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ATTRACTORS FOR A DOUBLE TIME-DELAYED 2D-NAVIER-STOKES MODEL

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Dedicated to the memory of Professor José Real
with our deepest admiration, gratitude, and love.

Abstract. In this paper, a double time-delayed 2D-Navier-Stokes model is considered. It includes delays in the convective and the forcing terms. Existence and uniqueness results and suitable dynamical systems are established. We also analyze the existence of pullback attractors for the model in several phase-spaces and the relationship among them.

1. **Introduction and statement of the problem.** The importance of physical models for fluid mechanics problems including delay terms is related, for instance, to real applications where devices to control properties of fluids (temperature, velocity, etc.) are inserted in domains and make a local influence on the behaviour of the system (e.g., cf. [19] for a wind-tunnel model).

The study of Navier-Stokes models including delay terms –existence, uniqueness, stationary solutions, exponential decay, existence of attractors, etcetera– was initiated in the references [5, 6, 7], and after that, many different questions, as dealing with unbounded domains, and models (for instance in three dimensions for modified terms) have been addressed (e.g., cf. [28, 11, 21, 23, 9] among others).

While the theory of linear viscoelasticity in fluid mechanics has often considered the inclusion of delay effects in the viscous part of the model (e.g. cf. [26]), the inclusion in other parts has not been investigated so often.

In the recent paper [18] a time-delayed term in the Burgers' equation was considered. Such a kind of delay in the trajectory that a particle should follow could present some obstacles to a rigorous physical interpretation. However, as many other simplified and/or approximative models in fluid mechanics (with truncations, as the globally modified Navier-Stokes equations, e.g. cf. [4, 15, 16, 27, 20]), this kind of effect may be interesting to study from the mathematical point of view.

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Consider a bounded domain $\Omega \subset \mathbb{R}^2$, $\tau \in \mathbb{R}$, and the non-autonomous functional Navier-Stokes model

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu \Delta u + (u(t-\rho(t)) \cdot \nabla) u + \nabla p = f(t) + g(t, u_t) \text{ in } \Omega \times (\tau, \infty), \\ \operatorname{div} u = 0 \text{ in } \Omega \times (\tau, \infty), \\ u = 0 \text{ on } \partial\Omega \times (\tau, \infty), \\ u(x, \tau) = u^\tau(x) \text{ in } \Omega, \\ u(x, \tau + s) = \phi(x, s) \text{ in } \Omega \times (-h, 0), \end{cases} \tag{1}$$

where $\nu > 0$ is the kinematic viscosity, $u = (u^1, u^2)$ is the velocity field of the fluid, p is the pressure, f is a non-delayed external force field, g is another external force with some hereditary characteristics, u_t denotes –as usual– the delay function $u_t(s) = u(t+s)$ where it has sense. The delay function ρ in the convective term is assumed to belong to $C^1(\mathbb{R}; [0, h])$ with $\rho'(t) \leq \rho^* < 1$ for all $t \in \mathbb{R}$, where $h > 0$ is fixed, and u^τ and ϕ are the initial data in τ and $(\tau - h, \tau)$ respectively.

Existence, uniqueness, some regularity features for this model, and some partial long-time estimates were studied in [25] in dimension two (see [12] for the case in dimension three). The interesting point of the model in dimension two is that the natural estimate of u' is in $L^{4/3}(V')$ (see below for the proper definitions), as the Navier-Stokes equations in three dimensions without delay does. This means that, without any additional assumption on the phase-space, the appearance of a delay –however small it be– in the nonlinear term has an important influence. Therefore, the study of existence of attractor (or pullback attractor) is more involved for problem (1), leading to the same kind of (lack of uniqueness or lack of continuity in time) troubles (e.g. cf. [1, 24, 14, 13] for multi-valued approaches).

Our approach in this paper is to modify the phase-space improving slightly the initial conditions, such that existence and uniqueness of solution hold. For the associated single-valued process, we will study the existence of pullback attractors in different universes and the relation among them.

The structure of the paper is as follows. In the rest of this section we recall the abstract setting of the problem with the standard functional spaces, and the definition of a weak solution to problem (1). In Section 2 we recall for completeness the proof of existence of weak solutions, and the uniqueness under additional assumptions in the phase-space (that are closely related to the existence of an energy equality). Continuity with respect to the initial data are also given. Section 3 provides a very briefly summary on the theory of minimal pullback attractors, that will be used in the last part of the paper. Our main results are given in Section 4, where estimates on the solutions, absorption, and asymptotic compactness are proved, leading to the existence of several minimal pullback attractors, in different phase-spaces. We also establish some relations among these families. Finally, Section 5 is devoted to expose the above results in the autonomous framework. This allows to simplify the statements and concentrate in the problem of a delay perturbation in the convective term. Existence of global attractors and relationship among them are so deduced.

We will consider the usual functional spaces to deal with the problem in an abstract setting (e.g. cf. [17, 29]). Let be

$$\mathcal{V} = \{u \in (C_0^\infty(\Omega))^2 : \operatorname{div} u = 0\};$$

H is the closure of $\mathcal{V}$ in $(L^2(\Omega))^2$ with the norm $|\cdot|$, and inner product $(\cdot,\cdot)$, where for u, $v \in (L^2(\Omega))^2$,

$$(u,v) = \sum_{j=1}^{2} \int_\Omega u_j(x) v_j(x) dx;$$

V is the closure of $\mathcal{V}$ in $(H_0^1(\Omega))^2$ with the norm $\|\cdot\|$ associated to the inner product $((\cdot,\cdot))$, where for u, $v \in (H_0^1(\Omega))^2$,

$$((u,v)) = \sum_{i,j=1}^{2} \int_\Omega \frac{\partial u_j}{\partial x_i} \frac{\partial v_j}{\partial x_i} dx.$$

We will use $\|\cdot\|_*$ for the norm in V' and $\langle\cdot,\cdot\rangle$ for the duality between V' and V. We consider every element $h \in H$ as an element of V', given by the equality $\langle h, v\rangle = (h,v)$ for all $v \in V$. It follows that $V \subset H \subset V'$, where the injections are dense and continuous, and, in fact, compact.

Define the operator $A : V \to V'$ as

$$\langle Au, v\rangle := ((u,v)) \quad \forall\, u, v \in V.$$

Let us denote

$$b(u,v,w) = \sum_{i,j=1}^{2} \int_\Omega u_i \frac{\partial v_j}{\partial x_i} w_j \mathrm{d}x,$$

for every functions $u, v, w : \Omega \to \mathbb{R}^2$ for which the right-hand side is well defined.

In particular, b has sense for all $u, v, w \in V$, and is a continuous trilinear form on $V \times V \times V$. For suitable u and v (for instance in V) it is also useful to denote $B(u,v)$ the operator of V' given by $\langle B(u,v), w\rangle = b(u,v,w)$ for any $w \in V$.

On other hand, let us recall that the operator b satisfies

$$b(u,v,v) = 0 \qquad \forall\, u \in V, v \in (H_0^1(\Omega))^2, \tag{2}$$

and since we are in dimension two there exists a constant $C > 0$, depending only on Ω, such that

$$|b(u,v,w)| \le C|u|^{1/2}\|u\|^{1/2}\|v\||w|^{1/2}\|w\|^{1/2} \qquad \forall\, u, v, w \in V. \tag{3}$$

Before continuing, for short, we introduce the notation $L^p_X = L^p(-h, 0; X)$, which will be used in the sequel for suitable choices of p and X. The norm in these spaces will be denoted by $\|\cdot\|_{L^p_X}$. On other hand, $C_H = C([-h,0]; H)$ will also be used, and the sup norm in C_H will be denoted by $|\cdot|_{C_H}$. Finally, $\overline{B}_E(0,\alpha)$ will denote the closed ball in a metric space E of center zero and radius α.

The second delay operator is $g : \mathbb{R} \times C_H \to (L^2(\Omega))^2$, and we assume that it satisfies the following assumptions:

(H1) for all $\xi \in C_H$, the function $\mathbb{R} \ni t \mapsto g(t,\xi) \in (L^2(\Omega))^2$ is measurable,
(H2) $g(t,0) = 0$, for all $t \in \mathbb{R}$,
(H3) there exists $L_g > 0$ such that for all $t \in \mathbb{R}$, and for all ξ, $\eta \in C_H$,

$$|g(t,\xi) - g(t,\eta)| \le L_g |\xi - \eta|_{C_H},$$

(H4) there exists $C_g > 0$ such that for all $\tau \le t$ and for all u, $v \in C([\tau - h, t]; H)$

$$\int_\tau^t |g(r,u_r) - g(r,v_r)|^2 dr \le C_g^2 \int_{\tau-h}^t |u(r) - v(r)|^2 dr.$$

Examples of fixed, variable, and distributed delay operators can be found, for instance, in [5, Section 3], [7, Sections 3.5 and 3.6], and [11, Section 3], and we omit them here just for the sake of brevity.

Remark 1. From (H1)–(H3), for $T > \tau$ and $u \in C([\tau - h, T]; H)$, the function $g_u : [\tau, T] \to (L^2(\Omega))^2$ given by $g_u(t) = g(t, u_t)$ is measurable and belongs to $L^\infty(\tau, T; (L^2(\Omega))^2)$. By using (H4), the mapping

$$C([\tau - h, T]; H) \ni u \mapsto \mathcal{G}(u) := g_u \in L^2(\tau, T; (L^2(\Omega))^2)$$

has a unique extension to a mapping $\widetilde{\mathcal{G}}$ which is uniformly continuous from $L^2(\tau - h, T; H)$ into $L^2(\tau, T; (L^2(\Omega))^2)$. We will still denote by $g(t, u_t) = \widetilde{\mathcal{G}}(u)(t)$ for each $u \in L^2(\tau - h, T; H)$, and therefore assumption (H4) will hold for all u, $v \in L^2(\tau - h, T; H)$.

Concerning the goal of finding solutions to problem (1), different choices are possible for the initial data.

Let us consider that $u^\tau \in H$, $\phi \in L^2_V$, and $f \in L^2_{loc}(\mathbb{R}; V')$.

Definition 1. A weak solution to (1) is a function $u \in L^\infty(\tau, T; H) \cap L^2(\tau - h, T; V)$ for all $T > \tau$, such that $u(\tau) = u^\tau$, $u_\tau = \phi$, and satisfies

$$\frac{d}{dt}(u(t), v) + \nu\langle Au(t), v\rangle + b(u(t - \rho(t)), u(t), v) = \langle f(t), v\rangle + (g(t, u_t), v) \quad \forall v \in V,$$

where the equation must be understood in the sense of $\mathcal{D}'(\tau, \infty)$.

Remark 2. Let us observe that if u is a weak solution to (1), from (3), in particular we have that there exists a constant $\widetilde{C} > 0$ such that for any $v \in V$,

$$|b(u(t - \rho(t)), u(t), v)| \leq \widetilde{C}\|u(t - \rho(t))\|\|u(t)\|^{1/2}|u(t)|^{1/2}\|v\|,$$

where we have used the continuous embedding of V into H.

Then, by Young inequality, we conclude that $B(u(\cdot - \rho(\cdot)), u(\cdot)) \in L^{4/3}(\tau, T; V')$.

Therefore, $u' \in L^{4/3}(\tau, T; V')$ too. So, $u \in C([\tau, T]; V')$ and in particular (e.g. cf. [29]) $u \in C_w([\tau, T]; H)$ for all $T > \tau$ (whence to impose an initial datum $u^\tau \in H$ is meaningful).

Although the above choice of phase-space will lead to an existence result (see Theorem 1 below), the well-posedness of the problem in the sense of Hadamard will require more regularity on the initial data, pointing out that the above was an unnatural choice (compare with Remark 3 and Theorem 2 below).

2. Existence of solutions, uniqueness, and continuity results.

We have the following result concerning existence of weak solutions. It is also worth mentioning that the delay in the convective term, even if h is small, does matter in the sense that uniqueness of solution to (1) is unknown (compare Remark 2 –essentially as the case without delay in dimension three– with Remark 3 and Theorem 2 below, where this difficulty is sorted out).

Theorem 1. *Consider $u^\tau \in H$, $\phi \in L^2_V$, $f \in L^2_{loc}(\mathbb{R}; V')$, and $g : \mathbb{R} \times C_H \to (L^2(\Omega))^2$ satisfying assumptions (H1)–(H4). Then, there exists at least one weak solution $u(\cdot; \tau, u^\tau, \phi)$ to (1).*

Proof. The existence of weak solution can be proved as in [25, Theorem 2.1], and we include its proof here just for the sake of clarity.

Consider a special basis of H formed by normalized eigenfunctions of the Stokes operator, $\{w_j\}_{j\geq 1}$, with corresponding eigenvalues $\{\lambda_j\}_{j\geq 1}$ being $0<\lambda_1\leq\lambda_2\leq\dots$ with $\lim_{j\to\infty}\lambda_j=\infty$. Pose the approximate problems (for each $k\geq 1$) of finding $u^k\in V_k:=\mathrm{span}[w_1,\dots,w_k]$ with $u^k(t)=\sum_{j=1}^k\gamma_{jk}(t)w_j$ such that

$$\begin{aligned}&\frac{d}{dt}(u^k(t),w_j)+\nu\langle Au^k(t),w_j\rangle+b(u^k(t-\rho(t)),u^k(t),w_j)\\ =\;&\langle f(t),w_j\rangle+(g(t,u^k_t),w_j),\quad a.e.\ t>\tau,\ \forall\, 1\leq j\leq k,\end{aligned}\tag{4}$$

fulfilled with the initial conditions

$$u^k(\tau)=P_ku^\tau \text{ and } u^k(\tau+s)=P_k\phi(s) \text{ in } s\in(-h,0),$$

where P_k is the orthogonal projector from H onto V_k.

It is well known (e.g. cf. [5]) that the above system of ordinary functional differential equations (the unknowns are $\{\gamma_{jk}\}_{j=1}^k$) is well-posed in some local interval $[\tau,t_k)$. We fix a value $T>\tau$ and will provide uniform estimates that will imply that actually it holds that $t_k=T$ and pass to the limit via compactness arguments, whence existence of a weak solution on (τ,T) will be ensured.

Indeed, multiplying each equation in (4) by $\gamma_{jk}(t)$ and summing from $j=1$ to k, we obtain

$$\frac{1}{2}\frac{d}{dt}|u^k(t)|^2+\nu\|u^k(t)\|^2=\langle f(t),u^k(t)\rangle+(g(t,u^k_t),u^k(t)),\quad a.e.\ t\in(\tau,t_k),$$

where we have used (2) to remove the nonlinear term b.

By integrating in time, from Hölder and Young inequalities, and the assumptions on the delay operator g, we obtain that

$$\begin{aligned}&|u^k(t)|^2+2\nu\int_\tau^t\|u^k(s)\|^2ds\\ \leq\;&|u^\tau|^2+\frac{1}{\nu}\int_\tau^t\|f(s)\|_*^2ds+\nu\int_\tau^t\|u^k(s)\|^2ds+\int_\tau^t|g(s,u^k_s)|^2ds+\int_\tau^t|u^k(s)|^2ds\\ \leq\;&|u^\tau|^2+C_g^2\int_{-h}^0|\phi(s)|^2ds+\frac{1}{\nu}\int_\tau^t\|f(s)\|_*^2ds+\nu\int_\tau^t\|u^k(s)\|^2ds\\ &+(1+C_g^2)\int_\tau^t|u^k(s)|^2ds\end{aligned}$$

for all $t\in[\tau,t_k)$.

So, we deduce that

$$\begin{aligned}&|u^k(t)|^2+\nu\int_\tau^t\|u^k(s)\|^2ds\\ \leq\;&|u^\tau|^2+C_g^2\int_{-h}^0|\phi(s)|^2ds+\frac{1}{\nu}\int_\tau^t\|f(s)\|_*^2ds+(1+C_g^2)\int_\tau^t|u^k(s)|^2ds\end{aligned}$$

for all $t\in[\tau,t_k)$.

Now, from Gronwall lemma, we conclude that $t_k=T$, and that $\{u^k\}$ is bounded in $L^\infty(\tau,T;H)\cap L^2(\tau-h,T;V)$. Moreover, from (3) (see also Remark 2) we have that $\{du^k/dt\}$ is bounded in $L^{4/3}(\tau,T;V')$, whence by compactness results, the Dominated Convergence Theorem, assumption (H4), and Remark 1, we may extract

a subsequence (relabelled the same) and ensure the existence of a function $u \in L^2(\tau-h,T;V)$ with $du/dt \in L^{4/3}(\tau,T;V')$ with $u_\tau = \phi$, such that

$$\begin{array}{rl} u^k \to u & \text{strongly in } L^2(\tau-h,T;H), \\ u^k \rightharpoonup u & \text{weakly in } L^2(\tau-h,T;V), \\ du^k/dt \rightharpoonup du/dt & \text{weakly in } L^{4/3}(\tau,T;V'), \\ u^k(\cdot-\rho(\cdot)) \to u(\cdot-\rho(\cdot)) & \text{strongly in } L^2(\tau,T;H), \\ g(\cdot,u^k_\cdot) \to g(\cdot,u_\cdot) & \text{strongly in } L^2(\tau,T;H). \end{array} \tag{5}$$

It is standard to pass to the limit in (4). Just for clarity, we point out how to deal with the delayed convective term, which is the novelty here. Indeed, it holds that

$$b(u^k(\cdot-\rho(\cdot)),u^k(\cdot),w_j) \to b(u(\cdot-\rho(\cdot)),u(\cdot),w_j) \quad \text{in } L^1(\tau,T),$$

since

$$\begin{aligned} & |b(u^k(t-\rho(t)),u^k(t),w_j) - b(u(t-\rho(t)),u(t),w_j)| \\ = \; & |b(u^k(t-\rho(t)),w_j,u^k(t)) - b(u(t-\rho(t)),w_j,u(t)) \pm b(u(t-\rho(t)),w_j,u^k(t))| \\ \leq \; & |b(u^k(t-\rho(t))-u(t-\rho(t)),w_j,u^k(t))| + |b(u(t-\rho(t)),w_j,u^k(t)-u(t))|. \end{aligned}$$

We will prove that the first addend in the right hand-side goes to zero in the $L^1(\tau,T)$ norm (the second addend follows analogously). Using (3), Hölder inequality, and the fact that w_j is an eigenfunction of the Stokes operator, we have that

$$\begin{aligned} & \int_\tau^T |b(u^k(t-\rho(t))-u(t-\rho(t)),w_j,u^k(t))|dt \\ \leq \; & C\lambda_j^{1/2}\|u^k(\cdot-\rho(\cdot))-u(\cdot-\rho(\cdot))\|^{1/2}_{L^2(\tau,T;H)}\|u^k(\cdot-\rho(\cdot))-u(\cdot-\rho(\cdot))\|^{1/2}_{L^2(\tau,T;V)} \\ & \times\|u^k(\cdot)\|^{1/2}_{L^2(\tau,T;H)}\|u^k(\cdot)\|^{1/2}_{L^2(\tau,T;V)}. \end{aligned}$$

From (5) the above goes to zero, and the claim is proved.

Thus, we conclude that u is a weak solution to (1) in the interval (τ,T).

By concatenation of solutions, it is clear that we obtain at least one global (defined on (τ,∞)) weak solution to (1). □

If we modify slightly the initial data we may improve the above result in the sense that we gain an energy equality (and therefore uniqueness of solution and continuity of the solutions with respect to initial data). So we will be in a good position to study the associated dynamical system (which will be continuous). Roughly speaking, what we do now is to impose on the initial data the same regularity as we expect for the weak solutions.

Remark 3. Suppose that $u^\tau \in H$ and $\phi \in L^2_V \cap L^\infty_H$. Then, we may improve the regularity for the operator $B(u(\cdot-\rho(\cdot)),u(\cdot))$ obtained in Remark 2. Indeed, from (3) we have that for any $v \in V$,

$$|b(u(t-\rho(t)),u(t),v)| \leq C|u(t-\rho(t))|^{1/2}\|u(t-\rho(t))\|^{1/2}\|v\||u(t)|^{1/2}\|u(t)\|^{1/2}. \tag{6}$$

Therefore, we can conclude now that $B(u(\cdot-\rho(\cdot)),u(\cdot)) \in L^2(\tau,T;V')$ for all $T>\tau$, and so $u' \in L^2(\tau,T;V')$ and $u \in C([\tau,T];H)$ for all $T>\tau$. Now, the following energy equality holds for any solution to (1),

$$\begin{aligned} & |u(t)|^2 + 2\nu\int_s^t \|u(r)\|^2 dr \\ = \; & |u(s)|^2 + 2\int_s^t \langle f(r),u(r)\rangle dr + 2\int_s^t (g(r,u_r),u(r))dr \quad \forall \tau \leq s \leq t. \end{aligned} \tag{7}$$

Next, we establish a uniqueness result for problem (1).

Theorem 2. *Consider $u^\tau \in H$, $\phi \in L^2_V \cap L^\infty_H$, $f \in L^2_{loc}(\mathbb{R}; V')$, and $g : \mathbb{R} \times C_H \to (L^2(\Omega))^2$ satisfying assumptions (H1)–(H4). Then, there exists a unique weak solution to (1), $u(\cdot; \tau, u^\tau, \phi) \in C([\tau, \infty); H)$, which satisfies the energy equality (7).*

Moreover, if for short we denote by $u(\cdot)$ and $v(\cdot)$ the corresponding solutions to (1) with respective initial data (u^τ, ϕ) and (v^τ, ψ), then

$$\begin{aligned}
\operatorname*{ess\,sup}_{s\in(t-h,t)} |u(s) - v(s)|^2 &\leq \left(\|\phi - \psi\|^2_{L^\infty_H} + (\lambda_1^{-1} + \frac{\nu}{2}) \int_{-h}^0 \|\phi(s) - \psi(s)\|^2 ds \right) \\
&\quad \times exp\left(\widehat{C} \int_\tau^t (\|u(s)\|^2 + 1) ds \right), \qquad (8)\\
\nu \int_\tau^t \|u(s) - v(s)\|^2 ds &\leq \left(\|\phi - \psi\|^2_{L^\infty_H} + (\lambda_1^{-1} + \frac{\nu}{2}) \int_{-h}^0 \|\phi(s) - \psi(s)\|^2 ds \right) \qquad (9)\\
&\times \left[1 + \widehat{C} exp\left(\widehat{C} \int_\tau^t (\|u(s)\|^2 + 1) ds \right) \int_\tau^t (\|u(s)\|^2 + 1) ds \right]
\end{aligned}$$

for all $t \geq \tau$, where $\widehat{C} = C^2 \nu^{-1} (1 - \rho^*)^{-1/2} + C_g^2 + 1$.

Proof. The existence of at least one weak solution was already proved in Theorem 1. The energy equality (7) was given in Remark 3 for any solution to (1). So, it only remains to check uniqueness, and estimates (8) and (9). Actually, we will obtain uniqueness as a by-product of (8).

Indeed, consider two solutions $u(\cdot)$ and $v(\cdot)$ to (1) with corresponding initial data (u^τ, ϕ) and (v^τ, ψ) respectively, and denote $w = u - v$. Then, from (2) and the energy equality for w, we obtain that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|w(t)|^2 + \nu\|w(t)\|^2 + b(w(t - \rho(t)), u(t), w(t)) \\
= \;& (g(t, u_t) - g(t, v_t), w(t)), \quad a.e.\ t > \tau. \qquad (10)
\end{aligned}$$

Now, as ϕ, $\psi \in L^2_V \cap L^\infty_H$, by (3) we have the following estimate for the trilinear term b,

$$\begin{aligned}
&|b(w(t - \rho(t)), u(t), w(t))| \\
\leq \;& C|w(t - \rho(t))|^{1/2} \|w(t - \rho(t))\|^{1/2} \|u(t)\| |w(t)|^{1/2} \|w(t)\|^{1/2} \\
\leq \;& C \operatorname*{ess\,sup}_{r\in(t-h,t)} |w(r)| \|u(t)\| \|w(t - \rho(t))\|^{1/2} \|w(t)\|^{1/2}.
\end{aligned}$$

Integrating in time (10) and using the above estimate, the assumptions on g, and Young and Hölder inequalities with a suitable constant (to be fixed later on), we deduce that

$$\begin{aligned}
&|w(t)|^2 + 2\nu \int_\tau^t \|w(s)\|^2 ds \\
\leq \;& |w(\tau)|^2 + \frac{C^2}{\varepsilon} \int_\tau^t \operatorname*{ess\,sup}_{r\in(s-h,s)} |w(r)|^2 \|u(s)\|^2 ds + \varepsilon \int_\tau^t \|w(s - \rho(s))\| \|w(s)\| ds \\
&+ 2\int_\tau^t (g(s, u_s) - g(s, v_s), w(s)) ds
\end{aligned}$$

$$\leq \quad |w(\tau)|^2 + \frac{C^2}{\varepsilon}\int_\tau^t \operatorname*{ess\,sup}_{r\in(s-h,s)} |w(r)|^2\|u(s)\|^2 ds + \frac{\nu}{2}\int_\tau^t \|w(s)\|^2 ds$$
$$+\frac{\varepsilon^2}{2\nu}\int_\tau^t \|w(s-\rho(s))\|^2 ds + \int_{\tau-h}^t |w(s)|^2 ds + C_g^2\int_\tau^t |w(s)|^2 ds \quad \forall t\geq\tau.$$

In particular, after a change of variable in the integral of $w(s-\rho(s))$, thanks to the upper bound on ρ', and choosing $\varepsilon^2=\nu^2(1-\rho^*)$, we arrive at

$$|w(t)|^2 + \nu\int_\tau^t \|w(s)\|^2 ds$$
$$\leq \quad |w(\tau)|^2 + (\lambda_1^{-1}+\nu/2)\int_{\tau-h}^\tau \|w(s)\|^2 ds$$
$$+\frac{C^2}{\varepsilon}\int_\tau^t \operatorname*{ess\,sup}_{r\in(s-h,s)} |w(r)|^2\|u(s)\|^2 ds + (1+C_g^2)\int_\tau^t |w(s)|^2 ds \quad \forall t\geq\tau. \tag{11}$$

Thus, neglecting the integral term in the left hand side above, putting $s\in(t-h,t)$ instead of t, and taking the essential supremum in the resulting left hand side, we conclude that

$$\operatorname*{ess\,sup}_{s\in(t-h,t)} |w(s)|^2 \quad \leq \quad \|w_\tau\|^2_{L^\infty_H} + (\lambda_1^{-1}+\nu/2)\int_{\tau-h}^\tau \|w(s)\|^2 ds$$
$$+\left(\frac{C^2}{\varepsilon}+C_g^2+1\right)\int_\tau^t (\|u(s)\|^2+1)\operatorname*{ess\,sup}_{r\in(s-h,s)} |w(r)|^2 ds$$

for all $t\geq\tau$, whence (8) holds by applying Gronwall lemma.

Finally, (9) is a consequence of (11) by using (8). □

Remark 4. It is worth mentioning that even with $\phi\in L^2_V$ alone, the regularisation of the equation means that after an elapsed time h the weak solution obtained in Theorem 1 becomes well-posed and continuous. The problem is that before that elapsed time we cannot guarantee uniqueness of solution. So a possible dynamical system in such phase-space $H\times L^2_V$ would be eventually multi-valued, which means that all the study of the asymptotic behaviour would be an open question (among many conditional results for this type of problems, we recall the seminal paper by J. M. Ball [1]).

3. **Existence and comparison of minimal pullback attractors.** We give a brief summary of some well-known abstract results on existence and comparison of minimal pullback attractors for dynamical systems (e.g. cf. [2, 3, 22, 8]).

Consider given a metric space (X,d_X), and let us denote $\mathbb{R}^2_d=\{(t,\tau)\in\mathbb{R}^2:\tau\leq t\}$.

A process $\mathcal{U}$ on X is a mapping $\mathbb{R}^2_d\times X\ni(t,\tau,x)\mapsto\mathcal{U}(t,\tau)x\in X$ such that $\mathcal{U}(\tau,\tau)x=x$ for any $(\tau,x)\in\mathbb{R}\times X$, and $\mathcal{U}(t,r)(\mathcal{U}(r,\tau)x)=\mathcal{U}(t,\tau)x$ for any $\tau\leq r\leq t$ and all $x\in X$.

A process $\mathcal{U}$ is said to be continuous if for any pair $\tau\leq t$, the mapping $\mathcal{U}(t,\tau):X\to X$ is continuous.

On other hand, a process $\mathcal{U}$ is said to be closed if for any $\tau\leq t$, and any sequence $\{x_n\}\subset X$, if $x_n\to x\in X$ and $\mathcal{U}(t,\tau)x_n\to y\in X$, then $\mathcal{U}(t,\tau)x=y$. It is clear that every continuous process is closed.

Let us denote by $\mathcal{P}(X)$ the family of all nonempty subsets of X, and consider a family of nonempty sets $\widehat{D}_0=\{D_0(t):t\in\mathbb{R}\}\subset\mathcal{P}(X)$.

Definition 2. We say that a process $\mathcal{U}$ on X is pullback $\widehat{D}_0$-asymptotically compact if for any $t \in \mathbb{R}$ and any sequences $\{\tau_n\} \subset (-\infty, t]$ and $\{x_n\} \subset X$ satisfying $\tau_n \to -\infty$ and $x_n \in D_0(\tau_n)$ for all n, the sequence $\{\mathcal{U}(t, \tau_n)x_n\}$ is relatively compact in X.

Denote

$$\Lambda(\widehat{D}_0, t) = \bigcap_{s \le t} \overline{\bigcup_{\tau \le s} \mathcal{U}(t, \tau) D_0(\tau)}^X \quad \forall\, t \in \mathbb{R},$$

where $\overline{\{\cdots\}}^X$ is the closure in X.

Given two subsets of X, $\mathcal{O}_1$ and $\mathcal{O}_2$, we denote by $\mathrm{dist}_X(\mathcal{O}_1, \mathcal{O}_2)$ the Hausdorff semi-distance in X between them, defined as

$$\mathrm{dist}_X(\mathcal{O}_1, \mathcal{O}_2) = \sup_{x \in \mathcal{O}_1} \inf_{y \in \mathcal{O}_2} d_X(x, y).$$

Let be given $\mathcal{D}$ a nonempty class of families parameterized in time $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$. The class $\mathcal{D}$ will be called a universe in $\mathcal{P}(X)$.

Definition 3. A process $\mathcal{U}$ on X is said to be pullback $\mathcal{D}$-asymptotically compact if it is pullback $\widehat{D}$-asymptotically compact for any $\widehat{D} \in \mathcal{D}$.

It is said that $\widehat{D}_0 = \{D_0(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ is pullback $\mathcal{D}$-absorbing for the process $\mathcal{U}$ on X if for any $t \in \mathbb{R}$ and any $\widehat{D} \in \mathcal{D}$, there exists a $\tau_0(t, \widehat{D}) \le t$ such that

$$\mathcal{U}(t, \tau) D(\tau) \subset D_0(t) \quad \forall\, \tau \le \tau_0(t, \widehat{D}).$$

Next result was proved in [8, Theorem 3.11].

Theorem 3. *Consider a closed process $\mathcal{U} : \mathbb{R}^2_d \times X \to X$, a universe $\mathcal{D}$ in $\mathcal{P}(X)$, and a family $\widehat{D}_0 = \{D_0(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ which is pullback $\mathcal{D}$-absorbing for $\mathcal{U}$, and assume also that $\mathcal{U}$ is pullback $\widehat{D}_0$-asymptotically compact.*

Then, the family $\mathcal{A}_{\mathcal{D}} = \{\mathcal{A}_{\mathcal{D}}(t) : t \in \mathbb{R}\}$ defined by $\mathcal{A}_{\mathcal{D}}(t) = \overline{\bigcup_{\widehat{D} \in \mathcal{D}} \Lambda(\widehat{D}, t)}^X$, has the following properties:

(a) *for any $t \in \mathbb{R}$, the set $\mathcal{A}_{\mathcal{D}}(t)$ is a nonempty compact subset of X, and $\mathcal{A}_{\mathcal{D}}(t) \subset \Lambda(\widehat{D}_0, t)$,*
(b) *$\mathcal{A}_{\mathcal{D}}$ is pullback $\mathcal{D}$-attracting, i.e., $\lim_{\tau \to -\infty} \mathrm{dist}_X(\mathcal{U}(t, \tau) D(\tau), \mathcal{A}_{\mathcal{D}}(t)) = 0$ for all $\widehat{D} \in \mathcal{D}$, and any $t \in \mathbb{R}$,*
(c) *$\mathcal{A}_{\mathcal{D}}$ is invariant, i.e., $\mathcal{U}(t, \tau)\mathcal{A}_{\mathcal{D}}(\tau) = \mathcal{A}_{\mathcal{D}}(t)$ for all $(t, \tau) \in \mathbb{R}^2_d$,*
(d) *if $\widehat{D}_0 \in \mathcal{D}$, then $\mathcal{A}_{\mathcal{D}}(t) = \Lambda(\widehat{D}_0, t) \subset \overline{D_0(t)}^X$ for all $t \in \mathbb{R}$.*

The family $\mathcal{A}_{\mathcal{D}}$ is minimal in the sense that if $\widehat{C} = \{C(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ is a family of closed sets such that for any $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \in \mathcal{D}$, $\lim_{\tau \to -\infty} \mathrm{dist}_X(\mathcal{U}(t, \tau) D(\tau), C(t)) = 0$, then $\mathcal{A}_{\mathcal{D}}(t) \subset C(t)$.

Remark 5. Under the assumptions of Theorem 3, the family $\mathcal{A}_{\mathcal{D}}$ is called the minimal pullback $\mathcal{D}$-attractor for the process $\mathcal{U}$.

If $\mathcal{A}_{\mathcal{D}} \in \mathcal{D}$, then it is the unique family of closed subsets in $\mathcal{D}$ that satisfies (b)–(c).

A sufficient condition for $\mathcal{A}_{\mathcal{D}} \in \mathcal{D}$ is to have that $\widehat{D}_0 \in \mathcal{D}$, the set $D_0(t)$ is closed for all $t \in \mathbb{R}$, and the family $\mathcal{D}$ is inclusion-closed (i.e., if $\widehat{D} \in \mathcal{D}$, and $\widehat{D}' = \{D'(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ with $D'(t) \subset D(t)$ for all t, then $\widehat{D}' \in \mathcal{D}$).

We will denote by $\mathcal{D}_F(X)$ the universe of fixed nonempty bounded subsets of X, i.e., the class of all families $\widehat{D}$ of the form $\widehat{D} = \{D(t) = D : t \in \mathbb{R}\}$ with D a fixed nonempty bounded subset of X.

Now, it is easy to conclude the following result.

Corollary 1. *Under the assumptions of Theorem 3, if the universe $\mathcal{D}$ contains the universe $\mathcal{D}_F(X)$, then both attractors, $\mathcal{A}_{\mathcal{D}_F(X)}$ and $\mathcal{A}_{\mathcal{D}}$, exist, and $\mathcal{A}_{\mathcal{D}_F(X)}(t) \subset \mathcal{A}_{\mathcal{D}}(t)$ for all $t \in \mathbb{R}$.*

Moreover, if for some $T \in \mathbb{R}$, the set $\cup_{t \le T} D_0(t)$ is a bounded subset of X, then $\mathcal{A}_{\mathcal{D}_F(X)}(t) = \mathcal{A}_{\mathcal{D}}(t)$ for all $t \le T$.

4. **Dynamical system associated to (1) and long-time behaviour.** In view of Theorems 1 and 2, we will apply the above abstract results in the phase-space $X = H \times (L^2_V \cap L^\infty_H)$, which is a Banach space with the norm $\|(\zeta, \phi)\|_X = |\zeta| + \|\phi\|_{L^2_V} + \|\phi\|_{L^\infty_H}$ for a pair $(\zeta, \phi) \in X$.

The first consequence after the Theorems 1 and 2 is the following

Corollary 2. *Consider given $f \in L^2_{loc}(\mathbb{R}; V')$ and $g : \mathbb{R} \times C_H \to (L^2(\Omega))^2$ satisfying (H1)–(H4). Then, the biparametric family of mappings $S(t,\tau) : H \times (L^2_V \cap L^\infty_H) \to H \times (L^2_V \cap L^\infty_H)$, with $(t,\tau) \in \mathbb{R}^2_d$, given by $S(t,\tau)(u^\tau, \phi) = (u(t), u_t)$ where u is the weak solution to (1), defines a continuous process.*

Now we introduce an additional assumption in order to obtain some energy estimates.

(H5) Assume that $\nu\lambda_1 > C_g$, and that there exists a value $\eta \in (0, 2(\nu\lambda_1 - C_g))$ such that for every $u \in L^2(\tau - h, t; H)$,

$$\int_\tau^t e^{\eta s} |g(s, u_s)|^2 \, ds \le C_g^2 \int_{\tau - h}^t e^{\eta s} |u(s)|^2 \, ds \quad \forall t \ge \tau.$$

We have the following result (cf. [10]), which proof is included only for the sake of completeness.

Lemma 1. *Consider given $f \in L^2_{loc}(\mathbb{R}; V')$ and $g : \mathbb{R} \times C_H \to (L^2(\Omega))^2$ satisfying conditions (H1)–(H5). Then, for any $(u^\tau, \phi) \in H \times (L^2_V \cap L^\infty_H)$, the following inequalities hold for the solution u to (1) for all $t \ge s \ge \tau$:*

$$|u(t)|^2 \le e^{-\eta(t-\tau)}(|u^\tau|^2 + C_g \|\phi\|^2_{L^2_H}) + \frac{e^{-\eta t}}{\beta} \int_\tau^t e^{\eta r} \|f(r)\|^2_* dr, \quad (12)$$

$$\nu \int_s^t \|u(r)\|^2 dr \le |u(s)|^2 + C_g \|u_s\|^2_{L^2_H} + \frac{1}{\nu} \int_s^t \|f(r)\|^2_* dr + 2C_g \int_s^t |u(r)|^2 dr, \quad (13)$$

where

$$\beta = 2\nu - (\eta + 2C_g)\lambda_1^{-1} > 0. \quad (14)$$

Proof. By the energy equality (7) and Young inequality, we have

$$\begin{aligned} &\frac{d}{dt}|u(t)|^2 + 2\nu\|u(t)\|^2 \\ \le\ & \beta\|u(t)\|^2 + \beta^{-1}\|f(t)\|^2_* + C_g|u(t)|^2 + C_g^{-1}|g(t, u_t)|^2, \quad a.e.\ t > \tau. \end{aligned}$$

Thus,

$$\begin{aligned} &\frac{d}{dt}\big(e^{\eta t}|u(t)|^2\big) + e^{\eta t}\big(2\nu - \beta - (\eta + C_g)\lambda_1^{-1}\big)\|u(t)\|^2 \\ \le\ & e^{\eta t}\beta^{-1}\|f(t)\|^2_* + e^{\eta t}C_g^{-1}|g(t, u_t)|^2, \quad a.e.\ t > \tau, \end{aligned}$$

and therefore, integrating in time above and using property (H5), we obtain

$$\begin{aligned} & e^{\eta t}|u(t)|^2 + \left(2\nu - \beta - (\eta + C_g)\lambda_1^{-1}\right) \int_\tau^t e^{\eta r}\|u(r)\|^2\, dr \\ \leq\ & e^{\eta\tau}|u^\tau|^2 + \beta^{-1}\int_\tau^t e^{\eta r}\|f(r)\|_*^2\, dr + C_g \int_{\tau-h}^t e^{\eta r}|u(r)|^2\, dr \\ \leq\ & e^{\eta\tau}\left(|u^\tau|^2 + C_g\int_{-h}^0 |\phi(r)|^2 dr\right) + \beta^{-1}\int_\tau^t e^{\eta r}\|f(r)\|_*^2\, dr + C_g\int_\tau^t e^{\eta r}|u(r)|^2\, dr, \end{aligned}$$

for all $t \geq \tau$, and from this last inequality and (14), in particular we deduce (12).

Finally, observing that

$$\begin{aligned} & \frac{d}{dt}|u(t)|^2 + 2\nu\|u(t)\|^2 \\ \leq\ & \nu\|u(t)\|^2 + \nu^{-1}\|f(t)\|_*^2 + C_g|u(t)|^2 + C_g^{-1}|g(t,u_t)|^2, \quad a.e.\ t > \tau, \end{aligned}$$

and integrating in $[s,t]$, by (H4) we conclude (13). □

At the light of the previous result, we will now define an appropriate concept of (tempered) universe for problem (1).

Definition 4. We will denote by $\mathcal{D}_\eta^{H,L_H^2}(H \times (L_V^2 \cap L_H^\infty))$ the class of all families of nonempty subsets $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{P}(H \times (L_V^2 \cap L_H^\infty))$ such that

$$\lim_{\tau\to-\infty}\left(e^{\eta\tau}\sup_{(\zeta,\varphi)\in D(\tau)}(|\zeta|^2 + \|\varphi\|_{L_H^2}^2)\right) = 0.$$

Observe that the above definition does not make the most use of the natural norm of (ζ,φ) in $H \times (L_V^2 \cap L_H^\infty)$, but just in $H \times L_H^2$. Another immediate observation is that the above universe is inclusion-closed.

We will denote by $\mathcal{D}_F(H \times (L_V^2 \cap L_H^\infty))$ the universe of fixed bounded sets in $H \times (L_V^2 \cap L_H^\infty)$.

As a consequence of Lemma 1 we have the following

Corollary 3. *Assume that $f \in L_{loc}^2(\mathbb{R}; V')$ satisfies*

$$\int_{-\infty}^0 e^{\eta r}\|f(r)\|_*^2 dr < \infty, \tag{15}$$

and $g : \mathbb{R} \times C_H \to (L^2(\Omega))^2$ fulfills conditions (H1)–(H5). Then, the family $\widehat{D}_0 = \{D_0(t) : t \in \mathbb{R}\} \subset \mathcal{P}(H \times (L_V^2 \cap L_H^\infty))$ defined by

$$D_0(t) = \overline{B}_H(0, R_H(t)) \times (\overline{B}_{L_V^2}(0, R_V(t)) \cap \overline{B}_{L_H^\infty}(0, R_H(t))),$$

where

$$\begin{aligned} R_H^2(t) &= 1 + \beta^{-1}e^{-\eta(t-2h)}\int_{-\infty}^t e^{\eta r}\|f(r)\|_*^2 dr, \\ R_V^2(t) &= \nu^{-1}\left[(1 + 3C_g h)R_H^2(t) + \nu^{-1}\|f\|_{L^2(t-h,t;V')}^2\right], \end{aligned}$$

is pullback $\mathcal{D}_\eta^{H,L_H^2}(H \times (L_V^2 \cap L_H^\infty))-$absorbing for the process S on $H \times (L_V^2 \cap L_H^\infty)$ (and therefore pullback $\mathcal{D}_F(H \times (L_V^2 \cap L_H^\infty))-$absorbing too), and $\widehat{D}_0$ belongs to $\mathcal{D}_\eta^{H,L_H^2}(H \times (L_V^2 \cap L_H^\infty))$.

Proof. Fix $t \in \mathbb{R}$. From (12) we deduce that for any $\widehat{D} \in \mathcal{D}_\eta^{H,L^2_H}(H \times (L^2_V \cap L^\infty_H))$ there exists $\tau(\widehat{D}, t) \leq t - 2h$ such that

$$|u(t;\tau,u^\tau,\phi)|^2 \leq 1 + \beta^{-1} e^{-\eta t} \int_{-\infty}^{t} e^{\eta r} \|f(r)\|_*^2 dr$$

for all $t \geq \tau$ with $\tau \leq \tau(\widehat{D}, t)$ and $(u^\tau, \phi) \in D(\tau)$.

In particular, we deduce that $\|u_t\|^2_{L^\infty_H} \leq R^2_H(t)$.

Now, putting $s = t - h$ in (13) and using the above estimate, an immediate computation leads to $\|u_t\|^2_{L^2_V} \leq R^2_V(t)$.

The fact that $\widehat{D}_0$ belongs to $\mathcal{D}_\eta^{H,L^2_H}(H \times (L^2_V \cap L^\infty_H))$ follows from the definition of R_H and (15) (cf. Definition 4). The proof is finished. □

Lemma 2. *Under the assumptions of Corollary 3, the process S is pullback $\mathcal{D}_\eta^{H,L^2_H}(H \times (L^2_V \cap L^\infty_H))-$asymptotically compact.*

Proof. Fix $t \in \mathbb{R}$, and consider a family $\widehat{D} \in \mathcal{D}_\eta^{H,L^2_H}(H \times (L^2_V \cap L^\infty_H))$, and sequences $\{\tau_n\} \subset (-\infty, t]$ with $\tau_n \to -\infty$, $\{(u^{\tau_n}, \phi^n)\}$ with $(u^{\tau_n}, \phi^n) \in D(\tau_n)$ for all n. Denote for short $u^n(\cdot) = u(\cdot; \tau_n, u^{\tau_n}, \phi^n)$.

Analogously to Corollary 3, from Lemma 1 we have that there exists $\tau(\widehat{D}, t) < t - 4h - 1$ such that the subsequence $\{u^n : \tau_n \leq \tau(\widehat{D}, t)\}$ is bounded in $L^\infty(t - 4h - 1, t; H) \cap L^2(t - 3h - 1, t; V)$, and thanks to (H4) and (6), $\{(u^n)'\}$ is bounded in $L^2(t - 2h - 1, t; V')$. Therefore, by the Aubin-Lions compactness lemma (e.g., cf. [17]), there exists $u \in L^\infty(t - 4h - 1, t; H) \cap L^2(t - 3h - 1, t; V)$ with $u' \in L^2(t - 2h - 1, t; V')$ such that, for a subsequence (relabelled the same), the following convergences hold,

$$\begin{array}{ll} u^n \overset{*}{\rightharpoonup} u & \text{weakly-star in } L^\infty(t-4h-1,t;H), \\ u^n \rightharpoonup u & \text{weakly in } L^2(t-3h-1,t;V), \\ (u^n)' \rightharpoonup u' & \text{weakly in } L^2(t-2h-1,t;V'), \\ u^n \to u & \text{strongly in } L^2(t-2h-1,t;H), \\ u^n(s) \to u(s) & \text{strongly in } H, \ a.e.\ s \in (t-2h-1,t). \end{array} \tag{16}$$

From (H4) we also have that

$$g(\cdot, u^n_\cdot) \to g(\cdot, u_\cdot) \quad \text{strongly in } L^2(t - h - 1, t; H).$$

In particular, observe that thanks to the above convergences $u \in C([t - 2h - 1, t]; H)$ is a weak solution to (1) in $(t - h - 1, t)$ with u_{t-h-1} as initial datum.

We also deduce from (16) that $\{u^n\}$ is equi-continuous on $[t-2h-1, t]$ with values in V'. From the boundedness of $\{u^n\}$ in $C([t - 2h - 1, t]; H)$ and the compactness of the injection of H into V', by the Ascoli-Arzelà theorem we conclude that a subsequence (relabelled the same) satisfies

$$u^n \to u \quad \text{strongly in } C([t - 2h - 1, t]; V'). \tag{17}$$

Using once more the boundedness of $\{u^n\}$ in $C([t - 2h - 1, t]; H)$, we have that for any sequence $\{s_n\} \subset [t - 2h - 1, t]$ with $s_n \to s_*$, it holds that

$$u^n(s_n) \rightharpoonup u(s_*) \quad \text{weakly in } H, \tag{18}$$

where we have used (17) to identify the weak limit.

Claim 1: $u^n \to u$ strongly in $C([t - h, t]; H)$.

If not, there would exist $\varepsilon > 0$, a value $t_* \in [t-h, t]$, and subsequences (relabelled the same) $\{u^n\}$ and $\{t_n\} \subset [t-h, t]$, with $\lim_n t_n = t_*$, such that

$$|u^n(t_n) - u(t_*)| \geq \varepsilon \quad \forall n \geq 1. \tag{19}$$

Moreover, from (18) we have that

$$|u(t_*)| \leq \liminf_{n\to\infty} |u^n(t_n)|. \tag{20}$$

From the energy equality (7) for u^n and for u, we deduce that the following functions are non-increasing in $[t-h-1, t]$:

$$\begin{aligned} J_n(s) &:= \frac{1}{2}|u^n(s)|^2 - \int_{t-h-1}^{s} \langle f(r), u^n(r)\rangle dr - \int_{t-h-1}^{s} (g(r, u^n_r), u^n(r))dr, \\ J(s) &:= \frac{1}{2}|u(s)|^2 - \int_{t-h-1}^{s} \langle f(r), u(r)\rangle dr - \int_{t-h-1}^{s} (g(r, u_r), u(r))dr. \end{aligned}$$

Moreover, J and J_n are continuous, and by the above convergences, we have that

$$J_n(s) \to J(s) \quad a.e.\ s \in (t-h-1, t).$$

Therefore, it is possible to choose $\{\tilde{t}_k\} \subset (t-h-1, t_*)$ satisfying $\lim_k \tilde{t}_k = t_*$ and

$$\lim_n J_n(\tilde{t}_k) = J(\tilde{t}_k) \quad \forall k.$$

Consider an arbitrary value $\delta > 0$. By the continuity of J, there exists k_δ such that

$$|J(\tilde{t}_k) - J(t_*)| < \delta/2 \quad \forall\, k \geq k_\delta.$$

Now, let us take $n(k_\delta)$ such that for all $n \geq n(k_\delta)$ it holds

$$t_n \geq \tilde{t}_{k_\delta} \quad \text{and} \quad |J_n(\tilde{t}_{k_\delta}) - J(\tilde{t}_{k_\delta})| < \delta/2.$$

Then, since all J_n are non-increasing, we deduce that for all $n \geq n(k_\delta)$

$$\begin{aligned} J_n(t_n) - J(t_*) &\leq J_n(\tilde{t}_{k_\delta}) - J(t_*) \\ &\leq |J_n(\tilde{t}_{k_\delta}) - J(t_*)| \\ &\leq |J_n(\tilde{t}_{k_\delta}) - J(\tilde{t}_{k_\delta})| + |J(\tilde{t}_{k_\delta}) - J(t_*)| < \delta. \end{aligned}$$

Therefore, as $\delta > 0$ is arbitrary, we obtain that $\limsup_{n\to\infty} J_n(t_n) \leq J(t_*)$, and consequently, by (16),

$$\limsup_{n\to\infty} |u^n(t_n)| \leq |u(t_*)|,$$

whence, jointly with (20) and (18), gives the strong convergence $u^n(t_n) \to u(t_*)$ in H, in contradiction with (19). Thus, Claim 1 is proved.

Claim 2: $u^n \to u$ strongly in $L^2(t-h, t; V)$.

Indeed, by using again the energy equality (7) satisfied by u and u^n, all the convergences in (16), and Claim 1, we conclude that $\|u^n_t\|_{L^2_V} \to \|u_t\|_{L^2_V}$. This convergence of the norms, jointly with the weak convergence already proved in (16), concludes this Claim 2.

The proof follows from Claims 1 and 2. □

From the above results, we may establish the main result of the paper.

Theorem 4. *Assume that $f \in L^2_{loc}(\mathbb{R};V')$ satisfies (15) and $g : \mathbb{R} \times C_H \to (L^2(\Omega))^2$ fulfills conditions (H1)–(H5). Then, there exist the minimal pullback attractors $\{\mathcal{A}_{\mathcal{D}_F(H\times(L^2_V\cap L^\infty_H))}(t)\}_{t\in\mathbb{R}}$ and $\{\mathcal{A}_{\mathcal{D}^{H,L^2_H}_\eta(H\times(L^2_V\cap L^\infty_H))}(t)\}_{t\in\mathbb{R}}$, both belonging to $\mathcal{D}^{H,L^2_H}_\eta(H\times(L^2_V\cap L^\infty_H))$ and the following relations hold:*

$$\mathcal{A}_{\mathcal{D}_F(H\times(L^2_V\cap L^\infty_H))}(t) \subset \mathcal{A}_{\mathcal{D}^{H,L^2_H}_\eta(H\times(L^2_V\cap L^\infty_H))}(t) \subset D_0(t) \quad \forall t \in \mathbb{R}. \tag{21}$$

Moreover, if f satisfies the stronger requirement

$$\sup_{s\le 0}\int_{s-1}^{s} \|f(r)\|_*^2 dr < \infty, \tag{22}$$

then both attractors coincide, i.e., (21) becomes an equality for all $t \in \mathbb{R}$.

Proof. Since the process S is continuous in $H \times (L^2_V \cap L^\infty_H)$ by Corollary 2, there exists a pullback absorbing family $\widehat{D}_0 \in \mathcal{D}^{H,L^2_H}_\eta(H \times (L^2_V \cap L^\infty_H))$ by Corollary 3, and the process S is pullback $\mathcal{D}^{H,L^2_H}_\eta(H \times (L^2_V \cap L^\infty_H))$−asymptotically compact by Lemma 2, the existence of $\mathcal{A}_{\mathcal{D}^{H,L^2_H}_\eta(H\times(L^2_V\cap L^\infty_H))}$ and $\mathcal{A}_{\mathcal{D}_F(H\times(L^2_V\cap L^\infty_H))}$ follows from Theorem 3 and Corollary 1 respectively.

Moreover, the inclusion relations in (21) also follow from Corollary 1 and Theorem 3 respectively.

The fact that $\mathcal{A}_{\mathcal{D}^{H,L^2_H}_\eta(H\times(L^2_V\cap L^\infty_H))}$ belongs to $\mathcal{D}^{H,L^2_H}_\eta(H \times (L^2_V \cap L^\infty_H))$ is due to Remark 5, since the pullback absorbing family $\widehat{D}_0 \in \mathcal{D}^{H,L^2_H}_\eta(H \times (L^2_V \cap L^\infty_H))$ has closed sections and this universe is inclusion-closed.

Finally, the last claim of the coincidence of both families of pullback attractors under assumption (22) follows from Corollary 1, taking into account that $\sup_{t\le T} R_H(t)$ and $\sup_{t\le T} R_V(t)$ are bounded for any $T \in \mathbb{R}$. □

4.1. **A slight improvement: phase-space involving C_H.** It is clear by the invariance of the minimal pullback attractors under the process S and from Remark 3 that the second component of any time section of $\mathcal{A}_{\mathcal{D}_F(H\times(L^2_V\cap L^\infty_H))}$ and $\mathcal{A}_{\mathcal{D}^{H,L^2_H}_\eta(H\times(L^2_V\cap L^\infty_H))}$ lives in C_H.

The goal of this section is to compare these two families of attractors with others associated to this problem, related to the space $L^2_V \cap C_H$.

In order to do so, we need to introduce some additional notation.

Analogously to Definition 4, let us introduce a new universe.

Definition 5. Denote by $\mathcal{D}^{C_H}_\eta(L^2_V \cap C_H)$ the class of all families of nonempty subsets $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{P}(L^2_V \cap C_H)$ such that

$$\lim_{\tau\to-\infty}\left(e^{\eta\tau}\sup_{\varphi\in D(\tau)}|\varphi|^2_{C_H}\right) = 0.$$

We will also denote by $\mathcal{D}_F(L^2_V \cap C_H)$ the universe of fixed bounded sets in $L^2_V \cap C_H$.

Observe that $\mathcal{D}^{C_H}_\eta(L^2_V \cap C_H)$ is inclusion-closed.

Let us also introduce the biparametric family of mappings

$$U(t,\tau) : L^2_V \cap C_H \to L^2_V \cap C_H$$

for any $(t,\tau) \in \mathbb{R}^2_d$, by $U(t,\tau)\phi = u_t(\cdot;\tau,\phi(0),\phi)$ for any $\phi \in L^2_V \cap C_H$.

It is clear that U is also a process on $L^2_V \cap C_H$, and that under the assumptions of Theorem 2, U is continuous.

Before establishing the main result of this section, we need an auxiliar lemma.

Lemma 3. *Under the assumptions of Corollary 3, let consider* $\widehat{D} \in \mathcal{D}_\eta^{H,L^2_H}(H \times (L^2_V \cap L^\infty_H))$ *and* $r \geq h$. *Then, the family* $\widehat{D}^{(r)} = \{D^{(r)}(\tau) : \tau \in \mathbb{R}\}$, *where* $D^{(r)}(\tau) = \{u_{\tau+r}(\cdot;\tau,u^\tau,\phi) : (u^\tau,\phi) \in D(\tau)\}$, *belongs to* $\mathcal{D}_\eta^{C_H}(L^2_V \cap C_H)$.

Proof. By Theorem 2 it is clear that $D^{(r)}(\tau) \subset L^2_V \cap C_H$.

Fix an arbitrary value $\tau \in \mathbb{R}$ and denote by u, for short, the solution to (1) with (arbitrary) initial data $(u^\tau,\phi) \in D(\tau)$. From (12) we can deduce that

$$\begin{aligned} &\sup_{u_{\tau+r}\in D^{(r)}(\tau)} \left(e^{\eta\tau}|u_{\tau+r}|^2_{C_H}\right) \\ \leq\ & e^{-\eta(r-h)}e^{\eta\tau} \sup_{(u^\tau,\phi)\in D(\tau)} (|u^\tau|^2 + C_g\|\phi\|^2_{L^2_H}) + \beta^{-1}e^{-\eta(r-h)}\int_\tau^{\tau+r} e^{\eta s}\|f(s)\|^2_* ds. \end{aligned}$$

Taking into account that f satisfies (15), the proof is finished. □

Now we may establish the following result.

Theorem 5. *Assume that* $f \in L^2_{loc}(\mathbb{R};V')$ *satisfies (15) and* $g : \mathbb{R}\times C_H \to (L^2(\Omega))^2$ *fulfills conditions (H1)–(H5). Then, there exist the minimal pullback attractors* $\{\mathcal{A}_{\mathcal{D}_F(L^2_V\cap C_H)}(t)\}_{t\in\mathbb{R}}$ *and* $\{\mathcal{A}_{\mathcal{D}_\eta^{C_H}(L^2_V\cap C_H)}(t)\}_{t\in\mathbb{R}}$ *in* $L^2_V \cap C_H$ *for the universes of fixed bounded sets and for those with tempered growth in* $L^2_V \cap C_H$.

Both pullback attractors belong to $\mathcal{D}_\eta^{C_H}(L^2_V \cap C_H)$ *and the following relations hold:*

$$\mathcal{A}_{\mathcal{D}_F(L^2_V\cap C_H)}(t) \subset \mathcal{A}_{\mathcal{D}_\eta^{C_H}(L^2_V\cap C_H)}(t) \quad \forall t \in \mathbb{R}, \tag{23}$$

$$j(\mathcal{A}_{\mathcal{D}_F(L^2_V\cap C_H)}(t)) \subset \mathcal{A}_{\mathcal{D}_F(H\times(L^2_V\cap L^\infty_H))}(t) \quad \forall t \in \mathbb{R}, \tag{24}$$

$$j(\mathcal{A}_{\mathcal{D}_\eta^{C_H}(L^2_V\cap C_H)}(t)) = \mathcal{A}_{\mathcal{D}_\eta^{H,L^2_H}(H\times(L^2_V\cap L^\infty_H))}(t) \quad \forall t \in \mathbb{R}, \tag{25}$$

where $j : L^2_V \cap C_H \to H \times (L^2_V \cap L^\infty_H)$ *is defined by* $j(\varphi) = (\varphi(0),\varphi)$.

Finally, if f *satisfies (22), then, the inclusions in (23) and (24) are in fact equalities for all* $t \in \mathbb{R}$.

Proof. From Corollary 3, we have that the family $\widehat{D}_1 = \{D_1(t) : t \in \mathbb{R}\} \subset \mathcal{P}(L^2_V \cap C_H)$ given by

$$D_1(t) = \overline{B}_{L^2_V}(0,R_V(t)) \cap \overline{B}_{C_H}(0,R_H(t))$$

is pullback $\mathcal{D}_\eta^{C_H}(L^2_V \cap C_H)-$absorbing for U on $L^2_V \cap C_H$.

It is also immediate to check that $\widehat{D}_1 \in \mathcal{D}_\eta^{C_H}(L^2_V \cap C_H)$, and its time sections are closed.

From Lemma 2 we have that U is pullback $\mathcal{D}_\eta^{C_H}(L^2_V \cap C_H)-$asymptotically compact.

Therefore, we may apply again Theorem 3 and Corollary 1 to conclude the existence of the minimal pullback attractors in the statement and the inclusion relation (23).

Relations (24) and (25) through the canonical embedding j from $L^2_V \cap C_H$ to $H \times (L^2_V \cap L^\infty_H)$ can be obtained by the construction of the attractors, arguments of minimality of minimal pullback attractors, and estimates after a time-shift of length h, in the same manner as in [23, Theorem 5] or [10, Theorem 23].

Indeed, in order to prove (24), fix an arbitrary value $t \in \mathbb{R}$, and observe that

$$\mathcal{A}_{\mathcal{D}_F(L^2_V \cap C_H)}(t) = \overline{\bigcup_{\substack{B \subset L^2_V \cap C_H \\ B \text{ bounded}}} \Lambda_{L^2_V \cap C_H}(B,t)}^{L^2_V \cap C_H},$$

where the symbol $\Lambda_{L^2_V \cap C_H}$ denotes the omega-limit construction with respect to the topology of the space $L^2_V \cap C_H$.

Analogously, we have that

$$\mathcal{A}_{\mathcal{D}_F(H \times (L^2_V \cap L^\infty_H))}(t) = \overline{\bigcup_{\substack{B \subset H \times (L^2_V \cap L^\infty_H) \\ B \text{ bounded}}} \Lambda_{H \times (L^2_V \cap L^\infty_H)}(B,t)}^{H \times (L^2_V \cap L^\infty_H)},$$

where the symbol $\Lambda_{H \times (L^2_V \cap L^\infty_H)}$ denotes the omega-limit construction with respect to the topology of the space $H \times (L^2_V \cap L^\infty_H)$.

Now, observe that for any bounded set $B \subset L^2_V \cap C_H$, since the operator j is clearly linear and continuous, then $j(B)$ is also bounded in $H \times (L^2_V \cap L^\infty_H)$.

If $x \in \Lambda_{L^2_V \cap C_H}(B,t)$, then there exist sequences $\{\tau_n\}$, with $\tau_n \leq t$ for all n, and $\lim_n \tau_n = -\infty$, and $\{x^n\} \subset B$, such that

$$x = \lim_{\tau_n \to -\infty} U(t,\tau_n)x^n \quad \text{in } L^2_V \cap C_H.$$

But this implies that

$$(x(0),x) = \lim_{\tau_n \to -\infty} S(t,\tau_n)(x^n(0),x^n) \quad \text{in } H \times (L^2_V \cap L^\infty_H),$$

whence we deduce that

$$j(\Lambda_{L^2_V \cap C_H}(B,t)) \subset \Lambda_{H \times (L^2_V \cap L^\infty_H)}(j(B),t)$$

for all bounded set $B \subset L^2_V \cap C_H$. Thus, (24) follows.

The inclusion to the right in (25) can be proved analogously. Let us now prove the inclusion to the left in (25). Indeed, for any $t \in \mathbb{R}$ and $\widehat{D} \in \mathcal{D}^{H,L^2_H}_\eta(H \times (L^2_V \cap L^\infty_H))$, we have that for any $\tau < t - h$

$$\begin{aligned}
& \text{dist}_{H \times (L^2_V \cap L^\infty_H)}(S(t,\tau)D(\tau), j(\mathcal{A}_{\mathcal{D}^{C_H}_\eta(L^2_V \cap C_H)}(t))) \\
= \; & \text{dist}_{H \times (L^2_V \cap L^\infty_H)}(S(t,\tau+h)(S(\tau+h,\tau)D(\tau)), j(\mathcal{A}_{\mathcal{D}^{C_H}_\eta(L^2_V \cap C_H)}(t))) \\
= \; & \text{dist}_{H \times (L^2_V \cap L^\infty_H)}(j(U(t,\tau+h)D^{(h)}(\tau)), j(\mathcal{A}_{\mathcal{D}^{C_H}_\eta(L^2_V \cap C_H)}(t))) \\
\leq \; & C(j)\text{dist}_{L^2_V \cap C_H}(U(t,\tau+h)D^{(h)}(\tau), \mathcal{A}_{\mathcal{D}^{C_H}_\eta(L^2_V \cap C_H)}(t)),
\end{aligned}$$

where we have used the notation introduced in Lemma 3 for the family $\widehat{D}^{(h)}$, which belongs to $\mathcal{D}^{C_H}_\eta(L^2_V \cap C_H)$, and once more the fact that j is a linear and continuous operator from $L^2_V \cap C_H$ to $H \times (L^2_V \cap L^\infty_H)$.Thus, we have that the right-hand side of the above inequality goes to zero when τ goes to $-\infty$, and so the left-hand side also does. Therefore, the inclusion

$$\mathcal{A}_{\mathcal{D}^{H,L^2_H}_\eta(H \times (L^2_V \cap L^\infty_H))}(t) \subset j(\mathcal{A}_{\mathcal{D}^{C_H}_\eta(L^2_V \cap C_H)}(t))$$

follows since $\mathcal{A}_{\mathcal{D}^{H,L^2_H}_\eta(H \times (L^2_V \cap L^\infty_H))}(t)$ is the minimal closed set in $H \times (L^2_V \cap L^\infty_H)$ that attracts any family $\widehat{D} \in \mathcal{D}^{H,L^2_H}_\eta(H \times (L^2_V \cap L^\infty_H))$ at time t in a pullback sense.

Last claim about the equalities in (23) and (24) when f also satisfies (22) follows again from Corollary 1 since then it holds that $\sup_{t\leq T} R_H(t)$ and $\sup_{t\leq T} R_V(t)$ are bounded for any $T \in \mathbb{R}$. This gives immediately the equality in (23). Then, combining this with the equality in (25) and the equality in (21), we conclude that (24) becomes an equality too, for all $t \in \mathbb{R}$. □

Remark 6. Under the assumptions of the above theorem, if besides f satisfies (22), then, for each $T \in \mathbb{R}$, the sets

$$\{\mathcal{A}_{\mathcal{D}_\eta^{C_H}(L_V^2\cap C_H)}(t)\}_{t\leq T} \quad \text{and} \quad \{\mathcal{A}_{\mathcal{D}_\eta^{H,L_H^2}(H\times(L_V^2\cap L_H^\infty))}(t)\}_{t\leq T}$$

are bounded in $L_V^2 \cap C_H$ and $H \times (L_V^2 \cap L_H^\infty)$ respectively.

5. **The autonomous case.** In this section we translate and adapt the previous results to the framework of time-independent forces. Observe that without an explicit dependence on time, the dynamical system then becomes autonomous, which means that only the elapsed time is important, rather than the pair of initial and final times. Actually, the autonomous results are just a particular case of all the previous exposition, but for some readers it might be a more clear exposition of the nature of the problem itself without the interferences of non-autonomous modifications. In particular, we will be able to state the existence of the global attractor for the cited (autonomous) dynamical system under suitable conditions.

Consider the functional Navier-Stokes model

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu\Delta u + (u(t-h)\cdot\nabla)u + \nabla p = f + g(u_t) \text{ in } \Omega\times(0,\infty), \\ \operatorname{div} u = 0 \text{ in } \Omega\times(0,\infty), \\ u = 0 \text{ on } \partial\Omega\times(0,\infty), \\ u(x,0) = u^0(x) \text{ in } \Omega, \\ u(x,s) = \phi(x,s) \text{ in } \Omega\times(-h,0), \end{cases} \tag{26}$$

where all the unknowns were already explained in the introduction of the paper ($h > 0$ is fixed and now $\rho \equiv h$). Observe too that f, the non-delayed external force field, and g, the external force with some hereditary characteristics, are time-independent. Let us also observe that in contrast to (1), here $\tau = 0$ (actually, since the problem is autonomous, the initial time is not relevant).

For the delay operator g we assume that $g : C_H \to (L^2(\Omega))^2$ satisfies (observe that the assumption (H1) holds trivially in this framework):

(H2') $g(0) = 0$,

(H3') there exists $L_g > 0$ such that for all ξ, $\eta \in C_H$,

$$|g(\xi) - g(\eta)| \leq L_g|\xi - \eta|_{C_H},$$

(H4') there exists $C_g > 0$ such that for all $0 \leq \tau \leq t$, and for all u, $v \in C([-h,t];H)$

$$\int_\tau^t |g(u_r) - g(v_r)|^2 dr \leq C_g^2 \int_{\tau-h}^t |u(r) - v(r)|^2 dr.$$

Then, the immediate translation of the first existence result (cf. Theorem 1) is the following

Theorem 6. *Consider $u^0 \in H$, $\phi \in L_V^2$, $f \in V'$, and $g : C_H \to (L^2(\Omega))^2$ satisfying assumptions (H2')–(H4'). Then, there exists at least one weak solution $u(\cdot;0,u^0,\phi)$ to (26).*

Of course, the concept of weak solution given in Definition 1 to problem (1) just needs to substitute $\tau = 0$ to be referred to problem (26).

The trouble of this result arises from the fact that (cf. Remark 2) $B(u(\cdot - h), u(\cdot)) \in L^{4/3}(0,T;V')$, which does not allow to apply an energy equality, and therefore uniqueness is unknown. However, if we improve slightly the initial data (cf. Remark 3) we gain $B(u(\cdot - h), u(\cdot)) \in L^2(0,T;V')$. We state these results precisely in the following

Theorem 7. *Consider $u^0 \in H$, $\phi \in L^2_V \cap L^\infty_H$, $f \in V'$, and $g : C_H \to (L^2(\Omega))^2$ satisfying assumptions (H2')–(H4'). Then, there exists a unique weak solution to (26), which additionally satisfies $u(\cdot; 0, u^0, \phi) \in C([0,\infty); H)$ and the energy equality (7) for all $0 \le s \le t$.*

Moreover, if for short we denote by $u(\cdot)$ and $v(\cdot)$ the corresponding solutions to (26) with respective initial data (u^0, ϕ) and (v^0, ψ), then the estimates (8) and (9) (with $\tau = 0$) hold for all $t \ge 0$, where $\widehat{C} = C^2\nu^{-1} + C_g^2 + 1$.

Once that a solution operator to problem (26) is suitably given, since continuous dependence with respect to initial data holds (by the previous theorem), and concatenation of solutions is clearly a solution too, one may use the standard results of (autonomous) dynamical systems (see e.g. [29] for a detailed exposition on concepts and results). Let us for the sake of brevity just include the very essential elements we need for our analysis.

Definition 6. A semi flow $\mathcal{S}$ on a metric space (X, d_X) is a mapping $\mathbb{R}_+ \times X \ni (t,x) \mapsto \mathcal{S}(t)x \in X$ such that $\mathcal{S}(0) = \mathrm{Id}_X$, and $S(t)S(s)x = S(t+s)x$ for any $t, s \ge 0$ and all $x \in X$.

It is said that the semi flow $\mathcal{S}$ is continuous if for any $t \in \mathbb{R}_+$, the mapping $\mathcal{S}(t) : X \to X$ is continuous.

The semi flow $\mathcal{S}$ is said to be asymptotically compact if for any bounded sequence $\{x_n\} \subset X$ and $\{t_n\} \subset \mathbb{R}_+$ with $\lim_n t_n = \infty$, the sequence $\{\mathcal{S}(t_n)x_n\}$ is relatively compact in X.

A subset $B_0 \subset X$ is said to be absorbing for the semi flow $\mathcal{S}$ if for any bounded subset $B \subset X$ there exists a time $T(B) \ge 0$ such that $\mathcal{S}(t)B := \cup_{b \in B}\mathcal{S}(t)b \subset B_0$ for all $t \ge T(B)$.

A subset $\mathcal{A} \subset X$ is said to be a global attractor for the semi flow $\mathcal{S}$ on X if it is compact, invariant (i.e. $\mathcal{S}(t)\mathcal{A} = \mathcal{A}$ for all $t \in \mathbb{R}_+$), and it attracts bounded sets of X, i.e. $\lim_{t\to\infty} \mathrm{dist}_X(\mathcal{S}(t)B, \mathcal{A}) = 0$ for all $B \subset X$ bounded.

Observe that from the definition of a global attractor for a semi flow, if it exists, it is unique. Moreover, it is the minimal closed set with the property of attracting all bounded sets, and the maximal compact invariant set.

With the above concepts, the basic result on existence of global attractor is the following.

Theorem 8. *(cf. [29]) Consider a semi flow $\mathcal{S}$ defined on a metric space (X, d_X), which is continuous. Then, there exists the global attractor $\mathcal{A}$ for $\mathcal{S}$ if and only if the semi flow is asymptotically compact and it has a bounded absorbing set $B_0 \subset X$. Moreover, then*

$$\mathcal{A} = \omega(B_0) := \bigcap_{t \ge 0} \overline{\bigcup_{s \ge t} \mathcal{S}(s)B_0}^{X}.$$

To apply the above result, as in Section 4 we consider the Banach space $X = H \times (L^2_V \cap L^\infty_H)$, wit the norm $\|(\zeta,\phi)\|_X = |\zeta| + \|\phi\|_{L^2_V} + \|\phi\|_{L^\infty_H}$ for a pair $(\zeta,\phi) \in X$.

After Corollary 2, but within the assumptions of Theorem 7, the family of mappings that form a continuous semi flow is now given by $\mathcal{S}(t) = S(t,0)$ from into X, and for any $t \in \mathbb{R}_+$, i.e. $\mathcal{S}(t) : H \times (L^2_V \cap L^\infty_H) \to H \times (L^2_V \cap L^\infty_H)$ given by $\mathcal{S}(t)(u^0,\phi) = (u(t), u_t)$ where u is the weak solution to (26).

In order to obtain asymptotic estimates, we impose this new condition:

(H5') Assume that $\nu\lambda_1 > C_g$, and that there exists a value $\eta \in (0, 2(\nu\lambda_1 - C_g))$ such that for any $0 \le \tau \le t$ and for every $u \in L^2(\tau - h, t; H)$,

$$\int_\tau^t e^{\eta s}|g(u_s)|^2\,ds \le C_g^2 \int_{\tau-h}^t e^{\eta s}|u(s)|^2\,ds \quad \forall t \ge \tau \ge 0.$$

The analogous result to Lemma 1 is the following

Lemma 4. *Consider given $f \in V'$ and $g : C_H \to (L^2(\Omega))^2$ satisfying conditions (H2')–(H5'). Then, for any $(u^0,\phi) \in H \times (L^2_V \cap L^\infty_H)$, the following inequalities hold for the solution u to (26) for all $t \ge s \ge 0$:*

$$\begin{aligned} |u(t)|^2 &\le e^{-\eta t}(|u^0|^2 + C_g\|\phi\|^2_{L^2_H}) + \frac{1}{\beta\eta}\|f\|^2_*, \\ \nu\int_s^t \|u(r)\|^2 dr &\le |u(s)|^2 + C_g\|u_s\|^2_{L^2_H} + \frac{1}{\nu}\|f\|^2_*(t-s) + 2C_g\int_s^t |u(r)|^2 dr, \end{aligned}$$

where β is given by (14).

Since condition (15) now is fulfilled trivially for a constant $f \in V'$, as a particular case of Corollary 3 we have the first ingredient for applying Theorem 8: the existence of an absorbing set.

Corollary 4. *Under the assumptions of Lemma 4, the set*

$$\widehat{B}_0 := \overline{B}_H(0,\widehat{R}_H) \times (\overline{B}_{L^2_V}(0,\widehat{R}_V) \cap \overline{B}_{L^\infty_H}(0,\widehat{R}_H)) \subset H \times (L^2_V \cap L^\infty_H),$$

where

$$\widehat{R}^2_H = 1 + (\beta\eta)^{-1}\|f\|^2_*, \quad \widehat{R}^2_V = \nu^{-1}[(1+3C_g h)\widehat{R}^2_H + \nu^{-1}h\|f\|^2_*],$$

is absorbing for the semi flow $\mathcal{S}$ on $H \times (L^2_V \cap L^\infty_H)$.

Second ingredient for applying Theorem 8 is the asymptotic compactness of $\mathcal{S}$, but this is again a consequence of the previously proved result in Section 4 (see Lemma 2; observe that the autonomous or non-autonomous formulation is not really a matter for the application of the energy method).

Lemma 5. *Under the assumptions of Lemma 4, the semi flow $\mathcal{S}$ is asymptotically compact.*

Main result of this section is the following

Theorem 9. *Assume that $f \in V'$, and $g : C_H \to (L^2(\Omega))^2$ satisfies conditions (H2')–(H5'). Then, there exists the global attractor $\mathcal{A}_{\mathcal{D}_F(H\times(L^2_V\cap L^\infty_H))}$ for $\mathcal{S}$ on $H \times (L^2_V \cap L^\infty_H)$.*

Since the adaptation of Section 4.1 is also obvious (we omit the details here just for the sake of brevity), and we may consider a natural semi flow $\widetilde{\mathcal{S}} : \mathbb{R}_+ \times L^2_V \cap C_H \to L^2_V \cap C_H$ given by $\widetilde{\mathcal{S}}(t) = U(t,0)$, and the continuity of this semi flow and absorption and asymptotic compactness properties are not difficult to obtain (inherited from

the proofs in that section), we conclude the following result (compare with Theorem 5).

Theorem 10. *Assume that $f \in V'$, and $g : C_H \to (L^2(\Omega))^2$ satisfies conditions (H2')–(H5'). Then, there exists the global attractor $\mathcal{A}_{\mathcal{D}_F(L^2_V \cap C_H)}$ for $\widetilde{\mathcal{S}}$ on $L^2_V \cap C_H$.*

Moreover, the following relation holds:

$$j(\mathcal{A}_{\mathcal{D}_F(L^2_V \cap C_H)}) = \mathcal{A}_{\mathcal{D}_F(H \times (L^2_V \cap L^\infty_H))}$$

where $j : L^2_V \cap C_H \to H \times (L^2_V \cap L^\infty_H)$ is defined by $j(\varphi) = (\varphi(0), \varphi)$.

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