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# EQUI-ATTRACTION AND THE CONTINUOUS DEPENDENCE OF ATTRACTORS ON TIME DELAYS

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Abstract. Under appropriate regularity conditions it is shown that the continuous dependence of the global attractors $\mathcal{A}_\tau$ of semi dynamical systems $S^{(\tau)}(t)$ in $C([-\tau, 0]; Z)$ with $Z$ a Banach space and time delay $\tau \in [T_*, T^*]$, where $T_* > 0$, is equivalent to the equi-attraction of the attractors. Examples and counter examples posed in this right framework are provided.

## 1. Introduction.

The upper semi continuous dependence of attractors on a parameter is a standard result in dynamical systems theory, see e.g. [5, 11, 13, 15, 16]. In general, lower semi continuous, and hence continuous, dependence does not hold without additional assumptions, which usually are given in terms of the structure of the attractor, such as its being Morse-Smale. In another approach, Li and Kloeden [14] showed recently that continuous dependence in a parameter is equivalent to the equi-attraction of the parametrized attractors. These results also apply to attractors of delay differential equations (DDE) with a fixed time delay.

On the other hand very little has appeared in the literature about the dependence of attractors of DDE on the time delay itself, a difficulty being that the attractors belong to different state spaces. An early paper on upper semi continuity for a concrete retarded nonlinear PDE is [1] (see also [2, 4] for the same question about inertial manifolds to deterministic and stochastic problems).

Kloeden [11] showed how the upper semi continuous dependence of attractors in the time delay can be formulated by embedding the different semi dynamical systems and their attractors in a common state space. See also [3, 9] for other results.

Our aim in this paper is to find an analogue of the equivalence of continuous dependence and equi-attraction in [14] (see also [12]) for the dependence of attractors

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of semi dynamical systems (SDS) generated DDE on the time delay. For this we first summarize the main ingredients from [14].

Let $\lambda$ be a parameter in a compact metric space $(\Lambda, D_\Lambda)$ and let $\{S_t^{(\lambda)},\ t \in \mathbb{R}_+\}_{\lambda\in\Lambda}$ be a family of SDS on a complete metric space $(X,d)$. Define $d(x,A) = \inf_{y\in A} d(x,y)$ for any $x \in X$ and $A \subset X$, and let $B_X(a,r)$ denote the open ball of $X$ with center $a$ and radius $r$; and $P(X)$ and $C(X)$ the classes of all nonempty and nonempty and closed subsets of $X$, respectively. In addition, denote the Hausdorff semidistance and Hausdorff distance on $X$, respectively, by

$$H_X^*(A,B) = \sup_{x\in A} d(x,B), \qquad H_X(A,B) = \max\{H_X^*(A,B), H_X^*(B,A)\}$$

for any closed nonempty subsets $A$ and $B$ of $X$.

**Definition 1.** *A nonempty compact subset $\mathcal{A}$ of $X$ is called a global attractor of an SDS $\{S_t,\ t \in \mathbb{R}_+\}$ on $X$ (i.e. a semi-group of mappings with $S_t : X \to X$ continuous for each fixed $t \geq 0$) if it is invariant, i.e. $S_t(\mathcal{A}) = \mathcal{A}$ for all $t \in \mathbb{R}_+$, and attracts bounded subsets $B$ of $X$, i.e.*

$$H^*(S_t(B), \mathcal{A}) \to 0 \quad as \quad t \to +\infty.$$

**Definition 2.** *Let $\{S_t^{(\lambda)},\ \lambda \in \Lambda\}$ be a family of SDS on X. It is said to be*

(i) **equi-dissipative** *on $X$ if there exists a bounded subset $\mathcal{U}$ of $X$ so that for any bounded subset $B \subset X$, there exists a $T_B \in \mathbb{R}_+$ independent of $\lambda \in \Lambda$ such that*

$$S_t^{(\lambda)}(B) \subset \mathcal{U}, \qquad t \geq T_B;$$

(ii) **eventually equi-compact** *(or uniformly compact for large t in* [14]*) if for any bounded subset $B$ of $X$, there exists a $T_B \in \mathbb{R}_+$ independent of $\lambda \in \Lambda$ such that $\bigcup_{\lambda\in\Lambda} S_t^{(\lambda)}(B)$ is relatively compact in $X$ for any $t \geq T_B$.*

**Theorem 3.** [14, Th.2.9] *Suppose that a family of SDS $\{S_t^{(\lambda)}, \lambda \in \Lambda\}$ on $X$ is equi-dissipative and eventually equi-compact and that $\mathcal{A}_\lambda$ is the global attractor of $S_t^{(\lambda)}$ for $\lambda \in \Lambda$. In addition, suppose that*

(A1) *for any $t \in \mathbb{R}_+$ fixed, $S_t^{(\lambda)}(x)$ is jointly continuous in $(x,\lambda)$ on $X \times \Lambda$.*

(A2) *$S_t^{(\lambda)}(x)$ is equi-continuous in $\lambda$ for $(t,x)$ in any bounded subset of $\mathbb{R}_+ \times X$.*
*Then $\{\mathcal{A}_\lambda\}$ is equi-attracting* **if and only if** *$\mathcal{A}_\lambda$ is continuous in $\lambda$ with respect to the Hausdorff distance.*

**Remark 4.** *The above equivalence also holds if* (A2) *is replaced by:*

(A2') *$S_t^{(\lambda)}(x)$ is equi-continuous in $\lambda$ for $t$ in any bounded subset of $\mathbb{R}_+$ and $x$ in any bounded subset of $\bigcup_{\lambda\in\Lambda} \mathcal{A}_\lambda$.*

**Theorem 5.** [14, Th.2.7] *Suppose that $\{S_t^{(\lambda)}, \lambda \in \Lambda\}$ is equi-dissipative and eventually equi-compact and that the assumptions* (A1) *and*

(A3) *For any bounded subset $B$ of $X$ and $T > 0$, $S_t^{(\lambda)}x$ is uniformly continuous in $x \in B$ uniformly w.r.t. $\lambda \in \Lambda$ and $t \leq T$, i.e.*

$$\forall\varepsilon > 0,\ \exists\delta > 0 : x, y \in B,\ d(x,y) < \delta \Rightarrow d\left(S_t^{(\lambda)}(x), S_t^{(\lambda)}(y)\right) < \varepsilon,\ \forall t \in [0,T], \lambda \in \Lambda.$$

*hold.*

*Then, if $\mathcal{A}_\lambda$ is continuous in $\lambda$, the family $\{\mathcal{A}_\lambda\}$ is uniformly Lyapunov stable, i.e. for any $\varepsilon > 0$, there exists $\delta > 0$ (independent of $\lambda$) such that for all $\lambda \in \Lambda$, if $d(x, \mathcal{A}_\lambda) < \delta$, then $d(S_t^{(\lambda)}x, \mathcal{A}_\lambda) < \varepsilon$ for all $t \in \mathbb{R}_+$.*

In Section 2 we reinterpret the above "equi" concepts for SDS generated by DDE with finite delay, where the finite delay is considered as the parameter and extend the SDS and their attractors (which are assumed to exist, see [11] for existence results) to a common state space. In Section 3 we interpret the above "equi" concepts for the SDS in their original state spaces. Finally, in Section 4 an example is given of a scalar DDE with attractors which are continuous and discontinuous in the time delay at different time delays.

2. **Extension to a common state space.** A delay differential equation in a Banach space $(Z, |\cdot|)$ with time delay $\tau > 0$ generates an SDS in the function space $C_\tau := C([-\tau, 0]; Z)$ of continuous functions $\phi : [-\tau, 0] \to Z$, which is a Banach space with the supremum norm $\|\cdot\|_\tau$. We denote this SDS in $C_\tau$ by $S^{(\tau)}$ and consider a family of such SDS for different, fixed values of the time delay $\tau \in [T_*, T^*]$, with $0 < T_* < T^* < \infty$. In addition, we assume that each SDS $S^{(\tau)}$ possesses a global attractor $\mathcal{A}_\tau$ in its state space $C_\tau$. Theorem 3 cannot be applied directly to this family, but can be after we represent then as SDS on the common state space $C_{T^*}$.

In order to translate the different SDS to the common space we project a solution $S_t^{(\tau)}\phi$ in function space $C([-\tau, 0]; Z)$ onto the base space $Z$ and then reconstitute it as a time dependent function taking values in the function space $C([-T^*, 0]; Z)$.

Let $\phi \in C_{T^*}$ and let $\phi|_{[-\tau,0]}$ be its truncation in $C_\tau$. Hence $S_t^{(\tau)}\phi|_{[-\tau,0]}$ is well defined for all $t \geq 0$. Define its projection $x : [-T^*, \infty) \times C_{T^*} \to Z$ in $Z$ by

$$x(t, \phi) := \begin{cases} \phi(t) & t \in [-T^*, 0], \\ S_t^{(\tau)}(\phi|_{[-\tau,0]})(0) & t > 0, \end{cases}$$

where $S_t^{(\tau)}\phi|_{[-\tau,0]}(0)$ is the value that takes the function $S_t^{(\tau)}(\phi|_{[-\tau,0]})$ in $Z$ at time 0. Finally, define $\widehat{S}_t^{(\tau)}(\phi) \in C_{T^*}$ for each $t \geq 0$ by

$$\widehat{S}_t^{(\tau)}(\phi)(s) := x(t + s, \phi), \quad s \in [-T^*, 0].$$

**Theorem 6.** *If $S^{(\tau)}$ be an SDS on $C_\tau$, then $\{\widehat{S}_t^{(\tau)}, t \in \mathbb{R}_+\}$ defines an SDS on $C_{T^*}$. Moreover, if $S^{(\tau)} : \mathbb{R}_+ \times C_\tau \to C_\tau$ is jointly continuous in $(t, \phi) \in \mathbb{R}_+ \times C_\tau$, then $\widehat{S}^{(\tau)} : \mathbb{R}_+ \times C_{T^*} \to C_{T^*}$ is jointly continuous in $(t, \phi) \in \mathbb{R}_+ \times C_{T^*}$.*

*Proof.* The initial condition property of an SDS follows directly from the definition for $t = 0$, specifically

$$\widehat{S}_0^{(\tau)}(\phi)(s) = x(s, \phi) = \phi(s), \quad s \in [-T^*, 0],$$

so $\widehat{S}_0^{(\tau)}(\phi) = \phi$ for all $\phi \in C([-T^*, 0]; \mathbb{R}^d)$.

To check the semi-group property, that is,

$$\widehat{S}_{t_1+t_2}^{(\tau)}(\phi) = \widehat{S}_{t_1}^{(\tau)}\widehat{S}_{t_2}^{(\tau)}(\phi), \quad \text{for all } t_1, t_2 \geq 0, \text{ and } \phi \in C_{T^*},$$

we consider two cases:

Case 1: $t_1 + s > 0$. We use the semi-group property of the SDS $S^{(\tau)}$ several times:

$$\begin{aligned}\widehat{S}^{(\tau)}_{t_1+t_2}(\phi)(s) &= x(t_1+t_2+s,\phi)\\ &= S^{(\tau)}_{t_1+t_2+s}(\phi|_{[-\tau,0]})(0)\\ &= S^{(\tau)}_{t_1+s}(S^{(\tau)}_{t_2}(\phi|_{[-\tau,0]}))(0)\\ &= x(t_1+s,\widehat{S}^{(\tau)}_{t_2}(\phi)) = \widehat{S}^{(\tau)}_{t_1}(\widehat{S}^{(\tau)}_{t_2}(\phi))(s).\end{aligned}$$

Case 2: $t_1 + s \leq 0$ (since $s \in [-T^*, 0]$, this case only holds if $T^* > t_1$). By the definitions we have

$$\widehat{S}^{(\tau)}_{t_1+t_2}(\phi)(s) = x(t_1+t_2+s,\phi), \tag{1}$$

as well as

$$\begin{aligned}\widehat{S}^{(\tau)}_{t_1}(\widehat{S}^{(\tau)}_{t_2}(\phi))(s) &= x(t_1+s,\widehat{S}^{(\tau)}_{t_2}(\phi))\\ &= \widehat{S}^{(\tau)}_{t_2}(\phi)(t_1+s) = x(t_1+t_2+s,\phi). \end{aligned}\tag{2}$$

Comparing (2) with (1) we obtain the desired semi-group property.

The continuity of $\widehat{S}^{(\tau)}_t$ from $C_{T^*}$ into $C_{T^*}$ for each fixed $t \in \mathbb{R}_+$ and the second assertion of the theorem can be proved similarly, so we prove just the latter.

Suppose that $\phi^{(n)} \to \bar{\phi}$ in $C_{T^*}$ and $t_n \to t$ in $\mathbb{R}_+$. Then $\phi^{(n)}|_{[-\tau,0]} \to \bar{\phi}|_{[-\tau,0]}$ in $C_\tau$ and hence $S^{(\tau)}_{t_n}(\phi^{(n)}|_{[-\tau,0]}) \to S^{(\tau)}_t(\bar{\phi}|_{[-\tau,0]})$ in $C_\tau$ for each $t_n \to t$ in $\mathbb{R}_+$, which means that

$$\begin{aligned}x(t_n+s,\phi^{(n)}) &= S^{(\tau)}_{t_n}(\phi^{(n)}|_{[-\tau,0]})(s)\\ &\to S^{(\tau)}_{t_n}(\bar{\phi}|_{[-\tau,0]})(s) = x(t+s,\bar{\phi})\end{aligned}$$

for all $s \in [-\tau, 0]$ (not only punctually, but uniformly in $[-\tau, 0]$). Concatenating as many intervals as necessary, we obtain in a finite number of steps that

$$\widehat{S}^{(\tau)}_{t_n}(\phi^{(n)})(s) = x(t_n+s,\phi^{(n)}) \to x(t+s,\bar{\phi}) = \widehat{S}^{(\tau)}_t(\bar{\phi})(s)$$

for all $s \in [-T^*, 0]$, i.e.

$$\widehat{S}^{(\tau)}_{t_n}(\phi^{(n)}) \to \widehat{S}^{(\tau)}_t(\bar{\phi})$$

in $C_{T^*}$ as $t_n \to t$ in $\mathbb{R}_+$ and $\phi^{(n)} \to \bar{\phi}$ in $C_{T^*}$. Hence the mapping $(t, \phi) \mapsto \widehat{S}^{(\tau)}_t(\phi)$ is continuous.

This completes the proof that $\widehat{S}^{(\tau)}$ is an SDS on $C_{T^*}$. □

The next step in our goal is to extend the attractors to the common state space and to ensure that the extended objects are indeed attractors for the extended SDS. Observe that it is not enough to have

$$H^*_{C_\tau}\left(S^{(\tau)}_{t-j\tau}(B_\tau),\mathcal{A}_\tau\right) < \varepsilon \quad \text{for} \quad j = 0,\ldots,n^*-1, \tag{3}$$

where $n^*$ is the first integer with $n^*\tau \geq T^*$. This does not ensure that there exists a corresponding concatenated set in $C_{T^*}$ satisfying the corresponding inequality there. To show this we use the compactness of the attractors and the continuity of the SDS.

**Theorem 7.** *Suppose that an SDS* $S^{(\tau)} : \mathbb{R}_+ \times C_\tau \to C_\tau$ *has a global attractor* $\mathcal{A}_\tau$*. Then, the extended SDS* $\widehat{S}^{(\tau)}$ *in* $C_{T^*}$ *given in* Theorem 6 *possesses a global attractor* $\widehat{\mathcal{A}}_\tau$ *in* $C_{T^*}$*, which is characterized by*

$$\widehat{\mathcal{A}}_\tau := \Big\{ \psi \in C_{T^*} \quad : \quad \exists \text{ entire trajectory } \bar{\Phi}_t^{(\tau)} \text{ of } S^{(\tau)} \text{ in } \mathcal{A}_\tau \\ \text{with } \psi(s) = \bar{\phi}(s) \ \forall s \in [-T^*, 0] \Big\}, \tag{4}$$

*where* $\bar{\phi}(t)$ *is the projection in* $Z$ *of the entire solution* $\bar{\Phi}_t^{(\tau)}$ *defined by* $\bar{\phi}(t) := \bar{\Phi}_t^{(\tau)}(0)$ *for all* $t \in \mathbb{R}$.

*Proof.* ¿From the strict invariance of $\mathcal{A}_\tau$ it is known that for each $\phi \in \mathcal{A}_\tau$ there exists at least one entire solution $\bar{\Phi}_t^{(\tau)}$ of the SDS $S^{(\tau)}$ in $\mathcal{A}_\tau$ with $\bar{\Phi}_0^{(\tau)} = \phi$, so the set $\widehat{\mathcal{A}}_\tau$ is well defined. The invariance of $\widehat{\mathcal{A}}_\tau$ under the extended SDS $\widehat{S}^{(\tau)}$ follows immediately from the definitions. The compactness of $\mathcal{A}_\tau$ in $C_{T^*}$ follows from the definitions and the fact that the backward extension of an SDS in a compact invariant set generates a multivalued semi-group with compact attainability sets [10].

It remains to prove that $\widehat{\mathcal{A}}_\tau$ is the global attractor for the extended SDS $\widehat{S}^\tau$.

Let $\varepsilon > 0$ be arbitrary and let $n^*$ be the first integer such that $n^*\tau \geq T^*$. For each $\chi \in \mathcal{A}_\tau$, define $\delta(\chi) := \min\{\delta_1(\chi), \ldots, \delta_{n^*}(\chi)\}$, where $\delta_j(\chi)$ for $j = 1, \ldots, n^*$ are such that the continuous maps $S_{j\tau}^{(\tau)}$ satisfy $H^*_{C_\tau}(S_{j\tau}^{(\tau)}(\chi), S_{j\tau}^{(\tau)}(\phi)) \leq \varepsilon$ for all $\phi \in B_{C_\tau}(\chi, \delta_j(\chi))$.

Since $\mathcal{A}_\tau$ is compact it has finite cover of open balls

$$\mathcal{A}_\tau \subset \bigcup_{i=1}^{k} B_{C_\tau}(x^{(i)}, \delta(x^{(i)})).$$

There thus exists an $\rho > 0$ with

$$B_{C_\tau}(\mathcal{A}_\tau, \rho) \subset \bigcup_{i=1}^{k} B_{C_\tau}(x^{(i)}, \delta(x^{(i)})). \tag{5}$$

Now consider a bounded set $B$ in $C_{T^*}$. By the attraction of $\mathcal{A}_\tau$ there exists $T = T(\rho, B|_{[-\tau,0]}) \geq 0$ such that

$$H^*_{C_\tau}\left(S_t^{(\tau)}(B|_{[-\tau,0]}), \mathcal{A}_\tau\right) \leq \rho \qquad \forall\, t \geq T = T(\rho, B|_{[-\tau,0]}).$$

Consider an arbitrary element $\varphi \in B$ and a $x^{(i_0)} \in C_\tau$ such that, by (5),

$$\left\| S_{j\tau}^{(\tau)} S_T^{(\tau)}(\varphi) - S_{j\tau}^{(\tau)}(x^{(i_0)}) \right\|_\tau \leq \varepsilon \quad \text{for} \quad j = 1, \ldots, n^*.$$

This implies that

$$H^*_{C_{T^*}}\left(\widehat{S}^{(\tau)}_{n^*\tau+T}(B), \widehat{\mathcal{A}}_\tau\right) \leq \varepsilon.$$

Thus $\widehat{\mathcal{A}}_\tau$ is the global attractor of $\widehat{S}^{(\tau)}$ in $C_{T^*}$. □

Finally, following [11], we recall that the continuous convergence of the attractors for different time delays is understood as

$$H_{C_{T^*}}\left(\widehat{\mathcal{A}}_{\tau'}, \widehat{\mathcal{A}}_{\tau}\right) \to 0 \quad \text{as} \quad \tau' \to \tau.$$

3. **The equi-properties for the original SDS.** We will now translate the concepts of equi-attraction, equi-dissipative and eventually equi-compact of the family of extended SDS $\{\widehat{S}_t^{(\tau)}, \tau \in [T_*, T^*]\}$ on the space $C_{T^*}$ in terms of the original SDS $S_t^{(\tau)}$ on their state spaces $C_\tau$. This is important as the properties will be verified here, especially when the SDS are generated by specific DDE.

3.1. **Equi-dissipativity.** The concept of equi-dissipativity in Definition 2, (i), in terms of the extended SDS reads: *there exists a bounded subset $\mathcal{U}$ of $C_{T^*}$ and for every bounded subset $\mathcal{B}$ of $C_{T^*}$ there exists a $T_{\mathcal{B}} \in \mathbb{R}_+$, which is independent of $\tau$, such that*

$$\widehat{S}_t^{(\tau)}(\mathcal{B}) \subset \mathcal{U} \quad \textit{for all} \quad t \geq T_{\mathcal{B}} \quad \textit{and } \tau \in [T_*, T^*]. \tag{6}$$

In terms of the original SDS and state space this implies that

$$S_t^{(\tau)}(\mathcal{B}|_{[-\tau,0]}) \subset \mathcal{U}|_{[-\tau,0]} \quad \text{for all} \quad t \geq T_{\mathcal{B}} \quad \text{and } \tau \in [T_*, T^*],$$

where the previous notation is used for the restricted sets, i.e.

$$\mathcal{B}|_{[-\tau,0]} = \{\phi|_{[-\tau,0]} \;:\; \phi \in \mathcal{B}\}, \quad \mathcal{U}|_{[-\tau,0]} = \{\psi|_{[-\tau,0]} \;:\; \psi \in \mathcal{U}\}.$$

The definition of equi-dissipativity has an equivalent form in terms of the underlying base space $Z$, namely:

**Lemma 8.** *A family of SDS $\{\widehat{S}^{(\tau)}, \tau \in [T_*, T^*]\}$ is equi-dissipative if and only if there exists a bounded subset $U$ of $Z$ such that for every bounded subset $B$ of $Z$ there exists a $T_B \in \mathbb{R}_+$, which is independent of $\tau$, such that*

$$S_t^{(\tau)}(\mathcal{B}|_{[-\tau,0]})(0) \subset U \quad \textit{for all} \quad t \geq T_B \quad \textit{and } \tau \in [T_*, T^*],$$

*where*

$$\mathcal{B} := \{\phi \in C_{T^*} \;:\; \phi(s) \in B \;\forall\, s \in [-T^*, 0]\}\,. \tag{7}$$

*Proof.* Starting with (6), we simply define

$$U = \{\phi(s) \in Z \;:\; \phi \in \mathcal{U},\, s \in [-T^*, 0]\}$$

with $T_B := T_{\mathcal{B}}$ corresponding to the bounded subset $\mathcal{B}$ of $C_{T^*}$ defined in (7).

In the other direction, following (7), we define

$$\mathcal{U} := \{\phi \in C_{T^*} \;:\; \phi(s) \in U, s \in [-T^*, 0]\}\,. \tag{8}$$

Given a bounded subset $\mathcal{B}$ of $C_{T^*}$ we define $T_{\mathcal{B}} := T_B + T^*$ corresponding to the set

$$B = \{\phi(s) \in Z \;:\; \phi \in \mathcal{B},\, s \in [-T^*, 0]\}\,.$$

(Note that the new set $\mathcal{B}$ defined by (7) in terms of this $B$ will contain and in general be larger than the original set $\mathcal{B}$). □

3.2. **Eventual equi-compactness.** We first note that the property of eventual equi-compactness in Definition 2, (ii), can be rewritten as: *for any bounded subset $B$ of $X$, there exists a $T_B \in \mathbb{R}_+$ independent of $\lambda \in \Lambda$ and a family of compact subsets $\{K(t), t \geq T_B\}$ of $X$ such that*

$$S_t^{(\lambda)}(B) \subset K(t) \quad \textit{for every} \;\; t \geq T_B.$$

We simply take $U(t)$ to be the closure of $\bigcup_{\lambda \in \Lambda} S_t^{(\lambda)}(B)$ in $X$. The compact sets $U(t)$ here need not be uniformly bounded in $t$ – if they were then we would also have equi-dissipativity.

In our situation this definition takes the form: *for every bounded subset $\mathcal{B}$ of $C_{T^*}$ there exists a $T_{\mathcal{B}} \in \mathbb{R}_+$, which is independent of $\tau$, and a family of compact subsets $\{\mathcal{U}(t), t \geq T_{\mathcal{B}}\}$ of $C_{T^*}$ such that*

$$\widehat{S}_t^{(\tau)}(\mathcal{B}) \subset \mathcal{U}(t) \qquad \textit{for all} \quad t \geq T_{\mathcal{B}} \quad \textit{and each}\ \tau \in [T_*, T^*].$$

In terms of the original dynamical systems this translates to

$$S_t^{(\tau)}(\mathcal{B}|_{[-\tau,0]}) \subset \mathcal{U}(t)|_{[-\tau,0]} \qquad \text{for all} \quad t \geq T_{\mathcal{B}} \quad \text{and each}\ \tau \in [T_*, T^*].$$

**Remark 9.** *For DDE with finite delay, when $Z$ is finite dimensional, compactness follows from the existence of a bounded absorbing family, thanks to Ascoli-Arzelà Theorem, if the right hand side of the DDE is a bounded map (i.e. it maps bounded sets onto bounded sets).*

3.3. **Joint and equi-continuity.** Theorem 3 requires that the family of SDS satisfies the continuity properties (A1) and (A2), i.e.

(A1) For any $t \in \mathbb{R}_+$ fixed, $S_t^{(\lambda)}(x)$ is jointly continuous in $(x, \lambda)$ on $X \times \Lambda$.
(A2) $S_t^{(\lambda)}(x)$ is equi-continuous in $\lambda$ for $(t, x)$ in any bounded subset of $\mathbb{R}_+ \times X$.

In our context the joint continuity property (A1) becomes: *for any $t \in \mathbb{R}_+$ fixed, $\widehat{S}_t^{(\tau)}(\phi)$ is jointly continuous in $(\tau, \phi)$* in $[T_*, T^*] \times C_{T^*}$.

Thus, if $(\tau_n, \phi^{(n)}) \to (\tau, \phi)$ in $[T_*, T^*] \times C_{T^*}$ as $n \to \infty$, then so too does

$$\widehat{S}_t^{(\tau_n)}(\phi^{(n)}) \to \widehat{S}_t^{(\tau)}(\phi) \qquad \text{as} \quad n \to \infty.$$

Recalling the projection notation introduced before Theorem 6

$$x^{(\tau)}(t+s, \phi) := \widehat{S}_t^{(\tau)}(\phi)(s), \qquad s \in [-T^*, 0],$$

joint continuity means that

$$x^{(\tau_n)}(t, \phi^{(n)}) \to x^{(\tau)}(t, \phi) \qquad \text{as} \quad n \to \infty,$$

in the base space $Z$ uniformly on the interval $[t - T^*, t]$ for each fixed $t \geq 0$. Thus it will also be uniform on all finite time intervals $[-T^*, T]$ with $T > 0$. This uniform joint convergence in $Z$ implies the function space joint continuity of condition (A1) above.

Similarly, the equi-continuity property (A2) becomes: *$\widehat{S}_t^{(\tau)}(\phi)$ is equi-continuous in $\tau$ for $(t, \phi)$ in any bounded subset $[T_1, T_2] \times \mathcal{B}$ of $\mathbb{R}_+ \times C_{T^*}$,* which is essentially uniform continuity in $\tau$, i.e. for every $\varepsilon > 0$ and bounded subset $[T_1, T_2] \times \mathcal{B}$ of $\mathbb{R}_+ \times C_{T^*}$ there exists $\delta = \delta(T_1, T_2, \mathcal{B}, \varepsilon) > 0$ such that

$$|\tau' - \tau| < \delta \Longrightarrow \left\| \widehat{S}_t^{(\tau')}(\phi) - \widehat{S}_t^{(\tau)}(\phi) \right\|_{T^*} < \varepsilon \qquad \forall (t, \phi) \in [T_1, T_2] \times \mathcal{B}.$$

In terms of the projections in the base space $Z$ this reads as

$$|\tau' - \tau| < \delta \Longrightarrow \left| x^{(\tau')}(t, \phi) - x^{(\tau)}(t, \phi) \right| < \varepsilon \qquad \forall (t, \phi) \in [T_1 - T^*, T_2] \times \mathcal{B},$$

which implies the function space equi-continuity condition (A2) above.

3.4. **Equi-attraction.** Suppose that each SDS $S_t^{(\tau)}$ on $C_\tau$ has global attractor $\mathcal{A}_\tau$ in $C_\tau$ for $\tau \in [T_*, T^*]$. Then, by Theorem 7, each extended SDS $\widehat{S}_t^{(\tau)}$ on $C_{T^*}$ has an attractor $\widehat{\mathcal{A}}_\tau$ in $C_{T^*}$, where $\widehat{\mathcal{A}}_\tau$ is defined in terms of $\mathcal{A}_\tau$ through (4).

These extended attractors are equi-attracting if for every $\varepsilon > 0$ and bounded subset $\mathcal{B}$ of $C_{T^*}$ there exists $T_{\varepsilon,\mathcal{B}} \in \mathbb{R}_+$ independent of $\tau \in [T_*, T^*]$ such that

$$H^*_{C_{T^*}}\left(\widehat{S}_t^{(\tau)}(\phi), \widehat{\mathcal{A}}_\tau\right) < \varepsilon \quad \text{for all} \quad t \geq T_{\varepsilon,\mathcal{B}},\ \phi \in \mathcal{B},\ \tau \in [T_*, T^*] \tag{9}$$

which obviously implies that

$$H^*_{C_\tau}\left(S_t^{(\tau)}(\phi|_{[-\tau,0]}), \mathcal{A}_\tau\right) < \varepsilon \qquad \text{for all} \quad t \geq T_{\varepsilon,\mathcal{B}},\ \phi \in \mathcal{B},\ \tau \in [T_*, T^*]. \tag{10}$$

Of course, one would like that (9) and (10) to be equivalent (perhaps with a slightly larger $T_{\varepsilon,\mathcal{B}}$). However, the value $\rho$ appearing in the proof of Theorem 7 depends on $\tau$ in a not necessarily uniform way. We will use property (A3) and borrow some ideas from Theorem 5 to obtain an equivalence.

**Remark 10.** *Condition* (A3) *in* Theorem 5 *for the extended SDS* $\widehat{S}^{(\tau)}$ *is equivalent to the following condition for the original semi dynamical systems* $S^{(\tau)}$ *:*

(A3') *For any bounded subset* $\mathcal{B}$ *of* $C_{T^*}$ *and* $T > 0$, $S_t^{(\tau)}(\chi|_{[-\tau,0]})$ *is uniformly continuous in* $\chi|_{[-\tau,0]} \in \mathcal{B}|_{[-\tau,0]}$ *uniformly w.r.t.* $\tau$ *and* $t \leq T$, *i.e.*

$$\forall \varepsilon > 0,\ \exists \delta > 0\ :\ \chi, \phi \in \mathcal{B},\ \|\chi|_{[-\tau,0]} - \phi|_{[-\tau,0]}\|_\tau < \delta$$

$$\Rightarrow \|S_t^{(\tau)}(\chi|_{[-\tau,0]}) - S_t^{(\tau)}(\phi|_{[-\tau,0]})\|_\tau < \varepsilon,\ \forall t \in [0, T],\ \tau \in [T_*, T^*]. \tag{11}$$

**Theorem 11.** *Let* $S^{(\tau)} : \mathbb{R}_+ \times C_\tau \to C_\tau$ *for* $\tau \in [T_*, T^*]$ *be a family of SDS with attractors* $\mathcal{A}_\tau$*, which is equi-dissipative and equi-attracting in the sense of* (10) *and also satisfies condition* (A3'). *Then the extended attractors* $\widehat{\mathcal{A}}_\tau$ *are equi-attracting.*

*Proof.* By the equi-dissipativeness there exists a bounded subset $U$ of $Z$ such that $\mathcal{A}_\tau \subset \mathcal{U}|_{[-\tau,0]}$ for all $\tau$, where the subset $\mathcal{U}$ of $C_{T^*}$ is defined from $U$ through (8). Consider any $\varepsilon > 0$ and the bounded set

$$\mathcal{B} = \{\phi \in C_{T^*}\ :\ \phi(s) \in B_Z(U, \varepsilon),\ s \in [-T^*, 0]\}.$$

It is enough to check (9) only with this bounded set. By the equi-attraction of $\{\mathcal{A}_\tau\}_\tau$, there exists $T_{\varepsilon,\mathcal{B}}$ independent of $\tau$, such that (10) holds. In particular, this implies that $\mathcal{B}|_{[-\tau,0]}$ is positively invariant for any $S_t^{(\tau)}$ with $t \geq T_{\varepsilon,\mathcal{B}}$, i.e.

$$S_t^{(\tau)}(\mathcal{B}|_{[-\tau,0]}) \subset \mathcal{B}|_{[-\tau,0]} \quad \forall t \geq T_{\varepsilon,\mathcal{B}}. \tag{12}$$

For the bounded set $\mathcal{B}$, by (A3'), there exists $\delta > 0$ depending on $\varepsilon$ such that (11) holds for $T = n_*T_*$, with $n_*$ the first integer such that $n_*T_* \geq T^*$. We will use (11) for $t = jT_*$ with $j = 1, \ldots, n_*$ , which ensures that we can cover any interval of length $T^*$ by delays of length $\tau \in [T_*, T^*]$.

Let $\rho = \min(\delta, \varepsilon)$. By the equi-attraction again, analogously to (10), there exists a time $T_{\rho,\mathcal{B}}$ (which we can take w.l.o.g. larger than $T_{\varepsilon,\mathcal{B}}$) such that

$$H^*_{C_\tau}(S_t^{(\tau)}(\mathcal{B}|_{[-\tau,0]}), \mathcal{A}_\tau) < \rho \quad \text{for all } \ t \geq T_{\rho,\mathcal{B}}, \ \tau \in [T_*, T^*]. \tag{13}$$

To finish the proof, take $\psi \in \mathcal{B}$. By (13), for any $t \geq T_{\rho,\mathcal{B}}$, there exists $\xi \in \mathcal{A}_\tau$ such that $\|S_t^{(\tau)}(\psi|_{[-\tau,0]}) - \xi\|_\tau < \rho \leq \delta$. Using (11) for $T = n_*T_*$ we have

$$\|S_{jT_*}^{(\tau)} S_t^{(\tau)}(\psi|_{[-\tau,0]}) - S_{jT_*}^{(\tau)}(\xi)\|_\tau < \varepsilon \quad \text{for } \ j = 1, \ldots, n_*.$$

This means that

$$H^*_{C_{T^*}}\left(\widehat{S}^{(\tau)}_{n_*T_*+t}(\psi), \widehat{\mathcal{A}}_\tau\right) < \varepsilon \quad \forall t \geq T_{\rho,\mathcal{B}},$$

which is the equi-attraction property (9) with $T_{\varepsilon,\mathcal{B}}$ replaced by $n_*T_* + T_{\rho,\mathcal{B}}$. □

**Remark 12.** *There is an equivalent condition to Assumption* (A3') *in the above result, though apparently is less restrictive, in which the uniform continuity for t in bounded intervals can be substituted by uniform continuity at a single time instant* $t_*$*, namely,*

(A3") *There exists* $t_* \in (0, T_*]$ *such that for any bounded subset* $\mathcal{B}$ *of* $C_{T^*}$*, the SDS* $S_{t_*}^{(\tau)}(\chi)$ *is uniformly continuous in* $\chi \in \mathcal{B}|_{[-\tau,0]}$ *uniformly w.r.t.* $\tau$*, i.e.*

$$\forall \varepsilon > 0, \ \exists \delta > 0 \, : \, \chi, \phi \in \mathcal{B}, \ \|\chi|_{[-\tau,0]} - \phi|_{[-\tau,0]}\|_\tau < \delta \tag{14}$$

$$\Rightarrow \|S_{T_*}^{(\tau)}(\chi|_{[-\tau,0]}) - S_{T_*}^{(\tau)}(\phi|_{[-\tau,0]})\|_\tau < \varepsilon, \quad \forall \tau \in [T_*, T^*].$$

*Actually, by* (12)*, the above uniform continuity of* $S_{t_*}^{(\tau)}$ *in* $\mathcal{B}|_{[-\tau,0]}$ *holds for all* $S_{jt_*}^{(\tau)}$ *with* $j = 1, \ldots, n_*$*, where* $n_*$ *now denotes the first integer such that* $n_*t_* \geq T^*$*.*

*Indeed, since* $S_{t_*}^{(\tau)}$ *is uniformly continuous in* $\mathcal{B}|_{[-\tau,0]}$*, for an arbitrary* $\varepsilon > 0$ *there exists* $\delta_1$ *such that* $\|S_{t_*}^{(\tau)}(\chi) - S_{t_*}^{(\tau)}(\phi)\|_\tau \leq \varepsilon$ *if* $\|\chi - \phi\|_\tau < \delta_1$*. For* $S_{2t_*}^{(\tau)}$*, choose* $\delta_2$ *associated with* $\varepsilon_2 = \delta_1$ *(property* (12) *plays an essential role here). Recursively, we conclude the claim in* $n_*$ *steps, with* $\delta = \min\{\delta_1, \ldots, \delta_{n_*}\}$*.*

4. **An example.** Li and Kloeden [14, Sec.3,Ex.3.2] gave the following example of a scalar ordinary differential equation to illustrate their results. Let $\lambda_0 = 2\sqrt{3}/9$ and $\Lambda = [0, \lambda_0]$ and let $f : \Lambda \times \mathbb{R} \to \mathbb{R}$ be given by

$$f(\lambda, x) = -x^3 + x + 4\sqrt{3}/9 - \lambda,$$

which is illustrated in Figure 1 below. In particular, for $\lambda < \lambda_0$ it has a single zero $x^{(\lambda^+)} > 0$ and for $\lambda = \lambda_0$ a new zero $x^{(\lambda_0^-)}$ appears.

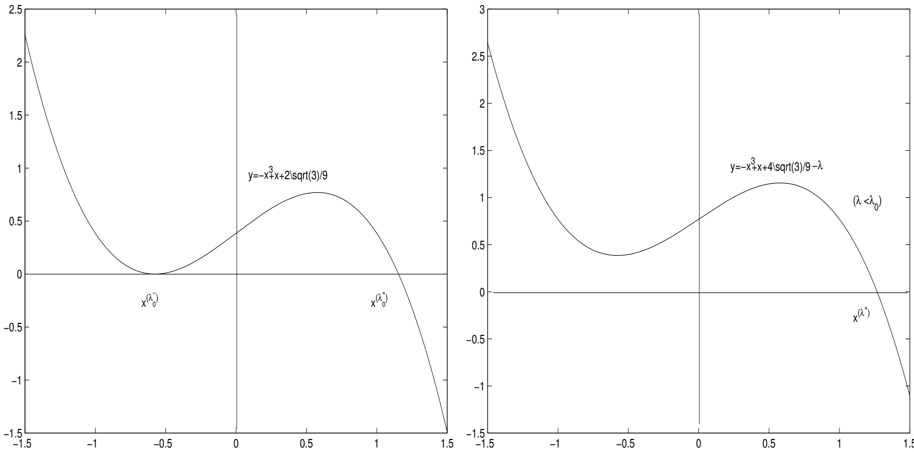


Fig. 1. Two equilibria at $\lambda_0$. One equilibrium for $\lambda < \lambda_0$.

Then the ODE

$$\frac{dx}{dt} = f(\lambda, x) \tag{15}$$

generates a semi dynamical system on the state space $Z = \mathbb{R}$ for each $\lambda \in \Lambda$. For $\lambda \in [0, \lambda_0)$ the global attractor is the singleton $\{x^{(\lambda^+)}\}$ and for $\lambda_0$ the global attractor is the interval $[x^{(\lambda_0^-)}, x^{(\lambda_0^+)}]$. The setvalued mapping $\lambda \mapsto \mathcal{A}_\lambda$ is obviously not continuous at $\lambda = \lambda_0$ (although it is continuous for all other values of $\lambda$).

We now introduce a delay DE variation of the above example with the time delay $\tau = \lambda$. Consider some $a > 0$, and let $g : \mathbb{R} \to [0, a]$ be a globally Lipschitz function with Lipschitz constant $L_g > 0$ and with $g(-\sqrt{3}/3) = 0$. We consider the scalar delay DE

$$\frac{d}{dt}x(t) = F_\lambda(x_t) := f(\lambda, x(t)) + g(x(t - \lambda)), \tag{16}$$

where $x_t \in C_\lambda = C([-\lambda, 0]; \mathbb{R})$ is defined as usual. We restrict attention to time delays $\lambda \in [\theta\lambda_0, \lambda_0]$ for some $0 < \theta < 1$, i.e. $\Lambda = [\theta\lambda_0, \lambda_0]$ here.

4.1. **Well defined SDS and existence of attractors.** We check now that each DDE generates a well posed problem.

Suppose that $\lambda \in [\theta\lambda_0, \lambda_0)$. Local Lipschitz continuity of the right hand side of the DDE (16) ensures the local existence and uniqueness of solutions. We will now show that they are in fact globally defined in time.

Let $\phi \in C_\lambda$ be an initial value. If $\phi(0) < x_\lambda$, then $F_\lambda$ is positive (since both $f$ and $g$ are), then the solution increases until $x_\lambda$. It does not matter whether the solution takes the value $x_\lambda$ or not. The important fact is that the solution cannot decrease, this would be absurd by the mean-value theorem. On the other hand, if $\phi(0) > f(\lambda, \cdot)^{-1}(-a)$, then $f$ being negative is more important than $g$, and the solution decreases. The critical point is then $f(\lambda, \cdot)^{-1}(-a)$. Whether or not a solution arrives at $\max f(\lambda, \cdot)^{-1}(-a)$, there is no blow-up. In particular, if $f(\lambda, \cdot)^{-1}(-a)$ is reached at some time $t^*$, i.e. $f(\lambda, x(t^*)) = -a$, then it cannot increase again (impossible by the mean-value theorem). Thus all solutions are globally defined and the DDE (16) generates an SDS $S^{(\lambda)}$ in $C_\lambda = C([-\lambda, 0]; \mathbb{R})$.

The above discussion also shows that the projection $S_t^{(\lambda)}(\phi)(0)$ of solutions in $\mathbb{R}$ are attracted by the subset $B_\lambda = [x_\lambda, f(\lambda, \cdot)^{-1}(-a)]$ of $\mathbb{R}$. Thus the subset $\mathcal{B}_\lambda$ of $C_\lambda$

defined in terms of $B_\lambda$ by (7) is an absorbing set for the SDS $S^{(\lambda)}$. The existence of a global attractor $\mathcal{A}_\lambda$ (contained in $\mathcal{B}_\lambda$) then follows from the compactness of the SDS operators $S_t^{(\lambda)}$ for $t > \lambda$ (and that of attractors $\widehat{\mathcal{A}}_\lambda$ for the SDS $\widehat{S}^{(\lambda)}$).

We observe that the extended semi dynamical systems $\widehat{S}^{(\lambda)}$ obtained here from Theorem 6 are generated by the solutions of the DDE $x'(t) = \widehat{F}_\lambda(x_t)$ with $\widehat{F}_\lambda : C_{\lambda_0} \to \mathbb{R}$ defined as $\widehat{F}_\lambda(\varphi) = F_\lambda(\varphi|_{[-\lambda,0]})$.

4.2. **Equi-properties and continuity.** Our first goal now is to prove equi-dissipativity for the SDS $\widehat{S}^{(\lambda)}$ (after that, eventual equi-compactness follows adding an elapsed time $\lambda_0$). We will obtain it for any set of parameters of the form $[\theta\lambda_0, \lambda]$ with $\lambda < \lambda_0$.

Fix $\lambda \in [\theta\lambda_0, \lambda_0)$. Observe that a solution with initial data $\phi$ such that $\phi(0) < x_\lambda$ reaches any value $x_\lambda - \varepsilon > \phi(0)$ in a time $t_\varepsilon$ which by the mean-value theorem is bounded by

$$t_\varepsilon \leq \frac{x_\lambda - \varepsilon - \phi(0)}{A} \quad \text{with} \quad A = \sup_{[\phi(0), x_\lambda - \varepsilon]} f(\lambda, \cdot).$$

The case $\phi(0) > \hat{f}(\lambda, \cdot)^{-1}(a)$ is similar.

Joining both analysis, and by the continuity of $f(\lambda, \cdot)$, the equi-dissipativity property holds in compact intervals of the parameter $[\theta\lambda_0, \lambda]$ with $\lambda < \lambda_0$.

Our second goal is to prove the required continuity properties. Let $\lambda$, $\lambda' \in \Lambda$ and consider the solutions of the initial value problems

$$\begin{cases} x'(t) = \widehat{F}_\lambda(x_t), \\ x_0 = \phi, \end{cases} \qquad \begin{cases} y'(t) = \widehat{F}_{\lambda'}(x_t), \\ y_0 = \psi \end{cases}$$

for initial values $\phi$, $\psi \in C([-\lambda_0, 0]; \mathbb{R})$.

Define $z(t) = x(t) - y(t)$. Then

$$z(t) = z(0) + \int_0^t (\widehat{F}_\lambda(x_s) - \widehat{F}_{\lambda'}(y_s))\mathrm{d}s,$$

and hence from the local and global properties of the functions $f$ and $g$ it follows that

$$\begin{aligned} |z(t)| &\leq |z(0)| + L_{f,loc} \int_0^t |x(s) - y(s)|\mathrm{d}s + |\lambda - \lambda'| \\ &\quad + \int_0^t |g(x(s-\lambda)) - g(y(s-\lambda))|\mathrm{d}s \\ &\leq |z(0)| + |\lambda - \lambda'| + \lambda L_g \|\varphi - \psi\| + (L_{f,loc} + L_g) \int_0^t |z(r)|\mathrm{d}r \\ &\quad + \int_0^t |g(y(s-\lambda)) - g(y(s-\lambda'))|\mathrm{d}s. \end{aligned} \tag{17}$$

(The local Lipschitz constant $L_{f,loc}$ of $f$ depends on an appropriately chosen region depending on the initial values $\phi$ and $\psi$).

It is clear from (17), the uniform continuity of the solution $y$ in a compact interval of time, and the Gronwall Lemma that condition (A1) on the continuity of the SDS $\widehat{S}^{(\lambda)}$ in $(\lambda, \phi)$ holds, i.e. the corresponding solutions are close at a fixed $t$ when $(\lambda', \psi)$ is close enough to $(\lambda, \phi)$.

The condition (A2) does not hold in bounded sets as initial data because of the last line in (17). But the boundedness of the SDS and the functionals $F_\lambda$ provide the equi-continuity of solutions of the attractors, which is the only requirement of (A2'), and is sufficient to apply the theory (recall Remark 4).

4.3. **Discontinuity of attractors at $\lambda_0$.** ¿From the discussion in subsection 4.1, the attractor $\mathcal{A}_\lambda$ SDS $S^{(\lambda)}$ for any $\lambda \in [\theta\lambda_0, \lambda_0)$ is contained in the sets

$$\mathcal{B}_\lambda = \left\{ \phi \in C_\lambda \ : \ \phi(s) \in \left[ x^{(\lambda^+)}, f(\lambda, \cdot)^{-1}(-a) \right] \ \ \forall s \in [-h, 0] \right\},$$

where $x^{(\lambda^+)} > 0$ for all $\lambda \in [\theta\lambda_0, \lambda_0]$. However, when the parameter $\lambda$ reaches the value $\lambda_0$, a negative equilibrium solution of the DDE (16) appears, which is the constant function in $C_{\lambda_0}$ identically equal to $x^{(\lambda_0^-)} = -\sqrt{3}/3$. Since the attractor $\widehat{\mathcal{A}}^{(\lambda_0)}$ contains this negative stationary solution as well as solutions with positive values and is a connected set we conclude that there exists a discontinuity in the setvalued mapping $\lambda \mapsto \widehat{\mathcal{A}}^{(\lambda)}$ at $\lambda_0$.

4.4. **Continuity of attractors for $\lambda \in [\theta\lambda_0, \lambda_0)$.** Let us now assume that the function $g$ in the DDE (16) also satisfies the condition

$$\operatorname{supp} g \bigcap \left[ x^{(\lambda^+)}, f(\lambda, \cdot)^{-1}(-a) \right] = \emptyset \quad \forall \lambda \in \Lambda. \tag{18}$$

Then by (18), for a sufficient small $\varepsilon > 0$, the set

$$B_\lambda^{(\varepsilon)} = \left[ x^{(\lambda^+)} - \varepsilon, f(\lambda, \cdot)^{-1}(-a) + \varepsilon \right].$$

is an absorbing set in $\mathbb{R}$ for the projected dynamics $S_t^{(\lambda)}(\phi)(0)$ (since the delay term has no influence near this set). The corresponding situation holds for the dynamics of the SDS with the subset $\mathcal{B}_\lambda^{(\varepsilon)}$ defined in terms of $B_\lambda^{(\varepsilon)}$ as in (7)).

In particular, the asymptotic behaviour of the DDE near the absorbing set reduces to that of the original ODE and the attractor consists of a single solution, i.e. $\mathcal{A}_\lambda = \{\phi_\lambda\}$ where $\phi_\lambda(s) \equiv x^{(\lambda^+)}$ for all $s \in [-\lambda, 0]\}$ and $\widehat{\mathcal{A}}_\lambda = \{\widehat{\phi}_\lambda\}$ where $\widehat{\phi}_\lambda(s) \equiv x^{(\lambda^+)}$ for all $s \in [-\lambda_0, 0]\}$.

Thus the attractors are continuous in $\lambda \in [\theta\lambda_0, \lambda_0)$ and we conclude they are equi-attracting as a consequence of Theorem 3.

**Remark 13.** *The above results have been presented, for simplicity, with the parameter $\lambda = \tau$ of the family of SDS $S^{(\tau)}$ denoting the influence on the phase space $C_\tau$, i.e. of the delay directly. It is not difficult to generalize these results to the case of parametric dependence of the delay. Namely, let $\Lambda$ be a compact set of $\mathbb{R}_+$, and consider $\rho \in C(\Lambda; [T_*, T^*])$, and a family of SDS $S^{(\lambda)} : \mathbb{R}_+ \times C_{\rho(\lambda)} \to C_{\rho(\lambda)}$. Using* Theorem 3 *and* Theorem 5 *one can obtain analogous results to the above ones. Indeed, an analogous and valid example would be $x'(t) = G_\lambda(x_t)$ with $G_\lambda(\phi) = f(\lambda, \phi(0)) + g(\phi(-\rho(\lambda)))$, for any $\phi \in C_{T^*}$.*

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