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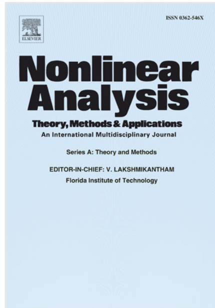

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Attractors for parametric delay differential equations without uniqueness and their upper semicontinuous behaviour

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Dedicated to the memory of Pedro Bellido, my partner's grandfather, with sorrow and tenderness.

Abstract

We prove existence of a global attractor $\mathbb{A}^{(\lambda)}$ under minimal assumptions for a general class of parameterized delay differential equations without uniqueness and posed in potentially different state spaces. Secondly, we establish the upper semicontinuity of the attractors with respect to the parameter λ.

Key words: Delay differential equations without uniqueness, multi-valued semiflows and attractors, upper semicontinuity of attractors.

1 Introduction

Delay differential equations (DDE for short) are of major interest in many fields of science. They appear in Biology, Economics, Physics, Chemistry, etc. There are many interesting questions concerning the qualitative behaviour of DDE, although most of attention has been paid to stability properties. Even when such results do not hold, it is still useful the study their long-time behaviour, and in particular the existence of attractors. There exists a wide literature on this topic, see for instance [6,7] and the references therein.

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A middle step between autonomous and non-autonomous models are DDE with parameters, which arise, for instance, when dealing with approximations or (singular) perturbations of the original model.

The aim of this paper is twofold:

First, we will prove the existence of global attractors for a general class of parameterized DDE which includes fixed, variable, and distributed delays. Each equation is posed in a potentially different state space, and we only make weak continuity assumptions on the right hand side which only allow us to prove existence but no uniqueness.

Our second goal is to study the behaviour of these attractors when varying the parameter. There are many results on upper semicontinuity in the literature about attractors for dynamical systems, and their perturbations and approximations (e.g. cf. [3,10] among many others). We obtain an upper semicontinuity result here w.r.t. to the parameter although, as commented before, we deal in principle with different state spaces.

The structure of the paper is as follows. In Section 2 we study the existence and estimates of solutions for a general class of parameterized delay differential equations. A multi-valued semiflow is then established, and the existence of an attractor for each value of the parameter is proved. In Section 3 we establish an embedding of all the problems in a common state space, and obtain a new family of attractors, which we relate with that obtained previously. Finally, we prove the upper semicontinuity of these attractors with respect to the parameter.

2 A general class of parametric DDEs

Let us introduce some notation which will be used all through the paper.

For a given metric space (X, d), $P(X)$, $C(X)$, and $K(X)$ will denote the class of all nonempty, nonempty and closed, and nonempty and compact subsets of X respectively. $B_X(a, r)$ will denote the open ball of X with center a and radius r. In addition, denote the Hausdorff semidistance by

$$H_X^*(A, B) = \sup_{x \in A} d(x, B)$$

for any subsets A, $B \in C(X)$.

In $\mathbb{R}^d$ ($d \in \mathbb{N}$), we denote by $|\cdot|$ the Euclidean norm. And for any $T > 0$

we will denote by $(C_T, \|\cdot\|_T)$ the Banach space $C([-T,0];\mathbb{R}^d)$ with the norm $\|\varphi\|_T = \sup_{t\in[-T,0]} |\varphi(t)|$. The usual notation for delay function will be a subscript: $x_t(s) = x(t+s)$ where it has sense.

Hypothesis 1 *Let $\Lambda \subset \mathbb{R}$ be a closed interval, and suppose that positive numbers $0 < T_* < T^*$, and functions τ, $\rho \in C(\Lambda;[T_*,T^*])$ (which will drive the delay effects) are given.*

Consider also the functions F_0, $F_1 \in C(\mathbb{R}^d;\mathbb{R}^d)$, and $b : [-\max_\Lambda \tau, 0]\times\mathbb{R}^d \to \mathbb{R}^d$, measurable w.r.t. its first variable and continuous w.r.t. the second variable, m_0, $m_1 \in L^1((-\max_\Lambda \tau, 0);\mathbb{R}_+)$, and α, $\beta > 0$, and k_1, $k_2 \geq 0$, such that

$$\begin{aligned}
|b(s,x)| &\leq m_1(s)|x| + m_0(s), && \forall x \in \mathbb{R}^d,\ a.e.\ s \in [-\max_\Lambda \tau, 0],\\
\langle x, F_0(x)\rangle &\leq -\alpha|x|^2 + \beta, && \forall x \in \mathbb{R}^d,\\
|F_1(x)|^2 &\leq k_1^2 + k_2^2|x|^2, && \forall x \in \mathbb{R}^d.
\end{aligned}$$

For convenience we introduce the following notation

$$M_\lambda^{\rho,\tau} = \max\{\rho(\lambda), \tau(\lambda)\}, \qquad m_i = \max_\Lambda \int_{-\tau(\lambda)}^0 m_i(s)\mathrm{d}s \quad \text{for} \quad i = 0, 1. \tag{1}$$

Under the above assumptions, consider (for each $\lambda \in \Lambda$) the functional

$$f(\lambda,\cdot) : C_{M_\lambda^{\rho,\tau}} \to \mathbb{R}^d$$

given by

$$f(\lambda,\varphi) = F_0(\varphi(0)) + F_1(\varphi(-\rho(\lambda))) + \int_{-\tau(\lambda)}^0 b(s,\varphi(s))\mathrm{d}s,$$

and the family of DDE

$$x'(t) = f(\lambda, x_t) = F_0(x(t)) + F_1(x(t-\rho(\lambda))) + \int_{-\tau(\lambda)}^0 b(s, x(t+s))\mathrm{d}s. \tag{2}$$

Remark 2

(i) Thanks to the continuity assumptions for F_0, F_1, b, τ and ρ, and using the dominated convergence theorem, it is not difficult to check that $f(\lambda,\cdot)$ is a continuous functional.

(ii) The results presented here can be extended to more general functionals depending on the parameter and/or different delay terms. However, for clarity in the presentation, we prefer to restrict to this case.

When necessary we will denote with superscript (λ) the parametric dependence of the problem. However, if no confusion is possible, we will just use the notation x for any solution instead of $x^{(\lambda)}$ since the estimates that we will obtain are uniform in λ (which is one of our main goals).

2.1 Semiflows and attractors for DDE$^{(\lambda)}$

Local existence of solutions is well know (cf. [7]) for finite delay differential equations provided the right hand side is a continuous functional, as observed in Remark 2(i). With a priori estimates from the following result we will obtain global (and not only local) solutions but no uniqueness.

The following notion from Dynamical Systems Theory will be necessary (cf. [17] and the references therein).

Definition 3 *A multi-valued map $\mathcal{G} : \mathbb{R}_+ \times X \to P(X)$ is called a multi-valued semiflow if*

a) $\mathcal{G}(0,\cdot) = \mathrm{Id}$ *(identity map)*

b) *For any pair t_1, $t_2 \geq 0$ and for all $x \in X$,*

$$\mathcal{G}(t_1+t_2,x) \subset \mathcal{G}(t_1,\mathcal{G}(t_2,x)), \quad \textit{where} \quad \mathcal{G}(t,A) = \bigcup_{a\in A} \mathcal{G}(t,a).$$

When the above inclusion is an equality, it is said that the multi-valued semiflow is strict.

Lemma 4 *Assume that* Hypothesis 1 *holds, and consider a local solution x to* (2)*, defined on an interval $[0,T_x)$. Then, there exist positive constants A, B, and δ such that x satisfies for all $t < T_x$:*

$$\mathrm{e}^{\delta t}|x(t)|^2 \leq |x(0)|^2 + \int_0^t \mathrm{e}^{\delta s}(A + B\|x_s\|^2_{M_\lambda^{\rho,\tau}})\mathrm{d}s. \tag{3}$$

Proof. From the equation, we easily obtain

$$
\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|x(t)|^2 \le & -\alpha|x(t)|^2 + \beta + \left(k_1^2 + k_2^2|x(t-\rho(\lambda))|^2\right)^{1/2}|x(t)| \\
& +\langle x(t), \int_{-\tau(\lambda)}^0 b(s, x(t+s))\mathrm{d}s\rangle \\
& \le -\alpha|x(t)|^2 + \beta + \frac{1}{2\varepsilon}\left(k_1^2 + k_2^2|x(t-\rho(\lambda))|^2\right) + \frac{\varepsilon}{2}|x(t)|^2 \\
& +\frac{\bar{\varepsilon}}{2}|x(t)|^2 + \frac{m_0^2}{2\bar{\varepsilon}} + |x(t)|\int_{-\tau(\lambda)}^0 m_1(s)|x(t+s)|\mathrm{d}s \\
& \le -\alpha|x(t)|^2 + \beta + \frac{1}{2\varepsilon}\left(k_1^2 + k_2^2|x(t-\rho(\lambda))|^2\right) + \frac{\varepsilon}{2}|x(t)|^2 \\
& +\frac{\bar{\varepsilon}}{2}|x(t)|^2 + \frac{m_0^2}{2\bar{\varepsilon}} + \|x(t)\|_{\tau(\lambda)}^2 m_1,
\end{aligned}
$$

where ε and $\bar{\varepsilon}$ are positive constants to fix later, and m_i were defined in (1).

We deduce

$$
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|x(t)|^2 \le -\left(\alpha - \frac{\varepsilon}{2} - \frac{\bar{\varepsilon}}{2}\right)|x(t)|^2 + \beta + \frac{k_1^2}{2\varepsilon} + \frac{k_2^2}{2\varepsilon}\|x_t\|_{\rho(\lambda)}^2 + \frac{m_0^2}{2\bar{\varepsilon}} + m_1\|x_t\|_{\tau(\lambda)}^2.
$$

Take $\delta \in (0, 2\alpha)$ such that $\varepsilon + \bar{\varepsilon} = 2\alpha - \delta$ and write

$$
A = 2\beta + \frac{k_1^2}{\varepsilon} + \frac{m_0^2}{\bar{\varepsilon}} \qquad \text{and} \qquad B = \frac{k_2^2}{\varepsilon} + 2m_1,
$$

so we rewrite the above inequality as

$$
\frac{\mathrm{d}}{\mathrm{d}t}|x(t)|^2 \le -\delta|x(t)|^2 + A + B\|x_t\|_{M_\lambda^{\rho,\tau}}^2. \tag{4}
$$

Multiplying (4) by $\mathrm{e}^{\delta t}$ we arrive to (3). ■

The next step in order to construct a multi-valued semiflow is clear. This proposition-definition follows from standard continuation results (cf. [7, Ch.2]).

Proposition 5 *Assume* Hypothesis 1 *holds. Then, the set*

$$
D(\psi) = \{x \ : \ x \textit{ is a global solution of } (2) \textit{ with } x_0 = \psi\}
$$

is nonempty, and the following multi-valued map is a multi-valued semiflow,

$$
\begin{aligned}
\mathbb{G}^{(\lambda)} : \mathbb{R}_+ \times C_{M_\lambda^{\rho,\tau}} &\to P(C_{M_\lambda^{\rho,\tau}}) \\
(t, \psi) &\mapsto \mathbb{G}(t, \psi) = \{x_t \ : \ x \in D(\psi)\}.
\end{aligned}
$$

The following two notions will be useful for our purpose.

Definition 6 *A multi-valued semiflow $\mathcal{G} : \mathbb{R}_+ \times X \to P(X)$ is called pointwise dissipative if there exists a bounded set $B \subset X$ that attracts the dynamics starting at all single points, i.e.*

$$\lim_{t \to +\infty} H^*(\mathcal{G}(t, x), B) = 0 \quad \forall x \in X.$$

It is called asymptotically compact if for any bounded set $B \subset X$ and any sequence $\{t_n\}$ with $t_n \to +\infty$, any sequence $\{\psi^{(n)}\}_n$ with $\psi^{(n)} \in \mathcal{G}(t_n, B)$, possesses a converging subsequence in X.

The following result was stated in [17] for complete metric spaces, but it really does not need the completeness.

Theorem 7 [cf. [17, Th.3]] *Let X be a metric space, and $\mathcal{G}$ be a pointwise dissipative and asymptotically compact multi-valued semiflow on X. Suppose that $\mathcal{G}(t, \cdot) : X \to C(X)$ is upper semicontinuous for any $t \geq 0$. Then $\mathcal{G}$ has a compact global attractor $\mathcal{A}$, that is, a compact invariant set, $\mathcal{G}(t)\mathcal{A} = \mathcal{A}$ for all $t \geq 0$, that attracts all bounded sets:*

$$\lim_{t \to +\infty} H^*(\mathcal{G}(t, B), \mathcal{A}) = 0 \qquad \forall B \text{ bounded.}$$

It is minimal among all closed sets attracting each bounded set.

We can establish now our main result in this section. For this, we borrow and adapt some ideas from Wang & Xu [18] and Ball [1] already used in [5] for infinite delay. This will lead to the starting estimates for Theorem 8 below, which improves the analogous result in [4]. Note that the additional condition (5) means that the dissipativity of F_0 dominates the effects of the other terms in the equation (2).

Theorem 8 *Assume the conditions in* Hypothesis 1. *If the following inequality holds,*

$$\alpha > k_2 + m_1, \tag{5}$$

then there exist constants A, B, and δ as in Lemma 4 *satisfying $\delta > B$. Moreover, the semiflow $\mathbb{G}^{(\lambda)}$ is pointwise attracted by the set*

$$B_0^{(\lambda)} = \{\psi \in C_{M_\lambda^{\rho,\tau}} \ : \ \|\psi\|^2_{M_\lambda^{\rho,\tau}} \leq K = \frac{A}{\delta - B}\},$$

that is, $\lim_{t \to +\infty} H^*_{C_{M_\lambda^{\rho,\tau}}}(\mathbb{G}^{(\lambda)}(t, \varphi), B_0^{(\lambda)}) = 0$ for all $\varphi \in C_{M_\lambda^{\rho,\tau}}$.

Proof. We start checking that it is posible to consider two constants δ and B in Lemma 4, which additionally satisfy $\delta > B$.

To take the smallest posible value of $B = \frac{k_2^2}{\varepsilon} + 2m_1$ in Lemma 4, we put the biggest possible divisor ε. Recall that we imposed the relation $\varepsilon + \bar{\varepsilon} = 2\alpha - \delta$, so we arrive to analyze the positive character of the function "$\delta - B$" given by $g(\delta) = \delta - 2m_1 - \frac{k_2^2}{2\alpha-\delta}$. This function is defined over the open interval $(0, 2\alpha)$.

Since $\lim_{\delta\to 2\alpha} g'(\delta) < 0$, it is clear that we must impose

$$\lim_{\delta\to 0} g'(\delta) > 0. \tag{6}$$

(Otherwise, $\max g = g(0) < 0$). The condition for (6) is $2\alpha > k_2$. In this case, the maximum of g is $2(\alpha - m_1 - k_2)$, therefore the first part of the theorem is proved.

Now, we proceed to prove the second statement in the theorem. Thanks to the first part, it has sense to consider the value $K = \frac{A}{\delta - B}$. We divide this proof in three steps.

Step 1: For any $R \geq 1$, $B_{C_{M_\lambda^{\rho,\tau}}}(0, \sqrt{RK})$, is positively invariant for the semi-flow $\mathbb{G}^{(\lambda)}$ associated with equation (2).

If not, there must be an initial datum ψ with $\|\psi\|^2_{M_\lambda^{\rho,\tau}} < RK$ and a solution x of (2) with $x_0 = \psi$ and a first time t_1 such that $\|x_{t_1}\|^2_{M_\lambda^{\rho,\tau}} = RK$, i.e. $|x(t_1)|^2 = RK$.

But from (3) we deduce that

$$\begin{aligned} |x(t_1)|^2 &< \mathrm{e}^{-\delta t_1} RK + \int_0^{t_1} \mathrm{e}^{-\delta(t_1 - s)}(A + BRK)\mathrm{d}s \\ &= \mathrm{e}^{-\delta t_1} RK + \frac{A + BRK}{\delta}(1 - \mathrm{e}^{-\delta t_1}). \end{aligned}$$

Observe that

$$\frac{A + BRK}{\delta} \leq \frac{R(A + BK)}{\delta} = RK,$$

which is a contradiction with $|x(t_1)|^2 = RK$.

Step 2: The closed ball $B_0^{(\lambda)} = \bar{B}_{C_{M_\lambda^{\rho,\tau}}}(0, \sqrt{K})$ attracts any solution of (2).

Consider a solution $x(\cdot)$ with initial data ψ with $\|\psi\|^2_{M_\lambda^{\rho,\tau}} = d \geq K$ (otherwise, the claim holds by Step 1).

Thanks to Step 1 we have that $|x(t)| \leq d$ for all $t \geq 0$. Therefore, $\limsup_{t\to+\infty} |x(t)|^2 = \sigma$ exists. Therefore,

$$\forall \varepsilon > 0 \quad \exists\, T_1(\varepsilon) > 0 \quad \text{such that} \quad |x(t)|^2 \leq \sigma + \varepsilon \quad \forall t \geq T_1(\varepsilon),$$
$$\Rightarrow \ \|x_t\|^2_{M^{\rho,\tau}_\lambda} \leq \sigma + \varepsilon \quad \forall t \geq T_1(\varepsilon) + M^{\rho,\tau}_\lambda. \tag{7}$$

Take now $T_2(\varepsilon)$ such that

$$\mathrm{e}^{-\delta t} d + \frac{A + Bd}{\delta} \left(\mathrm{e}^{-\delta T_2(\varepsilon)} - \mathrm{e}^{-\delta t} \right) \leq \varepsilon \ \forall t \geq T_2(\varepsilon). \tag{8}$$

So, for any $t \geq T_2(\varepsilon) + T_1(\varepsilon) + \max(\rho(\lambda), \tau(\lambda))$, from (3), splitting the integral in two parts,

$$|x(t)|^2 \leq \mathrm{e}^{-\delta t} |x(0)|^2 + \int_0^{t - T_2(\varepsilon)} \mathrm{e}^{-\delta(t-s)} (A + B \|x_s\|^2_{M^{\rho,\tau}_\lambda}) \mathrm{d}s$$
$$+ \int_{t - T_2(\varepsilon)}^{t} \mathrm{e}^{-\delta(t-s)} (A + B \|x_s\|^2_{M^{\rho,\tau}_\lambda}) \mathrm{d}s,$$

applying (8) to the first two terms in the sum (thanks to Step 1), and (7) to the last term, we obtain for all $t \geq T_2(\varepsilon) + T_1(\varepsilon) + \max(\rho(\lambda), \tau(\lambda))$:

$$|x(t)|^2 \leq \varepsilon + \frac{A + B(\sigma + \varepsilon)}{\delta} \left(1 - \mathrm{e}^{-\delta T(\varepsilon)} \right). \tag{9}$$

Passing to the limit as ε goes to zero, we deduce that

$$\sigma = \limsup_{t \to +\infty} |x(t)|^2 \leq \frac{A + B\sigma}{\delta},$$

in other words, $\sigma \leq \frac{A}{\delta - B} = K$, which proves the claim.

Step 3: We prove now the general result: the semiflow $\mathbb{G}^{(\lambda)}$ is pointwise dissipative, i.e. for any fixed initial data ψ, the set $G^{(\lambda)}(t, \psi)$ (possibly not a singleton) is attracted by $B_0^{(\lambda)}$.

Firstly let us denote (for an arbitrary $\eta > 0$)

$$B^{(\lambda)}_{0,\eta} = \{ \psi \in C_{M^{\rho,\tau}_\lambda} \ : \ \|\psi\|^2_{M^{\rho,\tau}_\lambda} \leq K + \eta \}.$$

We claim that $B^{(\lambda)}_{0,\eta}$ is absorbing for $\mathbb{G}(t, \psi)$ (since this will be proved for $\eta > 0$ arbitrarily small, we will obtain the main statement from this step).

We proceed by a contradiction argument. Assume that there exist a sequence of times $t_n \to +\infty$, and solutions $x^{(n)}_{t_n}$ with the same initial data $x^{(n)}_0 = \psi$ such that $x^{(n)}_{t_n} \notin B_{0,\eta}$.

Therefore, by the first step, we deduce that $x^{(n)}_t \notin B^{(\lambda)}_0$ for all $0 \leq t \leq t_n$. Besides this, we know that solutions are uniformly bounded since it is so for the (unique) initial datum. So, by the Ascoli-Arzelà Theorem and a diagonal procedure argument, we obtain the existence of a function $y \in C([0,+\infty);\mathbb{R}^d)$ and a subsequence (relabelled the same) such that

$$x^{(n)}|_{[0,T]} \to y|_{[0,T]} \quad \text{in } C([0,T];\mathbb{R}^d), \quad \forall T > 0.$$

In particular, extending y to $[-M_\lambda^{\rho,\tau}, 0]$ by ψ (denote this function again by y), we have that $x_t^n \to y_t$ for all $t \geq 0$. By standard arguments (cf. [7]) we deduce that y is solution of the problem, but on the other hand it satisfies

$$\|y_t\|^2_{M_\lambda^{\rho,\tau}} \geq K + \eta, \quad \forall t \geq 0.$$

This is a contradiction with the result of the second step since $B_0^{(\lambda)}$ attracts any solution, in particular y. ■

Remark 9 *Condition* (5)*, which will be sufficient to ensure the existence of attractors (see* Theorem 11 *below), improves* Theorem 35 *in* [4] *in the autonomous case. Even in the easiest situation* $2m_1eh \sim 1$ *in* [4]*, i.e. when one is forced to put* $\lambda \sim 2m_1 e$*, comparing* (26) *there with* (5) *here, our condition is less restrictive.*

The following result is an immediate consequence of the Ascoli-Arzelà Theorem, and its proof is similar to [4, Prop.10] or [5, Prop.2].

Proposition 10 *Consider* $T > 0$ *and a functional* $h : C_T \to \mathbb{R}^n$ *continuous, bounded (i.e. maps bounded sets onto bounded sets), and such that the DDE* $x'(t) = h(x_t)$ *generates a semiflow* $\mathcal{G}$*. If* $\mathcal{G}$ *satisfies the following boundedness condition,*

$$\forall R > 0,\ \exists M(R) > 0, \quad \text{such that} \quad \mathcal{G}(t, B_{C_T}(0,R)) \subset B_{C_T}(0, M(R)),$$

then $\mathcal{G}$ *has compact values, is upper semicontinuous and asymptotically compact.*

We can combine the above result with Theorem 7 and Theorem 8 to conclude the existence of attractors for the semiflows $\{\mathbb{G}^{(\lambda)}\}_{\lambda\in\Lambda}$ defined in Proposition 5.

Theorem 11 *Assume that* Hypothesis 1 *and* (5) *hold. Then, for each* $\lambda \in \Lambda$, (2) *generates a multi-valued semiflow* $\mathbb{G}^{(\lambda)} : \mathbb{R}_+ \times C_{M_\lambda^{\rho,\tau}} \to P(C_{M_\lambda^{\rho,\tau}})$*, which has compact values and is upper semicontinuous.*

Moreover, it possesses a global attractor $\mathbb{A}^{(\lambda)}$*, which satisfies a uniform bound (for all* λ*) on the Euclidean projected space* $\mathbb{R}^d$ *:*

$$\|\psi\|^2_{M_\lambda^{\rho,\tau}} \leq K, \qquad \forall \psi \in \mathbb{A}^{(\lambda)}, \tag{10}$$

where the constant K *is given in* Theorem 8.

Proof. By Theorem 8 we know that each $\mathbb{G}^{(\lambda)}$ is pointwise dissipative. Moreover, Step 1 gives the uniform boundedness condition in Proposition 10 to ensure compact values for $\mathbb{G}^{(\lambda)}$, upper semicontinuity, and asymptotic compactness.

The hypotheses of [17, Th.3] are satisfied, so we obtain the desired attractor $\mathbb{A}^{(\lambda)}$.

For the second statement, we follows the proof of Theorem 3 in [17]. Take any value $\varepsilon > 0$. The attractor $\mathbb{A}^{(\lambda)}$ coincides with the $\omega-$limit (in $\mathbb{G}^{(\lambda)}$) of the inflated ball $B_{C_{M_\lambda^{\rho,\tau}}}(B_0^{(\lambda)}, \varepsilon)$. Observe that, by Step 1, this set $B_{C_{M_\lambda^{\rho,\tau}}}(B_0^{(\lambda)}, \varepsilon)$ is positively invariant, so

$$\mathbb{A}^{(\lambda)} \subset B_{C_{M_\lambda^{\rho,\tau}}}(B_0^{(\lambda)}, \varepsilon).$$

The proof can be finished by taking into account the definition of $B_0^{(\lambda)} = B_{C_{M_\lambda^{\rho,\tau}}}(0, \sqrt{K})$. Observe that we have a uniform bound $K = \frac{A}{\delta - B}$ in the Euclidean projected space, which is independent of λ (see Lemma 4 and Theorem 8). ■

3 Upper semicontinuous dependence of the attractors on the parameter

Our aim now is to show an upper semicontinuous dependence on λ for the attractors $\mathbb{A}^{(\lambda)}$ obtained in Theorem 11. Observe that each $\mathbb{A}^{(\lambda)}$ lives in a potentially different state space.

Therefore, in order to compare the obtained attractors $\mathbb{A}^{(\lambda)}$ to our parametric problem (2), we need to do some adaptations.

First, a common state space is required for all the problems, independent of the parameter. We can achieve this by extending the multi-valued semiflows $\mathbb{G}^{(\lambda)} : \mathbb{R}_+ \times C_{M_\lambda^{\rho,\tau}} \to P(C_{M_\lambda^{\rho,\tau}})$.

The following result achieves this goal, and moreover, it ensures the existence of new attractors and establishes their relation with the obtained in the previous section.

Theorem 12 *Assume that* Hypothesis 1 *holds. Then, for each* $\lambda \in \Lambda$, *there exists a multi-valued semiflow* $\widehat{\mathbb{G}}^{(\lambda)} : \mathbb{R}_+ \times C_{T^*} \to P(C_{T^*})$ *that extends the multi-valued semiflow* $\mathbb{G}^{(\lambda)}$ *constructed in* Proposition 5 *in the following sense:*

$$\widehat{\mathbb{G}}^{(\lambda)}(t, \varphi)\Big|_{[-M_\lambda^{\rho,\tau},0]} = \mathbb{G}^{(\lambda)}(t, \varphi|_{[-M_\lambda^{\rho,\tau},0]}) \qquad \forall (t, \varphi) \in \mathbb{R}_+ \times C_{T^*}.$$

Moreover, the semiflow $\widehat{\mathbb{G}}^{(\lambda)}$ *possesses a global attractor* $\widehat{\mathbb{A}}^{(\lambda)}$*, which is related with the attractor* $\mathbb{A}^{(\lambda)}$ *obtained in* Theorem 11 *in the following way:*

$$\widehat{\mathbb{A}}^{(\lambda)} := \Big\{\psi \in C_{T^*} : \exists \text{ entire trajectory } \bar{\Phi}_t^{(\lambda)} \text{ of } \mathbb{G}^{(\lambda)} \text{ in } \mathbb{A}^{(\lambda)} \quad \text{with } \psi(s) = \bar{\phi}(s)\ \forall s \in [-T^*, 0]\Big\}, \tag{11}$$

where $\bar{\phi}(t)$ *is the projection in* $\mathbb{R}^d$ *of the entire solution* $\bar{\Phi}_t^{(\tau)}$ *defined by* $\bar{\phi}(t) := \bar{\Phi}_t^{(\tau)}(0)$ *for all* $t \in \mathbb{R}$.

Proof. For the first claim, we simply have to observe that it is possible to extend the definition of $f(\lambda, \cdot)$ to C_{T^*}. In other words, it is enough to consider an analogous DDE to (2) but instead of $f(\lambda, x_t)$, with right hand side $\widehat{f}(\lambda, \cdot) \in C(C_{T^*}; \mathbb{R}^d)$ defined as

$$\widehat{f}(\lambda, \widehat{\phi}) = f(\widehat{\phi}|_{[-M_\lambda^{\rho,\tau}, 0]}) \qquad \forall \widehat{\phi} \in C_{T^*}.$$

Actually, it is not difficult to check that

$$\widehat{f} \in C(\Lambda \times C_{T^*}; \mathbb{R}^d) \tag{12}$$

(see Remark 2(i)). The new parametric DDE

$$x'(t) = \widehat{f}(\lambda, x_t) \tag{13}$$

is settled as that in the above section, therefore we can apply Theorem 11 to ensure the existence of semiflow $\widehat{\mathbb{G}}^{(\lambda)}$ and attractor $\widehat{\mathbb{A}}^{(\lambda)} \in K(C_{T^*})$. Therefore, the second claim is proved.

Finally, the characterization (11) is a consequence of Theorem 14 given below. ■

Remark 13 *An abstract construction of an extended semiflow from a given one acting on delay phase spaces and without an explicit DDE can be done, see* [12, Th.6] *for a proof in the single-valued case.*

Theorem 14 *Suppose that a multi-valued semiflow* $\mathcal{G}^{(\tau)} : \mathbb{R}_+ \times C_\tau \to P(C_\tau)$ *has a global attractor* $\mathcal{A}^{(\tau)}$ *and that there exists an extended multi-valued semiflow*

$$\widehat{\mathcal{G}}^{(\tau)} : \mathbb{R}_+ \times C_T \to P(C_T)$$

in the sense given in Theorem 12*. Then,* $\widehat{\mathcal{G}}^{(\tau)}$ *has a global attractor* $\widehat{\mathcal{A}}^{(\tau)}$*, and it has the following characterization:*

$$\widehat{\mathcal{A}}^{(\tau)} := \Big\{ \psi \in C_{T^*} : \exists \ \textit{entire trajectory} \ \bar{\Phi}_t^{(\tau)} \textit{of} \ S^{(\tau)} \ \textit{in} \ \mathcal{A}^{(\tau)} \\ \textit{with} \ \psi(s) = \bar{\phi}(s) \ \forall s \in [-T^*, 0] \Big\},$$

where $\bar{\phi}(t)$ *is the projection in* $\mathbb{R}^d$ *of the entire solution* $\bar{\Phi}_t^{(\tau)}$ *defined by* $\bar{\phi}(t) := \bar{\Phi}_t^{(\tau)}(0)$ *for all* $t \in \mathbb{R}$.

Proof. The proof follows analogously to the proof of Theorem 7 in [12]. Although there the statement is single-valued, the arguments follow the same course changing *distance* there by *Hausdorff semidistance* here. ∎

Now we give the main result of this section, a comparison of the attractors $\widehat{\mathbb{A}}^{(\lambda)}$ obtained in Theorem 12 [all of them living in the same phase space C_{T^*}].

Theorem 15 *Assume that* Hypothesis 1 *holds. Then, the multi-valued map*

$$\Lambda \ni \lambda \mapsto \widehat{\mathbb{A}}^{(\lambda)} \in K(C_{T^*})$$

is upper semicontinuous, i.e. given $\lambda_0 \in \Lambda$,

$$H^*_{C_{T^*}}(\widehat{\mathbb{A}}^{(\lambda)}, \widehat{\mathbb{A}}^{(\lambda_0)}) \to 0 \qquad \textit{as} \quad \lambda \to \lambda_0.$$

Proof. Consider the set

$$\mathcal{K} = \bigcup_{\lambda \in E(\lambda_0)} \widehat{\mathbb{A}}^{(\lambda)},$$

where $E(\lambda_0) \subset \Lambda$ is a neighborhood of λ_0. Denote by $\overline{\mathcal{K}}$ its closure in C_{T^*}.

By the characterization in Theorem 12 of $\widehat{\mathbb{A}}^{(\lambda)}$ and the estimate (10), $\mathcal{K}$ is bounded in C_{T^*}. Moreover, by (12) and the Ascoli-Arzelà Theorem, $\overline{\mathcal{K}}$ is a compact set of C_{T^*}.

Now, using the invariance of the attractors under their own semiflows, the definition of $\mathcal{K}$, and a basic property of the Hausdorff semidistance, we arrive at the following inequality:

$$\begin{aligned} H^*_{C_{T^*}}(\widehat{\mathbb{A}}^{(\lambda)}, \widehat{\mathbb{A}}^{(\lambda_0)}) &= H^*_{C_{T^*}}(\widehat{\mathbb{G}}^{(\lambda)}(t, \widehat{\mathbb{A}}^{(\lambda)}), \widehat{\mathbb{A}}^{(\lambda_0)}) \\ &\leq H^*_{C_{T^*}}(\widehat{\mathbb{G}}^{(\lambda)}(t, \overline{\mathcal{K}}), \widehat{\mathbb{A}}^{(\lambda_0)}) \\ &\leq H^*_{C_{T^*}}(\widehat{\mathbb{G}}^{(\lambda)}(t, \overline{\mathcal{K}}), \widehat{\mathbb{G}}^{(\lambda_0)}(t, \overline{\mathcal{K}})) + H^*_{C_{T^*}}(\widehat{\mathbb{G}}^{(\lambda_0)}(t, \overline{\mathcal{K}}), \widehat{\mathbb{A}}^{(\lambda_0)}). \end{aligned}$$

As long as the second term in the last line is sufficiently small, which can be ensured provided that t is large enough, it only remains to prove that for an arbitrary $t \geq 0$ one can obtain

$$H^*_{C_{T^*}}(\widehat{\mathbb{G}}^{(\lambda)}(t, \overline{\mathcal{K}}), \widehat{\mathbb{G}}^{(\lambda_0)}(t, \overline{\mathcal{K}})) \to 0 \qquad \text{as} \quad \lambda \to \lambda_0. \tag{14}$$

We proceed by a contradiction argument. Suppose it is not so: then there exists $\varepsilon > 0$ and a sequence $\{\lambda^{(n)}\}_{n\geq 1} \subset \Lambda$ with $\lim_{n\to\infty} \lambda^{(n)} = \lambda_0$, such that

$$\sup_{x \in \widehat{\mathbb{G}}^{(\lambda^{(n)})}(t,\overline{\mathcal{K}})} d(x, \widehat{\mathbb{G}}^{(\lambda_0)}(t,\overline{\mathcal{K}})) \geq \varepsilon \qquad \forall n. \tag{15}$$

Fixing any positive value $\varepsilon' \in (0, \varepsilon)$, for each n, there exists $x^{(n)} \in \widehat{\mathbb{G}}^{(\lambda^{(n)})}(t, y^{(n)})$ with $y^{(n)} \in \overline{\mathcal{K}}$, such that

$$d(x^{(n)}, \widehat{\mathbb{G}}^{(\lambda_0)}(t,\overline{\mathcal{K}})) \geq \varepsilon'.$$

By definition, there exists a solution $\tilde{x}^{(n)}$ to the problem

$$\begin{aligned} \tilde{x}^{(n)}(s) = y^{(n)}(0) + \int_0^s \Big[& F_0(\tilde{x}^{(n)}(r)) + F_1(\tilde{x}^{(n)}(s - \rho(\lambda^{(n)}))) \\ & + \int_{-\tau(\lambda^{(n)})}^0 b(v, \tilde{x}^{(n)}(v+s))dv \Big] \mathrm{d}r \quad \forall s \in [0,t], \\ \tilde{x}_0^{(n)} = y^{(n)}, & \end{aligned} \tag{16}$$

with $x^{(n)}(\theta) = \tilde{x}^{(n)}(t+\theta)$ for all $\theta \in [-T^*, 0]$. Since $\overline{\mathcal{K}}$ is compact, there exists a converging subsequence (which we relabel the same) $y^{(n)} \to y \in \overline{\mathcal{K}}$.

Now we recall the positive invariance for any $\mathbb{G}^{(\lambda)}$ of any bounded ball proved in Step 1 in Theorem 8. Applying this for the extended semiflows $\widehat{\mathbb{G}}^{(\lambda)}$, and since $\mathcal{K}$ is bounded, we can deduce that all the solutions $\tilde{x}^{(n)}$ remain uniformly bounded. On other hand, the continuity of the map $\widehat{f}$ defined in (12) gives an upper bound of the derivatives of $\tilde{x}^{(n)}$. In summary, we have that $\{x^{(n)}\}_n$ is relatively compact. Now it is standard to continue the argument: extract a convergent subsequence (which we relabel the same) $x^{(n)} \to x \in C_{T^*}$, and passing to the limit in (16) using the dominated convergence theorem, we conclude that

$$x^{(n)} \to x \in \widehat{\mathbb{G}}^{(\lambda_0)}(t, y) \subset \widehat{\mathbb{G}}^{(\lambda_0)}(t, \overline{\mathcal{K}}),$$

which contradicts (15). ∎

Remark 16 *Observe that, although an upper semicontinuous condition on the semiflows w.r.t. the parameter like (C1) in* [16, Th.2, p.408] *is not possible here in a general bounded absorbing set $\mathcal{U}$, it is possible to circumvent this difficulty.*

The crucial point is to weaken the above condition. Instead of considering a general bounded absorbing set $\mathcal{U}$ as done in [16, Th.2]*, the upper semicontinuity result uses the compact set* $\overline{\mathcal{K}} = \overline{\bigcup_{\lambda \in E(\lambda_0)} \widehat{\mathbb{A}}^{(\lambda)}}$.

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