseteq$).

To allow for both continuous and discrete time we let $\mathbb{T}$ be either $\mathbb{R}$ or $\mathbb{Z}$, and obviously $\mathbb{T}^+$ be either $\mathbb{R}^+$ or $\mathbb{Z}^+$. For completeness we give the following well known definitions, see for example [5, 20, 21].

Definition 1 *An* autonomous semi-dynamical system *is a continuous mapping* $(t, x_0) \mapsto \varphi(t, x_0)$ *for* $t \in \mathbb{T}^+$ *and* $x_0 \in X$ *with the initial value and semi-group properties*

(i) $\varphi(0, x_0) = x_0$ for all $x_0 \in X$,
(ii) $\varphi(s+t, x_0) = \varphi(s, \varphi(t, x_0))$ for all s, $t \in \mathbb{T}^+$ and $x_0 \in X$.

Definition 2 *A subset A of X is said to be* φ-invariant *if $\varphi(t, A) = A$ for all $t \in \mathbb{T}^+$,* φ-positively invariant *if $\varphi(t, A) \subset A$ for all $t \in \mathbb{T}^+$, and* φ-negatively invariant *if $\varphi(t, A) \supset A$ for all $t \in \mathbb{T}^+$.*

The nonautonomous counterparts of these definitions will be given in Section 4.

The following result will be used repeatedly.

Lemma 1 *Let $\{A_n, n \in \mathbb{N}\}$ be a nested sequence of nonempty compact subsets of a metric space (X, d_X). Then $A = \cap_{n\geq 1} A_n$ is a nonempty compact subset of X and*

$$\mathrm{dist}_X(A_n, A) \to 0 \quad \text{as } n \to \infty. \tag{1}$$

Proof: Firstly we prove $A \neq \emptyset$. Take any $a_j \in A_j$ for all $j \in \mathbb{N}$. For any fixed $n \in \mathbb{N}$, the sequence $\{a_m\}_{m\geq n} \subset A_n$, and as A_n is compact, there exists a converging subsequence to some element $\bar{a} \in A_n$. Indeed, by a diagonal argument for subsequences, this can be reproduced recursively for any $n \in \mathbb{N}$. Therefore, we conclude that $\bar{a} \in A$.

Secondly, since A is closed and contained in A_1, which is compact, it is also compact.

The convergence (1) can be easily proved by contradiction. Suppose it is not so. Then there exists $\varepsilon > 0$ and a sequence $\{n'\} \subset \mathbb{N}$ such that for some $a_{n'} \in A_{n'}$ it holds:

$$\mathrm{d}(a_{n'}, A) \geq \varepsilon \quad \text{for all} \quad n'. \tag{2}$$

However, reasoning as before, we may extract a subsequence $\{a_{n''}\} \subset \{a_{n'}\}$ such that $a_{n''} \to \bar{a} \in A$, which contradicts (2). Thus, we conclude (1). □

2 Discrete time autonomous systems

A discrete time autonomous semi-dynamical system $\widetilde{\varphi}$ consists of iterations of a single continuous mapping $\varphi : X \to X$, i.e., so $\widetilde{\varphi}(n, x_0) = \varphi^n(x_0)$, and trajectories are sequences. We will first prove that a positively invariant compact subset contains an invariant compact subset and then consider the negatively invariant case.

Proposition 1 *Let A be a nonempty compact subset of X which is φ-positively invariant, i.e., $\varphi(A) \subset A$. Then there exists a maximal nonempty compact subset A_∞ of A which is φ-invariant, i.e., $\varphi(A_\infty) = A_\infty$.*

Proof: Since A is compact and φ continuous, the set $\varphi(A)$ is compact. Define $A_0 = A$ and $A_1 = \varphi(A_0)$, so $A_1 \subset A_0$. Then $A_2 = \varphi(A_1)$ is compact and contained in A_1 since $\varphi(A_1) = \varphi(\varphi(A_0)) \subset \varphi(A_0) \subset A_n$. Continuing in this way gives a nested sequence of nonempty compact subsets $A_{n+1} = \varphi(A_n) \subset A_n$ for $n = 0, 1, 2, \ldots$ Hence the set defined by

$$A_\infty = \bigcap_{n\geq 0} A_n$$

is a nonempty compact subset of A. Moreover, A_∞ is φ-invariant, since

1) If $\bar{a} \in A_\infty$, then $\bar{a} \in A_n$ for all $n \geq 0$ and $\varphi(\bar{a}) \in \varphi(A_n)$ for all $n \geq 0$. Hence

$$\varphi(\bar{a}) \in \bigcap_{n\geq 0} \varphi(A_n) = \bigcap_{n\geq 1} A_n = A_\infty,$$

from which it follows that $\varphi(A_\infty) \subset A_\infty$.

2) If $\bar{a} \in A_\infty$, then $\bar{a} \in A_{n+1} = \varphi(A_n)$ for all n, so there exist $b_n \in A_n$ such that $\varphi(b_n) = \bar{a}$. Now the $b_n \in A$, which is compact. Hence there exists a convergent subsequence $b_{n_j} \to \bar{b}$ in A. In fact, $\bar{b} \in A_\infty$, since

$$\mathrm{dist}_X\left(\bar{b}, A_\infty\right) \leq \mathrm{dist}_X\left(\bar{b}, b_{n_j}\right) + \mathrm{dist}_X\left(A_{n_j}, A_\infty\right) \to 0 \quad \text{as } j \to \infty,$$

where we have used Lemma 1. Moreover, by continuity $\bar{a} = \varphi(b_{n_j}) \to \varphi(\bar{b})$, so $\bar{a} = \varphi(\bar{b})$, which means that $A_\infty \subset \varphi(A_\infty)$.

The maximality of A_∞ is clear by its construction. $\square$

Proposition 2 *Let A be a nonempty compact subset of X which is φ-negatively invariant, i.e., $A \subset \varphi(A)$. Then there exists a maximal nonempty compact subset A_∞ of A which is φ-invariant, i.e., $\varphi(A_\infty) = A_\infty$.*

Proof: Define $A_0 = A$ and let A_{-1} be the maximal subset of A_0 such that $A_0 = \varphi(A_{-1})$. It is not difficult to characterize A_{-1} as $\{a \in A : \varphi(a) \in A\}$, or also equivalently as $A \cap \varphi^{-1}(A)$, which clearly is a closed set inside the compact set A. Therefore, A_{-1} is compact too.

Repeating this procedure gives a nested sequence of (maximal) nonempty compact subsets $A_{-n-1} \subset A_{-n} = \varphi(A_{-n-1})$ for $n = 0, 1, 2, \ldots$ Hence the set defined by

$$A_\infty = \bigcap_{n\geq 0} A_{-n}$$

is a nonempty compact subset of A. Moreover, A_∞ is φ-invariant by a similar argument to that in the proof of Proposition 1. Again the maximality is clear by construction. $\square$

For the discrete time systems being considered here, entire trajectories are bi-infinite sequences $\{x_n : n \in \mathbb{Z}\}$ such that $\varphi(x_n) = x_{n+1}$ for all $n \in \mathbb{Z}$. For completeness we state and prove the following well known result.

Proposition 3 *Let A_∞ be a φ-invariant set. Then for any point $\bar{a} \in A_\infty$ there exists an entire trajectory $\{x_n : n \in \mathbb{Z}\}$ (which is not necessarily unique) such that $x_0 = \bar{a}$ and $x_n \in A_\infty$ for all $n \in \mathbb{Z}$.*

Proof: Since $\varphi(A_\infty) = A_\infty$, there exists an $x_{-1} \in A_\infty$ (not necessarily unique) such that $\varphi(x_{-1}) = x_0 := \bar{a}$. Repeating this argument, there exists an $x_{-n-1} \in A_\infty$ such that $\varphi(x_{-n-1}) = x_{-n} \in A_\infty$ for each $n \in \mathbb{Z}^+$. The forward part of the trajectory is obtained by iterating the mapping φ starting at $x_0 = \bar{a}$, i.e., $x_{n+1} = \varphi(x_n)$ for $n \in \mathbb{Z}^+$. $\square$

3 Continuous time autonomous systems

In this section $\varphi : \mathbb{R}^+ \times X \to X$ is a continuous mapping, which satisfies the initial condition and semi-group properties. Indeed, the results stated here are particular cases of the non-autonomous ones (cf. Section 4). However, we consider that ideas about the proofs are clearly exposed if we start in this way.

Firstly, we will consider the simpler case of a positively invariant compact subset.

Proposition 4 *Let A be a nonempty compact subset of X which is φ-positively invariant, i.e., $\varphi(t, A) \subset A$ for all $t \in \mathbb{R}^+$. Then there exists a maximal nonempty compact subset A_∞ of A which is φ-invariant, i.e., $\varphi(t, A_\infty) = A_\infty$ for all $t \in \mathbb{R}^+$.*

Proof: Since A is compact and φ continuous, the set $\varphi(t, A)$ is compact for each $t \in \mathbb{R}^+$. Moreover, by the semi-group property

$$\varphi(s + t, A) = \varphi(s, \varphi(t, A)) \subset \varphi(s, A) \subset A$$

for all $s,\ t \in \mathbb{R}^+$, i.e., the $\varphi(t, A)$ are a nested family of nonempty compact subsets. Hence the set defined by

$$A_\infty = \bigcap_{t \geq 0} \varphi(t, A)$$

is a nonempty compact subset of A. Moreover, A_∞ is φ-invariant by a similar argument to that in the proof of Proposition 1, with some slight differences which are worth showing.

1) Fix $\tau > 0$. If $\bar{a} \in A_\infty$, then $\bar{a} \in \varphi(t, A)$ for all $t \geq 0$. Then $\varphi(\tau, \bar{a}) \in \varphi(\tau, \varphi(t, A)) = \varphi(\tau + t, A)$ for all $t \geq 0$. Hence

$$\varphi(\tau, \bar{a}) \in \bigcap_{t \geq 0} \varphi(\tau + t, A) = \bigcap_{t \geq \tau} \varphi(t, A) \subset \bigcap_{t \geq 0} \varphi(t, A) = A_\infty,$$

since $\varphi(\tau, A) = \varphi(t, \varphi(\tau - t, A)) \subset \varphi(t, A)$ for all $0 \leq t \leq \tau$, from which it follows that $\varphi(\tau, A_\infty) \subset A_\infty$ for any $\tau > 0$.

2) Fix $\tau > 0$. If $\bar{a} \in A_\infty$, then $\bar{a} \in \varphi(\tau + t_n, A) = \varphi(\tau, \varphi(t_n, A))$, where $t_n \to 0$ as $n \to \infty$. Hence there exist $b_n \in \varphi(t_n, A)$ such that $\varphi(\tau, b_n) = \bar{a}$ for all $n \in \mathbb{N}$. Now the $b_n \in A$, which is compact. Hence there exists a convergent subsequence $b_{n_j} \to \bar{b}$ in A. In fact, $\bar{b} \in A_\infty$, since

$$\operatorname{dist}_X\left(\bar{b}, A_\infty\right) \leq \operatorname{dist}_X\left(\bar{b}, b_{n_j}\right) + \operatorname{dist}_X\left(\varphi(t_{n_j}, A), A_\infty\right) \to 0 \quad \text{as } j \to \infty,$$

where we have used again Lemma 1. Moreover, by continuity $\bar{a} = \varphi(\tau, b_{n_j}) \to \varphi(\tau, \bar{b})$, so $\bar{a} = \varphi(\tau, \bar{b})$, which means that $A_\infty \subset \varphi(\tau, A_\infty)$.

Finally, we claim that A_∞ is the maximal φ-invariant set inside A. Indeed, consider $B \subset A$ with $\varphi(t, B) = B$. Then, $B = \varphi(t, B) \subset \varphi(t, A)$ for all $t \geq 0$. This implies that $B \subset A_\infty$. □

The negative invariant case is more complicated as one has to ensure that constructed subsets remain in the original set A.

Theorem 1 *Let A be a nonempty compact subset of X which is φ-negatively invariant, i.e., $A \subset \varphi(t, A)$ for all $t \in \mathbb{R}^+$. Then there exists a maximal nonempty compact subset A_∞ of A which is φ-invariant, i.e., $\varphi(t, A_\infty) = A_\infty$ for all $t \in \mathbb{R}^+$.*

Proof: We apply the result of Proposition 2 to the discrete time system formed by the time-1 mapping $\varphi(1, \cdot) : X \to X$. This gives us a nonempty compact subset $A_\infty^{(1)}$ of A which is the maximal $\varphi(1, \cdot)$-invariant subset of A, i.e., with $\varphi(1, A_\infty^{(1)}) = A_\infty^{(1)}$. The problem is that $\varphi(t, A_\infty^{(1)})$ may not be a subset of A for all $t \in (0, 1)$. Therefore we repeat the procedure for the discrete time system formed by the time-2^{-1} mapping $\varphi(2^{-1}, \cdot) : X \to X$ and obtain a nonempty compact subset $A_\infty^{(2)}$ of A which is the maximal $\varphi(2^{-1}, \cdot)$-invariant subset of A, i.e., with $\varphi(2^{-1}, A_\infty^{(2)}) = A_\infty^{(2)}$. By this and the semi-group property,

$$A_\infty^{(2)} = \varphi(2^{-1}, A_\infty^{(2)}) = \varphi\left(2^{-1}, \varphi(2^{-1}, A_\infty^{(2)})\right) = \varphi\left(1, A_\infty^{(2)}\right)$$

so $A_\infty^{(2)}$ is also a $\varphi(1, \cdot)$-invariant subset of A. But $A_\infty^{(1)}$ is the maximal compact $\varphi(1, \cdot)$-invariant subset of A, so $A_\infty^{(2)} \subset A_\infty^{(1)}$.

We repeat this procedure with the discrete time system formed by the time-2^{-n} mapping $\varphi(2^{-n}, \cdot) : X \to X$ and obtain a nonempty compact subset $A_\infty^{(n)}$ of A which is the maximal $\varphi(2^{-n}, \cdot)$-invariant subset of A, which is thus also $\varphi(2^{-n+1}, \cdot)$-invariant. Hence $A_\infty^{(n)} \subset A_\infty^{(n-1)}$ for $n = 1, 2, \ldots$ This is a nested family of nonempty compact subsets, so the set defined by

$$A_\infty = \bigcap_{n \geq 1} A_\infty^{(n)}$$

is a nonempty compact subset of A. Moreover, A_∞ is $\varphi(2^{-n}, \cdot)$-invariant for each $n = 0, 1, \ldots$, i.e., $\varphi(2^{-n}, A_\infty) = A_\infty$.

Indeed, the inclusion $\varphi(2^{-n}, A_\infty) \subset A_\infty$ follows easily from the definition of A_∞ and the $\varphi(2^{-n}, \cdot)$-invariance of the sets $A_\infty^{(m)}$ for m=n, $n+1, \ldots$ For the opposite inclusion, fix an element $x \in A_\infty$, then $x \in A_\infty^{(m)}$ for all m, and therefore there exist $y_m \in A_\infty^{(m)}$ such that $x = \varphi(2^{-n}, y_m)$. As $A_\infty^{(m)} \subset A$, from $\{y_m\}$ we may extract a convergent subsequence $y_{m'} \to \bar{y}$. Actually, $\bar{y} \in A_\infty$ (again by Lemma 1), and finally by continuity of $\varphi(2^{-n}, \cdot)$, we deduce that $x = \varphi(2^{-n}, \bar{y})$, which concludes the required inclusion and the equality $\varphi(2^{-n}, A_\infty) = A_\infty$.

Now, from this and the semi-group property it follows that $\varphi(j2^{-n}, A_\infty) = A_\infty$ for all $j = 0, \ldots, 2^n$ and for all $n = 1, 2, \ldots$, i.e., for all dyadic numbers in $[0,1]$.

By continuity of φ, it is not difficult to deduce that for any compact set $B \subset A$, it holds that

$$H_X\left(\varphi(\tau, B), \varphi(t, B)\right) \to 0 \quad \text{as } \tau \to t,$$

for dyadic $\tau \in [0,1]$ with $\tau \to t \in [0,1]$, where t is arbitrary. In particular, putting $B = A_\infty$, we deduce

$$H_X\left(\varphi(\tau, A_\infty), \varphi(t, A_\infty)\right) \to 0 \quad \text{as } \tau \to t$$

for dyadic $\tau \in [0,1]$ with $\tau \to t \in [0,1]$, where t is arbitrary. Finally,

$$H_X\left(\varphi(t, A_\infty), A_\infty\right) \leq H_X\left(\varphi(t, A_\infty), \varphi(\tau, A_\infty)\right) + H_X\left(\varphi(\tau, A_\infty), A_\infty\right)$$

gives $\varphi(t, A_\infty) = A_\infty$ for all $t \in [0,1]$, and hence for all $t \in \mathbb{R}^+$, since $H_X\left(\varphi(\tau, A_\infty), A_\infty\right) = 0$ for all dyadic $\tau \in [0,1]$.

We conclude now proving that A_∞ is the maximal φ-invariant set in A. Assume that $B \subset A$ satisfies that $\varphi(t, B) = B$ for all $t \geq 0$. Then, by Proposition 2 $B \subset A_\infty^{(1)}$, $A_\infty^{(2)}, \ldots$ and thus $B \subset \bigcap_{n \geq 1} A_\infty^{(n)} = A_\infty$. □

4 Nonautonomous dynamical systems

Solution mappings of nonautonomous differential equations provide one of the main motivations for the process definition of an abstract nonautonomous dynamical system on a metric state space (X, d_X), [1, 5, 20]. Recall that to allow for both continuous and discrete time, we denote by $\mathbb{T}$ either $\mathbb{R}$ or $\mathbb{Z}$ and define $\mathbb{T}_2^{\geq} := \{(t,s) \in \mathbb{T}^2 : t \geq s\}$.

Definition 3 *A* process *is a continuous mapping* $(t, t_0, x_0) \mapsto \phi(t, t_0, x_0)$ *for* $(t, t_0) \in \mathbb{T}_2^{\geq}$ *and* $x_0 \in X$ *with the initial value and evolution properties*
(i) $\phi(t_0, t_0, x_0) = x_0$ for all $t_0 \in \mathbb{T}$ and $x_0 \in X$,
(ii) $\phi(t_2, t_0, x_0) = \phi(t_2, t_1, \phi(t_1, t_0, x_0))$ for all $t_0 \leq t_1 \leq t_2$ in $\mathbb{T}$ and $x_0 \in X$.

A process is often also called a *two-parameter semi-group* on X in contrast with the one-parameter semi-group of an autonomous semi-dynamical system.

Definition 4 *A family of nonempty compact sets* $\mathcal{A} = \{A(t) : t \in \mathbb{T}\}$ *of* X *said to be* ϕ-invariant *if*

$$\phi(t, t_0, A(t_0)) = A(t) \quad \textit{for all } (t, t_0) \in \mathbb{T}_2^{\geq},$$

ϕ-positively invariant *if*

$$\phi(t, t_0, A(t_0)) \subset A(t) \quad \textit{for all } (t, t_0) \in \mathbb{T}_2^{\geq}$$

and ϕ-negatively invariant *if*

$$\phi(t, t_0, A(t_0)) \supset A(t) \quad \textit{for all } (t, t_0) \in \mathbb{T}_2^{\geq}.$$

It follows from the above definition that the set-valued mapping $t \mapsto A(t) = \phi(t, t_0, A(t_0))$ is continuous in $t \in \mathbb{R}$ in the Hausdorff metric H_X for a ϕ-invariant family of nonempty compact sets $\mathcal{A}$ of a continuous time process.

For positive invariant sets we can consider the continuous and discrete time cases together.

Proposition 5 *Let* $\mathcal{A} = \{A(t) : t \in \mathbb{T}\}$ *be a family of nonempty compact subsets of* X *which is positively invariant for the process* ϕ*, i.e.,* $\phi(t, t_0, A(t_0)) \subset A(t)$ *for all* $(t, t_0) \in \mathbb{T}_2^{\geq}$*. Then there exists a family of nonempty compact subsets* $\mathcal{A}_\infty = \{A_\infty(t) : t \in \mathbb{T}\}$ *contained in* $\mathcal{A}$ *in the sense that* $A_\infty(t) \subset A(t)$ *for all* $t \in \mathbb{T}$*, which is* ϕ*-invariant, i.e.,* $\phi(t, t_0, A_\infty(t_0)) = A_\infty(t)$ *for all* $(t, t_0) \in \mathbb{T}_2^{\geq}$*.*

Moreover, $\mathcal{A}_\infty$ *is the maximal* ϕ*-invariant family contained in* $\mathcal{A}$*, i.e., any other* ϕ*-invariant family* $\mathcal{B} = \{B(t) : t \in \mathbb{T}\}$ *with* $B(t) \subset A(t)$ *for all* $t \in \mathbb{T}$*, satisfies that* $B(t) \subset A_\infty(t)$ *for all* $t \in \mathbb{T}$*.*

Proof: Since $\mathcal{A}$ is a family of compact sets and the process ϕ is continuous, the set $\phi(t, t_0, A(t_0))$ is compact for all $(t, t_0) \in \mathbb{T}_2^{\geq}$. Moreover, by the two-parameter semi-group property we have that

$$\phi(t, s_0, A(s_0)) = \phi(t, t_0, \phi(t_0, s_0, A(s_0))) \subset \phi(t, t_0, A(t_0)) \subset A(t) \quad \text{for all } s_0 \leq t_0 \leq t. \tag{3}$$

So, for fixed $t \in \mathbb{T}$, the sets $\phi(t, t_0, A(t_0))$, for $t_0 \leq t$, are a nested family of nonempty compact subsets of $A(t)$. Hence the set defined by

$$A_\infty(t) = \bigcap_{t_0 \leq t} \phi(t, t_0, A(t_0))$$

is a nonempty compact subset of $A(t)$ for each $t \in \mathbb{T}$. Moreover, $\mathcal{A}_\infty = \{A_\infty(t) : t \in \mathbb{T}\}$ is ϕ-invariant, since

1) If $\bar{a} \in A_\infty(t_0)$, then $\bar{a} \in \phi(t_0, s_0, A(s_0))$ for all $s_0 \leq t_0$. Hence

$$\phi(t, t_0, \bar{a}) \subset \phi(t, t_0, \phi(t_0, s_0, A(s_0))) = \phi(t, s_0, A(s_0)),$$

for any $t \geq t_0$ and any $s_0 \leq t_0$. So, using the nested character proved in (3),

$$\phi(t,t_0,\bar{a}) \in \bigcap_{s_0 \leq t_0} \phi(t,s_0,A(s_0)) \subset \bigcap_{s_0 \leq t} \phi(t,s_0,A(s_0)) = A_\infty(t).$$

Thus $\phi(t,t_0,A_\infty(t_0)) \subset A_\infty(t)$.

2) If $\bar{a} \in A_\infty(t)$, then $\bar{a} \in \phi(t,s_n,A(s_n)) = \phi(t,t_0,\phi(t_0,s_n,A(s_n)))$ for all $s_n \leq t_0 \leq t$. Hence there exist $b_n \in \phi(t_0,s_n,A(s_n)) \subset A(t_0)$ such that $\phi(t,t_0,b_n) = \bar{a}$. Now the $b_n \in A(t_0)$, which is compact, so there exists a convergent subsequence $b_{n_j} \to \bar{b}$ in $A(t_0)$. Moreover, we can choose the $s_n \to -\infty$. In fact, $\bar{b} \in A_\infty(t_0)$, since

$$\mathrm{dist}_X\left(\bar{b}, A_\infty(t_0)\right) \leq \mathrm{dist}_X\left(\bar{b}, b_{n_j}\right) + \mathrm{dist}_X\left(\phi(t_0,s_{n_j},A(s_{n_j})), A_\infty(t_0)\right) \to 0 \quad \text{as } j \to \infty,$$

by Lemma 1. Finally, by continuity $\bar{a} = \phi(t,t_0,b_{n_j}) \to \phi(t,t_0,\bar{b})$, so $\bar{a} = \phi(t,t_0,\bar{b})$, which means that $A_\infty(t) \subset \phi(t,t_0,A_\infty(t_0))$.

The maximality of $\mathcal{A}_\infty$ as ϕ-invariant family inside $\mathcal{A}$ follows from its construction. Indeed, consider a ϕ-invariant family $\mathcal{B}$ with $B(t) \in A(t)$ for all $t \in \mathbb{T}$. Then $\phi(t,t_0,B(t_0)) = B(t) \subset \phi(t,t_0,A(t_0))$ for all $t_0 \leq t$, whence $B(t) \subset A_\infty(t)$. □

We consider the negative invariant case first for discrete time processes and then for continuous time processes.

Proposition 6 *Let $\mathcal{A} = \{A(n) : n \in \mathbb{Z}\}$ be a family of nonempty compact subsets of X which is ϕ-negatively invariant for a discrete time process ϕ, i.e., $A(n) \subset \phi(n,n_0,A(n_0))$ for all $(n,n_0) \in \mathbb{Z}^{\geq}_2$. Then there exists a maximal family of nonempty compact subsets $\mathcal{A}_\infty = \{A_\infty(n), n \in \mathbb{Z}\}$ of $\mathcal{A}$, which is ϕ-invariant, i.e., $\phi(n,n_0,A_\infty(n_0)) = A_\infty(n)$ for all $(n,n_0) \in \mathbb{Z}^{\geq}_2$.*

Proof: Define $A_0^{(j)} \equiv A(j)$ for all $j \in \mathbb{Z}$. Fix $n \in \mathbb{Z}$ and let $A_{-1}^{(n)}$ be the maximal subset of $A_0^{(n-1)}$ such that $A_0^{(n)} = \phi(n,n-1,A_{-1}^{(n)})$. Since $A_0^{(n-1)}$ is nonempty and compact and $\phi(n,n-1,\cdot)$ continuous, the set $A_{-1}^{(n)}$, which similarly as done in Proposition 2 can be characterized as $A_0^{(n-1)} \cap \phi(n,n-1,\cdot)^{-1}(A_0^{(n)})$, is nonempty and compact too.

Repeating recursively this procedure gives a sequence of nonempty compact subsets $A_{-j}^{(n)} \subset A(n-j)$ for all $j \geq 0$, with $A_{-j-1}^{(n)}$ defined as the maximal subset in $A(n-j-1)$ such that $A_{-j}^{(n)} = \phi(n-j,n-j-1,A_{-j-1}^{(n)})$, and hence in particular

$$\phi\left(n,n-j,A_{-j}^{(n)}\right) = A_0^{(n)} \qquad \text{for} \quad j = 0,1,2,\ldots$$

Similarly, we can define nonempty compact subsets sets $A_{-j}^{(n+k)}$ for j, $k \in \mathbb{Z}^+$.

Now we claim that for a fixed $n \in \mathbb{Z}$ the following relation holds:

$$A_{-k-1}^{(n+k+1)} \subset A_{-k}^{(n+k)} \text{ for each } k \in \mathbb{Z}^+, \tag{4}$$

i.e., a nested family of nonempty compact subsets of $A(n)$. To see this consider the case $k = 1$, recall that $A_{-1}^{(n+1)}$ is the maximal subset of $A(n)$ with $\phi(n+1, n, A_{-1}^{(n+1)}) = A(n+1)$, and by construction $A(n+1) \supset A_{-1}^{(n+2)} = \phi(n+1, n, A_{-2}^{(n+2)})$, so $A_{-2}^{(n+2)} \subset A_{-1}^{(n+1)}$. Reasoning similarly and recursively, (4) is proved. Hence the set defined by

$$A_\infty(n) = \bigcap_{k \geq 0} A_{-k}^{(n+k)}$$

is a nonempty compact subset of $A(n)$ for each $n \in \mathbb{Z}$.

Moreover, the family of nonempty compact subsets $\mathcal{A}_\infty = \{A_\infty(n), n \in \mathbb{Z}\}$ is ϕ-invariant, since

1) If $\bar{a} \in A_\infty(n_0)$, then $\bar{a} \in A_{-k}^{(n_0+k)}$ and $\phi(n, n_0, \bar{a}) \in \phi\left(n, n_0, A_{-k}^{(n_0+k)}\right)$ for all $k \geq 0$. Moreover, for $k \geq n - n_0$ and $l = k - n + n_0 \geq 0$,

$$A_{-k}^{(n_0+k)} = A_{-[l+n-n_0]}^{(n_0+[l+n-n_0])} = A_{-l-n+n_0}^{(n+l)}.$$

But

$$\phi\left(n, n_0, A_{-l-n+n_0}^{(n+l)}\right) = A_{-l}^{(n+l)}$$

by construction, so

$$\phi(n, n_0, \bar{a}) \in \phi\left(n, n_0, A_{-k}^{(n_0+k)}\right) = \phi\left(n, n_0, A_{-l-n+n_0}^{(n+l)}\right) = A_{-l}^{(n+l)}.$$

Hence

$$\phi(n, n_0, \bar{a}) \in \bigcap_{l \geq 0} A_{-l}^{(n+l)} = A_\infty(n),$$

which means that $\phi\left(n, n_0, A_\infty(n_0)\right) \subset A_\infty(n)$.

2) If $\bar{a} \in A_\infty(n)$, then $\bar{a} \in A_{-l}^{(n+l)}$ for all $l \geq 0$. But

$$A_{-l}^{(n+l)} = A_{-l}^{(n_0+[l+n-n_0])} = A_{-[k-n+n_0]}^{(n_0+k)}$$

for $k = l + n - n_0 \geq n - n_0$. Moreover,

$$\phi\left(n, n_0, A_{-k}^{(n_0+k)}\right) = A_{-[k-n+n_0]}^{(n_0+k)},$$

so $\bar{a} \in \phi\left(n, n_0, A_{-k}^{(n_0+k)}\right)$. Hence there exist $b_k \in A_{-k}^{(n_0+k)} \subset A(n_0)$ such that $\phi(n, n_0, b_k) = \bar{a}$. Now the $b_k \in A(n_0)$, which is compact, so there exists a convergent subsequence $b_{k_j} \to \bar{b}$ in $A(n_0)$. In fact, $\bar{b} \in A_\infty(n_0)$, since

$$\operatorname{dist}_X\left(\bar{b}, A_\infty(n_0)\right) \leq \operatorname{dist}_X\left(\bar{b}, b_{k_j}\right) + \operatorname{dist}_X\left(A_{-k_j}^{(n_0+k_j)}, A_\infty(n_0)\right) \to 0 \quad \text{as } j \to \infty.$$

Finally, by continuity $\bar{a} = \phi(n, n_0, b_{k_j}) \to \phi(n, n_0, \bar{b})$, so $\bar{a} = \phi(n, n_0, \bar{b})$, which means that $A_\infty(n) \subset \phi(n, n_0, A_\infty(n_0))$.

The maximality of the family $\mathcal{A}_\infty$ is clear by construction, analogously as in Proposition 5. $\square$

Theorem 2 *Let $\mathcal{A} = \{A(t) : t \in \mathbb{R}\}$ be a family of nonempty compact subsets of X which is ϕ-negatively invariant for a continuous time process ϕ, i.e., $A(t) \subset \phi(t, t_0, A(t_0))$ for all $(t, t_0) \in \mathbb{R}_2^{\geq}$. Then there exists a family of nonempty compact subsets $\mathcal{A}_\infty = \{A_\infty(t) : t \in \mathbb{R}\}$ with $A_\infty(t) \subset A(t)$ for all $t \in \mathbb{R}$, which is ϕ-invariant, i.e., $\phi(t, t_0, A_\infty(t_0)) = A_\infty(t)$ for all $(t, t_0) \in \mathbb{R}_2^{\geq}$.*

Moreover, $\mathcal{A}_\infty$ is the maximal ϕ-invariant family contained in $\mathcal{A}$.

Proof: The proof generalizes that of Theorem 1 to processes. We first consider the process restricted the dyadic numbers in $\mathbb{R}$. Let $\mathbb{T}_0 = \mathbb{Z}$ and $\mathbb{D}_n = \left\{d_j^{(n)} := j2^{-n} : j = 0, 1, \ldots, 2^n - 1\right\}$, then define

$$\mathbb{T}_n := \mathbb{Z} + \mathbb{D}_n = \left\{k + d_j^{(n)} \;:\; k \in \mathbb{Z}, d_j^{(n)} \in \mathbb{D}_n\right\}, \qquad n = 1, 2, \ldots$$

We apply the result of Proposition 6 to the discrete time system formed by the restriction $\phi\big|_{\mathbb{T}_0}$ of the mapping ϕ to the time set $\mathbb{T}_0$. This gives us a family $\mathcal{A}_\infty^{(0)} = \{A_\infty^{(0)}(t) : t \in \mathbb{T}_0\}$ of nonempty compact subsets, with $A_\infty^{(0)}(t) \subset A(t)$ for all $t \in \mathbb{T}_0$, which is the maximal $\phi\big|_{\mathbb{T}_0}$-invariant family of subsets of $\{A(t) : t \in \mathbb{T}_0\}$, i.e., with $\phi(n+1, n, A_\infty^{(0)}(n)) = A_\infty^{(0)}(n+1)$ for any $n \in \mathbb{Z}$.

The problem is, as before, that $\phi(n+t, n, A_\infty^{(0)}(n))$ may not be a subset of $A(n+t)$ for all $t \in (0,1)$. Therefore we repeat the procedure for the discrete time system formed by the restriction $\phi\big|_{\mathbb{T}_1}$ of the mapping ϕ to the time set $\mathbb{T}_1$ and obtain a family $\mathcal{A}_\infty^{(1)} = \{A_\infty^{(1)}(t) : t \in \mathbb{T}_1\}$ of nonempty compact sets, which is the maximal $\phi\big|_{\mathbb{T}_1}$-invariant family with $A_\infty^{(1)}(t) \subset A(t)$ for all $t \in \mathbb{T}_1$, i.e., with $\phi\left(t_{j+1}^{(1)}, t_j^{(1)}, A_\infty^{(1)}(t_j^{(1)})\right) = A_\infty^{(1)}(t_{j+1}^{(1)})$ for every $t_j^{(1)}$, $t_{j+1}^{(1)} \in \mathbb{T}_1$ with $t_{j+1}^{(1)} - t_j^{(1)} = 2^{-1}$. By this and the semi-group property,

$$\begin{aligned} A_\infty^{(1)}(m+1) &= \phi\left(m+1, m+2^{-1}, A_\infty^{(1)}(m+2^{-1})\right) \\ &= \phi\left(m+1, m+2^{-1}, \phi(m+2^{-1}, m, A_\infty^{(1)}(m))\right) = \phi\left(m+1, m, A_\infty^{(1)}(m)\right) \end{aligned}$$

for all $m \in \mathbb{Z}$, so $\{A_\infty^{(1)}(t) : t \in \mathbb{T}_0\}$ is also a $\phi\big|_{\mathbb{T}_0}$-invariant family of compact subsets of $\{A(t) : t \in \mathbb{T}_0\}$. But $\mathcal{A}_\infty^{(0)}$ is the maximal $\phi\big|_{\mathbb{T}_0}$-invariant family of compact subset of $\{A(t) : t \in \mathbb{T}_0\}$, so $A_\infty^{(1)}(t) \subset A_\infty^{(0)}(t)$ for all $t \in \mathbb{T}_0 \cap \mathbb{T}_0 = \mathbb{T}_0$.

We repeat this procedure with the the discrete time system formed by the restriction $\phi\big|_{\mathbb{T}_n}$ of the mapping ϕ to the time set $\mathbb{T}_n$ and obtain a family $\mathcal{A}_\infty^{(n)} = \{A_\infty^{(n)}(t) : t \in \mathbb{T}_n\}$ composed by nonempty compact subsets of $\{A(t) : t \in \mathbb{T}_n\}$, which is the maximal $\phi\big|_{\mathbb{T}_n}$-invariant family of subsets of $\{A(t) : t \in \mathbb{T}_n\}$, which is thus also $\phi\big|_{\mathbb{T}_{n-1}}$-invariant. Hence $A_\infty^{(n)}(t) \subset A_\infty^{(n-1)}(t)$ for all $t \in \mathbb{T}_{n-1} \cap \mathbb{T}_n = \mathbb{T}_{n-1}$, for $n = 1, 2, \ldots$

Thus for each $t_l \in \mathbb{T}_l$ for an arbitrary $l \in \mathbb{N}$, the subsets $A_\infty^{(n)}(t_l)$, $n \geq l$, are nonempty, compact and nested. Hence the set defined by

$$A_\infty(t_l) = \bigcap_{n \geq l} A_\infty^{(n)}(t_l)$$

is a nonempty compact subset of $A(t_l)$. In this way we obtain a family $\mathcal{A}_\infty^{(dyadic)} = \{A_\infty(t_d) : t_d \in \cup_{l \geq 0} \mathbb{T}_l\}$ of nonempty compact subsets of X with $A_\infty(t_d) \subset A(t_d)$ for all $t_d \in \cup_{l \geq 0} \mathbb{T}_l$.

Moreover, by Proposition 2, the family $\mathcal{A}_\infty^{(dyadic)}$ is $\phi\big|_{\mathbb{T}_n}$-invariant for each $n = 0, 1, \ldots$, i.e., with

$$\phi\left(t_{j+1}^{(n)}, t_j^{(n)}, A_\infty(t_j^{(n)})\right) = A_\infty(t_{j+1}^{(n)})$$

for every $t_j^{(n)}$, $t_{j+1}^{(n)} \in \mathbb{T}_n$ with $t_{j+1}^{(n)} - t_j^{(n)} = 2^{-n}$. From this and the semi-group property it follows that $\phi(t_1, t_0, A_\infty(t_0)) = A_\infty(t_1)$ for all dyadic numbers $t_0 \leq t_1$ in $\mathbb{R}$.

Finally, for non-dyadic t, we define $A_\infty(t)$ by

$$A_\infty(t) = \phi\left(t, t_0, A_\infty(t_0)\right),$$

for an arbitrary dyadic $t_0 < t$. This definition is independent of the choice of t_0 (by the semi-group property and the invariance for the dyadic numbers).

Define the family $\mathcal{A}_\infty = \{A_\infty(t) : t \in \mathbb{R}\}$. We check that it is $\phi-$invariant. Indeed, it only remains to show the equality $A_\infty(t) = \phi\left(t, s, A_\infty(s)\right)$ for the case of s non-dyadic. The desired result follows from the definition of $A_\infty(s)$ and the semi-group property, i.e.,

$$\phi\left(t, s, A_\infty(s)\right) = \phi\left(t, s, \phi\left(s, t_0, A_\infty(t_0)\right)\right) = \phi\left(t, t_0, A_\infty(t_0)\right) = A_\infty(t),$$

where t_0 ($< s$) is dyadic, but otherwise arbitrary.

Now we check that $A_\infty(t) \subset A(t)$ for t non dyadic. Since both sets are compact and ϕ is a process, this follows from the following estimates being $\{t_d\}$ a sequence of dyadic values with $t_d > t$ and decreasing to t:

$$\begin{aligned}\mathrm{dist}_X(A_\infty(t), A(t)) &= \lim_{t_d \downarrow t} \mathrm{dist}_X(\phi(t_d, t, A_\infty(t)), \phi(t_d, t, A(t))) \\ &\leq \lim_{t_d \downarrow t} \mathrm{dist}_X(A_\infty(t_d), A(t_d)) = 0,\end{aligned}$$

where we have used the ϕ-negatively invariant character of $\mathcal{A}$ and the ϕ-invariance of $\mathcal{A}_\infty$. The limit is zero because for any t_d we have that $A_\infty(t_d) \subset A(t_d)$.

Finally, the maximality of $\mathcal{A}_\infty$ as ϕ-invariant family inside $\mathcal{A}$ follows by construction. Indeed, consider a ϕ-invariant family $\mathcal{B} = \{B(t) : t \in \mathbb{R}\}$ with $B(t) \subset A(t)$ for all $t \in \mathbb{R}$.

Then, the family $\{B(t) : t \in \mathbb{T}_0\}$ is $\phi\big|_{\mathbb{T}_0}$-invariant, and by construction we have $B(t) \subset A_\infty^{(0)}(t)$ for all $t \in \mathbb{T}_0$. We can repeat this analysis with times in $\mathbb{T}_1, \ldots$ Therefore $B(t_d) \subset A_\infty(t_d)$ for all $t_d \in \cup_{l \geq 0} \mathbb{T}_l$. The invariance of $\mathcal{B}$ and the definition of $A_\infty(t)$ for any non dyadic t implies $B(t) \subset A_\infty(t)$ and concludes the proof. □

5 Relatively invariant sets

Important dynamics is often restricted to a lower dimensional subset such as a stable or unstable invariant manifold. The above results carry over to this case by the observation that the dynamical system restricted to such a manifold is a dynamical system in its own right.

Consider a discrete time autonomous semi-dynamical system given by a continuous mapping $\varphi : X \to X$ and let M be a nonempty subset of X such that $\varphi(M) \subset M$. Then the restriction φ to M is a continuous mapping $\varphi|_M : M \to M$, where continuity is considered in the subspace topology, i.e., (M, d_X) is a metric subspace of (X, d_X). Similar considerations also hold for continuous time autonomous semi-dynamical system. Thus Proposition 1 and Proposition 2 carry over to $\varphi|_M$ and we have the following results. To include discrete and continuous time systems in the same statement, we define $\varphi(n, x) = \varphi^n(x)$ in the discrete time case.

Corollary 1 *Let φ be an autonomous semi-dynamical system on a metric space (X, d_X) for the time set $\mathbb{T}$ and let M be a nonempty subset of X which is φ-positively invariant, i.e., $\varphi(t, M) \subset M$ for all $t \in \mathbb{T}^+$. In addition, let A be a nonempty compact subset of M which is φ-positively invariant or φ-negatively invariant.*

Then there exists a maximal nonempty compact subset A_∞ of A, and hence of M, which is φ-invariant, i.e., $\varphi(t, A_\infty) = A_\infty$ for all $t \in \mathbb{T}^+$.

The nonautonomous case can be generalized directly in the same way. However, since nonautonomous invariant manifolds typically depend on time we will allow both the subsets M to depend on time and the set A too. (In fact, for the proofs one does not have to use the restricted system at all).

Corollary 2 *Let ϕ be a process on a metric space (X, d_X) for the time set $\mathbb{T}$ and let $\mathcal{M} = \{M(t) : t \in \mathbb{T}\}$ be a family of nonempty closed subsets of X which is ϕ-positively invariant, i.e., $\phi(t, t_0, M(t_0)) \subset M(t)$ for all $(t, t_0) \in \mathbb{T}_2^{\geq}$. In addition, let $\mathcal{A} = \{A(t) : t \in \mathbb{T}\}$ be a family of nonempty compact subsets of X with $A(t) \subset M(t)$ for each $t \in \mathbb{T}$ which is ϕ-positively invariant, i.e., $\phi(t, t_0, A(t_0)) \subset A(t)$ for all $(t, t_0) \in \mathbb{T}_2^{\geq}$, or ϕ-negatively invariant, i.e., $A(t) \subset \phi(t, t_0, A(t_0))$ for all $(t, t_0) \in \mathbb{T}_2^{\geq}$.*

Then there exists a maximal family of nonempty compact subsets $\mathcal{A}_\infty = \{A_\infty(t) : t \in \mathbb{T}\}$ contained in $\mathcal{A}$, and hence in $\mathcal{M}$, which is ϕ-invariant, i.e., $\phi(t, t_0, A_\infty(t_0)) = A_\infty(t)$ for all $(t, t_0) \in \mathbb{T}_2^{\geq}$.

These two results can then be used to obtain the existence of entire solutions taking values in the given compact subsets of M and $\mathcal{M}$, respectively.

Example 1 *Consider a process ϕ on X. Suppose that $X = X_1 \times X_2$ and that*

$$\phi(t, t_0, x_0) = (\phi_1(t, t_0, (x_{01}, x_{02})), \phi_2(t, t_0, (x_{01}, x_{02}))) ,$$

where $x_0 = (x_{01}, x_{02})$. Let $m : \mathbb{T} \times X_1 \to X_2$ be continuous and define $\mathcal{M} = \{M(t) : t \in \mathbb{T}\}$ by

$$M(t) = \{(x_1, m(t, x_1)) : \ x_1 \in X_1\} , \qquad t \in \mathbb{T},$$

which are obviously nonempty closed subsets of X. Then $\mathcal{M}$ is ϕ-positively invariant if

$$\phi_2 (t, t_0, (x_1, m(t_0, x_1))) = m (t, \phi_1 (t, t_0, (x_1, m(t_0, x_1))))$$

for all $x_1 \in X_1$ and $(t, t_0) \in \mathbb{T}_2^{\geq}$.

6 Bifurcation in a nonautonomous system

There are at present few general, theoretical results about bifurcations in nonautonomous dynamical systems, e.g., [7, 8, 10, 11, 14, 15, 16]. The above results allow us to make a preliminary investigation to show that what could be considered to be a bifurcation has occurred. This will be illustrated in terms of modifications of a simple example of a pitch fork bifurcation in a scalar ordinary differential equation.

The zero steady state solution of the autonomous semi-dynamical system generated by the differential equation

$$\frac{dx}{dt} = \nu x - x^3$$

undergoes a supercritical bifurcation at $\nu = 0$ to produce two locally asymptotically stable steady state solutions $\pm\sqrt{\nu}$ for $\nu > 0$, with the zero steady state solution now unstable.

Now let $b : \mathbb{R} \to \mathbb{R}$ be a nonconstant continuous function with bounded values, specifically with $0 < \alpha \le b(t) \le \beta$ for all $t \in \mathbb{R}$. Then the scalar ordinary differential equation

$$\frac{dx}{dt} = \nu x - b(t)x^3 \tag{5}$$

is nonautonomous and generates a process. This process has the zero solution for all ν, which is asymptotically stable for $\nu < 0$ and unstable for $\nu > 0$. There are no other steady state solutions. Note that

$$\frac{d}{dt}x^2 = 2\nu x^2 - 2b(t)x^4,$$

so

$$2\nu x^2 - 2\beta x^4 \le \frac{d}{dt}x^2 \le 2\nu x^2 - 2\alpha x^4$$

Let $\nu > 0$. Then

$$2\nu x^2\left(1 - \frac{\beta}{\nu}x^2\right) \le \frac{d}{dt}x^2 \le 2\nu x^2\left(1 - \frac{\alpha}{\nu}x^2\right),$$

from which it follows that

$$\frac{d}{dt}x^2 < 0 \qquad \text{for} \quad x^2 > \frac{\nu}{\alpha}$$

and

$$\frac{d}{dt}x^2 > 0 \qquad \text{for} \quad 0 < x^2 < \frac{\nu}{\beta}.$$

Hence the sets

$$A^- = \left[-\sqrt{\frac{\nu}{\alpha}}, -\sqrt{\frac{\nu}{\beta}}\right], \qquad A^+ = \left[\sqrt{\frac{\nu}{\beta}}, \sqrt{\frac{\nu}{\alpha}}\right]$$

are each positively invariant with respect to the process and so each contains an invariant family $\mathcal{A}^{\pm}_{\infty} = \{A^{\pm}_{\infty}(t) : t \in \mathbb{R}\}$. In particular, each interval contains at least one entire solution of the process. One can also conclude that the families $\mathcal{A}^{\pm}_{\infty}$ are local pullback (nonautonomous) attractors for the process, see [9, 16] for definitions and details. These are candidates for the counterparts of the bifurcating steady state solutions in the autonomous case and provide an indication that some kind of nonautonomous bifurcation has occurred.

In fact, the differential equation (5) is a Bernoulli equation and can be solved explicitly. Following the analysis in [9] one can show that each subintervals contains just one entire solution which is locally asymptotically stable. These entire solutions are given explicitly by

$$\xi^{\pm}_{\nu}(t) = \pm\frac{1}{\sqrt{2\int_{-\infty}^{t} b(s)e^{-2\nu(t-s)}\,ds}}$$

i.e., the sets in $\mathcal{A}^{\pm}_{\infty}$ are singleton sets $A^{\pm}_{\infty}(t) = \{\xi^{\pm}_{\nu}(t)\}$ for all $t \in \mathbb{R}$.

This detailed structure cannot, however, be obtained from the above analysis using invariant subsets. Nevertheless, such an analysis gives useful information when no alternative finer analysis is possible, for example for the following modification of the differential equation (5). Let $g : \mathbb{R}^2 \to [1, 2]$ be continuously differentiable and consider the nonautonomous differential equation

$$\frac{dx}{dt} = \nu x - b(t)x^3 + \varepsilon x g(t, x),$$

where $\varepsilon > 0$ is very small. Proceeding as above, we obtain

$$2(\nu + \varepsilon)x^2 - 2\beta x^4 \leq \frac{d}{dt}x^2 \leq 2(\nu + 2\varepsilon)x^2 - 2\alpha x^4$$

and the positively invariant intervals

$$A^- = \left[-\sqrt{\frac{\nu + 2\varepsilon}{\alpha}}, -\sqrt{\frac{\nu + \varepsilon}{\beta}} \right], \qquad A^+ = \left[\sqrt{\frac{\nu + \varepsilon}{\beta}}, \sqrt{\frac{\nu + 2\varepsilon}{\alpha}} \right],$$

each of which contains an invariant family $\mathcal{A}^{\pm}_{\infty} = \{A^{\pm}_{\infty}(t) : t \in \mathbb{R}\}$ of nonempty compact subsets which is a pullback attractor. The additional term destabilizes the zero solution, which loses stability now for some $\nu \in (-\varepsilon, 0)$.

A more complicated situation occurs for the nonautonomous system (where $\varepsilon > 0$ is assumed again to be very small, particularly such that $\alpha\varepsilon^2 < 1$)

$$\frac{dx}{dt} = \nu x - b(t)x^3 + \varepsilon,$$

which has no trivial solution.

For an initial condition $x_0 \geq \varepsilon$, one can check that the sign of x' for the corresponding solution through x_0 is negative, provided $\nu < -1 + \alpha\varepsilon^2$ (< 0), since $x' < \varepsilon(\nu - \alpha\varepsilon^2 + 1)$. It follows from the above and the positive sign of x' at any $x \leq 0$ that the interval $[0, \varepsilon]$ is then absorbing and positively invariant and thus contains a nontrivial entire solution.

On the other hand, by examining the signs of x' at $x = \pm\, \varepsilon$, one sees that the interval $[-\varepsilon, \varepsilon]$ is negatively invariant for $\nu > 1 + \beta\varepsilon^2$ and thus contains a nontrivial entire solution. One can also show that there exist positively invariant absorbing sets on both the positive and negative sides of this interval. These positively invariant absorbing sets also contain entire solutions different from that contained in the interval $[-\varepsilon, \varepsilon]$ near the origin.

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