H^2-boundedness of the pullback attractors for non-autonomous 2D-Navier-Stokes equations in bounded domains

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Abstract

We prove some regularity results for the pullback attractors of a non-autonomous 2D-Navier-Stokes model in a bounded domain Ω of $\mathbb{R}^2$. We establish a general result about $(H^2(\Omega))^2 \cap V$-boundedness of invariant sets for the associate evolution process. Then, as a consequence, we deduce that, under adequate assumptions, the pullback attractors of the non-autonomous 2D-Navier-Stokes equations are bounded not only in V but also in $(H^2(\Omega))^2$.

Key words: 2D-Navier-Stokes equations, pullback attractors, invariant sets, H^2-regularity.
Mathematics Subject Classifications (2010): 35B41, 35B65, 35Q30

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This work has been partially supported by Ministerio de Ciencia e Innovación (Spain) under project MTM2008-00088, and Junta de Andalucía grant P07-FQM-02468. J.G.-L. is a fellow of Programa de FPU del Ministerio de Educación.
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1 Introduction and setting of the problem

Let us consider the following problem for a non-autonomous 2D-Navier-Stokes system:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu \triangle u + (u \cdot \nabla) u + \nabla p = f(t) \ \text{ in } \ \Omega \times (\tau, +\infty)\,, \\ \nabla \cdot u = 0 \ \text{ in } \ \Omega \times (\tau, +\infty)\,, \\ u = 0 \ \text{ on } \ \partial\Omega \times (\tau, +\infty)\,, \\ u(x,\tau) = u_\tau(x), \quad x \in \Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded open set, with regular boundary $\partial\Omega$, the number $\nu > 0$ is the kinematic viscosity, u is the velocity field of the fluid, p the pressure, $\tau \in \mathbb{R}$ is a given initial time, u_τ is the initial velocity field, and $f(t)$ a given external force field.

To set our problem in the abstract framework, we consider the following usual abstract spaces (see [1] and [2–4]):

$$\mathcal{V} = \left\{ u \in (C_0^\infty(\Omega))^2 : \operatorname{div} u \ = 0 \right\},$$

$H =$ the closure of $\mathcal{V}$ in $(L^2(\Omega))^2$ with inner product $(\cdot,\cdot)$ and associate norm $|\cdot|$, where for $u, v \in (L^2(\Omega))^2$,

$$(u,v) = \sum_{j=1}^{2} \int_\Omega u_j(x) v_j(x) \mathrm{d}x,$$

$V =$ the closure of $\mathcal{V}$ in $(H_0^1(\Omega))^2$ with scalar product $((\cdot,\cdot))$ and associate norm $\|\cdot\|$, where for $u, v \in (H_0^1(\Omega))^2$,

$$((u,v)) = \sum_{i,j=1}^{2} \int_\Omega \frac{\partial u_j}{\partial x_i} \frac{\partial v_j}{\partial x_i} \mathrm{d}x.$$

We also consider the operator $A : V \to V'$ defined by $\langle Au, v \rangle = ((u,v))$. Denoting $D(A) = (H^2(\Omega))^2 \cap V$, then $Au = -P\Delta u, \forall\, u \in D(A)$, is the Stokes operator (P is the ortho-projector from $(L^2(\Omega))^2$ onto H).

Now we define the continuous trilinear form b on $V \times V \times V$ by

$$b(u,v,w) = \sum_{i,j=1}^{2} \int_\Omega u_i \frac{\partial v_j}{\partial x_i} w_j \,\mathrm{d}x, \quad \forall\, u, v, w \in V.$$

It is well known that

$$b(u,v,v)=0 \qquad \text{for all } u,v\in V. \tag{2}$$

We remember (see [2] or [3]) that there exists a constant $C_1>0$ only dependent on Ω such that

$$|b(u,v,w)|\le C_1|u|^{1/2}\|u\|^{1/2}\|v\||w|^{1/2}\|w\|^{1/2}, \quad \forall\, u,v,w\in V, \tag{3}$$

$$|b(u,v,w)|\le C_1|Au|\|v\||w|, \quad \forall u\in D(A),\ v\in V,\ w\in H, \tag{4}$$

and

$$|b(u,v,w)|\le C_1|u|^{1/2}|Au|^{1/2}\|v\||w|, \quad \forall u\in D(A),\ v\in V,\ w\in H. \tag{5}$$

Assume that $u_\tau\in H$ and $f\in L^2_{loc}(\mathbb{R};H)$.

Definition 1.1 *A solution of (1) is a function* $u\in C([\tau,T];H)\cap L^2(\tau,T;V)$ *for all* $T>\tau$*, with* $u(\tau)=u_\tau$*, such that for all* $v\in V$*,*

$$\frac{d}{dt}(u(t),v)+\nu((u(t),v))+b(u(t),u(t),v)=(f(t),v),$$

where the equation must be understood in the sense of $\mathcal{D}'(\tau,+\infty)$*.*

Under the conditions above (e.g. cf. [2] or [3]), there exists a unique solution $u(\cdot)=u(\cdot;\tau,u_\tau)$ of (1). Moreover, this solution u satisfies that $u\in C([\tau+\varepsilon,T];V)\cap L^2(\tau+\varepsilon,T;(H^2(\Omega))^2)$ for every $\varepsilon>0$ and $T>\tau+\varepsilon$. In fact, if $u_\tau\in V$, then $u\in C([\tau,T];V)\cap L^2(\tau,T;(H^2(\Omega))^2)$ for every $T>\tau$.

Therefore, we can define a process $U=\{U(t,\tau),\ \tau\le t\}$ in H as

$$U(t,\tau)u_\tau=u(t;\tau,u_\tau) \quad \forall u_\tau\in H, \quad \forall\tau\le t, \tag{6}$$

and the restriction of this process to V is a process in V.

A pullback attractor for the process U defined by (6) (cf. [5–7]) is a family $\hat{\mathcal{A}}=\{\mathcal{A}(t):\ t\in\mathbb{R}\}$ of compact subsets of H such that

a) (invariance) $U(t,\tau)\mathcal{A}(\tau)=\mathcal{A}(t)$ for all $\tau\le t$,
b) (pullback attraction) $\lim\limits_{\tau\to-\infty}\sup\limits_{u_\tau\in B}\inf\limits_{v\in\mathcal{A}(t)}|U(t,\tau)u_\tau-v|=0$, for all $t\in\mathbb{R}$, for any bounded subset $B\subset H$.

It can be proved (see [9]) that, under the above conditions, if in addition f satisfies

$$\int_{-\infty}^{0} e^{\mu r} |f(r)|^2 \, dr < +\infty,$$

for some $0 < \mu < 2\nu\lambda_1$, where λ_1 denotes the first eigenvalue of the Stokes operator A, then there exists a pullback attractor for the process U defined by (6).

Several studies on this model have already been published (cf. [5], [8,9]). However, as far as we know, no one refers to the H^2-regularity we will consider in this paper.

In the next section we prove some results which, in particular, imply that, under suitable assumptions, any pullback attractor $\hat{\mathcal{A}}$ for U satisfies that $\mathcal{A}(t)$ is a bounded subset of $(H^2(\Omega))^2 \cap V$, for every $t \in \mathbb{R}$ (for similar results for reaction-diffusion equations see [10], and for related results for Navier-Stokes equations see [11]).

2 H^2-boundedness of invariant sets

In this section we prove that, under suitable assumptions, any family of bounded subsets of H which is invariant for the process U, is in fact bounded in $(H^2(\Omega))^2 \cap V$.

First, we recall a result (cf. [2]) which will be used below.

Lemma 2.1 *Let X, Y be Banach spaces such that X is reflexive, and the inclusion $X \subset Y$ is continuous. Assume that $\{u_n\}$ is a bounded sequence in $L^\infty(t_0, T; X)$ such that $u_n \rightharpoonup u$ weakly in $L^q(t_0, T; X)$ for some $q \in [1, +\infty)$ and $u \in C^0([t_0, T]; Y)$.*

Then, $u(t) \in X$ and $\|u(t)\|_X \leq \liminf_{n \geq 1} \|u_n\|_{L^\infty(t_0,T;X)}$, for all $t \in [t_0, T]$.

For each integer $n \geq 1$, we denote by $u_n(t) = u_n(t; \tau, u_\tau)$ the Galerkin approximation of the solution $u(t; \tau, u_\tau)$ of (1), which is given by

$$u_n(t) = \sum_{j=1}^{n} \gamma_{nj}(t) w_j,$$

and is the solution of

$$\begin{cases} \dfrac{d}{dt}(u_n(t), w_j) + \nu((u_n(t), w_j)) + b(u_n(t), u_n(t), w_j) = (f(t), w_j), \\ (u_n(\tau), w_j) = (u_\tau, w_j) \qquad j = 1, ..., n, \end{cases} \tag{7}$$

where $\{w_j : j \geq 1\} \subset V$ is the Hilbert basis of H formed by the eigenvectors of the Stokes operator A. Observe that by the regularity of Ω, all the w_j belong to $(H^2(\Omega))^2$.

We first prove the following result.

Proposition 2.2 *Assume that $f \in L^2_{loc}(\mathbb{R}; H)$. Then, for any bounded set $B \subset H$, any $\tau \in \mathbb{R}$, any $\varepsilon > 0$ and any $t > \tau + \varepsilon$, the following three properties are satisfied:*

i) The set $\{u_n(r; \tau, u_\tau) : r \in [\tau + \varepsilon, t], u_\tau \in B, n \geq 1\}$, is a bounded subset of V.
ii) The set of functions $\{u_n(\cdot; \tau, u_\tau) : u_\tau \in B, n \geq 1\}$, is a bounded subset of $L^2(\tau + \varepsilon, t; D(A))$.
iii) The set of time derivatives functions $\{u'_n(\cdot; \tau, u_\tau) : u_\tau \in B, n \geq 1\}$, is a bounded subset of $L^2(\tau + \varepsilon, t; H)$.

Proof.

Let us fix a bounded set $B \subset H$, $\tau \in \mathbb{R}$, $\varepsilon > 0$, $t > \tau + \varepsilon$, and $u_\tau \in B$.

Multiplying by $\gamma_{nj}(t)$ in (7), summing from $j = 1$ to n, and using (2), we obtain

$$\frac{1}{2}\frac{d}{d\theta} |u_n(\theta)|^2 + \nu \|u_n(\theta)\|^2 = (f(\theta), u_n(\theta)), \quad \text{a.e. } \theta > \tau. \tag{8}$$

Observing that

$$\begin{aligned} |(f(\theta), u_n(\theta))| &\leq \frac{1}{2\nu\lambda_1} |f(\theta)|^2 + \frac{\nu\lambda_1}{2} |u_n(\theta)|^2 \\ &\leq \frac{1}{2\nu\lambda_1} |f(\theta)|^2 + \frac{\nu}{2} \|u_n(\theta)\|^2, \end{aligned}$$

from (8) we deduce

$$\frac{d}{d\theta} |u_n(\theta)|^2 + \nu \|u_n(\theta)\|^2 \leq \frac{1}{\nu\lambda_1} |f(\theta)|^2,$$

and integrating between τ and r,

$$\begin{aligned} &|u_n(r)|^2 + \nu \int_\tau^r \|u_n(\theta)\|^2 \, d\theta \\ &\qquad \leq |u_\tau|^2 + \frac{1}{\nu\lambda_1} \int_\tau^t |f(\theta)|^2 \, d\theta, \ \ \forall r \in [\tau, t], \ \ \forall n \geq 1. \end{aligned} \tag{9}$$

Now, multiplying in (7) by $\lambda_j \gamma_{nj}(t)$, where λ_j is the eigenvalue associated to the eigenvector w_j, and summing from $j = 1$ to n, we obtain

$$\frac{1}{2}\frac{d}{d\theta}\left\|u_n(\theta)\right\|^2 + \nu\left|Au_n(\theta)\right|^2 + b(u_n(\theta), u_n(\theta), Au_n(\theta)) = (f(\theta), Au_n(\theta))\,, \quad (10)$$

a.e. $\theta > \tau$. Observe that

$$|\,(f(\theta), Au_n(\theta))\,| \leq \frac{1}{\nu}\left|f(\theta)\right|^2 + \frac{\nu}{4}\left|Au_n(\theta)\right|^2,$$

and by (5) and Young's inequality,

$$\begin{aligned} |b(u_n(\theta), u_n(\theta), Au_n(\theta))| &\leq C_1|u_n(\theta)|^{1/2}\|u_n(\theta)\||Au_n(\theta)|^{3/2} \\ &\leq \frac{\nu}{4}\left|Au_n(\theta)\right|^2 + C^{(\nu)}|u_n(\theta)|^2\|u_n(\theta)\|^4, \end{aligned} \quad (11)$$

where $C^{(\nu)} = 27C_1^4(4\nu^3)^{-1}$.

Thus, from (10) we deduce

$$\frac{d}{d\theta}\left\|u_n(\theta)\right\|^2 + \nu\left|Au_n(\theta)\right|^2 \leq \frac{2}{\nu}\left|f(\theta)\right|^2 + 2C^{(\nu)}|u_n(\theta)|^2\|u_n(\theta)\|^4, \quad (12)$$

a.e. $\theta > \tau$.

From this inequality, in particular we deduce

$$\begin{aligned} \|u_n(r)\|^2 \leq \|u_n(s)\|^2 &+ \frac{2}{\nu}\int_\tau^t \left|f(\theta)\right|^2\, d\theta \\ &+ 2C^{(\nu)}\int_s^r |u_n(\theta)|^2\|u_n(\theta)\|^4\, d\theta \end{aligned}$$

for all $\tau \leq s \leq r \leq t$, and therefore, by Gronwall's lemma,

$$\|u_n(r)\|^2 \leq \left(\|u_n(s)\|^2 + \frac{2}{\nu}\int_\tau^t \left|f(\theta)\right|^2\, d\theta\right)\exp\left(2C^{(\nu)}\int_\tau^t |u_n(\theta)|^2\|u_n(\theta)\|^2\, d\theta\right)$$

for all $\tau \leq s \leq r \leq t$.

Integrating this last inequality for s between $\tau + \varepsilon/2$ and r, we obtain

$$\begin{aligned} (r - \tau - \frac{\varepsilon}{2})\,\|u_n(r)\|^2 \leq &\left(\int_\tau^t \|u_n(s)\|^2\, ds + \frac{2(t-\tau)}{\nu}\int_\tau^t \left|f(\theta)\right|^2\, d\theta\right) \\ &\times \exp\left(2C^{(\nu)}\int_\tau^t |u_n(\theta)|^2\|u_n(\theta)\|^2\, d\theta\right) \end{aligned}$$

for all $\tau + \varepsilon/2 \leq r \leq t$, and in particular,

$$\|u_n(r)\|^2 \leq \frac{2}{\varepsilon}\left(\int_\tau^t \|u_n(s)\|^2\,ds + \frac{2(t-\tau)}{\nu}\int_\tau^t |f(\theta)|^2\ d\theta\right) \tag{13}$$
$$\times \exp\left(2C^{(\nu)}\int_\tau^t |u_n(\theta)|^2\|u_n(\theta)\|^2\,d\theta\right)$$

for all $\tau+\varepsilon \leq r \leq t$, for any $n \geq 1$.

From (9) and (13), the assertion in i) holds. Moreover, by (12),

$$\nu\int_{\tau+\varepsilon}^t |Au_n(\theta)|^2\ d\theta \leq \|u_n(\tau+\varepsilon)\|^2 + \frac{2}{\nu}\int_\tau^t |f(\theta)|^2\ d\theta$$
$$+2C^{(\nu)}\int_{\tau+\varepsilon}^t |u_n(\theta)|^2\|u_n(\theta)\|^4\,d\theta,$$

and therefore, by i), the assertion in ii) holds.

On the other hand, multiplying by the derivative $\gamma'_{nj}(t)$ in (7), and summing from $j=1$ till n, we obtain

$$|u'_n(\theta)|^2 + \frac{\nu}{2}\frac{d}{d\theta}\|u_n(\theta)\|^2 + b(u_n(\theta),u_n(\theta),u'_n(\theta)) = (f(\theta),u'_n(\theta))\,, \tag{14}$$

a.e. $\theta > \tau$.

Observing that

$$|\,(f(\theta),u'_n(\theta))\,| \leq \frac{1}{4}|u'_n(\theta)|^2 + |f(\theta)|^2,$$

and by (4)

$$|b(u_n(\theta),u_n(\theta),u'_n(\theta))| \leq C_1|Au_n(\theta)|\|u_n(\theta)\||u'_n(\theta)|$$
$$\leq \frac{1}{4}|u'_n(\theta)|^2 + C_1^2|Au_n(\theta)|^2\|u_n(\theta)\|^2,$$

we obtain from (14)

$$|u'_n(\theta)|^2 + \nu\frac{d}{d\theta}\|u_n(\theta)\|^2 \leq 2|f(\theta)|^2 + 2C_1^2|Au_n(\theta)|^2\|u_n(\theta)\|^2.$$

Integrating this last inequality, we deduce that

$$\int_{\tau+\varepsilon}^t |u'_n(\theta)|^2\,d\theta \leq \nu\|u_n(\tau+\varepsilon)\|^2 + 2\int_\tau^t |f(\theta)|^2\,d\theta$$
$$+2C_1^2\sup_{\theta\in[\tau+\varepsilon,t]}\|u_n(\theta)\|^2\int_{\tau+\varepsilon}^t |Au_n(\theta)|^2\,d\theta,$$

and therefore iii) follows from i) and ii).

■

Corollary 2.3 *Assume that* $f \in L^2_{loc}(\mathbb{R}; H)$. *Then, for any bounded set* $B \subset H$, *any* $\tau \in \mathbb{R}$, *any* $\varepsilon > 0$, *and any* $t > \tau + \varepsilon$, *the set* $\bigcup_{r\in[\tau+\varepsilon,t]} U(r,\tau)B$ *is a bounded subset of* V.

Proof. This is a straightforward consequence of Lemma 2.1, assertion i) in Proposition 2.2, and the well known fact (e.g. cf. [1–4]) that for all $u_\tau \in B$ the Galerkin approximations $u_n(\cdot;\tau,u_\tau)$ converge weakly to $u(\cdot;\tau,u_\tau)$ in $L^2(\tau,t;V)$, and $u(\cdot;\tau,u_\tau) \in C([\tau,t];H)$. ■

Assuming additional regularity for the time derivative of f, we can improve the above results.

Proposition 2.4 *Assume that* $f \in W^{1,2}_{loc}(\mathbb{R}; H)$. *Then, for any bounded set* $B \subset H$, *any* $\tau \in \mathbb{R}$, *any* $\varepsilon > 0$, *and any* $t > \tau+\varepsilon$, *the following two properties are satisfied:*

iv) The set of time derivatives $\{u_n'(r;\tau,u_\tau) : \ r \in [\tau+\varepsilon,t], \ u_\tau \in B, \ n \geq 1\}$, *is a bounded subset of* H.

v) The set $\{u_n(r;\tau,u_\tau) : \ r \in [\tau+\varepsilon,t], \ u_\tau \in B, \ n \geq 1\}$ *is a bounded subset of* $D(A)$.

Proof. Let us fix a bounded set $B \subset H$, $\tau \in \mathbb{R}$, $\varepsilon > 0$, $t > \tau+\varepsilon$, and $u_\tau \in B$.

As we are assuming that $f \in W^{1,2}_{loc}(\mathbb{R}; H)$, we can differentiate with respect to time in (7), and then, multiplying by $\gamma_{nj}'(t)$, and summing from $j = 1$ to n, we obtain

$$\frac{1}{2}\frac{d}{d\theta}\left|u_n'(\theta)\right|^2 + \nu\left\|u_n'(\theta)\right\|^2 + b(u_n'(\theta), u_n(\theta), u_n'(\theta)) = (f'(\theta), u_n'(\theta))$$

a.e. $\theta > \tau$.

From this inequality, taking into account that

$$|\,(f'(\theta), u_n'(\theta))\,| \leq \frac{\nu}{2}\left\|u_n'(\theta)\right\|^2 + \frac{1}{2\nu\lambda_1}|f'(\theta)|^2,$$

and by (3)

$$\begin{aligned}|b(u_n'(\theta), u_n(\theta), u_n'(\theta))| &\leq C_1|u_n'(\theta)|\|u_n'(\theta)\|\|u_n(\theta)\| \\ &\leq \frac{\nu}{2}\left\|u_n'(\theta)\right\|^2 + \frac{C_1^2}{2\nu}|u_n'(\theta)|^2\|u_n(\theta)\|^2,\end{aligned}$$

we deduce

$$\frac{d}{d\theta}\left|u_n'(\theta)\right|^2 \leq \frac{1}{\nu\lambda_1}|f'(\theta)|^2 + \frac{C_1^2}{\nu}|u_n'(\theta)|^2\|u_n(\theta)\|^2.$$

Integrating in the last inequality,

$$|u_n'(r)|^2 \le |u_n'(s)|^2 + \frac{1}{\nu\lambda_1}\int_\tau^t |f'(\theta)|^2\,d\theta + \frac{C_1^2}{\nu}\int_s^r |u_n'(\theta)|^2\|u_n(\theta)\|^2\,d\theta,$$

for all $\tau \le s \le r \le t$.

Thus, by Gronwall's inequality,

$$|u_n'(r)|^2 \le \left(|u_n'(s)|^2 + \frac{1}{\nu\lambda_1}\int_\tau^t |f'(\theta)|^2\,d\theta\right)\exp\left(\frac{C_1^2}{\nu}\int_{\tau+\varepsilon/2}^t \|u_n(\theta)\|^2\,d\theta\right),$$

for all $\tau+\varepsilon/2 \le s \le r \le t$.

Now, integrating this inequality with respect to s between $\tau+\varepsilon/2$ and r, we obtain

$$\begin{aligned}(r-\tau-\varepsilon/2)\,|u_n'(r)|^2 \le &\left(\int_{\tau+\varepsilon/2}^t |u_n'(s)|^2\,ds + \frac{t-\tau}{\nu\lambda_1}\int_\tau^t |f'(\theta)|^2\,d\theta\right)\\ &\times\exp\left(\frac{C_1^2}{\nu}\int_{\tau+\varepsilon/2}^t \|u_n(\theta)\|^2\,d\theta\right),\end{aligned}$$

for all $\tau+\varepsilon/2 \le r \le t$, and any $n \ge 1$. In particular, thus,

$$\begin{aligned}|u_n'(r)|^2 \le \frac{2}{\varepsilon}&\left(\int_{\tau+\varepsilon/2}^t |u_n'(s)|^2\,ds + \frac{t-\tau}{\nu\lambda_1}\int_\tau^t |f'(\theta)|^2\,d\theta\right)\\ &\times\exp\left(\frac{C_1^2}{\nu}\int_{\tau+\varepsilon/2}^t \|u_n(\theta)\|^2\,d\theta\right),\end{aligned}$$

for all $\tau+\varepsilon \le r \le t$, and any $n \ge 1$.

From this inequality and properties i) and iii) in Proposition 2.2, we obtain iv).

On the other hand, multiplying again in (7) by $\lambda_j\gamma_{nj}(t)$, and summing once more from $j=1$ to n, we obtain

$$(u_n'(r), Au_n(r)) + \nu\,|Au_n(r)|^2 + b(u_n(r), u_n(r), Au_n(r)) = (f(r), Au_n(r))\,, \quad (15)$$

for all $r \ge \tau$. But

$$|\,(u_n'(r), Au_n(r))\,| \le \frac{2}{\nu}\,|u_n'(r)|^2 + \frac{\nu}{8}\,|Au_n(r)|^2\,,$$

and

$$|\,(f(r),Au_n(r))\,| \le \frac{2}{\nu}\,|f(r)|^2 + \frac{\nu}{8}\,|Au_n(r)|^2\,.$$

Therefore, taking into account (11), we deduce from (15) that

$$\frac{\nu}{2}\,|Au_n(r)|^2 \le \frac{2}{\nu}(|u_n'(r)|^2 + |f(r)|^2) + C^{(\nu)}|u_n(r)|^2\|u_n(r)\|^4 \tag{16}$$

for all $r \ge \tau$.

Thus, since in particular $f \in C(\mathbb{R};H)$, from i) in Proposition 2.2, iv) and inequality (16), we deduce v). ■

As a direct consequence of the above, we can now establish our main results.

Theorem 2.5 *Assume that $f \in W^{1,2}_{loc}(\mathbb{R};H)$. Then, for any bounded set $B \subset H$, any $\tau \in \mathbb{R}$, any $\varepsilon > 0$, and any $t > \tau+\varepsilon$, the set $\bigcup_{r\in[\tau+\varepsilon,t]} U(r,\tau)B$ is a bounded subset of $D(A) = (H^2(\Omega))^2 \cap V$.*

Proof. This follows from Lemma 2.1, Proposition 2.4, and the well known facts that $u_n(\cdot;\tau,u_\tau)$ converges weakly to $u(\cdot;\tau,u_\tau)$ in $L^2(\tau,t;V)$, and $u(\cdot;\tau,u_\tau)$ belongs to $C([\tau+\varepsilon,t];V)$. ■

Theorem 2.6 *Assume that $f \in L^2_{loc}(\mathbb{R};H)$, and $\hat{\mathcal{A}} = \{\mathcal{A}(t) :\ t \in \mathbb{R}\}$ is a family of bounded subsets of H, such that $U(t,\tau)\mathcal{A}(\tau) = \mathcal{A}(t)$ for any $\tau \le t$. Then:*

i) For any $T_1 < T_2$, the set $\bigcup_{t\in[T_1,T_2]} \mathcal{A}(t)$ is a bounded subset of V.

ii) If moreover $f' \in L^2_{loc}(\mathbb{R};H)$, then for any $T_1 < T_2$, the set $\bigcup_{t\in[T_1,T_2]} \mathcal{A}(t)$ is a bounded subset of $(H^2(\Omega))^2 \cap V$.

Proof. It is enough to observe that if $\tau < T_1 - 1$ is fixed, then

$$\bigcup_{t\in[T_1,T_2]} \mathcal{A}(t) \subset \bigcup_{t\in[\tau+1,T_2]} U(t,\tau)\mathcal{A}(\tau).$$

Now, apply Corollary 2.3 and Theorem 2.5. ■

Acknowledgments

The authors would like to thank the referees for their careful reading on a previous version of the paper.

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