Programa de Doctorado "Matemáticas"

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Resultados teóricos y numéricos de control para EDPs lineales y no lineales

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Theoretical and numerical control results for linear and nonlinear PDEs

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"Ce que j'aime dans les mathématiques appliquées, c'est qu'elles ont pour ambition de donner du monde des systèmes une représentation qui permette de comprendre et d'agir. Et, de toutes les représentations, la représentation mathématique, lorsqu'elle est possible, est celle qui est la plus souple et la meilleure. Du coup, ce qui m'intéresse, c'est de savoir jusqu'où on peut aller dans ce domaine de la modélisation des systèmes, c'est d'atteindre les limites."

Jacques-Louis Lions

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Abstract

This Thesis concerns the theoretical and numerical control of linear and nonlinear PDEs.

First, in Chapter 2, we analyze the local null controllability of the Burgers-α model. The state is the solution to a regularized Burgers equation, where the transport term is of the form zy_x, $z = (Id - \alpha^2 \frac{\partial^2}{\partial x^2})^{-1} y$ and $\alpha > 0$ is a small parameter. We also prove some results concerning the behavior of the null controls and associated states as $\alpha \to 0^+$.

Secondly, in Chapter 3, we deal with the distributed and boundary controllability of the so called Leray-α model. We prove that the Leray-α equations are locally null controllable, with controls bounded independently of α. We also prove that, if the initial data are sufficiently small, the controls converge as $\alpha \to 0^+$ to a null control of the Navier-Stokes equations.

Then, in Chapter 4, we study the boundary controllability of inviscid incompressible fluids for which thermal effects are important. We establish the simultaneous global exact controllability of the velocity field and the temperature for 2D and 3D flows. When the diffusion coefficient in the heat equation is positive, we present some additional results concerning the exact controllability of the velocity field and the local null controllability of the temperature.

In Chapter 5, we deal with the numerical computation of null controls for the linear heat equation. The main idea is to minimize, over the class of admissible null controls, a quadratic weighted functional that involves only the control. The optimality conditions of the problem are reformulated as a mixed formulation involving both the state and its adjoint. We prove the well-posedness of the mixed formulation and, then, we discuss several numerical experiments.

Finally, Chapter 6 deals with some strategies designed to solve numerically the null controllability problem for the two-dimensional heat and Stokes equations and the problem of local exact controllability to the trajectories of the Navier-Stokes equations. The main idea is to minimize over the class of admissible null controls a functional that involves weighted integrals of the state and the control, with weights that blow up near the final time. The associated optimality conditions can be viewed as a differential system in the variables $(\mathbf{x}, t)$ that is second–order in time and fourth–order in space, completed with appropriate boundary conditions. We present several mixed formulations of the problems and, then, appropriate mixed finite element approximations that rely on Lagrangian C^0 spaces.

Resumen

En esta Tesis presentamos resultados teóricos y numéricos de control para EDPs lineales y no lineales.

Primeramente, en el Capítulo 2, analizamos el control nulo local del modelo de Burgers-α. El estado es una solución de una ecuación de Burgers regularizada, donde el término de transporte es de la forma zy_x, $z = (Id - \alpha^2 \frac{\partial^2}{\partial x^2})^{-1} y$ y $\alpha > 0$ es un parámetro pequeño. También probamos algunos resultados sobre el comportamiento de los controles nulos y estados asociados cuando $\alpha \to 0^+$.

En segundo lugar, en el Capítulo 3, nos preocupamos por el control distribuido y frontera del llamado modelo de Leray-α. Probamos que las ecuaciones de Leray-α son controlables a cero localmente, con controles acotados independientemente de α. También probamos que, si los datos iniciales son suficientemente pequeños, los controles convergen cuando $\alpha \to 0^+$ a un control nulo de las ecuaciones de Navier-Stokes.

Después, en el Capítulo 4, estudiamos el control frontera de fluidos incompresibles no viscosos para los que los efectos térmicos son importantes. Establecemos simultáneamente la controlabilidad exacta global del campo de velocidades y de la temperatura para flujos 2D y 3D. Cuando el coeficiente de difusión de calor es positivo, presentamos algunos resultados adicionales sobre la controlabilidad exacta para el campo de velocidades y control nulo local de la temperatura.

En el Capítulo 5, nos preocupamos por el cálculo numérico de los controles nulos para la ecuación del calor lineal. La idea principal es minimizar sobre la clase de controles nulos admisibles un funcional promediado que involucra sólo el control. Las condiciones de optimalidad del problema se reformulan como una ecuación variacional mixta que hacen aparecer tanto al estado como a su adjunto. Probamos el buen planteamiento de la formulación mixta y luego discutimos varios experimentos numéricos.

Finalmente, el Capítulo 6 trata de algunas estrategias diseñadas para resolver numéricamente el problema de control nulo para las ecuaciones bidimensionales del calor y Stokes y el problema de control local exacto a trayectorias para las ecuaciones de Navier-Stokes. La idea principal es minimizar sobre la clase de controles nulos admisibles un funcional que contiene integrales promediadas del estado y del control, con pesos que explotan cerca del tiempo final. Las condiciones de optimalidad asociadas pueden ser vistas como un sistema diferencial en las variables $(\mathbf{x}, t)$ que es de segundo orden en tiempo y cuarto orden en espacio, completado con condiciones de frontera adecuadas. Presentamos varias formulaciones mixtas del sistema y, a continuación, aproximaciones basadas en elementos finitos Lagrangianos apropiados en espacios de funciones C^0.

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Chapter 1

Introduction

Introducción

Esta Tesis está dedicada al análisis y control de varios problemas de valores iniciales y de contorno para sistemas de ecuaciones en derivadas parciales con origen en Física, Ingeniería, Biología y otras Ciencias.

1.1 Generalidades

Los sistemas de control son aquéllos en cuyo comportamiento es posible influir. La teoría de control es un área de las Matemáticas que determina y estudia los principios fundamentales que subyacen en el análisis y diseño de los sistemas de control.

Para presentar algunos conceptos de la teoría de control, consideremos el sistema dinámico

$$\frac{\partial u}{\partial t} = F(t,u,v), \quad u \in \mathcal{H}, \ v \in \mathcal{U}_{ad}. \tag{1.1}$$

Aquí, t representa el tiempo y u representa el *estado* (la variable que deseamos controlar). La dinámica del sistema (1.1) depende de la variable v, llamada *control*, que actúa sobre la evolución del estado. La cuestión general es :

> *Fijado un tiempo T y un estado inicial u_0, ¿es posible encontrar un control v tal que la solución asociada de* (1.1) *con $u(0) = u_0$ cumpla una propiedad deseada?*

Cuando queremos influir en la evolución de un estado de modo a alcanzar un comportamiento deseado en un tiempo final, se dice que estamos frente a un *problema de controlabilidad*. Las propiedades de controlabilidad del sistema (1.1) pueden ser muy diferentes, dependiendo de la naturaleza del problema estudiado. A continuación, presentamos algunas de las nociones relacionadas :

- Diremos que el sistema posee la propiedad de *controlabilidad aproximada* si, partiendo de un estado inicial arbitrario, la solución del sistema puede ser conducida arbitrariamente cerca (con respecto a una determinada topología) de un estado deseado arbitrario.

- Diremos que el sistema posee la propiedad de *controlabilidad exacta* si, partiendo de un estado inicial arbitrario, la solución del sistema puede ser conducida exactamente a estado deseado arbitrario.

- Diremos que el sistema posee la propiedad de *controlabilidad nula* si, partiendo de un estado inicial arbitrario, la solución del sistema puede ser conducida exactamente a cero.

- Diremos que el sistema posee la propiedad de *controlabilidad exacta a trayectorias* si, partiendo de un estado inicial arbitrario, la solución del sistema puede ser conducida exactamente a cualquier trayectoria del sistema.

En el caso de una ecuación en derivadas parciales lineal, utilizando argumentos propios del Análisis Funcional, podemos reescribir las propiedades de controlabilidad en términos de propiedades de las soluciones. Por ejemplo, se puede ver que la controlabilidad aproximada para un sistema es equivalente a una propiedad de *continuación única*, la controlabilidad exacta para un sistema es equivalente a una *desigualdad de observabilidad fuerte* y la controlabilidad nula para un sistema es equivalente a una *desigualdad de observabilidad débil*, para más detalles véase [30]. En el caso de ecuaciones escalares y de algunos sistemas, las desigualdades de observabilidad suelen ser consecuencias de apropiadas *desigualdades de Carleman*. Éstas se pueden ver como desigualdades de energía promediadas que permiten acotar integrales promediadas globales por integrales promediadas locales. Otras herramientas que conducen a desigualdades de observabilidad son las *desigualdades de Ingham*, el *método de los multiplicadores*, el *análisis microlocal*, el *método de momentos*, etc.

Sin embargo, cuando estamos frente a una ecuación no lineal, el tema se vuelve más delicado, los argumentos son mucho más complicados y los resultados, cuando se consiguen, casi siempre son locales. Esto se debe a que las técnicas utilizadas casi siempre se basan en linealizar, probar un resultado análogo para un problema lineal apropiado y, luego, recuperar el resultado para el problema no lineal. En general, las técnicas reposan sobre alguno de los dos siguientes argumentos : aplicar un teorema de punto-fijo o aplicar un teorema de función inversa.

1.2 Resultados previos

En esta Tesis, una de las principales herramientas que se utilizan para obtener los resultados de controlabilidad son las desigualdades de Carleman. Éstas fueran introducidas, en el marco de la controlabilidad nula, por Andrei Vladimirovich Fursikov y Oleg Yur'evich Imanuvilov, véase [62]. También, se puede mencionar el trabajo de Gilles Lebeau y Luc Robbiano [92], dedicado a la ecuación de calor lineal, en el cual combinaron el método de Russell, las propiedades de una transformada integral y una desigualdad de Carleman para ecuaciones elípticas para entonces deducir la controlabilidad nula de la ecuación del calor. Por otro lado, la controlabilidad aproximada para la ecuación del calor semilinear fue probada por Caroline Fabre, Jean-Pierre Puel y Enrique Zuazua, véase [43].

Los primeros resultados locales de controlabilidad nula para la ecuación de Burgers fueron obtenidos por Fursikov e Imanuvilov, véase [61]. Otros trabajos que trataron la controlabilidad para la ecuación de Burgers son [17, 37, 45, 62, 74, 80]. Entre ellos, podemos destacar la referencia [45], donde los autores probaron un resultado óptimo de controlabilidad nula para la ecuación de Burgers.

En el contexto de las ecuaciones de Navier-Stokes, Jacques-Louis Lions, en [97], formuló una conjectura sobre la controlabilidad aproximada global, frontera y distribuida; desde entonces, la controlabilidad de esas ecuaciones y sus variantes ha despertado el interés de muchos investigadores, pero hasta el presente momento sólo resultados parciales son conocidos.

En [60], Fursikov e Imanuvilov probaron un resultado local sobre la controlabilidad exacta a trayectorias C^∞ de las ecuaciones de Navier-Stokes a partir de una desigualdad de Carleman y un teorema de función inversa. Algunos años después, Enrique Fernández-Cara, Sergio Guerrero, Oleg Yur'evich Imanuvilov y Jean-Pierre Puel mejoraron este resultado, relajando la regularidad de las trayectorias a L^∞, véase [46]. Algún tiempo después, inspirado en [46, 60], Guerrero prueba un resultado local sobre la controlabilidad exacta a trayectorias para el sistema de Boussinesq, véase [73]. Finalmente, Fernández-Cara, Guerrero, Imanuvilov y Puel prueban un resultado local sobre la controlabilidad exacta a trayectorias de los sistemas N-dimensionales de Navier-Stokes y Boussinesq con un número reducido de controles escalares bajo algunas condiciones geométricas, véase [47]. En las referencias [12, 32, 33, 47, 53], podemos encontrar resultados posteriores sobre la controlabilidad para este tipo de ecuaciones con un número reducido de controles escalares.

En relación a los fluidos no viscosos podemos destacar los trabajos de Jean-Michel Coron, véase [27, 29], donde se prueba un resultado global de control frontera exacto para las ecuaciones de Euler, en el caso bidimensional. También destacamos los trabajos de Olivier Glass, donde se prueba un resultado análogo en el caso tridimensional, véase [65, 66]. El principal argumento en ambos casos consiste en aplicar el *método do retorno*, combinado con las propiedades de invarianza de escala para las ecuaciones de Euler.

Finalmente, en relación a la controlabilidad numérica, las primeras contribuciones se deben a Roland Glowinski y Jacques-Louis Lions, véase [70] donde, basándose en argumentos de dualidad, reemplazan el problema original consistente a buscar controles de norma mínima (un problema con restricciones) por un problema sin restricciones (el dual). Este punto de vista ha sido explotado numéricamente por Craig Alan Carthel, Roland Glowiski y Jacques-Louis Lions, véase [13]. Se ha podido observar que el control de norma mínima oscila mucho cerca del tiempo final T. Para evitar este comportamiento oscilatorio, Enrique Fernández-Cara y Arnaud Diego Münch, véase [49, 48], introdujeron una aproximación diferente, que consiste en resolver directamente el sistema de optimalidad al que conduce la formulación promediada de Fursikov-Imanuvilov.

1.3 Contenido de la Tesis

En esta Tesis, presentamos resultados teóricos y numéricos de control (locales y globales, en un sentido que se dirá más adelante) para problemas no lineales gobernados por ecuaciones en derivadas parciales con origen en Física, Ingeniería, Biología y otras Ciencias. Todos los resultados constan en artículos publicados o aceptados para publicación, mencionaremos la respectiva referencia al final de la descripción de cada capítulo. Para un mejor entendimiento y agrupación de los temas tratados, la Tesis está dividida en dos partes :

1.3.1 Parte I. Resultados teóricos sobre el control de algunos modelos con origen en mecánica de fluidos

En los Capítulos 2 y 3, presentaremos resultados sobre el control uniforme (respecto a un parámetro regularizante) de algunos modelos viscosos no lineales que pueden ser observados como regularizaciones de las ecuaciones de Burgers y Navier-Stokes. Estos modelos fueron introducidos para describir flujos turbulentos, como alternativa a los clásicos modelos de Reynolds. El interés por conseguir resultados de control uniforme de estos modelos está justificado por tanto por su origen. Debido a la presencia de términos no lineales, los resultados aquí obtenidos son locales.

En el Capítulo 4, presentaremos resultados sobre el control de algunos modelos de fluidos incompresibles no viscosos donde los efectos térmicos son importantes. Los modelos de fluidos con efectos térmicos son muy relevantes para el estudio y descripción de la turbulencia atmosférica y oceanográfica, así como en el contexto de fenómenos astrofísicos, donde la rotación y estratificación desempeñan un papel dominante (véase, e.g. [107]). Debido a la amplia gama de aplicaciones, es muy importante comprender bien las propiedades de control de estos modelos, es decir, saber si podemos o no influir en su comportamiento de modo a alcanzar una situación deseada.

En lo que sigue, describiremos con detalle el contenido de los capítulos de esta parte.

Capítulo 2 : Controlabilidad del sistema Burgers-α

El Capítulo 2 de esta Tesis está dedicado al análisis de la controlabilidad uniforme de una familia de sistemas regularizados.

Para una descripción más detallada de los resultados obtenidos, consideremos la longitud espacial, $L > 0$ y el tiempo final $T > 0$. Sea $(a, b) \subset (0, L)$ un intervalo abierto no vacío, llamado *dominio de control*.

Consideremos la siguiente ecuación de Burgers con control distribuido :

$$\begin{cases} y_t - y_{xx} + yy_x = v1_{(a,b)} & \text{en} \quad (0, L) \times (0, T), \\ y(0, \cdot) = y(L, \cdot) = 0 & \text{en} \quad (0, T), \\ y(\cdot, 0) = y_0 & \text{en} \quad (0, L). \end{cases} \tag{1.2}$$

En (1.2), la función $y = y(x, t)$ puede ser interpretada como una velocidad unidimensional de un flujo e $y_0 = y_0(x)$ es la velocidad inicial. La función $v = v(x, t)$ es el control que actúa sobre el sistema y $1_{(a,b)}$ denota la función característica de (a, b).

En este capítulo vamos a considerar un sistema parecido a (1.2), donde el término de transporte es de la forma zy_x, donde z es la solución de una ecuación elíptica gobernada por y. Más precisamente, consideramos la siguiente versión *regularizada* de (1.2), donde

$\alpha > 0$:

$$\begin{cases} y_t - y_{xx} + z y_x = v 1_{(a,b)} & \text{en} \quad (0,L)\times(0,T), \\ z - \alpha^2 z_{xx} = y & \text{en} \quad (0,L)\times(0,T), \\ y(0,\cdot) = y(L,\cdot) = z(0,\cdot) = z(L,\cdot) = 0 & \text{en} \quad (0,T), \\ y(\cdot,0) = y_0 & \text{en} \quad (0,L). \end{cases} \tag{1.3}$$

Este sistema es conocido como *sistema de Burgers-α*.

Notemos que (1.3) es diferente de (1.2), pues aparecen términos no lineales que son no locales en la variable espacial, dependientes de un parámetro α.

En el análisis del control de EDPs, este tipo de ecuaciones no son muy comunes. De hecho, en general, cuando tratamos con no linealidades no locales, no parece ser fácil transmitir la información proporcionada por controles localmente soportados a todo el dominio de manera satisfactoria. Por tanto, la principal novedad del capítulo 2 es que sus resultados garantizan el control uniforme de un tipo de ecuaciones parabólicas no lineales que son no locales.

De hecho, probamos los siguientes resultados para el sistema de Burgers-α:

Teorema 1.1 (Control nulo local uniforme). *Para cada $T > 0$, el sistema (1.3) es localmente controlable a cero en el tiempo T. Más precisamente, existe $\delta > 0$ (independiente de α pero dependiente de T) tal que, para cualquier $y_0 \in H_0^1(0,L)$ con $\|y_0\|_\infty \le \delta$, existen controles $v_\alpha \in L^\infty((a,b)\times(0,T))$ y estados asociados (y_α, z_α) satisfaciendo*

$$y_\alpha(\cdot,T) = 0 \quad \textit{en} \quad (0,L). \tag{1.4}$$

Además, tenemos

$$\limsup_{\alpha\to 0^+} \|v_\alpha\|_\infty < +\infty. \tag{1.5}$$

Teorema 1.2 (Control para tiempo largo). *Para cada $y_0 \in H_0^1(0,L)$ con $\|y_0\|_\infty < \pi/L$, el sistema (1.3) es controlable a cero para tiempo largo. En otras palabras, existen $T > 0$ (independiente de α), controles $v_\alpha \in L^\infty((a,b)\times(0,T))$ y estados asociados (y_α, z_α) que verifican (1.4) y (1.5).*

Teorema 1.3 (Control local en el límite). *Sea $T > 0$ fijo y sea $\delta > 0$ la constante proporcionada por el Teorema 1.1. Supongamos que $y_0 \in H_0^1(0,L)$, con $\|y_0\|_{L^\infty} \le \delta$, sea (v_α) una familia de controles nulos para (1.3) que verifican (1.5) y sea (y_α, z_α) la familia de estados asociados satisfaciendo (1.4). Entonces, al menos para una subsucesión, tenemos*

$$\begin{gathered} v_\alpha \to v \quad \textit{débil-}* \quad \textit{en} \quad L^\infty((a,b)\times(0,T)), \\ z_\alpha \to y \quad \textit{y} \quad y_\alpha \to y \quad \textit{débil-}* \quad \textit{en} \quad L^\infty((0,L)\times(0,T)) \end{gathered} \tag{1.6}$$

cuando $\alpha \to 0^+$, donde (y,v) es un par estado-control para (1.2) que verifica (1.4).

Notemos que los dos primeros resultados proporcionan controles en $L^\infty((a,b)\times(0,T))$ y no solamente en $L^2((a,b)\times(0,T))$. Esto es muy conveniente no sólo para (1.2) y (1.3), sino también en otros problemas intermedios que aparecen en las demostraciones. De esta manera obtenemos mejores estimaciones para los estados y resulta más fácil probar existencia y convergencia.

Ahora explicaremos la metodología de la prueba de estos tres resultados.

Empecemos con el Teorema 1.1. En términos generales, fijamos $\overline{y}$, encontramos z solución de :

$$\begin{cases} z - \alpha^2 z_{xx} = \overline{y}, & \text{in} \quad (0,L)\times(0,T), \\ z(0,\cdot) = z(L,\cdot) = 0 & \text{in} \quad (0,T), \end{cases}$$

y controlamos exactamente a cero el sistema lineal :

$$\begin{cases} y_t - y_{xx} + z y_x = v 1_{(a,b)} & \text{in} \quad (0,L)\times(0,T), \\ y(0,\cdot) = y(L,\cdot) = 0 & \text{in} \quad (0,T), \\ y(\cdot,0) = y_0 & \text{in} \quad (0,L). \end{cases} \tag{1.7}$$

De este modo, queda definida una aplicación $\Lambda_\alpha(\overline{y}) = y$, en espacios apropiados. Entonces, la tarea es resolver la ecuación de punto fijo $y = \Lambda_\alpha(y)$.

Muchos teoremas de punto fijo pueden ser aplicados. En este capítulo, preferimos usar el *Teorema del punto fijo de Schauder*, aunque otros resultados también conducen a la buena conclusión; por ejemplo, un argumento basado en *Teorema del punto fijo de Kakutani*, como en [38], es posible.

Notemos que, para lograr el Teorema 1.1, necesitamos garantizar una cierta uniformidad en α para la aplicación Λ_α. Esto se consigue gracias a que el *principio del máximo* puede ser aplicado y garantiza buenas estimaciones uniformes en $L^\infty((0,L)\times(0,T))$.

Como mencionamos antes, para conseguir buenas propiedades para Λ_α, es muy conveniente que el control pertenezca a L^∞. Esto se puede lograr de varias formas; por ejemplo, utilizando una desigualdad de observabilidad "mejorada" para las soluciones del sistema adjunto de (1.7) y argumentando como en [38]. Aquí, optamos por utilizar otras técnicas que se basan en la regularidad de los estados y se utilizaron originalmente en [7]; véase también [8]. La idea que hay por detrás es, básicamente, construir un control en $L^\infty(a,b)$ para (1.7) a partir de un control en L^2 soportado en un conjunto abierto más pequeño que (a,b). Para eso, se utiliza de forma esencial las propiedades de regularidad local que tienen las ecuaciones parabólicas.

La prueba del Teorema 1.2 es parecida. Es suficiente reemplazar la hipótesis "y_0 es pequeño" por una hipótesis en la que imponemos un tiempo T suficientemente grande. De nuevo, esto hace posible aplicar un argumento de punto fijo.

Más precisamente, aceptemos que, si $y_0 \in H_0^1(0,L)$ y $\|y_0\|_\infty < \pi/L$, entonces la solución y_α asociada al sistema (1.3), con control v igual a cero satisface

$$\|y_\alpha(\cdot\,,t)\|_{H_0^1} \leq C(y_0)e^{-\frac{1}{2}((\pi/L)^2-\|y_0\|_\infty^2)t},$$

donde $C(y_0)$ es una constante que depende solamente de $\|y_0\|_\infty$ y $\|y_0\|_{H_0^1}$. Recordemos que π/L es la raíz cuadrada del primer autovalor del operador Laplaciano.

Entonces, si inicialmente tomamos $v \equiv 0$, el estado $y_\alpha(\cdot, t)$ se hace pequeño cuando t es grande. En una segunda etapa, como $\|y_\alpha(\cdot, t)\|_{H_0^1}$ es suficientemente pequeño, podemos aplicar el Teorema 1.1 y conducir el estado exactamente a cero.

El Teorema 1.3 es una consecuencia casi inmediata. Este teorema proporciona un resultado relacionado con el control nulo local en el límite. La idea es usar la estimación uniforme de los controles v_α para obtener buenas convergencias débiles de los estados asociados y_α. Así, gracias al *Lema de Aubin-Lions*, podemos conseguir convergencias fuertes del estado y del término regularizante z_α (cuyo límite coincide con el de y_α) y, finalmente, ver que el estado límite es solución de una ecuación de Burgers controlada a cero.

Este capítulo está basado en el artículo [3], en colaboración con F. D. Araruna y E. Fernández-Cara.

Capítulo 3 : Control nulo local uniforme del modelo Leray-α

A continuación, estudiamos las propiedades de controlabilidad uniforme de una familia de sistemas regularizados análogos en dimensión superior a uno.

Para presentar con detalles los logros de este capítulo, sea $\Omega \subset \mathbb{R}^N$ ($N = 2$ ó 3) un dominio acotado cuya frontera Γ es de clase C^2. Sean $\omega \subset \Omega$ un conjunto abierto no vacío (llamado *dominio de control interno*), $\gamma \subset \Gamma$ un subconjunto abierto y no vacío de Γ (llamado *dominio de control frontera*) y $T > 0$. Fijemos las siguientes notaciones $Q := \Omega \times (0,T)$, $\Sigma := \Gamma \times (0,T)$ y denotemos $\mathbf{n} = \mathbf{n}(\mathbf{x})$ el vector normal exterior al dominio Ω en los puntos $\mathbf{x} \in \Gamma$.

El modelo que trataremos en este capítulo es una variante regularizada de las conocidas ecuaciones de Navier-Stokes :

$$\begin{cases} \mathbf{y}_t - \Delta\mathbf{y} + (\mathbf{y}\cdot\nabla)\mathbf{y} + \nabla p = \mathbf{f} & \text{en} \quad Q, \\ \nabla\cdot\mathbf{y} = 0 & \text{en} \quad Q, \\ \mathbf{y} = \mathbf{0} & \text{sobre} \quad \Sigma, \\ \mathbf{y}(\cdot,0) = \mathbf{y}_0 & \text{en} \quad \Omega, \end{cases} \tag{1.8}$$

donde $\mathbf{y}$ representa el campo de velocidades, p representa la presión, $\mathbf{f} = \mathbf{f}(\mathbf{x}, t)$ es un término de fuerza e $\mathbf{y}_0 = \mathbf{y}_0(\mathbf{x})$ es el dato inicial.

Con el objetivo de probar la existencia de solución para las ecuaciones de Navier-Stokes, Leray, en [95], tuvo la idea de formular un *modelo de turbulencia cerrado* sin aumentar la disipación viscosa. De ese modo, Leray introdujo una variante "regularizada" de las ecuaciones de Navier-Stokes, modificando el término no lineal como sigue :

$$\begin{cases} \mathbf{y}_t - \Delta\mathbf{y} + (\mathbf{z}\cdot\nabla)\mathbf{y} + \nabla p = \mathbf{f} & \text{en} \quad Q, \\ \nabla\cdot\mathbf{y} = 0 & \text{en} \quad Q, \end{cases}$$

donde $\mathbf{y}$ y $\mathbf{z}$ están relacionados por

$$\mathbf{z} = \phi_\alpha * \mathbf{y}$$

y ϕ_α es un núcleo regularizante. Al menos formalmente, las ecuaciones de Navier-Stokes pueden ser recuperadas en el límite cuando $\alpha \to 0^+$, con $\mathbf{z} \to \mathbf{y}$.

Aquí, consideramos un núcleo especial, asociado al operador $\mathbf{Id} + \alpha^2\mathbf{A}$, donde $\mathbf{A}$ es el *operador de Stokes*. Esto lleva a las siguientes ecuaciones de Navier-Stokes regularizadas, conocidas como *sistema de Leray-α* (véase [19]) :

$$\begin{cases} \mathbf{y}_t - \Delta\mathbf{y} + (\mathbf{z}\cdot\nabla)\mathbf{y} + \nabla p = \mathbf{f} & \text{en} \quad Q, \\ \mathbf{z} - \alpha^2\Delta\mathbf{z} + \nabla\pi = \mathbf{y} & \text{en} \quad Q, \\ \nabla\cdot\mathbf{y} = 0, \ \nabla\cdot\mathbf{z} = 0 & \text{en} \quad Q, \\ \mathbf{y} = \mathbf{z} = \mathbf{0} & \text{sobre} \quad \Sigma, \\ \mathbf{y}(\cdot,0) = \mathbf{y}_0 & \text{en} \quad \Omega. \end{cases} \tag{1.9}$$

En este capítulo, estudiaremos el control de los siguientes sistemas :

$$\begin{cases} \mathbf{y}_t - \Delta\mathbf{y} + (\mathbf{z}\cdot\nabla)\mathbf{y} + \nabla p = \mathbf{v}1_\omega & \text{en} \quad Q, \\ \mathbf{z} - \alpha^2\Delta\mathbf{z} + \nabla\pi = \mathbf{y} & \text{en} \quad Q, \\ \nabla\cdot\mathbf{y} = 0, \ \nabla\cdot\mathbf{z} = 0 & \text{en} \quad Q, \\ \mathbf{y} = \mathbf{z} = \mathbf{0} & \text{sobre} \quad \Sigma, \\ \mathbf{y}(\cdot,0) = \mathbf{y}_0 & \text{en} \quad \Omega, \end{cases} \tag{1.10}$$

y

$$\begin{cases} \mathbf{y}_t - \Delta\mathbf{y} + (\mathbf{z}\cdot\nabla)\mathbf{y} + \nabla p = \mathbf{0} & \text{en} \quad Q, \\ \mathbf{z} - \alpha^2\Delta\mathbf{z} + \nabla\pi = \mathbf{y} & \text{en} \quad Q, \\ \nabla\cdot\mathbf{y} = 0, \ \nabla\cdot\mathbf{z} = 0 & \text{en} \quad Q, \\ \mathbf{y} = \mathbf{z} = \mathbf{h}1_\gamma & \text{sobre} \quad \Sigma, \\ \mathbf{y}(\cdot,0) = \mathbf{y}_0 & \text{en} \quad \Omega, \end{cases} \tag{1.11}$$

donde $\mathbf{v} = \mathbf{v}(\mathbf{x},t)$ (respectivamente $\mathbf{h} = \mathbf{h}(\mathbf{x},t)$) representa el control, actuando solamente en un pequeño conjunto ω (respectivamente sobre γ) durante todo el intervalo de tempo $(0,T)$. El símbolo 1_ω (respectivamente 1_γ) representa la función característica de ω (respectivamente γ).

En las aplicaciones, el *control interno* $\mathbf{v}1_\omega$ puede ser interpretado como un campo de fuerzas, mientras que el *control frontera* $\mathbf{h}1_\gamma$ es la traza del campo de velocidades sobre Σ.

Observación 1.1. Es natural suponer que los estados $\mathbf{y}$ y $\mathbf{z}$ satisfacen las mismas condiciones de frontera sobre Σ, dado que, en el límite, debemos tener $\mathbf{z} = \mathbf{y}$. Consecuentemente, vamos a suponer que el control frontera $\mathbf{h}1_\gamma$ actúa simultáneamente sobre ambos estados $\mathbf{y}$ y $\mathbf{z}$. □

Antes de presentar los resultados de este capítulo, recordemos algunas definiciones de espacios usuales en el contexto de los fluidos incompresibles :

$$\mathbf{H} := \{\, \mathbf{u} \in \mathbf{L}^2(\Omega) : \nabla \cdot \mathbf{u} = 0 \text{ en } \Omega,\ \mathbf{u} \cdot \mathbf{n} = 0 \text{ sobre } \Gamma \,\},$$
$$\mathbf{V} := \{\, \mathbf{u} \in \mathbf{H}_0^1(\Omega) : \nabla \cdot \mathbf{u} = 0 \text{ en } \Omega \,\}.$$

Observación 1.2. Se puede probar que para todo $\mathbf{y}_0 \in \mathbf{H}$ y todo $\mathbf{v} \in \mathbf{L}^2(\omega \times (0,T))$, existe una única solución débil $(\mathbf{y}, p, \mathbf{z}, \pi)$ de (1.10) que satisface (entre otras cosas)

$$\mathbf{y}, \mathbf{z} \in C^0([0,T]; \mathbf{H}).$$

Esto marca una diferencia evidente con las ecuaciones de Navier-Stokes. □

Los resultados logrados en este capítulo garantizan el control de (1.10) y (1.11) y determinan la manera en la que estos sistemas controlados se comportan cuando $\alpha \to 0^+$. Más precisamente, logramos probar en el Capitulo 3 resultados de control uniforme, interno y frontera, para un tipo de sistema no lineal, no local y en dimensión superior a uno, donde el *principio del máximo* no puede ser aplicado.

Primeramente, logramos probar los siguientes resultados de control interno y frontera para el sistema de Leray-α :

Teorema 1.4 (Control interno uniforme). *Existe $\epsilon > 0$ (independiente de α) tal que, para cada $\mathbf{y}_0 \in \mathbf{H}$ con $\|\mathbf{y}_0\| \le \epsilon$, existen controles $\mathbf{v}_\alpha \in L^\infty(0,T; \mathbf{L}^2(\omega))$ tales que las soluciones asociadas de (1.10) verifican*

$$\mathbf{y}(\cdot, T) = \mathbf{0} \quad \text{en} \quad \Omega. \tag{1.12}$$

Además, tenemos

$$\limsup_{\alpha \to 0^+} \|\mathbf{v}_\alpha\|_{L^\infty(\mathbf{L}^2)} < +\infty. \tag{1.13}$$

Teorema 1.5 (Control frontera uniforme). *Existe $\delta > 0$ (independiente de α) tal que, para cada $\mathbf{y}_0 \in \mathbf{H}$ con $\|\mathbf{y}_0\| \le \delta$, existen controles $\mathbf{h}_\alpha \in L^\infty(0,T; \mathbf{H}^{-1/2}(\gamma))$ con $\int_\gamma \mathbf{h}_\alpha \cdot \mathbf{n}\, d\Gamma = 0$ y soluciones asociadas de (1.11) que verifican*

$$\mathbf{y}, \mathbf{z} \in C^0([0,T]; \mathbf{L}^2(\Omega))$$

y (1.12). Además, tenemos

$$\limsup_{\alpha \to 0^+} \|\mathbf{h}_\alpha\|_{L^\infty(H^{-1/2})} < +\infty. \tag{1.14}$$

Las demostraciones de estos resultados se basan en argumentos de punto fijo. La idea base ha sido aplicada en muchos otros problemas de control no lineal. Sin embargo, en los presentes casos, encontramos dos dificultades específicas :

- Para encontrar espacios y aplicaciones de punto fijo adecuadas al *Teorema de punto fijo de Schauder*, el dato inicial $\mathbf{y}_0$ debe ser suficientemente regular. Consecuentemente, debemos establecer *propiedades de regularidad* (¡uniformes en α!) para (1.10) y (1.11).

- Para la prueba de (1.13) y (1.14), son necesarias estimaciones cuidadosas de los controles nulos y estados asociados para algunos problemas lineales.

Ahora explicaremos con más detalle los argumentos utilizados.

Comencemos con el Teorema 1.4. Como dijimos anteriormente, la idea es aplicar un argumento de punto fijo. Al contrario del caso de las ecuaciones de Navier-Stokes, no es suficiente trabajar con controles en $\mathbf{L}^2(\omega \times (0,T))$. De hecho, necesitamos un espacio $\mathbf{Y}$ para $\mathbf{y}$ que asegure $\mathbf{z}$ en $\mathbf{L}^\infty(Q)$ y un espacio $\mathbf{X}$ para $\mathbf{v}$ que garantice que la solución para el sistema de Oseen con velocidad $\mathbf{z}$ pertenece a un compacto de $\mathbf{Y}$. Además, queremos estimaciones en $\mathbf{Y}$ y $\mathbf{X}$ independientes de α.

Primeramente, necesitamos probar un resultado de regularidad que mantenga la pequeñez, independiente de α, en un espacio más regular. Esto se puede ver en el siguiente lema que está inspirado en un resultado de Constantin y Foias para las ecuaciones de Navier-Stokes, véase [25] :

Lema 1.1. *Existe una función continua $\phi : \mathbb{R}_+ \mapsto \mathbb{R}_+$, con $\phi(s) \to 0$ cuando $s \to 0^+$, satisfaciendo las siguientes propiedades :*

a) Para $\mathbf{f} = \mathbf{0}$, $\mathbf{y}_0 \in \mathbf{H}$ y $\alpha > 0$, existen tiempos arbitrariamente pequeños $t^* \in (0, T/2)$ tales que la correspondiente solución para (1.9) satisface $\|\mathbf{y}_\alpha(\cdot, t^*)\|^2_{D(\mathbf{A})} \le \phi(\|\mathbf{y}_0\|)$.

b) El conjunto de estos tiempos t^* tiene medida positiva.

De ese modo, sólo necesitamos considerar el caso en que el estado inicial $\mathbf{y}_0$ pertenece a $D(\mathbf{A})$ y posee una norma suficientemente pequeña en $D(\mathbf{A})$.

Fijemos σ con $N/4 < \sigma < 1$. Entonces, para cada $\overline{\mathbf{y}} \in L^\infty(0,T; D(\mathbf{A}^\sigma))$, sea $(\mathbf{z}, \pi)$ la única solución para

$$\begin{cases} \mathbf{z} - \alpha^2 \Delta \mathbf{z} + \nabla \pi = \overline{\mathbf{y}} & \text{en} \quad Q, \\ \nabla \cdot \mathbf{z} = 0 & \text{en} \quad Q, \\ \mathbf{z} = \mathbf{0} & \text{sobre} \quad \Sigma. \end{cases}$$

Así, tenemos que $\mathbf{z} \in L^\infty(0,T; D(\mathbf{A}^\sigma))$ y, en particular, $\mathbf{z} \in \mathbf{L}^\infty(Q)$. Además, la siguiente estimación es válida :

$$\|\mathbf{z}\|_{L^\infty(0,T;D(\mathbf{A}^\sigma))} \le \|\overline{\mathbf{y}}\|_{L^\infty(0,T;D(\mathbf{A}^\sigma))}.$$

Ahora, asociamos a $\mathbf{z}$ el control $\mathbf{v}$ de norma mínima en $L^\infty(0,T;\mathbf{L}^2(\omega))$ y la correspondiente solución $(\mathbf{y},p)$ para

$$\begin{cases} \mathbf{y}_t - \Delta\mathbf{y} + (\mathbf{z}\cdot\nabla)\mathbf{y} + \nabla p = \mathbf{v}1_\omega & \text{en} \quad Q, \\ \nabla\cdot\mathbf{y} = 0 & \text{en} \quad Q, \\ \mathbf{y} = \mathbf{0} & \text{sobre} \quad \Sigma, \\ \mathbf{y}(\cdot,0) = \mathbf{y}_0, \ \ \mathbf{y}(\cdot,T) = \mathbf{0} & \text{en} \quad \Omega. \end{cases}$$

Notemos que para garantizar el control nulo uniforme de este sistema lineal es importante que $\mathbf{z}\in\mathbf{L}^\infty(Q)$.

Como $\mathbf{y}_0\in D(\mathbf{A})$, $\mathbf{z}\in\mathbf{L}^\infty(Q)$ y $\mathbf{v}\in L^\infty(0,T;\mathbf{L}^2(\omega))$, tenemos:

$$\|\mathbf{y}\|_{L^\infty(D(\mathbf{A}^{\sigma'}))} \le C(\|\mathbf{y}_0\|_{D(\mathbf{A})} + \|\mathbf{v}\|_{L^\infty(\mathbf{L}^2(\omega))})e^{C\|\overline{\mathbf{y}}\|^2_{L^\infty(D(\mathbf{A}^\sigma))}},$$

donde $\sigma < \sigma' < 1$.

De ese modo, definimos

$$\mathbf{W} = \{\, \mathbf{w}\in L^\infty(0,T;D(\mathbf{A}^{\sigma'})) : \mathbf{w}_t \in L^2(0,T;\mathbf{H}) \,\}$$

y consideramos la bola cerrada

$$\mathbf{K} = \{\, \overline{\mathbf{y}}\in L^\infty(0,T;D(\mathbf{A}^{\sigma})) : \|\overline{\mathbf{y}}\|_{L^\infty(D(\mathbf{A}^\sigma))} \le 1 \,\}$$

y la aplicación $\widetilde{\Lambda}_\alpha : L^\infty(0,T;D(\mathbf{A}^\sigma)) \mapsto \mathbf{W}$, con $\widetilde{\Lambda}_\alpha(\overline{\mathbf{y}}) = \mathbf{y}$.

Finalmente, definimos Λ_α como la restricción de $\widetilde{\Lambda}_\alpha$ a $\mathbf{K}$. Así, tomando el dato inicial suficientemente pequeño, independiente de α, tenemos que Λ_α envía $\mathbf{K}$ en sí mismo. Además, es claro que $\Lambda_\alpha : \mathbf{K} \mapsto \mathbf{K}$ satisface las hipótesis del *Teorema de punto fijo de Schauder*. Consecuentemente, Λ_α posee al menos un punto fijo en $\mathbf{K}$.

La prueba del Teorema 1.5 es semejante. De nuevo, usamos un argumento de punto fijo. Al contrario del caso de la controlabilidad interna, trabajaremos en un espacio $\widetilde{\mathbf{Y}}$ de funciones definidas en un dominio extendido.

Sea $\widetilde{\Omega}$, con $\Omega \subset \widetilde{\Omega}$ y $\partial\widetilde{\Omega}\cap\Gamma = \Gamma\setminus\gamma$ tal que $\partial\widetilde{\Omega}$ es de clase C^2 (véase Fig. 1.1). Sea $\omega\subset\widetilde{\Omega}\setminus\overline{\Omega}$ un subconjunto abierto y no vacío e introduzcamos $\widetilde{Q} := \widetilde{\Omega}\times(0,T)$ y $\widetilde{\Sigma} := \partial\widetilde{\Omega}\times(0,T)$. Denotamos espacios y operadores asociados al dominio extendido por $\widetilde{\mathbf{H}}$, $\widetilde{\mathbf{V}}$, $\widetilde{\mathbf{A}}$, etc.

De ese modo, tomando inicialmente $\mathbf{h}_\alpha \equiv \mathbf{0}$ y aplicando el Lema 1.1 a la solución de (1.11), solamente necesitamos considerar el caso en el que el dato inicial $\mathbf{y}_0$ pertenece a $\mathbf{V}$ y posee una norma suficientemente pequeña en $\mathbf{V}$.

Sin embargo, esto no es suficiente para aplicar un teorema de punto fijo que nos garantice el Teorema 1.5. Con tal fin, usamos el siguiente resultado, similar al Lema 1.1 :

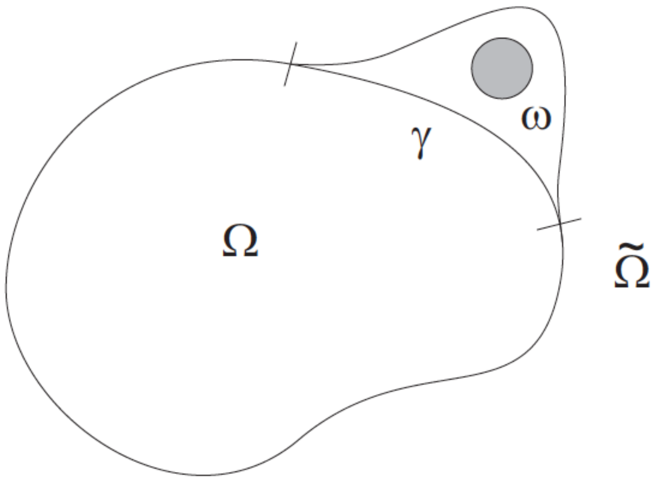

Figure 1.1: El dominio extendido

Lema 1.2. *Existe una función continua $\phi : \mathbb{R}_+ \mapsto \mathbb{R}_+$ satisfaciendo $\phi(s) \to 0$ cuando $s \to 0^+$ con la siguiente propriedad :*

a) Para cualesquier $\mathbf{y}_0 \in \mathbf{V}$ y $\alpha > 0$, existen tiempos $T_0 \in (0,T)$, controles $\mathbf{h}_\alpha \in L^2(0,T_0;\mathbf{H}^{1/2}(\Gamma))$ con $\int_\gamma \mathbf{h}_\alpha \cdot \mathbf{n}\, d\Gamma \equiv 0$, soluciones asociadas $(\mathbf{y}_\alpha, p_\alpha, \mathbf{z}_\alpha, \pi_\alpha)$ para (1.11) en $\Omega \times (0,T_0)$ y tiempos arbitrariamente pequeños $t^* \in (0,T/2)$ tales que los estados $\mathbf{y}_\alpha$ pueden ser extendidos a $\widetilde{\Omega} \times (0,T_0)$ y sus extensiones satisfacen $\|\tilde{\mathbf{y}}_\alpha(\cdot,t^*)\|^2_{D(\widetilde{\mathbf{A}})} \leq \phi(\|\mathbf{y}_0\|_\mathbf{V})$.

b) El conjunto de estos tiempos t^* tiene medida positiva.

c) Los controles $\mathbf{h}_\alpha$ son uniformemente acotados, i.e.,

$$\|\mathbf{h}_\alpha\|_{L^\infty(0,T_0;\mathbf{H}^{1/2}(\gamma))} \leq C.$$

Así, solo necesitamos considerar el caso en que el estado inicial $\mathbf{y}_0$ es tal que su extensión $\tilde{\mathbf{y}}_0$ a $\widetilde{\Omega}$ pertenece a $D(\widetilde{\mathbf{A}})$ y posee una norma suficientemente pequeña en $D(\widetilde{\mathbf{A}})$.

En consecuencia, todo se reduce a probar la existencia de $(\tilde{\mathbf{y}}, \tilde{p}, \mathbf{z}, \pi, \tilde{\mathbf{v}})$, con $\tilde{\mathbf{v}} \in$

$L^\infty(0,T;\mathbf{L}^2(\omega))$, satisfaciendo

$$\begin{cases} \tilde{\mathbf{y}}_t - \Delta\tilde{\mathbf{y}} + (\tilde{\mathbf{z}}\cdot\nabla)\tilde{\mathbf{y}} + \nabla\tilde{p} = \tilde{\mathbf{v}}1_\omega & \text{en} \quad \widetilde{Q}, \\ \mathbf{z} - \alpha^2\Delta\mathbf{z} + \nabla\pi = \tilde{\mathbf{y}} & \text{en} \quad Q, \\ \nabla\cdot\tilde{\mathbf{y}} = 0 & \text{en} \quad \widetilde{Q}, \\ \nabla\cdot\mathbf{z} = 0 & \text{en} \quad Q, \\ \tilde{\mathbf{y}} = \mathbf{0} & \text{sobre} \quad \widetilde{\Sigma}, \\ \mathbf{z} = \tilde{\mathbf{y}} & \text{sobre} \quad \Sigma, \\ \tilde{\mathbf{y}}(\cdot,0) = \tilde{\mathbf{y}}_0, \ \ \tilde{\mathbf{y}}(\cdot,T) = \mathbf{0} & \text{en} \quad \widetilde{\Omega}, \end{cases} \tag{1.15}$$

donde $\tilde{\mathbf{z}}$ es la extensión de $\mathbf{z}$ tal que $\tilde{\mathbf{z}} = \tilde{\mathbf{y}}$ en $\widetilde{\Omega}\backslash\Omega$. Obviamente, la restricción de $(\tilde{\mathbf{y}},\tilde{p})$ a Q, denotada $(\mathbf{y},p)$, el par $(\mathbf{z},\pi)$ y la traza lateral $\mathbf{h} := \tilde{\mathbf{y}}|_{\gamma\times(0,T)}$ satisfacen (1.11) y (1.12).

Igual que antes, fijemos $\sigma\in(N/4,1)$. Entonces para cada $\overline{\mathbf{y}}\in L^\infty(0,T;D(\widetilde{\mathbf{A}}^\sigma))$, sea $(\mathbf{w},\pi)$ la única solución de

$$\begin{cases} \mathbf{w} - \alpha^2\Delta\mathbf{w} + \nabla\pi = \alpha^2\Delta\overline{\mathbf{y}} & \text{en} \quad Q, \\ \nabla\cdot\mathbf{w} = 0 & \text{en} \quad Q, \\ \mathbf{w} = \mathbf{0} & \text{sobre} \quad \Sigma. \end{cases}$$

Usando resultados de regularidad para el problema de Stokes estacionario, se puede probar que $\mathbf{w}\in\mathbf{L}^\infty(Q)$ y

$$\|\mathbf{w}\|^2_{L^\infty(0,T;D(\mathbf{A}^\sigma))} \le C\|\overline{\mathbf{y}}\|^2_{L^\infty(0,T;D(\widetilde{\mathbf{A}}^\sigma))},$$

donde C es independiente de α.

Sea $\tilde{\mathbf{w}}$ la extensión por cero de $\mathbf{w}$ y sea $\tilde{\mathbf{z}} := \overline{\mathbf{y}} + \tilde{\mathbf{w}}$. Ahora, asociamos a $\tilde{\mathbf{z}}$ el control $\tilde{\mathbf{v}}$ de norma mínima en $L^\infty(0,T;\mathbf{L}^2(\omega))$ y la correspondiente solución $(\tilde{\mathbf{y}},\tilde{p})$ para :

$$\begin{cases} \tilde{\mathbf{y}}_t - \Delta\tilde{\mathbf{y}} + (\tilde{\mathbf{z}}\cdot\nabla)\tilde{\mathbf{y}} + \nabla\tilde{p} = \tilde{\mathbf{v}}1_\omega & \text{en} \quad \widetilde{Q}, \\ \nabla\cdot\tilde{\mathbf{y}} = 0 & \text{en} \quad \widetilde{Q}, \\ \tilde{\mathbf{y}} = \mathbf{0} & \text{sobre} \quad \widetilde{\Sigma}, \\ \tilde{\mathbf{y}}(\cdot,0) = \tilde{\mathbf{y}}_0, \ \ \tilde{\mathbf{y}}(\cdot,T) = \mathbf{0} & \text{en} \quad \widetilde{\Omega}. \end{cases}$$

Dado que $\tilde{\mathbf{y}}_0\in D(\widetilde{\mathbf{A}})$, $\tilde{\mathbf{z}}\in\mathbf{L}^\infty(\widetilde{Q})$ y $\tilde{\mathbf{v}}\in L^\infty(0,T;\mathbf{L}^2(\omega))$, tenemos :

$$\|\tilde{\mathbf{y}}\|_{L^\infty(D(\widetilde{\mathbf{A}}^\beta))} \le C(\|\tilde{\mathbf{y}}_0\|_{D(\widetilde{\mathbf{A}})} + \|\tilde{\mathbf{v}}\|_{L^\infty(\mathbf{L}^2(\omega))})e^{C\|\overline{\mathbf{y}}\|_{L^\infty(0,T;D(\widetilde{\mathbf{A}}^\sigma))}}$$

donde $\sigma<\beta<1$.

Entonces, podemos definir una aplicación $\widetilde{\Lambda}_\alpha(\overline{\mathbf{y}}) = \tilde{\mathbf{y}}$ y, procediendo como en el teorema anterior, podemos garantizar la existencia de un punto fijo $\tilde{\mathbf{y}}\in L^\infty(0,T;D(\widetilde{\mathbf{A}}^\sigma))$.

Finalmente, probamos resultados relacionados a la controlabilidad en el límite, es decir, cuando $\alpha \to 0^+$.

Mostramos que los controles nulos para (1.10) pueden ser elegidos de tal modo que convergen a controles nulos internos de las ecuaciones de Navier-Stokes

$$\begin{cases} \mathbf{y}_t - \Delta\mathbf{y} + (\mathbf{y}\cdot\nabla)\mathbf{y} + \nabla p = \mathbf{v}1_\omega & \text{en} \quad Q, \\ \nabla\cdot\mathbf{y} = 0 & \text{en} \quad Q, \\ \mathbf{y} = \mathbf{0} & \text{sobre} \quad \Sigma, \\ \mathbf{y}(\cdot,0) = \mathbf{y}_0 & \text{en} \quad \Omega. \end{cases} \tag{1.16}$$

También, probamos que los controles nulos para (1.11) pueden ser elegidos de modo que converjan a controles nulos frontera de las ecuaciones de Navier-Stokes

$$\begin{cases} \mathbf{y}_t - \Delta\mathbf{y} + (\mathbf{y}\cdot\nabla)\mathbf{y} + \nabla p = \mathbf{0} & \text{en} \quad Q, \\ \nabla\cdot\mathbf{y} = 0 & \text{en} \quad Q, \\ \mathbf{y} = \mathbf{h}1_\gamma & \text{sobre} \quad \Sigma, \\ \mathbf{y}(\cdot,0) = \mathbf{y}_0 & \text{en} \quad \Omega. \end{cases} \tag{1.17}$$

Más precisamente, probamos los resultados siguientes :

Teorema 1.6 (Convergencia del control interno)**.** *Sea $\epsilon > 0$ la constante proporcionada por el Teorema 1.4. Supongamos que $\mathbf{y}_0 \in \mathbf{H}$ con $\|\mathbf{y}_0\| \le \epsilon$, sea $(\mathbf{v}_\alpha)$ la familia de controles nulos internos para (1.10) satisfaciendo (1.13) y sea $(\mathbf{y}_\alpha, p_\alpha, \mathbf{z}_\alpha, \pi_\alpha)$ la familia de estados asociados satisfaciendo (1.12). Entonces, al menos para una subsución, tenemos*

$$\mathbf{v}_\alpha \to \mathbf{v} \quad \textit{débil-}\star \quad \textit{en} \quad L^\infty(0,T;\mathbf{L}^2(\omega)),$$
$$\mathbf{z}_\alpha \to \mathbf{y} \quad \textit{y} \quad \mathbf{y}_\alpha \to \mathbf{y} \quad \textit{fuertemente en} \quad \mathbf{L}^2(Q)$$

cuando $\alpha \to 0^+$, donde $(\mathbf{y},\mathbf{v})$ es, junto con alguna presión p, un par estado-control para (1.16) que verifica (1.12).

Teorema 1.7 (Convergencia del control frontera)**.** *Sea $\delta > 0$ la constante proporcionado por el Teorema 1.5. Supongamos que $\mathbf{y}_0 \in \mathbf{H}$ con $\|\mathbf{y}_0\| \le \delta$, sea $(\mathbf{h}_\alpha)$ la familia de controles nulos frontera para (1.11) satisfaciendo (1.14) y sea $(\mathbf{y}_\alpha, p_\alpha, \mathbf{z}_\alpha, \pi_\alpha)$ la familia de estados asociados satisfaciendo (1.12). Entonces, al menos para una subsución, tenemos*

$$\mathbf{h}_\alpha \to \mathbf{h} \quad \textit{débil-}\star \quad \textit{en} \quad L^\infty(0,T;H^{-1/2}(\gamma)),$$
$$\mathbf{z}_\alpha \to \mathbf{y} \quad \textit{y} \quad \mathbf{y}_\alpha \to \mathbf{y} \quad \textit{fuertemente en} \quad \mathbf{L}^2(Q)$$

cuando $\alpha \to 0^+$, donde $(\mathbf{y},\mathbf{h})$ es, junto con alguna presión p, un par estado-control para (1.17) que satisface (1.12).

Las pruebas de los Teoremas 1.6 y 1.7 son casi inmediatas. Estos resultados proporcionan una relación entre los controles nulos para el sistema de Leray-α y los controles nulos para las ecuaciones de Navier-Stokes. Más precisamente, vemos que el control nulo para las ecuaciones de Navier-Stokes puede ser obtenido como límite de controles nulos para el sistema de Leray-α. La idea de las demostraciones es usar la estimación uniforme de los controles nulos para obtener buenas convergencias débiles del estado asociado $\mathbf{y}_\alpha$. Y así, finalmente, ver que el estado límite es una solución controlada a cero para las ecuaciones de Navier-Stokes.

Este capítulo está basado en el artículo [2], en colaboración con F. D. Araruna y E. Fernández-Cara.

Capítulo 4 : Sobre la controlabilidad frontera de fluidos de Euler incompresibles con efectos de calor de Boussinesq

En el Capítulo 3 estudiamos propiedades globales de controlabilidad para una clase de sistemas de leyes de conservación con origen en mecánica de los fluidos.

Detallemos los resultados de este capítulo y sus respectivas pruebas. Para ello, sea $\Omega \subset \mathbb{R}^N$ $(N = 2,\ 3)$ un dominio acotado simplemente conexo de clase C^∞ y denotemos Γ_0 (llamado *dominio de control frontera*) un subconjunto abierto no vacío de la frontera Γ de Ω.

En este capítulo, estudiamos el control frontera del sistema

$$\begin{cases} \mathbf{y}_t + (\mathbf{y}\cdot\nabla)\mathbf{y} = -\nabla p + \vec{\mathbf{k}}\theta & \text{en} \quad \Omega\times(0,T), \\ \nabla\cdot\mathbf{y} = 0 & \text{en} \quad \Omega\times(0,T), \\ \theta_t + \mathbf{y}\cdot\nabla\theta = \kappa\Delta\theta & \text{en} \quad \Omega\times(0,T), \\ \mathbf{y}(\mathbf{x},0) = \mathbf{y}_0(\mathbf{x}),\ \theta(\mathbf{x},0) = \theta_0(\mathbf{x}) & \text{en} \quad \Omega. \end{cases} \tag{1.18}$$

Este sistema modela el comportamiento de un fluido no viscoso, homogéneo e incompresible con efectos térmicos. Más precisamente,

- El campo $\mathbf{y}$ y el escalar p representan el campo de velocidades y la presión de un fluido no viscoso incompresible en un dominio casi impermeable Ω que es observado durante el intervalo de tiempo $(0,T)$.

- La función θ proporciona una distribución de temperatura del fluido.

- El término $\vec{\mathbf{k}}\theta$ puede ser interpretado como la densidad de la *fuerza de flotación* que actúa en la dirección del vector $\vec{\mathbf{k}}$ (un vector no nulo de $\mathbb{R}^N$).

- La constante no negativa $\kappa \geq 0$ es el coeficiente de difusión del calor.

Cuando $\kappa = 0$, el sistema (1.18) es llamado de *Sistema de Boussinesq no viscoso incompresible*. Cuando $\kappa > 0$, el sistema (1.18) es llamado de *Sistema de Boussinesq no viscoso incompresible difusor de calor*.

Ahora, vamos formular el problema de control frontera para el sistema (1.18). Primeramente, observemos que es natural considerar una condición homogénea, conocida como *condición de impermeabilidad*, fuera de la región donde el control actuará, i.e. :

$$\mathbf{y} \cdot \mathbf{n} = 0 \quad \text{sobre} \quad (\Gamma \backslash \Gamma_0) \times (0, T). \tag{1.19}$$

Por otro lado, en el dominio de control debemos elegir condiciones de frontera no homogéneas. Estas condiciones serán los controles. Véase la figura (1.2).

Notemos que, cuando $\kappa = 0$, como condiciones de frontera no homogéneas podemos elegir la componente normal del campo velocidad sobre el dominio de control y todo el campo velocidad $\mathbf{y}$ e la temperatura θ sobre la sección de entrada de flujo, es decir, solamente donde $\mathbf{y} \cdot \mathbf{n} < 0$, véase por ejemplo [99]. Así, en este caso, podemos asumir que los controles son como sigue :

$$\begin{cases} v := \mathbf{y} \cdot \mathbf{n} \text{ sobre } \Gamma_0 \times (0, T), \text{ con } \displaystyle\int_{\Gamma_0} \mathbf{y} \cdot \mathbf{n} \, d\Gamma = 0; \\ \mathbf{h} := \mathbf{y} \text{ y } w := \theta \text{ en los puntos de } \Gamma_0 \times (0, T) \text{ donde } \mathbf{y} \cdot \mathbf{n} < 0. \end{cases}$$

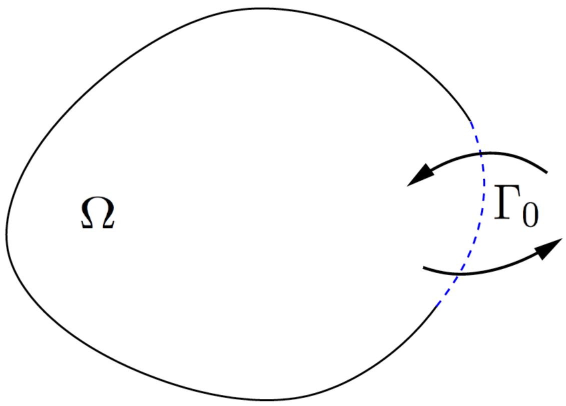

Figure 1.2: El dominio de control

Observemos que como la ecuación de la temperatura θ es de transporte, la única región donde se puede fijar condiciones de contorno es donde las partículas, que viajan con velocidad $\mathbf{y}$, están entrando en el dominio, es decir, sobre los puntos de la frontera donde $\mathbf{y} \cdot \mathbf{n} < 0$.

En el caso $\kappa > 0$, la situación es distinta. En este caso, se deben fijar condiciones de contorno para la temperatura sobre toda la frontera lateral $\Gamma \times (0, T)$. De ese modo, consideremos un subconjunto abierto no vacío $\gamma \subset \Gamma$, llamado *dominio de control térmico* (donde las partículas no están necesariamente entrando). Por tanto, podemos considerar una condición homogénea para la temperatura fuera del dominio de control térmico, es decir,

$$\theta = 0 \quad \text{sobre} \quad (\Gamma \backslash \gamma) \times (0, T). \tag{1.20}$$

Entonces, podemos asumir que los controles son como sigue :

$$\begin{cases} v := \mathbf{y} \cdot \mathbf{n} \text{ sobre } \Gamma_0 \times (0,T), \text{ con } \displaystyle\int_{\Gamma_0} \mathbf{y} \cdot \mathbf{n}\, d\Gamma = 0; \\ \mathbf{h} := \mathbf{y} \text{ en cualquier punto de } \Gamma_0 \times (0,T) \text{ satisfaciendo } \mathbf{y} \cdot \mathbf{n} < 0; \\ w := \theta \text{ en cualquier punto de } \gamma \times (0,T). \end{cases}$$

De ese modo, podemos formular los problemas de controlabilidad exacta. En realidad, estos problemas pueden ser formulados de dos formas distintas.

a) Formulación estándar : Dados $T > 0$ y datos iniciales y finales $\mathbf{y}_0$, $\mathbf{y}_T$, θ_0 y θ_T en espacios adecuados, ¿ podemos encontrar controles $(v, \mathbf{h}, w)$ tales que la solución asociada $(\mathbf{y}, p, \theta)$ satisface

$$\mathbf{y}(\cdot, T) = \mathbf{y}_T, \quad \theta(\cdot, T) = \theta_T \quad \text{en} \quad \Omega\,?$$

b) Formulación alternativa : Dados $T > 0$ y datos iniciales y finales $\mathbf{y}_0$, $\mathbf{y}_T$, θ_0 y θ_T en espacios adecuados, ¿ podemos encontrar una solución del sistema satisfaciendo

$$\mathbf{y} \cdot \mathbf{n} = 0 \quad \text{sobre} \quad (\partial\Omega \setminus \Sigma) \times [0, T]$$

que conduce el estado inicial $(\mathbf{u}_0, \theta_0)$ al estado final $(\mathbf{u}_T, \theta_T)$ en el tiempo T ?

En la formulación b) el sistema está sobre-determinado. De ese modo, los controles pueden ser obtenidos a partir de las trazas de $\mathbf{y}$ y θ. En esta Tesis, usamos la formulación b).

De ahora en adelante, supongamos que $\alpha \in (0, 1)$ y definamos los espacios

$$\begin{aligned} \mathbf{C}_0^{m,\alpha}(\bar{\Omega}; \mathbb{R}^N) &:= \{\, \mathbf{u} \in \mathbf{C}^{m,\alpha}(\bar{\Omega}; \mathbb{R}^N) : \nabla \cdot \mathbf{u} = 0 \text{ en } \bar{\Omega},\ \mathbf{u} \cdot \mathbf{n} = 0 \text{ sobre } \Gamma \,\}, \\ \mathbf{C}(m, \alpha, \Gamma_0) &:= \{\, \mathbf{u} \in \mathbf{C}^{m,\alpha}(\bar{\Omega}; \mathbb{R}^N) : \nabla \cdot \mathbf{u} = 0 \text{ en } \bar{\Omega},\ \mathbf{u} \cdot \mathbf{n} = 0 \text{ sobre } \Gamma \setminus \Gamma_0 \,\}, \end{aligned}$$

donde $\mathbf{C}^{m,\alpha}(\overline{\Omega}; \mathbb{R}^N)$ denota el espacio de las funciones de $\mathbf{C}^m(\overline{\Omega}; \mathbb{R}^N)$ cuyas derivadas de orden m son Hölder-continuas con exponente α.

Los principales resultados de este capítulo son los siguientes :

Teorema 1.8. *Si $\kappa = 0$, entonces el sistema de Boussinesq no viscoso incompresible (1.18) es exactamente controlable para (Ω, Γ_0) para cualquier tiempo $T > 0$. En otras palabras, dados $\mathbf{y}_0, \mathbf{y}_T \in \mathbf{C}(2, \alpha, \Gamma_0)$ y $\theta_0, \theta_T \in C^{2,\alpha}(\bar{\Omega})$, existen $\mathbf{y} \in C^0([0, T]; \mathbf{C}(1, \alpha, \Gamma_0))$, $\theta \in C^0([0, T]; C^{1,\alpha}(\bar{\Omega}))$ y $p \in \mathcal{D}'(\Omega \times (0, T))$ verificando (1.18)–(1.19) y*

$$\mathbf{y}(\mathbf{x}, T) = \mathbf{y}_1(\mathbf{x}), \quad \theta(\mathbf{x}, T) = \theta_1(\mathbf{x}) \ \text{ en } \ \Omega. \tag{1.21}$$

Teorema 1.9. *Si $\kappa > 0$, entonces el sistema (1.18) es localmente exactamente–nulo controlable para $(\Omega, \Gamma_0, \gamma)$. En otras palabras, datos $T > 0$ e $\mathbf{y}_0, \mathbf{y}_T \in \mathbf{C}_0^{m,\alpha}(\bar{\Omega}; \mathbb{R}^N)$, existe $\eta > 0$, dependiendo de $\mathbf{y}_0$, tal que, para cada $\theta_0 \in C^{2,\alpha}(\bar{\Omega})$ con*

$$\theta_0 = 0 \quad \textit{sobre} \quad \Gamma \setminus \gamma, \quad \|\theta_0\|_{2,\alpha} \leq \eta,$$

existen $\mathbf{y} \in C^0([0,T]; \mathbf{C}^{1,\alpha}(\bar{\Omega}; \mathbb{R}^N)$, $\theta \in C^0([0,T]; C^{1,\alpha}(\bar{\Omega}))$ y $p \in \mathcal{D}'(\Omega \times (0,T))$ satisfaciendo (1.18)–(1.20) y también

$$\mathbf{y}(\mathbf{x}, T) = \mathbf{y}_T(\mathbf{x}), \quad \theta(\mathbf{x}, T) = 0 \quad \textit{en} \quad \Omega. \tag{1.22}$$

El significado del resultado de Teorema 1.8 ($\kappa = 0$) es que podemos conducir el fluido velocidad–temperatura desde cualquier dato inicial $(\mathbf{y}_0, \theta_0)$ exactamente a cualquier dato final $(\mathbf{y}_T, \theta_T)$, actuando solamente sobre una parte arbitrariamente pequeña Γ_0 de la frontera durante un intervalo de tiempo arbitrariamente pequeño $(0, T)$.

Por otro lado, el significado del resultado de Teorema 1.9 ($\kappa > 0$) es que podemos conducir el fluido velocidad–temperatura desde cualquier dato inicial $(\mathbf{y}_0, \theta_0)$, satisfaciendo una condición de pequeñez para la temperatura inicial, exactamente a cualquier dato final de la forma $(\mathbf{y}_T, 0)$, actuando solamente sobre trozos arbitrariamente pequeños Γ_0 y γ de la frontera durante un intervalo de tiempo arbitrariamente pequeño $(0, T)$. Observemos que, en el caso $\kappa > 0$, el objetivo es conducir el sistema a un estado de la forma $(\mathbf{y}_T, 0)$. Esto es lo razonable : debido al *efecto regularizante* de la ecuación de la temperatura, no podemos esperar la controlabilidad exacta a toda temperatura final.

En el contexto de fluidos no viscosos incompresibles sin efectos térmicos, podemos citar los resultados de control global obtenidos por J.-M. Coron [27, 29] y O. Glass [65, 66, 67]. Las pruebas de los resultados anteriores están basadas en las técnicas y argumentos de [29] y [67].

Presentaremos a continuación un esquema de las mismas.

Empecemos con la prueba de Teorema 1.8. Lo primero que observamos es que hay invarianza de escala de tiempo en las ecuaciones del sistema (1.18). Más precisamente, podemos observar que si $(\mathbf{y}, p, \theta)(\mathbf{x}, t)$ es solución del sistema (1.18) en $\bar{\Omega} \times [0, T]$ entonces $\mathbf{u}_\varepsilon(\mathbf{x}, t) := \varepsilon \mathbf{u}(\mathbf{x}, \varepsilon t)$, $(p_\varepsilon, \theta_\varepsilon)(\mathbf{x}, t) := \varepsilon^2 (p, \theta)(\mathbf{x}, \varepsilon t)$ es solución del sistema (1.18) en $\bar{\Omega} \times [0, T/\varepsilon]$. Por tanto, el Teorema 1.8 es una consecuencia del siguiente resultado local :

Proposición 1.1. *Supongamos que $\kappa = 0$. Existe $\delta > 0$ tal que, para cualquier $\mathbf{y}_0 \in \mathbf{C}(2, \alpha, \Gamma_0)$ y cualquier $\theta_0 \in C^{2,\alpha}(\bar{\Omega})$ con*

$$\max \{ \|\mathbf{y}_0\|_{2,\alpha}, \|\theta_0\|_{2,\alpha} \} \leq \delta,$$

existen $\mathbf{y} \in C^0([0,T]; \mathbf{C}(1, \alpha, \Gamma_0))$, $\theta \in C^0([0,1]; C^{1,\alpha}(\bar{\Omega}))$ y $p \in \mathcal{D}'(\Omega \times (0,T))$ satisfaciendo

(1.18)– (1.19) *y*

$$\mathbf{y}(\mathbf{x},T)=\mathbf{0},\ \theta(\mathbf{x},T)=0 \quad en \quad \Omega. \tag{1.23}$$

Aquí tenemos un resumen de la prueba :

- Primero, construimos una "buena" trayectoria que conecta $(\mathbf{0},0,0)$ a $(\mathbf{0},0,0)$.
- Entonces, aplicamos el metodo de extensión de David L. Russell [111].
- Después, usamos un *Teorema de punto fijo* y deducimos un resultado local de control exacto.

La idea que hay detrás es que la ecuación satisfecha por la vorticidad es una ecuación de transporte. Así tenemos que la vorticidad sigue el flujo en el caso $N=2$ y su soporte sigue el flujo en el caso $N=3$. Con esta idea presente, podemos extender nuestro problema a un dominio conteniendo Ω y construir una trayectoria $(\overline{\mathbf{y}},\overline{p},\overline{\theta})$ que saca partículas fuera de Ω. De ese modo, construimos una aplicación de punto fijo bien definida en una bola centrada en $\overline{\mathbf{y}}$ (los elementos de esta bola también sacan las partículas). Finalmente, aplicamos el *Teorema de punto fijo de Banach* a una de las iteradas de la aplicación de punto fijo, debido a que no se puede garantizar que esta aplicación sea una contracción.

La prueba del Teorema 1.9 se hace de manera parecida. Dividamos la prueba en dos etapas :

Etapa 1 : Primeramente, observamos que sólo necesitamos controlar la temperatura a cero manteniendo la regularidad del campo de velocidades; para esto necesitamos que la temperatura inicial sea suficientemente pequeña. Esto está establecido en el siguiente resultado :

Proposición 1.2. *Para cada* $\mathbf{y}_0\in\mathbf{C}_0^{m,\alpha}(\bar\Omega;\mathbb{R}^N)$ *existen* $T^*\in(0,T)$ *y* $\eta>0$ *tales que, si* $\theta_0\in C^{2,\alpha}(\bar\Omega)$, $\theta_0=0$ *sobre* $\Gamma\backslash\gamma$ *y* $\|\theta_0\|_{2,\alpha}\le\eta$, *entonces el sistema*

$$\begin{cases}\mathbf{y}_t+(\mathbf{y}\cdot\nabla)\mathbf{y}=-\nabla p+\vec{\mathbf{k}}\,\theta & en \quad \Omega\times(0,T^*),\\ \nabla\cdot\mathbf{y}=0 & en \quad \Omega\times(0,T^*),\\ \theta_t+\mathbf{y}\cdot\nabla\theta=\kappa\Delta\theta & en \quad \Omega\times(0,T^*),\\ \mathbf{y}\cdot\mathbf{n}=0 & sobre \quad \Gamma\times(0,T^*),\\ \theta=0 & sobre \quad (\Gamma\setminus\gamma)\times(0,T^*),\\ \mathbf{y}(\mathbf{x},0)=\mathbf{y}_0(\mathbf{x}),\ \theta(\mathbf{x},0)=\theta_0(\mathbf{x}) & en \quad \Omega,\end{cases} \tag{1.24}$$

posee al menos una solución $\mathbf{y}\in C^0([0,T^*];\mathbf{C}^{2,\alpha}(\bar\Omega;\mathbb{R}^N))$, $\theta\in C^0([0,T^*];C^{2,\alpha}(\bar\Omega))$ *y* $p\in\mathcal{D}'(\Omega\times(0,T^*))$ *tal que*

$$\theta(\mathbf{x},T^*)=0 \quad en \quad \Omega.$$

Etapa 2 : En esta etapa, mantenemos la temperatura igual a cero y solamente controlamos el campo velocidad como en el Teorema 1.8, conduciendo el campo velocidad al estado deseado.

Este capítulo está basado en el artículo [52], en colaboración con E. Fernández-Cara y M. C. Santos.

1.3.2 Parte II. Resultados numéricos sobre el control de varias ecuaciones y sistemas parabólicos

La segunda parte de la Tesis está dedicada al desarollo de estrategias para la controlabilidad numérica de las ecuaciones lineales del calor y Stokes y para las ecuaciones no lineales de Navier-Stokes. El objetivo es calcular aproximaciones de controles que conducen la solución de un estado inicial prescrito a un estado deseado (nulo o sobre una trayectoria) en un tiempo positivo dado.

Para aproximar un problema de control, lo que se suele hacer es relacionarlo con un problema de control óptimo, i.e. minimizar cierto funcional que, en la mayoría de los casos, está relacionado con el coste de la controlabilidad.

En general, es muy complicado aproximar directamente el problema de minimizar el funcional coste (que viene dado por una norma del control), debido a que el espacio donde estamos minimizando hay serias restricciones. Recurriendo a la teoría de dualidad, se puede llegar a un problema equivalente, ahora sin restricciones, conocido como *problema dual*. En el problema dual se minimiza sobre un espacio "grande", lo cual genera otra vez importantes dificultades a la hora de aproximar. Sin embargo, esto se puede solucionar penalizando el problema dual. Podemos destacar que éste es un problema muy difícil de tratar numéricamente (además de la alta sensibilidad del problema dual con respecto al parámetro de penalización). En realidad, el control obtenido muestra un comportamiento altamente oscilatorio y singular cerca del tiempo final, produciendo inestabilidades numéricas.

Nuestra estrategia es diferente. Buscaremos controles que no tengan este comportamiento oscilatorio singular cerca del tiempo final. De ese modo, minimizaremos un funcional que involucra integrales promediadas del estado y control (o solamente del control) sobre una clase de controles nulos admisibles, con pesos que explotan cerca del tiempo final. Para estos problemas, las condiciones de optimalidad asociadas pueden ser vistas como formulaciones débiles de problemas diferenciales. Gracias a que los pesos tienen estas características, las variables minimizadas van a cero con comportamiento inverso al de los pesos. Los métodos propuestos en esta parte de la Tesis son robustos para una amplia clase de datos.

Complementaremos los resultados presentando varios experimentos numéricos.

En lo que sigue, describiremos con detalle el contenido de los capítulos de esta última parte.

Capítulo 5: Una formulación mixta para la aproximación de controles en el espacio L^2 con pesos para la ecuación del calor lineal

El Capítulo 5 de esta Tesis se dedica al cálculo numérico de controles nulos para la ecuación del calor lineal. Más precisamente, presentamos un método para aproximar el control de norma mínima, donde la norma es una integral promediada. Las condiciones de optimalidad del problema son escritas como una formulación mixta que involucra tanto el estado como su adjunto. Probamos que la formulación mixta está bien propuesta (en particular, la condición inf-sup) y luego discutimos varios experimentos numéricos.

Detallemos el contenido de este capítulo. Para eso, sea $\Omega \subset \mathbb{R}^N$, un dominio acotado cuya frontera Γ es de clase C^2. Sean $\omega \subset \Omega$ un conjunto abierto y no vacío (llamado de nuevo *dominio de control*) y $T > 0$. En lo que sigue, para cualquier $\tau > 0$ denotemos por Q_τ, Σ_τ y q_τ los conjuntos $\Omega \times (0,\tau)$, $\Gamma \times (0,\tau)$ y $\omega \times (0,\tau)$, respectivamente.

Consideramos el problema de controlabilidad nula para la ecuación del calor

$$\begin{cases} y_t - \nabla \cdot (c(x)\nabla y) + d(x,t)y = v\, 1_\omega, & \text{en} \quad Q_T, \\ y = 0, & \text{sobre} \quad \Sigma_T, \\ y(x,0) = y_0(x), & \text{en} \quad \Omega. \end{cases} \tag{1.25}$$

Aquí, suponemos que $c := (c_{i,j}) \in C^1(\overline{\Omega}; \mathcal{M}_N(\mathbb{R}))$ con $(c(x)\xi,\xi) \geq c_0|\xi|^2$ en $\overline{\Omega}$ $(c_0 > 0)$, $d \in L^\infty(Q_T)$ e $y_0 \in L^2(\Omega)$; $v = v(x,t)$ es el *control* (una función en $L^2(q_T)$) e $y = y(x,t)$ es el estado asociado. Además, 1_ω es la función característica asociada al conjunto ω.

Usaremos la siguiente notación :

$$L\, y := y_t - \nabla \cdot (c(x)\nabla y) + d(x,t)y, \qquad L^\star \varphi := -\varphi_t - \nabla \cdot (c(x)\nabla \varphi) + d(x,t)\varphi.$$

Para cualquier $y_0 \in L^2(\Omega)$ y $v \in L^2(q_T)$, existe exactamente una solución y de (1.25), con la regularidad $y \in C^0([0,T]; L^2(\Omega)) \cap L^2(0,T; H_0^1(\Omega))$ (véase [98, 15]). Por lo tanto, dado un tiempo final $T > 0$, el problema es el siguiente : para cada $y_0 \in L^2(\Omega)$, encontrar $v \in L^2(q_T)$ tal que la correspondiente solución para (1.25) satisface

$$y(\mathbf{x},T) = 0 \quad \text{en} \quad \Omega. \tag{1.26}$$

Recordemos que la controlabilidad de EDPs es un importante área de investigación que ha motivado muchos artículos en los últimos años, entre ellos [90, 96, 112] y [30]. En particular, citamos a [62] y [92] donde el control nulo de (1.25) está probado.

La controlabilidad numérica es también una cuestión fundamental, dado que no es posible en general conseguir expresiones explícitas de los controles. Debido a las fuertes propiedades regularizantes del núcleo del calor, la aproximación numérica de controles nulos es una cuestión delicada. Lo mismo vale para la teoría de problemas inversos cuando ecuaciones y sistemas parabólicos está involucrados (véase [40]). Esto ha sido exhibido numéricamente en [13], donde se hizo el uso de argumentos de dualidad y se consideró el control de norma mínima : el problema es

$$\begin{cases} \text{Minimizar} \;\; J_1(y,v) := \dfrac{1}{2}\displaystyle\iint_{q_T} |v(x,t)|^2\,dx\,dt \\ \text{Sujeto a} \;\; (y,v) \in \mathcal{C}(y_0,T), \end{cases} \tag{1.27}$$

donde $\mathcal{C}(y_0;T)$ denota la variedad lineal

$$\mathcal{C}(y_0;T) := \{\,(y,v) : v \in L^2(q_T),\; y \text{ resuelve } (1.25) \text{ y satisface } (1.26)\,\}.$$

Las primeras contribuciones a la controlabilidad numérica se deben a Glowinski y Lions en [70] (actualizado en [71]) y se basan en argumentos de dualidad que permiten reemplazar el problema original de minimización con restricciones por un problema de minimización sin restricciones (dual) que es *a priori* más fácil. El problema dual asociado a (1.27) es :

$$\min_{\varphi_T \in \mathcal{H}} J_1^\star(\varphi_T) := \frac{1}{2}\iint_{q_T} |\varphi(x,t)|^2\,dx\,dt + \int_\Omega y_0(x)\varphi(x,0)\,dx, \tag{1.28}$$

donde la variable φ resuelve la ecuación del calor retrógrada :

$$L^\star\varphi = 0 \quad \text{en} \quad Q_T, \qquad \varphi = 0 \quad \text{sobre} \quad \Sigma_T; \quad \varphi(\cdot,T) = \varphi_T \quad \text{en} \quad \Omega \tag{1.29}$$

y el espacio de Hilbert $\mathcal{H}$ es el completado de $\mathcal{D}(\Omega)$ con respecto a la norma

$$\|\varphi_T\|_{\mathcal{H}} := \|\varphi\|_{L^2(q_T)}.$$

En vista de la propiedad de continuación única para (1.29), la aplicación $\varphi_T \mapsto \|\varphi_T\|_{\mathcal{H}}$ es una norma Hilbertiana en $\mathcal{D}(\Omega)$. La coercividad del funcional $J_1^\star$ en $\mathcal{H}$ es una consecuencia de la llamada *desigualdad de observabilidad*

$$\|\varphi(\cdot,0)\|^2_{L^2(\Omega)} \le C \iint_{q_T} |\varphi(x,t)|^2\,dx\,dt \quad \forall \varphi_T \in \mathcal{H}, \tag{1.30}$$

donde φ resuelve (1.29). Esta desigualdad es cierta para una constante $C = C(\omega,T)$ adecuada y, a su vez, es consecuencia de desigualdades de Carleman apropiadas; véase [62].

La minimización de $J_1^\star$ está numéricamente mal-puesta, básicamente por cuenta de la inmensidad del espacio completado $\mathcal{H}$. El control de norma mínima oscila altamente cerca del tiempo final T, propiedad que es difícil de capturar numéricamente. Nos referimos a [6, 87, 101, 105], donde este fenómeno es abordado bajo varios puntos de vista.

Además, en la práctica la minimización efectiva de $J_1^\star$ requiere encontrar una aproximación finito dimensional y conforme de $\mathcal{H}$ tal que la correspondiente solución adjunta satisfaga (1.29) que, en general, es imposible conseguir con ayuda de aproximaciones polinómicas a trozos. El "truco" descrito inicialmente en [70], consiste primero en introducir una aproximación discreta y consistente de (1.25) y luego minimizar el correspondiente funcional conjugado discreto. Sin embargo, esto requiere previamente algunas desigualdades de observabilidad discretas uniformes, que es un tema delicado, dado que depende fuertemente de las aproximaciones usadas (nos referimos a [10, 42, 123] y sus referencias). Todavía está abierto el caso general de la ecuación del calor con coeficientes no constantes. Esto y la talla de $\mathcal{H}$ han hecho que muchos autores se planteen relajar el problema de controlabilidad, precisamente, la restricción (1.26). Mencionemos las referencias [10, 13, 123] y notablemente [9, 50, 88] para algunas realizaciones numéricas.

En [49] (véase también [48] en un caso semi-lineal), se introduce una aproximación diferente que permite resultados más generales y consiste en resolver directamente un sistema de optimalidad. En concreto, el siguiente problema extremal (introducido inicialmente por Fursikov e Imanuvilov en [62]) es considerado :

$$\begin{cases} \text{Minimizar } \; J(y,v) := \dfrac{1}{2}\displaystyle\iint_{Q_T} \rho^2|y|^2\,dx\,dt + \dfrac{1}{2}\iint_{q_T} \rho_0^2|v|^2\,dx\,dt \\ \text{Sujeto a } \; (y,v) \in \mathcal{C}(y_0,T). \end{cases} \tag{1.31}$$

Los pesos $\rho = \rho(x,t)$ y $\rho_0 = \rho_0(x,t)$ son continuos y positivos y pertenecen a $L^\infty(Q_{T-\delta})$ para todo $\delta > 0$ (así, pueden explotar cuando $t \to T^-$). Bajo estas condiciones, el problema extremal (1.31) está bien puesto (véase [49]).

Por otra parte, la aparición explícita del término y en el funcional permite resolver directamente las condiciones de optimalidad asociadas a (1.31): definiendo el espacio de Hilbert P como el completado del espacio lineal

$$P_0 = \{q \in C^\infty(\overline{Q_T}) : q = 0 \text{ sobre } \Sigma_T\}$$

con respecto al producto escalar

$$(p,q)_P := \iint_{Q_T} \rho^{-2} L^\star p\, L^\star q\,dx\,dt + \iint_{q_T} \rho_0^{-2}\,p\,q\,dx\,dt, \tag{1.32}$$

el par óptimo (y,v) para J está caracterizado como sigue :

$$y = \rho^{-2}L^\star p \quad \text{en} \quad Q_T, \qquad v = -\rho_0^{-2} p\, 1_\omega \quad \text{en} \quad Q_T \tag{1.33}$$

donde la variable adicional $p \in P$ es la única solución del siguiente problema :

$$(p,q)_P = \int_\Omega y_0(x)\,q(x,0)\,dx, \quad \forall q \in P,\; p \in P. \tag{1.34}$$

El buen planteamiento de esta formulación se asegura siempre que los pesos ρ_0, ρ sean de tipo Carleman (en particular, ρ y ρ_0 deben explotar exponencialmente cuando $t \to T^-$); este comportamiento específico cerca de T refuerza el control nulo e impide al control oscilar cerca de la hora final.

La búsqueda de un control v en la variedad $\mathcal{C}(y_0, T)$ se reduce a resolver la formulación variacional (elíptica) (1.34). En [49], la aproximación de (1.34) se lleva a cabo en el marco de la teoría de los elementos finitos, a través de una discretización del dominio espacio-tiempo Q_T. En la práctica, una aproximación p_h de p se obtiene de manera directa mediante la inversión de una matriz definida positiva y simétrica, en contraste con los métodos utilizados recurriendo a la dualidad. Por otra parte, una gran ventaja de esta aproximación es que una aproximación conforme P_h de P conduce a la convergencia fuerte de p_h hacia p en P, y consecuentemente a partir de (1.33), a una convergencia fuerte en $L^2(q_T)$ de $v_h := -\rho_0^{-2} p_h 1_\omega$ hacia v, un control nulo para (1.25). Vale la pena mencionar que, para cualquier $h > 0$, v_h no es *a priori* un control exacto de un sistema de dimensión finita (lo cual no es necesario en absoluto en la práctica), sino una aproximación de v en el sentido de la norma L^2.

La formulación variacional (1.34), que no es más que una escritura de las condiciones de optimalidad (1.33), se obtiene suponiendo que los pesos ρ y ρ_0 son estrictamente positivos en Q_T y q_T, respectivamente. En particular, esta aproximación no se aplica para el problema de control de norma L^2 mínima, simplemente tomando $\rho := 0$ y $\rho_0 := 1$. La principal razón de este capítulo es adaptar esta aproximación para cubrir el caso $\rho := 0$ y, así, obtener directamente una aproximación v_h del control de norma L^2 ponderada mínima. Para ello, adaptamos la idea desarrollada en [24] dedicada a la ecuación de ondas.

Primeramente, para evitar la minimización del funcional conjugado $J^\star$ con respecto al estado final φ_T por un proceso iterado, presentamos una forma directa de aproximar el control de norma mínima basado en la aproximación primal desarrollada en [49]. Empecemos por analizar el caso penalizado y escribamos la restricción $L^\star\varphi = 0$ como una igualdad en $L^2(Q_T)$.

Sea $\rho_\star \in \mathbb{R}_\star^+$ y sea $\rho_0 \in \mathcal{R}$ definido por

$$\mathcal{R} := \{w : w \in C(Q_T); w \geq \rho_\star > 0 \text{ en } Q_T; w \in L^\infty(Q_{T-\delta}) \ \forall \delta > 0\} \tag{1.35}$$

entonces, en particular, el peso ρ_0 puede explotar cuando $t \to T^-$. En primer lugar, consideramos el caso de la controlabilidad aproximada. Para cualquier $\varepsilon > 0$, el problema es el siguiente :

$$\begin{cases} \text{Minimizar } \ J_\varepsilon(y, v) := \dfrac{1}{2} \displaystyle\iint_{q_T} \rho_0^2 |v|^2 \, dt + \dfrac{1}{2\varepsilon} \|y(\cdot, T)\|^2_{L^2(\Omega)} \\ \text{Sujeto a } \ (y, v) \in \mathcal{A}(y_0; T) := \{\, (y, v) : v \in L^2(q_T), \ y \text{ resolve } (1.25) \,\} \end{cases} \tag{1.36}$$

El correspondiente problema conjugado está dado por

$$\begin{cases} \text{Minimizar } J^\star_\varepsilon(\varphi_T) := \dfrac{1}{2}\displaystyle\iint_{q_T} \rho_0^{-2}|\varphi(x,t)|^2 dx\, dt + \dfrac{\varepsilon}{2}\|\varphi_T\|^2_{L^2(\Omega)} + (y_0, \varphi(\cdot,0))_{L^2(\Omega)} \\ \text{Sujeto a } \varphi_T \in L^2(\Omega) \end{cases} \tag{1.37}$$

donde φ resuelve (1.29).

Como la solución φ de (1.29) está completa y unívocamente determinada por los datos φ_T, la idea principal de la reformulación es mantener φ como variable principal.

Así, se introduce el espacio lineal $\Phi^0 := \{\varphi \in C^2(\overline{Q_T}),\ \varphi = 0 \text{ sobre } \Sigma_T\}$. Para cualquier $\eta > 0$, definamos la forma bilineal

$$(\varphi, \overline{\varphi})_{\Phi^0} := \iint_{q_T} \rho_0^{-2} \varphi\, \overline{\varphi}\, dx\, dt + \varepsilon(\varphi(\cdot,T), \overline{\varphi}(\cdot,T))_{L^2(\Omega)} + \eta \iint_{Q_T} L^\star\varphi\, L^\star\overline{\varphi}\, dx\, dt, \ \forall \varphi, \overline{\varphi} \in \Phi^0.$$

Para cualquier $\varepsilon > 0$, sea Φ_ε el completado de Φ^0 para este producto escalar. Denotemos la norma sobre Φ_ε por

$$\|\varphi\|^2_{\Phi_\varepsilon} := \|\rho_0^{-1}\varphi\|^2_{L^2(q_T)} + \varepsilon\|\varphi(\cdot,T)\|^2_{L^2(\Omega)} + \eta\|L^\star\varphi\|^2_{L^2(Q_T)}, \quad \forall \varphi \in \Phi_\varepsilon. \tag{1.38}$$

Así, definamos el subconjunto cerrado W_ε de Φ_ε por

$$W_\varepsilon = \{\varphi \in \Phi_\varepsilon : L^\star\varphi = 0 \text{ en } L^2(Q_T)\}.$$

Entonces, definamos el siguiente problema extremal :

$$\min_{\varphi \in W_\varepsilon} \hat{J}^\star_\varepsilon(\varphi) := \frac{1}{2}\iint_{q_T} \rho_0^{-2}|\varphi(x,t)|^2 dx\, dt + \frac{\varepsilon}{2}\|\varphi(\cdot,T)\|^2_{L^2(\Omega)} + (y_0, \varphi(\cdot,0))_{L^2(\Omega)}. \tag{1.39}$$

Las estimaciones de energía para la ecuación del calor implican que, para cualquier $\varphi \in W_\varepsilon$, $\varphi(\cdot,0) \in L^2(\Omega)$, el funcional $\hat{J}^\star_\varepsilon$ está bien definido sobre W_ε. Además, dado que para cualquier $\varphi \in W_\varepsilon$, $\varphi(\cdot,T)$ pertenece a $L^2(\Omega)$, el problema (1.39) es equivalente al problema de minimización (1.37).

Consideremos la siguiente formulación mixta : encontrar $(\varphi_\varepsilon, \lambda_\varepsilon) \in \Phi_\varepsilon \times L^2(Q_T)$ solución de

$$\begin{cases} a_\varepsilon(\varphi_\varepsilon, \overline{\varphi}) + b(\overline{\varphi}, \lambda_\varepsilon) = l(\overline{\varphi}), & \forall \overline{\varphi} \in \Phi_\varepsilon \\ b(\varphi_\varepsilon, \overline{\lambda}) = 0, & \forall \overline{\lambda} \in L^2(Q_T), \end{cases} \tag{1.40}$$

donde

$$\begin{aligned} a_\varepsilon : \Phi_\varepsilon \times \Phi_\varepsilon \to \mathbb{R}, &\quad a_\varepsilon(\varphi, \overline{\varphi}) := \iint_{q_T} \rho_0^{-2} \varphi \, \overline{\varphi} \, dx \, dt + \varepsilon(\varphi(\cdot, T), \overline{\varphi}(\cdot, T))_{L^2(\Omega)} \\ b : \Phi_\varepsilon \times L^2(Q_T) \to \mathbb{R}, &\quad b(\varphi, \lambda) := -\iint_{Q_T} L^\star \varphi \, \lambda \, dx \, dt \\ l : \Phi_\varepsilon \to \mathbb{R}, &\quad l(\varphi) := -(y_0, \varphi(\cdot, 0))_{L^2(\Omega)}. \end{aligned}$$

Tenemos el siguiente resultado :

Teorema 1.10. *(i) La formulación mixta (1.40) está bien-planteada.*

(ii) La única solución $(\varphi_\varepsilon, \lambda_\varepsilon) \in \Phi_\varepsilon \times L^2(Q_T)$ es el punto de silla del Lagrangiano $\mathcal{L}_\varepsilon : \Phi_\varepsilon \times L^2(Q_T) \to \mathbb{R}$ definido por

$$\mathcal{L}_\varepsilon(\varphi, \lambda) := \frac{1}{2} a_\varepsilon(\varphi, \varphi) + b(\varphi, \lambda) - l(\varphi). \tag{1.41}$$

(iii) La función óptima φ_ε es el minimizador de $\hat{J}_\varepsilon^\star$ sobre W_ε mientras que el multiplicador óptimo $\lambda_\varepsilon \in L^2(Q_T)$ es el estado controlado de la ecuación del calor (1.25) en el sentido débil.

El Teorema 1.10 reduce la búsqueda de un control aproximado a la resolución de la formulación mixta (1.40) o, equivalentemente, a la búsqueda de un punto de silla para $\mathcal{L}_\varepsilon$. En general, es conveniente "incrementar" el Lagrangiano (véase [57]) y considerar en su lugar el Lagrangiano $\mathcal{L}_{\varepsilon,r}$, definido para cualquier $r > 0$ por

$$\begin{cases} \mathcal{L}_{\varepsilon,r}(\varphi, \lambda) := \frac{1}{2} a_{\varepsilon,r}(\varphi, \varphi) + b(\varphi, \lambda) - l(\varphi), \\ a_{\varepsilon,r}(\varphi, \varphi) := a_\varepsilon(\varphi, \varphi) + r \iint_{Q_T} |L^\star \varphi|^2 \, dx \, dt. \end{cases}$$

Como $a_\varepsilon(\varphi, \varphi) = a_{\varepsilon,r}(\varphi, \varphi)$ sobre W_ε y la función φ_ε tal que $(\varphi_\varepsilon, \lambda_\varepsilon)$ es el punto de silla de $\mathcal{L}_\varepsilon$ verifica $\varphi_\varepsilon \in W_\varepsilon$, los Lagrangianos $\mathcal{L}_\varepsilon$ y $\mathcal{L}_{\varepsilon,r}$ comparten el mismo punto de silla.

En el próximo resultado, presentamos el problema extremal correspondiente involucrando solamente la variable λ_ε, i.e. la variable primal del problema.

Proposición 1.3. *Para cualquier $r > 0$, se tiene la igualdad*

$$\sup_{\lambda \in L^2(Q_T)} \inf_{\varphi \in \Phi_\varepsilon} \mathcal{L}_{\varepsilon,r}(\varphi, \lambda) = -\inf_{\lambda \in L^2(Q_T)} J_{\varepsilon,r}^{\star\star}(\lambda) \; + \; \mathcal{L}_{\varepsilon,r}(\varphi^0, 0),$$

donde $\varphi^0 \in \Phi_\varepsilon$ es la única solución de

$$a_{\varepsilon,r}(\varphi^0, \overline{\varphi}) = l(\overline{\varphi}), \quad \forall \overline{\varphi} \in \Phi_\varepsilon,$$

$J^{\star\star}_{\varepsilon,r} : L^2(Q_T) \to L^2(Q_T)$ es el funcional definido por

$$J^{\star\star}_{\varepsilon,r}(\lambda) := \frac{1}{2}\iint_{Q_T} (\mathcal{A}_{\varepsilon,r}\lambda)\,\lambda\,dx\,dt - b(\varphi^0,\lambda)$$

y $\mathcal{A}_{\varepsilon,r} : L^2(Q_T) \mapsto L^2(Q_T)$ es el operador lineal definido por

$$\mathcal{A}_{\varepsilon,r}\lambda := L^\star\varphi, \quad \forall\lambda \in L^2(Q_T)$$

donde $\varphi \in \Phi_\varepsilon$ es la única solución de

$$a_{\varepsilon,r}(\varphi,\overline{\varphi}) = -b(\overline{\varphi},\lambda), \quad \forall\overline{\varphi} \in \Phi_\varepsilon.$$

Para cualquier $r > 0$, el operador $\mathcal{A}_{\varepsilon,r}$ es fuertemente elíptico y simétrico. Se trata de un isomorfismo de $L^2(Q_T)$ en sí mismo.

Vamos a considerar el caso límite que corresponde a $\varepsilon = 0$, i.e. a la controlabilidad nula. El funcional conjugado $J^\star$ correspondiente a este caso está dado por (1.28), con el peso ρ_0^{-2} en el primer término, precisamente

$$\min_{\varphi_T\in\mathcal{H}} J^\star(\varphi_T) := \frac{1}{2}\iint_{q_T} \rho_0^{-2}(x,t)|\varphi(x,t)|^2\,dxdt + (y_0,\varphi(\cdot,0))_{L^2(\Omega)}, \tag{1.42}$$

donde la variable φ resolve la ecuación del calor retrógrada (1.29) y $\mathcal{H}$ está definido como el completado del espacio $L^2(\Omega)$ con respecto a la norma $\|\varphi_T\|_{\mathcal{H}} := \|\rho_0^{-1}\varphi\|_{L^2(q_T)}$.

Sea $\rho \in \mathcal{R}$. Procediendo como antes, consideramos de nuevo el espacio $\widetilde{\Phi}_0 = \{\varphi \in C^2(\overline{Q_T}) : \varphi = 0 \text{ sobre } \Sigma_T\}$ y entonces, para cualquier $\eta > 0$, definimos la forma bilineal

$$(\varphi,\overline{\varphi})_{\widetilde{\Phi}_{\rho_0,\rho}} := \iint_{q_T} \rho_0^{-2}\varphi\,\overline{\varphi}\,dx\,dt + \eta\iint_{Q_T} \rho^{-2}L^\star\varphi\,L^\star\overline{\varphi}\,dx\,dt, \quad \forall\varphi,\overline{\varphi} \in \widetilde{\Phi}_0.$$

Sea $\widetilde{\Phi}_{\rho_0,\rho}$ el completado de $\widetilde{\Phi}_0$ para este producto interno. Denotemos la norma sobre $\widetilde{\Phi}_{\rho_0,\rho}$ por

$$\|\varphi\|^2_{\widetilde{\Phi}_{\rho_0,\rho}} := \|\rho_0^{-1}\varphi\|^2_{L^2(q_T)} + \eta\|\rho^{-1}L^\star\varphi\|^2_{L^2(Q_T)}, \quad \forall\varphi \in \widetilde{\Phi}_{\rho_0,\rho}. \tag{1.43}$$

Finalmente, definimos el conjunto cerrado $\widetilde{W}_{\rho_0,\rho} \subset \widetilde{\Phi}_{\rho_0,\rho}$ por

$$\widetilde{W}_{\rho_0,\rho} = \{\varphi \in \widetilde{\Phi}_{\rho_0,\rho} : \rho^{-1}L^\star\varphi = 0 \text{ in } L^2(Q_T)\}.$$

Tenemos entonces el siguiente problema extremal :

$$\min_{\varphi\in\widetilde{W}_{\rho_0,\rho}} \hat{J}^\star(\varphi) = \frac{1}{2}\iint_{q_T} \rho_0^{-2}|\varphi(x,t)|^2 dx\,dt + (y_0,\,\varphi(\cdot,0))_{L^2(\Omega)}. \tag{1.44}$$

Para cualquier $\varphi \in \widetilde{W}_{\rho_0,\rho}$ con $L^\star\varphi = 0$ y $\|\varphi\|_{\widetilde{W}_{\rho_0,\rho}} = \|\rho_0^{-1}\varphi\|_{L^2(q_T)}$ tenemos que $\varphi(\cdot,T)$

pertenece al espacio abstrato $\mathcal{H}$. Consecuentemente, los problemas extremales (1.44) y (1.42) son equivalentes.

Entonces, consideremos la siguiente formulación mixta : encontrar $(\varphi, \lambda) \in \widetilde{\Phi}_{\rho_0,\rho} \times L^2(Q_T)$ solución de

$$\left\{ \begin{array}{rcll} \tilde{a}(\varphi, \overline{\varphi}) + \tilde{b}(\overline{\varphi}, \lambda) & = & \tilde{l}(\overline{\varphi}), & \forall \overline{\varphi} \in \widetilde{\Phi}_{\rho_0,\rho} \\ \tilde{b}(\varphi, \overline{\lambda}) & = & 0, & \forall \overline{\lambda} \in L^2(Q_T), \end{array} \right. \tag{1.45}$$

donde

$$\tilde{a} : \widetilde{\Phi}_{\rho_0,\rho} \times \widetilde{\Phi}_{\rho_0,\rho} \to \mathbb{R}, \quad \tilde{a}(\varphi, \overline{\varphi}) = \iint_{q_T} \rho_0^{-2} \varphi\, \overline{\varphi}\, dx\, dt$$
$$\tilde{b} : \widetilde{\Phi}_{\rho_0,\rho} \times L^2(Q_T) \to \mathbb{R}, \quad \tilde{b}(\varphi, \lambda) = -\iint_{Q_T} \rho^{-1} L^\star \varphi\, \lambda\, dx\, dt$$
$$\tilde{l} : \widetilde{\Phi}_{\rho_0,\rho} \to \mathbb{R}, \quad \tilde{l}(\varphi) = -(y_0, \varphi(\cdot, 0))_{L^2(\Omega)}.$$

Proposición 1.4 ([62]). *Sean los pesos $\rho^c, \rho_0^c \in \mathcal{R}$ (véase (1.35)) definidos como sigue :*

$$\begin{aligned} &\rho^c(x,t) := \exp\left(\frac{\beta(x)}{T-t}\right), \quad \beta(x) := K_1\left(e^{K_2} - e^{\beta_0(x)}\right), \\ &\rho_0^c(x,t) := (T-t)^{3/2} \rho^c(x,t), \end{aligned} \tag{1.46}$$

donde las K_i son constantes positivas suficientemente grandes (sólo dependiendo de T, c_0 y $\|c\|_{C^1(\overline{\Omega})}$) tal que

$$\beta_0 \in C^\infty(\overline{\Omega}),\ \beta > 0 \quad in \quad \Omega,\ \beta = 0 \quad sobre \quad \partial\Omega, \quad Supp(\nabla\beta) \subset \overline{\Omega} \setminus \omega.$$

Entonces, existe una constante $C > 0$, sólo dependiendo de ω, T, tal que

$$\|\varphi(\cdot, 0)\|_{L^2(\Omega)} \le C \|\varphi\|_{\widetilde{\Phi}_{\rho_0^c,\rho^c}}, \quad \forall \varphi \in \widetilde{\Phi}_{\rho_0^c,\rho^c}. \tag{1.47}$$

La proposición anterior permite probar el siguiente resultado de existencia y unicidad :

Teorema 1.11. *Sean $\rho_0 \in \mathcal{R}$ y $\rho \in \mathcal{R} \cap L^\infty(Q_T)$ y supongamos que existe una constante positiva K tal que*

$$\rho_0 \le K \rho_0^c, \quad \rho \le K \rho^c \quad en \quad Q_T. \tag{1.48}$$

Entonces, tenemos :

(i) La formulación mixta (1.45) definida sobre $\widetilde{\Phi}_{\rho_0,\rho} \times L^2(Q_T)$ está bien planteada.

(ii) La única solución $(\varphi, \lambda) \in \widetilde{\Phi}_{\rho_0,\rho} \times L^2(Q_T)$ es el único punto de silla del Lagrangiano

$\widetilde{\mathcal{L}} : \widetilde{\Phi} \times L^2(Q_T) \mapsto \mathbb{R}$, *definido por*

$$\widetilde{\mathcal{L}}(\varphi, \lambda) = \frac{1}{2}\tilde{a}(\varphi, \varphi) + \tilde{b}(\varphi, \lambda) - \tilde{l}(\varphi). \tag{1.49}$$

(iii) La función óptima φ *es el minimizador de* $\hat{J}^\star$ *sobre* $\widetilde{\Phi}_{\rho_0,\rho}$ *mientras que* $\rho^{-1}\lambda \in L^2(Q_T)$ *es el estado controlado de la ecuación del calor (1.25) en el sentido débil.*

Por tanto, discretizando el problema (1.45), podemos llegar al siguiente sistema matricial lineal

$$\begin{pmatrix} A_{r,h} & B_h^T \\ B_h & 0 \end{pmatrix}_{\mathbb{R}^{n_h+m_h, n_h+m_h}} \begin{pmatrix} \{\varphi_h\} \\ \{\lambda_h\} \end{pmatrix}_{\mathbb{R}^{n_h+m_h}} = \begin{pmatrix} L_h \\ 0 \end{pmatrix}_{\mathbb{R}^{n_h+m_h}}. \tag{1.50}$$

La matriz $A_{r,h}$ es simétrica y definida positiva para todo $h > 0$ y todo $r > 0$. Por otro lado, la matriz de orden $m_h + n_h$ en (1.72) es simétrica pero no es definida positiva. El sistema (1.72) es resuelto usando el método directo de descomposición LU.

Presentaremos a continuación un experimento numérico del Capítulo 5. Una vez que una aproximación (φ_h, λ_h) es obtenida, una aproximación del control v está dada por $v_h = \rho_0^{-2}\varphi_h 1_\omega$, dado que el estado está dado por $\rho^{-1}\lambda$, simplemente usamos $\rho^{-1}\lambda_h$ como una aproximación de y.

Los cálculos han sido realizados con *Matlab*. Usamos elementos finitos C^1 de *Bogner-Fox-Schmit*, definidos para rectángulos para la variable φ y elementos finitos afines en (x,t) para la variable λ. Hemos tomado $\Omega = (0,1)$, $c := 1/10$, $d := 0$, $\omega = (0.2, 0.5)$, $y_0(x) \equiv \sin(\pi x)$.

En el espíritu de [49], consideramos la siguiente elección para el peso $\rho_0 \in \mathcal{R}$:

$$\rho_0(x,t) := (T-t)^{3/2} \exp\left(\frac{K_1}{(T-t)}\right), \quad (x,t) \in Q_T, \qquad K_1 := \frac{3}{4}. \tag{1.51}$$

Así, ρ_0 explota exponencialmente cuando $t \to T^-$. Esto permite conseguir un comportamiento suave del correspondiente control $v := \rho_0^{-2}\varphi\, 1_\omega$.

Algunas visualizaciones de las aproximaciones del estado y del control pueden ser observados en las Figuras 1.3–1.4.

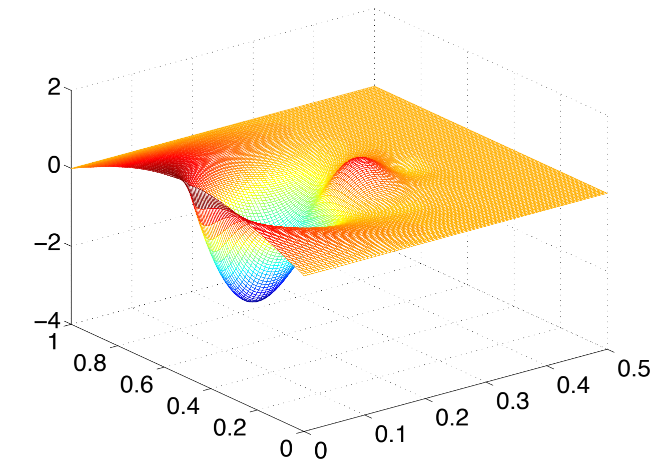

Figure 1.3: $\omega = (0.2, 0.5)$; Aproximación $\rho^{-1}\lambda_h$ de la solución controlada y sobre Q_T - $r = 1$ y $h = 8.83 \times 10^{-3}$.

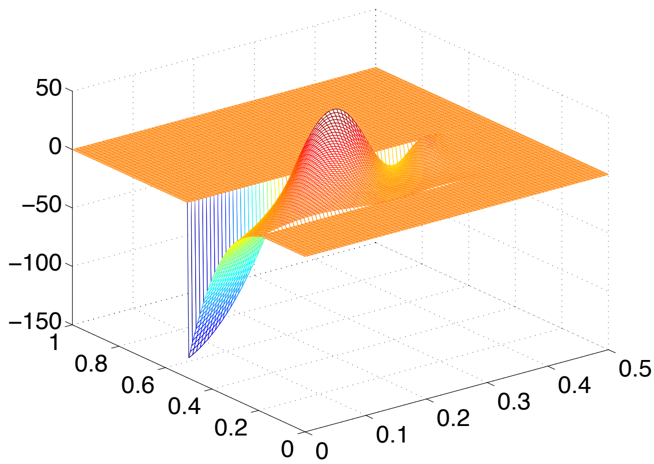

Figure 1.4: $\omega = (0.2, 0.5)$; Aproximación $v_h = \rho_0^{-2}\varphi_h$ del control nulo v sobre Q_T - $r = 1$ y $h = 8.83 \times 10^{-3}$.

Este capítulo está basado en el artículo [104], en colaboración con A. Münch.

Capítulo 6: Sobre la controlabilidad numérica de las ecuaciones bidimensionales del calor, Stokes y Navier-Stokes

El Capítulo 6 de esta Tesis tiene como objetivo presentar estrategias que permitan resolver numéricamente el problema de control nulo para las ecuaciones bidimensionales del calor, Stokes y Navier-Stokes.

Para una descripción más detallada de los logros de este capítulo, introduzcamos, un dominio acotado $\Omega \subset \mathbb{R}^2$ cuya frontera Γ es de clase C^2. Sean $\omega \subset \Omega$ un conjunto abierto y no vacío (llamado de nuevo *dominio de control*) y $T > 0$. En lo que sigue, para cualquier $\tau > 0$, denotemos de nuevo por Q_τ, Σ_τ y q_τ los conjuntos $\Omega \times (0, \tau)$, $\Gamma \times (0, \tau)$ y $\omega \times (0, \tau)$, respectivamente.

Este capítulo está dedicado al control nulo global para la ecuación del calor

$$\begin{cases} y_t - \nu\Delta y + G(\mathbf{x}, t)y = v1_\omega & \text{en} \quad Q_T, \\ y = 0 & \text{sobre} \quad \Sigma_T, \\ y(\cdot, 0) = y_0 & \text{en} \quad \Omega \end{cases} \tag{1.52}$$

y la ecuación de Stokes

$$\begin{cases} \mathbf{y}_t - \nu\Delta \mathbf{y} + \nabla\pi = \mathbf{v}1_\omega & \text{en} \quad Q_T, \\ \nabla \cdot \mathbf{y} = 0 & \text{en} \quad Q_T, \\ \mathbf{y} = \mathbf{0} & \text{sobre} \quad \Sigma_T, \\ \mathbf{y}(\cdot, 0) = \mathbf{y}_0 & \text{en} \quad \Omega \end{cases} \tag{1.53}$$

y el control local exacto a trayectorias para las ecuaciones de Navier-Stokes

$$\begin{cases} \mathbf{y}_t - \nu\Delta \mathbf{y} + (\mathbf{y} \cdot \nabla)\mathbf{y} + \nabla\pi = \mathbf{v}1_\omega & \text{en} \quad Q_T, \\ \nabla \cdot \mathbf{y} = 0 & \text{en} \quad Q_T, \\ \mathbf{y} = \mathbf{0} & \text{sobre} \quad \Sigma_T, \\ \mathbf{y}(\cdot, 0) = \mathbf{y}_0 & \text{en} \quad \Omega. \end{cases} \tag{1.54}$$

Aquí, $v = v(\mathbf{x}, t)$ y $\mathbf{v} = \mathbf{v}(\mathbf{x}, t)$ representan los controles (se supone que actúan sobre ω durante todo el intervalo de tiempo $(0, T)$; el símbolo 1_ω representa la función característica de ω). Además, $\nu > 0$ y suponemos que $G \in L^\infty(Q_T)$.

El problema de control nulo para (1.52) en el tiempo T es el siguiente :

Dado $y_0 \in L^2(\Omega)$, encontrar $v \in L^2(q_T)$ tal que la solución asociada de (1.52) satisface

$$y(\cdot, T) = 0 \quad \textit{en} \quad \Omega. \tag{1.55}$$

Recordemos la definición de algunos espacios usuales en el contexto de fluidos incompresibles :

$$\mathbf{H} := \left\{ \boldsymbol{\varphi} \in \mathbf{L}^2(\Omega) : \nabla \cdot \boldsymbol{\varphi} = 0 \text{ en } \Omega, \ \boldsymbol{\varphi} \cdot \mathbf{n} = 0 \text{ sobre } \Gamma \right\}.$$

El control nulo para (1.53) en el tiempo T es el siguiente :

Dado $\mathbf{y}_0 \in \mathbf{H}$*, encontrar* $\mathbf{v} \in \mathbf{L}^2(q_T)$ *tal que la solución asociada de* (1.53) *satisface*

$$\mathbf{y}(\cdot, T) = \mathbf{0} \quad \textit{en} \quad \Omega. \tag{1.56}$$

Ahora vamos a introducir el concepto de control exacto a las trayectorias para las ecuaciones de Navier-Stokes. La idea es que, aunque no sea posible alcanzar cualquier elemento del espacio de los estados exactamente, podemos tratar de alcanzar (en un tiempo finito T) cualquier estado sobre cualquier trayectoria.

Así, sea $(\overline{\mathbf{y}}, \overline{\pi})$ una solución de las ecuaciones de Navier-Stokes sin control :

$$\begin{cases} \overline{\mathbf{y}}_t - \nu\Delta\overline{\mathbf{y}} + (\overline{\mathbf{y}} \cdot \nabla)\overline{\mathbf{y}} + \nabla\overline{\pi} = \mathbf{0} & \text{en} \quad Q_T, \\ \nabla \cdot \overline{\mathbf{y}} = 0 & \text{en} \quad Q_T, \\ \overline{\mathbf{y}} = \mathbf{0} & \text{sobre} \quad \Sigma_T, \\ \overline{\mathbf{y}}(\cdot, 0) = \overline{\mathbf{y}}_0 & \text{en} \quad \Omega. \end{cases} \tag{1.57}$$

Buscaremos controles $\mathbf{v} \in \mathbf{L}^2(q_T)$ tales que las soluciones asociadas de (1.54) satisfacen :

$$\mathbf{y}(\mathbf{x}, T) = \overline{\mathbf{y}}(\mathbf{x}, T) \quad \textit{en} \quad \Omega.$$

Más precisamente, el problema de control exacto a trayectorias para las ecuaciones de Navier-Stokes es el siguiente :

Dado $\mathbf{y}_0 \in \mathbf{H}$ *y dada una trayectoria* $(\overline{\mathbf{y}}, \overline{\pi})$*, encontrar* $\mathbf{v} \in \mathbf{L}^2(q_T)$ *tal que la solución asociada a* (1.54) *satisface*

$$\mathbf{y}(\cdot, T) = \overline{\mathbf{y}}(\cdot, T) \textit{ en } \Omega. \tag{1.58}$$

Empezaremos describiendo la estrategia para el cálculo de los controles nulos para la ecuación del calor.

Fijemos la notación

$$Ly := y_t - \Delta y + G(\mathbf{x}, t)y, \quad L^*p := -p_t - \Delta p + G(\mathbf{x}, t)p$$

y denotemos ρ, β y ρ_i los pesos dados por

$$\rho(\mathbf{x},t) := e^{\frac{\beta(\mathbf{x})}{T-t}},\ \beta(\mathbf{x}) := K_1\left(e^{K_2} - e^{\beta_0(\mathbf{x})}\right),\ \rho_i(\mathbf{x},t) := (T-t)^{\frac{3}{2}-i}\rho(\mathbf{x},t), \tag{1.59}$$

donde $i = 0, 1, 2$, y K_1 y K_2 son constantes positivas suficientemente grandes (dependiendo solamente de T) y $\beta_0 = \beta_0(\mathbf{x})$ es una función acotada regular que es positiva en Ω, se anula sobre Γ y satisface

$$|\nabla\beta_0| > 0 \text{ en } \overline{\Omega}\setminus\omega.$$

La idea principal para resolver numéricamente el problema de control nulo para (1.52) es considerar el problema extremal :

$$\begin{cases} \text{Minimizar } \ J(y,v) = \dfrac{1}{2}\left(\displaystyle\iint_{Q_T}\rho^2|y|^2\,d\mathbf{x}\,dt + \iint_{q_T}\rho_0^2|v|^2\,d\mathbf{x}\,dt\right) \\ \text{Sujeto a } \ (y,v)\in\mathcal{H}(y_0,T). \end{cases} \tag{1.60}$$

Aquí, para cualquier $y_0 \in L^2(\Omega)$ y cualquier $T > 0$, la variedad linear $\mathcal{H}(y_0,T)$ está dada por

$$\mathcal{H}(y_0,T) := \{(y,v) : v\in L^2(q_T),\ (y,v) \text{ satisface } (1.52) \text{ y } (1.55)\}.$$

El buen planteamiento (1.60) es consecuencia de una adecuada *desigualdad de Carleman* para la ecuación del calor.

Más precisamente, vamos a introducir el espacio

$$P_0 := \{p\in C^2(\overline{Q}_T) : p = 0 \text{ sobre } \Sigma_T\}. \tag{1.61}$$

Entonces tenemos :

Proposición 1.5. *Existe una constante positiva C, sólo dependiente de Ω, ω y T, tal que*

$$\begin{aligned}\iint_{Q_T}\left[\rho_2^{-2}(|p_t|^2+|\Delta p|^2)+\rho_1^{-2}|\nabla p|^2+\rho_0^{-2}|p|^2\right]d\mathbf{x}\,dt \\ \leq C\iint_{Q_T}(\rho^{-2}|L^*p|^2+\rho_0^{-2}|p|^2 1_\omega)\,d\mathbf{x}\,dt\end{aligned} \tag{1.62}$$

para todo $p\in P_0$.

Vamos introducir el producto escalar en P_0

$$k(p,p') := \iint_{Q_T}\left(\rho^{-2}L^*p\,L^*p' + 1_\omega\rho_0^{-2}p\,p'\right)d\mathbf{x}\,dt \quad \forall p,p'\in P_0. \tag{1.63}$$

Sea P el completado de P_0 con respecto a este producto escalar. Entonces se puede probar que la solución de (1.60) está caracterizada, en términos de una variable dual, como sigue :

$$y = \rho^{-2} L^* p, \quad v = -\rho_0^{-2} p\big|_{q_T}, \tag{1.64}$$

donde p es la única solución de la siguiente ecuación variacional en P:

$$\begin{cases} \displaystyle\iint_{Q_T} \left(\rho^{-2} L^* p\, L^* p' + 1_\omega \rho_0^{-2} p\, p'\right)\, d\mathbf{x}\, dt = \int_\Omega y_0(\mathbf{x})\, p'(\mathbf{x}, 0)\, d\mathbf{x} \\ \forall\, p' \in P;\ p \in P. \end{cases} \tag{1.65}$$

Introduciendo la forma linear ℓ_0, con

$$\langle \ell_0, p \rangle := \int_\Omega y_0(\mathbf{x})\, p(\mathbf{x}, 0)\, d\mathbf{x} \quad \forall p \in P, \tag{1.66}$$

vemos que (1.65) puede ser rescrito en la forma

$$k(p, p') = \langle \ell_0, p' \rangle \quad \forall p' \in P;\ p \in P. \tag{1.67}$$

Vamos a denotar P_h un subespacio de P de dimensión finita. Una aproximación natural de (1.67) es la siguiente :

$$k(p_h, p_h') = \langle \ell_0, p_h' \rangle \quad \forall p_h' \in P_h;\ p_h \in P_h. \tag{1.68}$$

Así, para resolver numéricamente el problema variacional (1.67), es suficiente construir un espacio de dimensión finita explícitamente $P_h \subset P$. Notemos sin embargo que esto es posible pero no simple desde el punto de vista numérico. La razón es que, si $p \in P$, entonces $\rho^{-1} L^* p$ debe pertenecer a $L^2(Q_T)$ y $\rho_0^{-1} p\big|_{q_T}$ debe pertenecer a $L^2(q_T)$. De la desigualdad de Carleman (1.62), vemos también que p debe poseer derivadas en tiempo de primer orden y derivadas en espacio de segundo orden en $L^2_{\text{loc}}(Q_T)$. Por tanto, un aproximación basada en una triagulación estándar de Q_T requiere un espacio P_h de funciones que debe ser C^0 en $(\mathbf{x}, t)$ y C^1 en $\mathbf{x}$ y esto puede ser complejo y muy costoso. Espacios de este tipo están construidos por ejemplo en [21]. Por ejemplo, un buen comportamiento es observado para los llamados elementos finitos de *Argyris*, *Bell* o *Bogner-Fox-Schmit*; Podemos citar [49] y el capítulo 5 para aproximaciones numéricas de este tipo en el contexto de una dimensión espacial.

Nuestro objetivo es evitar el uso de elementos finitos C^1. Para lograr esto, vamos introducir multiplicadores y, en consecuencia, adecuadas formulaciones mixtas.

Consideremos la nueva variable

$$z := L^* p \tag{1.69}$$

y al nuevo espacio $Z := L^2(\rho^{-1}; Q_T)$. Entonces $z \in Z$ y $L^* p - z = 0$ (una igualdad en Z).

Notemos que esta identidad puede también ser escrita en la forma

$$\iint_{Q_T} (z - L^*p)\,\psi \; d\mathbf{x}\,dt = 0 \quad \forall \psi \in C_0^\infty(Q_T);$$

Entonces, vamos hacer una integración por partes, i.e.

$$\iint_{Q_T} \{[z + p_t - G(\mathbf{x},t)p]\,\psi - \nabla p \cdot \nabla \psi\} \; d\mathbf{x}\,dt = 0 \quad \forall \psi \in C_0^\infty(Q_T).$$

Consideremos también los espacios

$$\begin{aligned} Y &:= \left\{ \lambda : \iint_{Q_T} \left(\rho_2^2|\lambda|^2 + \rho_1^2|\nabla\lambda|^2\right) d\mathbf{x}\,dt < +\infty,\ \lambda\big|_{\Sigma_T} = 0 \right\}, \\ R &:= \left\{ p : \iint_{Q_T} \left[\rho_2^{-2}|p_t|^2 + \rho_1^{-2}|\nabla p|^2 + \rho_0^{-2}|p|^2\right] d\mathbf{x}\,dt < +\infty,\ p\big|_{\Sigma_T} = 0 \right\}, \\ X &:= Z \times R,\ W := X \times Y \end{aligned}$$

y las formas bilineales $\alpha(\cdot\,,\cdot) : X \times X \mapsto \mathbb{R}$ y $\beta(\cdot\,,\cdot) : X \times Y \mapsto \mathbb{R}$, con

$$\alpha((z,p),(z',p')) := \iint_{Q_T} \left(\rho^{-2} z\, z' + \rho_0^{-2} p\, p' 1_\omega\right) \, d\mathbf{x}\,dt$$

y

$$\beta((z,p),\lambda) := \iint_{Q_T} \left[(z + p_t - G(\mathbf{x},t)\,p)\,\lambda - \nabla p \cdot \nabla \lambda\right] \, d\mathbf{x}\,dt$$

y la forma lineal $\ell : R \mapsto \mathbb{R}$, con

$$\langle \ell, (z,p) \rangle := \int_\Omega y_0(\mathbf{x})\, p(\mathbf{x},0)\, d\mathbf{x}. \tag{1.70}$$

Entonces $\alpha(\cdot\,,\cdot)$, $\beta(\cdot\,,\cdot)$ y ℓ están bien-definidas y son continuas. Vamos a considerar la formulación mixta

$$\left\{ \begin{array}{l} \alpha((z,p),(z',p')) + \beta((z',p'),\lambda) = \langle \ell, (z',p') \rangle, \\ \beta((z,p),\lambda') = 0, \\ \qquad \forall (z',p',\lambda') \in W;\ (z,p,\lambda) \in W. \end{array} \right. \tag{1.71}$$

Es fácil ver que existe como máximo una solución para (1.71). Sin embargo, desafortunadamente, una prueba rigurosa de la existencia de solución para (1.71) es, que sepamos, desconocida. En la práctica, lo que necesitamos probar es que la siguiente condición inf-sup es válida :

$$\inf_{\lambda \in Y} \sup_{(z,p) \in X} \frac{\beta((z,p),\lambda)}{\|(z,p)\|_X \|\lambda\|_Y} > 0.$$

Por tanto, discretizando el problema (1.71), podemos llegar al siguiente sistema matricial lineal

$$\begin{pmatrix} A_h & B_h^T \\ B_h & 0 \end{pmatrix}_{\mathbb{R}^{n_h+m_h,n_h+m_h}} \begin{pmatrix} \{(z_h,p_h)\} \\ \{\lambda_h\} \end{pmatrix}_{\mathbb{R}^{n_h+m_h}} = \begin{pmatrix} L_h \\ 0 \end{pmatrix}_{\mathbb{R}^{n_h+m_h}}. \tag{1.72}$$

donde las matrices A_h, B_h y L_h están relacionadas con α, β y ℓ, respectivamente.

Si n_h y m_h son grandes, la matriz de coeficientes en (1.72) está mal-condicionada. Por esta razón, es conveniente resolver (1.72) usando un método iterado. Entre varias posibilidades, hemos podido comprobar que una buena elección es el llamado *Algoritmo de Arrow-Hurwicz*:

ALG (Arrow-Hurwicz):

(i) *Initializar*
Fijemos $r, s > 0$. Sean $(z_h^{(0)}, p_h^{(0)}) = (0,0)$ y $\lambda_h^{(0)} = 0$.
Para $k \geq 0$, supongamos que $(z_h^{(k)}, p_h^{(k)})$ y $\lambda_h^{(k)}$ son conocidos. Entonces:

(ii) *Avance para* (z_h, p_h): sea $(z_h^{(k+1)}, p_h^{(k+1)})$ definido por

$$(z_h^{(k+1)}, p_h^{(k+1)}) = (z_h^{(k)}, p_h^{(k)}) - r\left[A_h(z_h^{(k)}, p_h^{(k)}) - L_h + B_h^T\lambda_h^{(k)}\right].$$

(iii) *Avance para* $\hat{\lambda}_h$: sea $\hat{\lambda}_h^{(k+1)}$ definido por

$$\lambda_h^{(k+1)} = \lambda_h^{(k)} + rsB_h(z_h^{(k+1)}, p_h^{(k+1)}).$$

Comprobar la convergencia. Si el test de parada no es satisfecho, cambiar k por $k+1$ y volver a la etapa (ii).

Observación 1.3. La principal ventaja de **ALG 1** con respecto a otros algoritmos es que no necesitamos invertir ninguna matriz. Todo funciona bien incluso si A_h está mal-condicionada. La desventaja es que tenemos que elegir buenos valores para los parámetros r y s y, obviamente, esto necesita un trabajo extra. □

Presentaremos a continuación un experimento numérico del Capítulo 6. Una vez que una aproximación (z_h, p_h) es obtenida, una aproximación del control v está dada por $v_h = -\rho_0^{-2} p_h 1_\omega$, dado que el estado está dado por $\rho^{-2}z$, simplemente usamos $\rho^{-2}z_h$ como una aproximación de y.

Los cálculos han sido realizados con *Freefem++*, véase [77]. Usamos elementos finitos de P_2-Lagrange en $(\mathbf{x}, t)$ para todas las variables p, z y λ. Hemos tomado $\Omega = (0,1) \times (0,1)$, $\omega = (0.2, 0.6) \times (0.2, 0.6)$, $G(\mathbf{x}, t) :\equiv 1$, $y_0(\mathbf{x}) \equiv 1000$.

El dominio computacional y el mallado pueden ser vistos en la Figura 1.5. Algunas visualizaciones de las aproximaciones del control y del estado pueden ser observados en la Figura 1.6.

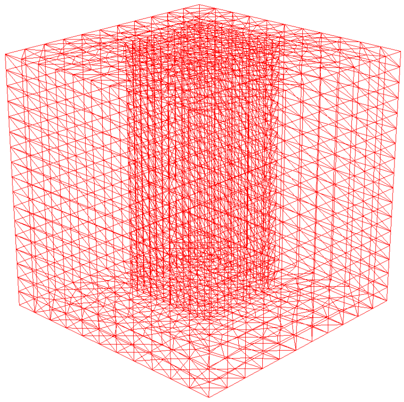

Figure 1.5: El dominio y el mallado. Número de vértices: 2800. Número de elementos (tetraedros): 14094. Número total de variables: 20539.

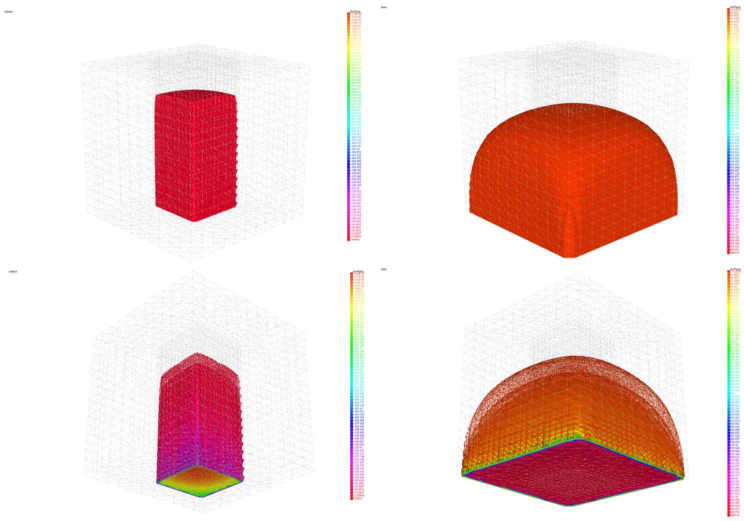

Figure 1.6: Visualización de los conjuntos $\{(\mathbf{x},t) : v_h(\mathbf{x},t) = 0\}$ (**Izquierda**) y $\{(\mathbf{x},t) : y_h(\mathbf{x},t) = 0\}$ (**Derecha**). Valores mínimo (máximo) de v_h y y_h: -1146.44 y -6.32 (resp. 7.69 y 1006.33).

Ahora, vamos a describir la estrategia para el cálculo de los controles nulos para las ecuaciones de Stokes.

La idea es parecida: se introduce un problema extremal donde se minimiza un coste promediado y entonces se llega a un problema de cuarto orden en las variables duales, que proporciona una caracterización de la solución del problema extremal.

Las variables duales, denotadas por $(\mathbf{p}, \sigma)$, pertenecen a un espacio similar a P, denotado por $\mathbf{\Phi}$. Las variables duales satisfacen varias propiedades que hacen considerablemente difícil construir espacios de elementos finitos $\mathbf{\Phi}_h \subset \mathbf{\Phi}$ explícitamente. A parte de la dificultad de evitar elementos finitos que sean C^1 en la variable espacial ahora tenemos una dificultad adicional, pues las funciones del espacio $\mathbf{\Phi}$ tienen divergencia cero.

En este capítulo presentamos varios problemas mixtos que eliminan algunas (o todas) estas dificultades. Al final, podemos llegar a una formulación mixta donde no hay ninguna de las dos dificultades citadas arriba.

Vamos a presentar algunos resultados numéricos correspondientes a los datos $\Omega = (0,1) \times (0,1)$, $\omega = (0.2, 0.6) \times (0.2, 0.6)$, $T = 1$, $\nu = 1$, $\mathbf{y}_0(\mathbf{x}) \equiv \nabla \times \psi(\mathbf{x})$ con $\psi(x_1, x_2) \equiv M(x_1 x_2)^2[(1 - x_1)(1 - x_2)]^2$ y $M = 100$.

De nuevo, los cálculos han sido realizado con el software *Freefem++*, usando aproximaciones P_2-Lagrange en $(\mathbf{x}, t)$ para todas las variables. También usamos el algoritmo de Arrow-Hurwicz para resolver el sistema lineal. El control y el estado calculados han sido representados en las Figuras 1.7-1.8.

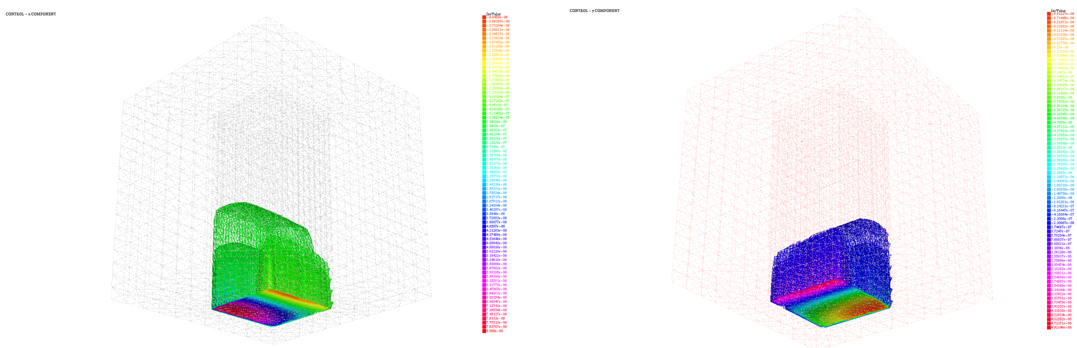

Figure 1.7: El control calculado: primera componente (**Izquierda**) y segunda componente (**Derecha**). Valores Mínimo (máximo) de la primera y segunda componentes de $\mathbf{v}_h$: -4.04×10^{-6} y -9.91×10^{-6} (resp. 8.09×10^{-6} y 4.92×10^{-6}).

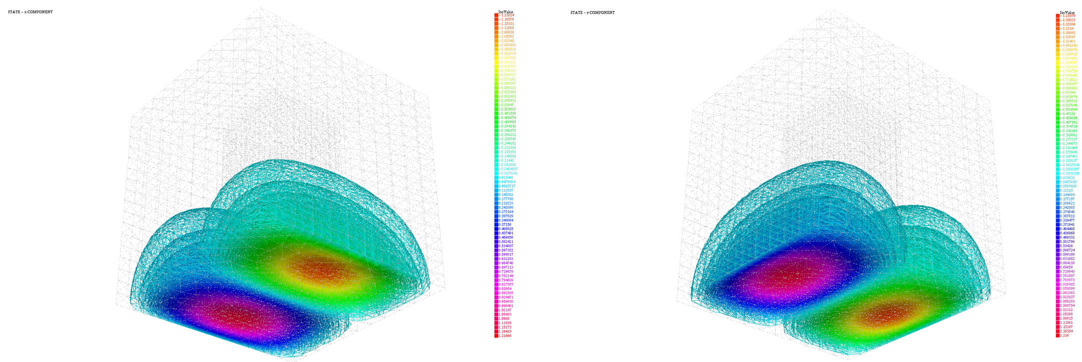

Figure 1.8: El estado calculado: primera componente (**Izquierda**) y segunda componente (**Derecha**). Valores Mínimo (máximo) de la primera y segunda componentes de $\mathbf{y}_h$: -1.22 y -1.22 (resp. 1.22 y 1.22).

Finalizamos el resumen de este capítulo presentando una estrategia para calcular controles exactos para las ecuaciones de Navier-Stokes.

La idea es parecida a las anteriores. Lo primero que se hace es reformular el problema de control exacto a trayectorias como un problema de control a cero. Más precisamente, sean $\mathbf{y} = \overline{\mathbf{y}} + \mathbf{u}$ y $\pi = \overline{\pi} + q$. Teniendo en cuenta que $(\overline{\mathbf{y}}, \overline{\pi})$ resuelve (1.57), llegamos a que

$$\begin{cases} \mathbf{u}_t - \nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\overline{\mathbf{y}} + ((\overline{\mathbf{y}}+\mathbf{u})\cdot\nabla)\mathbf{u} + \nabla q = \mathbf{v}1_\omega & \text{en} \quad Q_T, \\ \nabla\cdot\mathbf{u} = 0 & \text{en} \quad Q_T, \\ \mathbf{u} = \mathbf{0} & \text{sobre} \quad \Sigma_T, \\ \mathbf{u}(0) = \mathbf{u}_0 := \mathbf{y}_0 - \overline{\mathbf{y}}_0 & \text{en} \quad \Omega. \end{cases} \tag{1.73}$$

De este modo, reducimos el problema de controlabilidad exacta a trayectorias a un problema de controlabilidad a cero para las soluciones $(\mathbf{u}, q)$ del problema no lineal (1.73).

Por tanto, una estrategia completamente natural consiste en aplicar el algoritmo siguiente:

ALG 2 (Punto-Fijo):

(i) Elegir $\mathbf{u}_0$ suficientemente regular.

(ii) Entonces, para $n \geq 0$ y $\mathbf{u}^n \in \mathbf{W}$ dados, calcular $\mathbf{u}^{n+1} = F(\mathbf{u}^n)$, i.e. hallar la única solución $(\mathbf{u}^{n+1}, \mathbf{v}^{n+1})$ del problema extremal

$$\text{Minimizar} \;\; J(\mathbf{u}^n; \mathbf{u}^{n+1}, \mathbf{v}^{n+1}) = \frac{1}{2}\iint_{Q_T} \rho^2|\mathbf{u}^{n+1}|^2 \, d\mathbf{x}\, dt + \frac{1}{2}\iint_{q_T} \rho_0^2|\mathbf{v}^{n+1}|^2 \, d\mathbf{x}\, dt$$

sujeto a $\mathbf{v}^{n+1} \in \mathbf{L}^2(q_T)$ y

$$\begin{cases} \mathbf{u}_t^{n+1} - \nu\Delta\mathbf{u}^{n+1} + (\mathbf{u}^{n+1}\cdot\nabla)\overline{\mathbf{y}} + ((\overline{\mathbf{y}}+\mathbf{u}^n)\cdot\nabla)\mathbf{u}^{n+1} + \nabla q^{n+1} = \mathbf{v}^{n+1}1_\omega, \\ \nabla\cdot\mathbf{u}^{n+1} = 0, \\ \mathbf{u}^{n+1} = \mathbf{0}, \\ \mathbf{u}^{n+1}(0) = \mathbf{u}_0, \quad \mathbf{u}^{n+1}(T) = \mathbf{0}. \end{cases}$$

Éste es el clásico método de punto-fijo. Empezamos de un $\mathbf{u}^0$ dado y, a continuación, resolvemos un problema de control a cero para un sistema linealizado en cada etapa. Así, producimos una sucesión $\{\mathbf{u}^n, \mathbf{v}^n\}$ que se espera que converga a una solución del problema de control nulo.

Vamos a presentar un experimento numérico donde la trayectoria es el flujo de Poiseuille $\overline{\mathbf{y}}_P$.

Tomemos como datos $\Omega = (0,5)\times(0,1)$, $\omega = (1,2)\times(0,1)$, $T = 2$, $\nu = 1$, $\overline{\mathbf{y}}_P(x_1,x_2) := (4x_2(1-x_2),0)$, $\mathbf{y}_0(\mathbf{x}) \equiv \overline{\mathbf{y}}_p + M(\nabla\times\psi)(\mathbf{x})$ con $\psi(x_1,x_2) \equiv (x_1x_2)^2[(1-x_1)(1-x_2)]^2$ y $M > 0$ suficientemente pequeño.

El dominio computacional y la correspondiente triangulación se visualizan en la Figura 1.9. El estado controlado se visualiza en las Figuras 1.10 y 1.11.

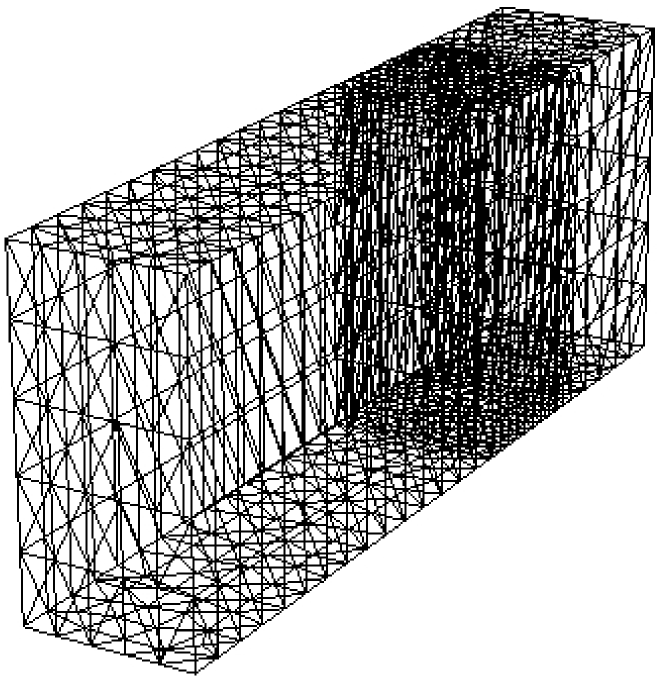

Figure 1.9: Test de Poiseuille – El dominio y el mallado. Número de vértices: 1830. Número de elementos (tetraedros): 7830. Número total de variables: 12810.

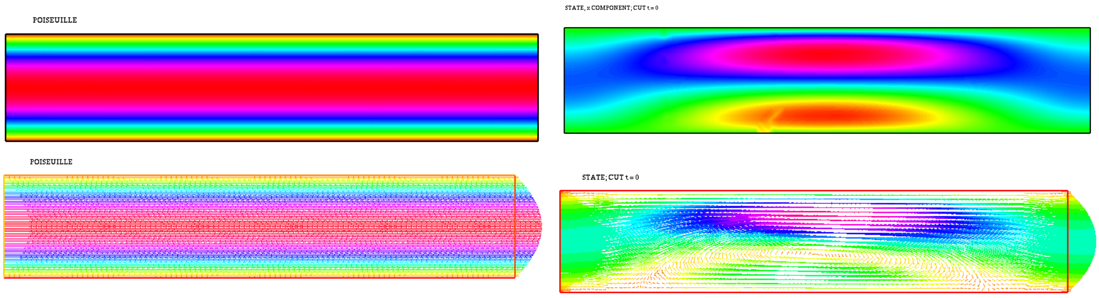

Figure 1.10: Test de Poiseuille – El estado final deseado (**Izquierda**) y el estado inicial (**Derecha**).

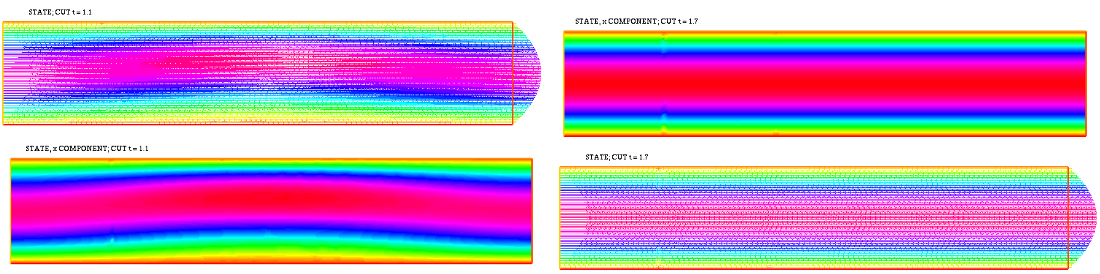

Figure 1.11: Test de Poiseuille – El estado en el tiempo $t = 1.1$ (**Izquierda**) y el estado en el tiempo $t = 1.7$ ((**Derecha**).

Este capítulo está basado en el artículo [51], en colaboración con E. Fernández-Cara y A. Münch.

Part I

Theoretical results about the control of nonlinear regularized viscous models and applications

Chapter 2

On the control of the Burgers-α model

On the control of the Burgers-α model

Fágner D. Araruna, Enrique Fernández-Cara and Diego A. Souza

Abstract. This work is devoted to prove the local null controllability of the Burgers-α model. The state is the solution to a regularized Burgers equation, where the transport term is of the form zy_x, $z = (Id - \alpha^2 \frac{\partial^2}{\partial x^2})^{-1} y$ and $\alpha > 0$ is a small parameter. We also prove some results concerning the behavior of the null controls and associated states as $\alpha \to 0^+$.

2.1 Introduction and main results

Let $L > 0$ and $T > 0$ be positive real numbers. Let $(a, b) \subset (0, L)$ be a (small) nonempty open subset which will be referred as the *control domain*.

We will consider the following controlled system for the Burgers equation:

$$\begin{cases} y_t - y_{xx} + yy_x = v1_{(a,b)} & \text{in } (0, L) \times (0, T), \\ y(0, \cdot) = y(L, \cdot) = 0 & \text{in } (0, T), \\ y(\cdot, 0) = y_0 & \text{in } (0, L). \end{cases} \tag{2.1}$$

In (2.1), the function $y = y(x, t)$ can be interpreted as a one-dimensional velocity of a fluid and $y_0 = y_0(x)$ is an initial datum. The function $v = v(x, t)$ (usually in $L^2((a, b) \times (0, T))$) is the control acting on the system and $1_{(a,b)}$ denotes the characteristic function of (a, b).

In this paper, we will also consider a system similar to (2.1), where the transport term is of the form zy_x, where z is the solution to an elliptic problem governed by y. Namely, we consider the following *regularized* version of (2.1), where $\alpha > 0$:

$$\begin{cases} y_t - y_{xx} + zy_x = v1_{(a,b)} & \text{in } (0, L) \times (0, T), \\ z - \alpha^2 z_{xx} = y & \text{in } (0, L) \times (0, T), \\ y(0, \cdot) = y(L, \cdot) = z(0, \cdot) = z(L, \cdot) = 0 & \text{in } (0, T), \\ y(\cdot, 0) = y_0 & \text{in } (0, L). \end{cases} \tag{2.2}$$

This will be called in this paper the *Burgers-α system*. It is a particular case of the systems introduced in [79] to describe the balance of convection and stretching in the dynamics of one-dimensional nonlinear waves in a fluid with small viscosity. It can also be viewed as a simplified 1D version of the so called *Leray-α system*, introduced to describe turbulent flows as an alternative to the classical averaged Reynolds models, see [55]; see also [19]. By considering a special kernel associated to the Green's function for the Helmholtz operator, this model compares successfully with empirical data from turbulent channel and pipe flows for a wide range of Reynolds numbers, at least for periodic boundary conditions, see [19] (the Leray-α system is also closely related to the systems treated by Leray in [95] to prove the existence of solutions to the Navier-Stokes equations; see [78]).

Other references concerning systems of the kind (2.2) in one and several dimensions are [18, 63] and [116, 120], respectively for numerical and optimal control issues.

Let us present the notations used along this work. The symbols C, $\hat{C}$ and C_i, $i = 0, 1, \ldots$ stand for generic positive constants (usually depending on a, b, L and T). For any $r \in [1, \infty]$ and any given Banach space X, $\| \cdot \|_{L^r(X)}$ will denote the usual norm in $L^r(0, T; X)$. In particular, the norms in $L^r(0, L)$ and $L^r((0, L) \times (0, T))$ will be denoted by $\| \cdot \|_r$. We will also need the Hilbert space $K^2(0, L) := H^2(0, L) \cap H_0^1(0, L)$.

The null controllability problems for (2.1) and (2.2) at time $T > 0$ are the following:

> *For any $y_0 \in H_0^1(0, L)$, find $v \in L^2((a, b) \times (0, T))$ such that the associated solution to (2.1) (resp. (2.2)) satisfies*

$$y(\cdot, T) = 0 \quad \textit{in} \quad (0, L). \tag{2.3}$$

Recently, important progress has been made in the controllability analysis of linear and semilinear parabolic equations and systems. We refer to the works [38, 43, 54, 62, 122, 124]. In particular, the controllability of the Burgers equation has been analyzed in [17, 37, 45, 62, 74, 80]. Consequently, it is natural to try to extend the known results to systems like (2.2). Notice that (2.2) is different from (2.1) at least in two aspects: first, the occurrence of nonlocal in space nonlinearities; secondly, the fact that a small parameter α appears.

Our first main results are the following:

Theorem 2.1. *For each $T > 0$, the system (2.2) is locally null-controllable at time T. More precisely, there exists $\delta > 0$ (independent of α) such that, for any $y_0 \in H_0^1(0, L)$ with $\|y_0\|_\infty \leq \delta$, there exist controls $v_\alpha \in L^\infty((a, b) \times (0, T))$ and associated states (y_α, z_α) satisfying (2.3). Moreover, one has*

$$\|v_\alpha\|_\infty \leq C \quad \forall \alpha > 0. \tag{2.4}$$

Theorem 2.2. *For each $y_0 \in H_0^1(0, L)$ with $\|y_0\|_\infty < \pi/L$, the system (2.2) is null-controllable at large time. In other words, there exist $T > 0$ (independent of α), controls $v_\alpha \in L^\infty((a, b) \times (0, T))$ and associated states (y_α, z_α) satisfying (2.3) and (2.4).*

Recall that π/L is the square root of the first eigenvalue of the Dirichlet Laplacian in this case. On the other hand, notice that these results provide controls in $L^\infty((a, b) \times (0, T))$ and not only in $L^2((a, b) \times (0, T))$. In fact, this is very convenient not only in (2.1) and (2.2), but also in some intermediate problems arising in the proofs, since this way we obtain better estimates for the states and the existence and convergence assertions are easier to establish.

The main novelty of these results is that they ensure the control of a kind of nonlocal nonlinear parabolic equations. This makes the difference with respect to other previous works, such as [43] or [38, 54]. This is not frequent in the analysis of the controllability of PDEs. Indeed, in general when we deal with nonlocal nonlinearities, it does not seem easy to transmit the information furnished by locally supported controls to the whole domain in a satisfactory way.

We will also prove a result concerning the controllability in the limit, as $\alpha \to 0^+$. More precisely, the following holds:

Theorem 2.3. *Let $T > 0$ be given and let $\delta > 0$ be the constant furnished by Theorem 2.1. Assume that $y_0 \in H_0^1(0,L)$ with $\|y_0\|_\infty \leq \delta$, let v_α be a null control for (2.2) satisfying (2.4) and let (y_α, z_α) be an associated state satisfying (2.3). Then, at least for a subsequence, one has*

$$\begin{aligned} v_\alpha \to v \quad \text{weakly-}\star \text{ in} \quad L^\infty((a,b)\times(0,T)), \\ z_\alpha \to y \text{ and } y_\alpha \to y \quad \text{weakly-}\star \text{ in} \quad L^\infty((0,L)\times(0,T)) \end{aligned} \tag{2.5}$$

as $\alpha \to 0^+$, where (y,v) is a state-control pair for (2.1) that verifies (2.3).

The rest of this paper is organized as follows. In Section 2.2, we prove some results concerning the existence, uniqueness and regularity of the solution to (2.2). Sections 2.3, 2.4, and 2.5 deal with the proofs of Theorems 2.1, 2.2 and 2.3, respectively. Finally, in Section 2.6, we present some additional comments and questions.

2.2 Preliminaries

In this Section, we will first establish a result concerning global existence and uniqueness for the Burgers-α system

$$\begin{cases} y_t - y_{xx} + z y_x = f & \text{in} \quad (0,L)\times(0,T), \\ z - \alpha^2 z_{xx} = y & \text{in} \quad (0,L)\times(0,T), \\ y(0,\cdot) = y(L,\cdot) = z(0,\cdot) = z(L,\cdot) = 0 & \text{in} \quad (0,T), \\ y(\cdot,0) = y_0 & \text{in} \quad (0,L). \end{cases} \tag{2.6}$$

It is the following:

Proposition 2.1. *Assume that $\alpha > 0$. Then, for any $f \in L^\infty((0,L)\times(0,T))$ and $y_0 \in H_0^1(0,L)$, there exists exactly one solution (y_α, z_α) to (2.6), with*

$$\begin{gathered} y_\alpha \in L^2(0,T;H^2(0,L)) \cap C^0([0,T];H_0^1(0,L)), \\ z_\alpha \in L^2(0,T;H^4(0,L)) \cap L^\infty(0,T;H_0^1(0,L)\cap H^3(0,L)), \\ y_{\alpha,t} \in L^2((0,L)\times(0,T)),\ z_{\alpha,t} \in L^2\big(0,T;H^2(0,L)\big). \end{gathered}$$

Furthermore, the following estimates hold:

$$
\begin{aligned}
\|y_{\alpha,t}\|_2 + \|y_\alpha\|_{L^2(H^2)} + \|y_\alpha\|_{L^\infty(H_0^1)} &\le C(\|y_0\|_{H_0^1} + \|f\|_2)e^{C(M(T))^2},\\
\|z_\alpha\|^2_{L^\infty(L^2)} + 2\alpha^2\|z_\alpha\|^2_{L^\infty(H_0^1)} &\le \|y_\alpha\|^2_{L^\infty(L^2)},\\
2\alpha^2\|z_{\alpha,x}\|^2_{L^\infty(L^2)} + \alpha^4\|z_{\alpha,xx}\|^2_{L^\infty(L^2)} &\le \|y_\alpha\|^2_{L^\infty(L^2)},\\
\|y_\alpha\|_\infty &\le M(T),\\
\|z_\alpha\|_\infty &\le M(T),
\end{aligned}
\tag{2.7}
$$

where $M(t) := \|y_0\|_\infty + t\|f\|_\infty$.

Proof. EXISTENCE: We will reduce the proof to the search of a fixed point of an appropriate mapping Λ_α.

Thus, for each $\overline{y} \in L^\infty((0,L)\times(0,T))$, let $z = z(x,t)$ be the unique solution to

$$
\begin{cases}
z - \alpha^2 z_{xx} = \overline{y}, & \text{in} \quad (0,L)\times(0,T),\\
z(0,\cdot) = z(L,\cdot) = 0 & \text{in} \quad (0,T).
\end{cases}
\tag{2.8}
$$

Since $\overline{y} \in L^\infty((0,L)\times(0,T))$, it is clear that $z \in L^\infty(0,T;K^2(0,L))$. Then, thanks to the Sobolev embedding, we have $z,\ z_x \in L^\infty((0,L)\times(0,T))$ and the following is satisfied:

$$
\begin{aligned}
\|z\|^2_{L^\infty(L^2)} + 2\alpha^2\|z\|^2_{L^\infty(H_0^1)} &\le \|\overline{y}\|^2_{L^\infty(L^2)},\\
2\alpha^2\|z_x\|^2_{L^\infty(L^2)} + \alpha^4\|z_{xx}\|^2_{L^\infty(L^2)} &\le \|\overline{y}\|^2_{L^\infty(L^2)},\\
\|z\|_\infty &\le \|\overline{y}\|_\infty.
\end{aligned}
\tag{2.9}
$$

From this z, we can obtain y as the unique solution to the linear problem

$$
\begin{cases}
y_t - y_{xx} + z y_x = f & \text{in} \quad (0,L)\times(0,T),\\
y(0,\cdot) = y(L,\cdot) = 0 & \text{in} \quad (0,T),\\
y(\cdot,0) = y_0 & \text{in} \quad (0,L).
\end{cases}
\tag{2.10}
$$

Since $z, f \in L^\infty((0,L)\times(0,T))$ and $y_0 \in H_0^1(0,L)$, it is clear that

$$
\begin{gathered}
y \in L^2(0,T;K^2(0,L)) \cap C^0([0,T];H_0^1(0,L)),\\
y_t \in L^2((0,L)\times(0,T))
\end{gathered}
$$

and we have the following estimate:

$$
\|y_t\|_2 + \|y\|_{L^2(H^2)} + \|y\|_{L^\infty(H_0^1)} \le C(\|y_0\|_{H_0^1} + \|f\|_2)e^{C\|z\|^2_\infty}.
$$

Indeed, this can be easily deduced, for instance, from a standard *Galerkin approximation* and *Gronwall's Lemma*; see for instance [36].

We will use the following result, whose proof is given below, after the proof of this Theorem.

Lemma 2.1. *The solution y to (2.10) satisfies*

$$\|y\|_\infty \leq M(T). \tag{2.11}$$

Now, we introduce the Banach space

$$W = \{w \in L^\infty(0,T;H_0^1(0,L)) : w_t \in L^2((0,L)\times(0,T))\}, \tag{2.12}$$

the closed ball

$$K = \{w \in L^\infty((0,L)\times(0,T)) : \|w\|_\infty \leq M(T)\}$$

and the mapping $\tilde{\Lambda}_\alpha$, with $\tilde{\Lambda}_\alpha(\overline{y}) = y$ for all $\overline{y} \in L^\infty((0,L)\times(0,T))$. Obviously $\tilde{\Lambda}_\alpha$ is well defined and, in view of Lemma 2.1, maps the whole space $L^\infty((0,L)\times(0,T))$ into $W \cap K$.

Let us denote by Λ_α the restriction to K of $\tilde{\Lambda}_\alpha$. Then, thanks to Lemma 2.1, Λ_α maps K into itself. Moreover, it is clear that $\Lambda_\alpha : K \mapsto K$ satisfies the hypotheses of *Schauder's Fixed Point Theorem*. Indeed, this nonlinear mapping is continuous and compact (the latter is a consequence of the fact that, if B is bounded in $L^\infty((0,L)\times(0,T))$, then $\Lambda_\alpha(B)$ is bounded in W and therefore it is relatively compact in the space $L^\infty((0,L)\times(0,T))$, in view of the classical results of the Aubin-Lions' kind, see for instance [117]). Consequently, Λ_α possesses at least one fixed point in K.

This immediately achieves the proof of existence.

UNIQUENESS: Let (z'_α, y'_α) be another solution to (2.6) and let us introduce $u := y_\alpha - y'_\alpha$ and $m := z_\alpha - z'_\alpha$. Then

$$\begin{cases} u_t - u_{xx} + z_\alpha u_x = -m y'_{\alpha,x} & \text{in } (0,L)\times(0,T), \\ m - \alpha^2 m_{xx} = u & \text{in } (0,L)\times(0,T), \\ u(0,\cdot) = u(L,\cdot) = m(0,\cdot) = m(L,\cdot) = 0 & \text{in } (0,T), \\ u(\cdot,0) = 0 & \text{in } (0,L). \end{cases}$$

Since $y'_\alpha \in L^2(0,T;H^2(0,L))$, thanks to the Sobolev embedding, we have $y'_\alpha \in L^2(0,T;C^1[0,L])$. Therefore, we easily get from the first equation of the previous system that

$$\frac{1}{2}\frac{\partial}{\partial t}\|u\|_2^2 + \|u_x\|_2^2 \leq \|z_\alpha\|_\infty \|u_x\|_2 \|u\|_2 + \|y'_{\alpha,x}\|_\infty \|m\|_2 \|u\|_2.$$

Since $\|m\|_2 \leq \|u\|_2$, we have

$$\frac{\partial}{\partial t}\|u\|_2^2 + \|u_x\|_2^2 \leq \Big(\|z_\alpha\|_\infty^2 + 2\|y'_{\alpha,x}\|_\infty\Big)\|u\|_2^2.$$

Therefore, in view of Gronwall's Lemma, we necessarily have $u \equiv 0$. Accordingly, we also obtain $m \equiv 0$ and uniqueness holds. □

Let us now return to Lemma 2.1 and establish its proof.

Proof of Lemma 2.1. Let y be the solution to (2.10) and let us set $w = (y - M(t))_+$. Notice that $w(x,0) \equiv 0$ and $w(0,t) \equiv w(L,t) \equiv 0$.

Let us multiply the first equation of (2.10) by w and let us integrate on $(0, L)$. Then we obtain the following for all t:

$$\int_0^L (y_t w + z y_x w)\, dx + \int_0^L y_x w_x\, dx = \int_0^L f w\, dx.$$

This can also be written in the form

$$\int_0^L (w_t w + z w_x w)\, dx + \int_0^L |w_x|^2\, dx = \int_0^L (f - M_t) w\, dx$$

and, consequently, we obtain the identity

$$\frac{1}{2}\frac{\partial}{\partial t}\|w\|_2^2 + \|w_x\|_2^2 - \frac{1}{2}\int_0^L z_x |w|^2\, dx = \int_0^L (f - \|f\|_\infty) w\, dx$$

and, therefore,

$$\frac{1}{2}\frac{\partial}{\partial t}\|w\|_2^2 + \|w_x\|_2^2 - \frac{1}{2}\int_0^L z_x |w|^2\, dx \le 0. \tag{2.13}$$

Since $z_x \in L^\infty((0,L)\times(0,T))$, it follows by (2.13) that

$$\frac{\partial}{\partial t}\|w\|_2^2 \le \|z_x\|_\infty \|w\|_2^2.$$

Then, using again Gronwall's Lemma, we see that $w \equiv 0$.

Analogously, if we introduce $\widetilde{w} = (y + M(t))_-$, similar computations lead to the identity $\widetilde{w} \equiv 0$. Therefore, y satisfies (2.11) and the Lemma is proved. □

We will now see that, when f is fixed and $\alpha \to 0^+$, the solution to (2.6) converges to the solution to the Burgers system

$$\begin{cases} y_t - y_{xx} + y y_x = f & \text{in} \quad (0,L)\times(0,T), \\ y(0,\cdot) = y(L,\cdot) = 0 & \text{on} \quad (0,T), \\ y(\cdot,0) = y_0 & \text{in} \quad (0,L). \end{cases} \tag{2.14}$$

Proposition 2.2. *Assume that $y_0 \in H_0^1(0,L)$ and $f \in L^\infty((0,L)\times(0,T))$ are given. For each $\alpha > 0$, let (y_α, z_α) be the unique solution to (2.6). Then*

$$z_\alpha \to y \quad \textit{and} \quad y_\alpha \to y \quad \textit{strongly in} \quad L^2(0,T;H_0^1(0,L)) \tag{2.15}$$

as $\alpha \to 0^+$, where y is the unique solution to (2.14).

Proof. Since (y_α, z_α) is the solution to (2.6), we have (2.7). Therefore, there exists y such that, at least for a subsequence, we have

$$\begin{aligned} y_\alpha \to y \quad &\text{weakly in} \quad L^2(0,T;H^2(0,L)),\\ y_\alpha \to y \quad &\text{weakly-}\star\text{ in} \quad L^\infty(0,T;H_0^1(0,L)),\\ (y_\alpha)_t \to y_t \quad &\text{weakly in} \quad L^2((0,L)\times(0,T)). \end{aligned} \tag{2.16}$$

The Hilbert space

$$Y = \{\, w \in L^2(0,T;K^2(0,L)) : w_t \in L^2((0,L)\times(0,T))\,\}$$

is compactly embedded in $L^2(0,T;H_0^1(0,L))$. Consequently,

$$y_\alpha \to y \quad \text{strongly in} \quad L^2(0,T;H_0^1(0,L)). \tag{2.17}$$

Let us see that y is the unique solution to (2.14).

Using the second equation in (2.6), we have

$$(z_\alpha - y) - \alpha^2 (z_\alpha - y)_{xx} = (y_\alpha - y) + \alpha^2 y_{xx}.$$

Multiplying this equation by $-(z_\alpha - y)_{xx}$ and integrating in $(0,L)\times(0,T)$, we obtain

$$\begin{aligned} &\int_0^T\!\!\int_0^L |(z_\alpha - y)_x|^2\,dx\,dt + \alpha^2 \int_0^T\!\!\int_0^L |(z_\alpha - y)_{xx}|^2\,dx\,dt\\ &= \int_0^T\!\!\int_0^L (y_\alpha - y)_x (z_\alpha - y)_x\,dx\,dt - \alpha^2 \int_0^T\!\!\int_0^L y_{xx}(z_\alpha - y)_{xx}\,dx\,dt, \end{aligned}$$

whence

$$\int_0^T\!\!\int_0^L |(z_\alpha - y)_x|^2\,dx\,dt \le \int_0^T\!\!\int_0^L |(y_\alpha - y)_x|^2\,dx\,dt + \alpha^2 \|y_{xx}\|_2^2.$$

This shows that

$$z_\alpha \to y \quad \text{strongly in} \quad L^2(0,T;H_0^1(0,L)) \tag{2.18}$$

and, consequently,

$$z_\alpha (y_\alpha)_x \to y y_x \quad \text{strongly in} \quad L^1((0,L)\times(0,T)). \tag{2.19}$$

Finally, for each $\psi \in L^\infty(0,T;H_0^1(0,L))$, we have

$$\int_0^T\!\!\int_0^L \left((y_{\alpha,t}\psi + y_{\alpha,x}\psi_x + z_\alpha y_{\alpha,x}\psi \right)\,dx\,dt = \int_0^T\!\!\int_0^L f\psi\,dx\,dt. \tag{2.20}$$

Using (2.16) and (2.19), we can take limits in all terms and find that

$$\int_0^T\int_0^L (y_t\psi + y_x\psi_x + yy_x\psi)\,dx\,dt = \int_0^T\int_0^L f\psi\,dx\,dt, \tag{2.21}$$

that is, y is the unique solution to (2.14).

This proves that (2.15) holds at least for a subsequence. But, in view of uniqueness, not only a subsequence but the whole sequence converges. □

Remark 2.1. In fact, a result similar to Proposition 2.2 can also be established with varying f and y_0. More precisely, if we introduce data f_α and $y_{0,\alpha}$ with

$$f_\alpha \to f \quad \text{weakly-}\star \text{ in} \quad L^\infty((0,L)\times(0,T))$$

and

$$(y_{0,\alpha} \to y_0 \quad \text{weakly-}\star \text{ in} \quad L^\infty(0,L),$$

then we find that the associated solutions (y_α, z_α) satisfy again (2.15). □

To end this Section, we will now recall a result dealing with the null controllability of general parabolic linear systems of the form

$$\begin{cases} y_t - y_{xx} + Ay_x = v1_{(a,b)} & \text{in} \quad (0,L)\times(0,T), \\ y(0,\cdot) = y(L,\cdot) = 0 & \text{in} \quad (0,T), \\ y(\cdot,0) = y_0 & \text{in} \quad (0,L). \end{cases} \tag{2.22}$$

where $y_0 \in L^2(0,L)$, $A \in L^\infty((0,L)\times(0,T))$ and $v \in L^2((a,b)\times(0,T))$.

It is well known that there exists exactly one solution y to (2.22), with

$$y \in C^0([0,T];L^2(0,L)) \cap L^2(0,T;H_0^1(0,L)).$$

Related to controllability result, we have the following:

Theorem 2.4. *The linear system (2.22) is null controllable at any time $T > 0$. In other words, for each $y_0 \in L^2(0,L)$ there exists $v \in L^2((a,b)\times(0,T))$ such that the associated solution to (2.22) satisfies (2.3). Furthermore, the extremal problem*

$$\begin{cases} \text{Minimize } \dfrac{1}{2}\displaystyle\int_0^T\int_a^b |v|^2\,dx\,dt \\ \text{Subject to: } v \in L^2((a,b)\times(0,T)), (2.22), (2.3) \end{cases} \tag{2.23}$$

possesses exactly one solution $\hat{v}$ satisfying

$$\|\hat{v}\|_2 \le C_0\|y_0\|_2, \tag{2.24}$$

where

$$C_0 = e^{C_1(1+1/T+(1+T)\|A\|_\infty^2)}$$

and C_1 only depends on a, b and L.

The proof of this result can be found in [84].

2.3 Local null controllability of the Burgers-α model

In this Section, we present the proof of Theorem 2.1.

Roughly speaking, we fix $\overline{y}$, we solve (2.8), we control exactly to zero the linear system (2.22) with $A = z$ and we set $\Lambda_\alpha(\overline{y}) = y$. Then the task is to solve the fixed point equation $y = \Lambda_\alpha(y)$.

Several fixed point theorems can be applied. In this paper, we have preferred to use Schauder's Fixed Point Theorem, although other results also lead to the good conclusion; for instance, an argument relying on *Kakutani's fixed point Theorem*, like in [38], is possible.

As mentioned above, in order to get good properties for Λ_α, it is very appropriate that the control belongs to L^∞. This can be achieved by several ways; for instance, using an "improved" observability estimate for the solutions to the adjoint of (2.22) and arguing as in [38]. We have preferred here to use other techniques that rely on the regularity of the states and were originally used in [7]; see also [8].

Let $y_0 \in H_0^1(0, L)$ and a', a'', b' and b'' be given, with $0 < a < a' < a'' < b'' < b' < b < L$. Let θ and η satisfy

$$\theta \in C^\infty([0, T]),\ \theta \equiv 1 \text{ in } [0, T/4],\ \theta \equiv 0 \text{ in } [3T/4, T],$$

$$\eta \in \mathcal{D}(a, b),\ \eta \equiv 1 \text{ in a neighborhood of } [a', b'],\ 0 \leq \eta \leq 1.$$

As in the proof of Proposition 2.1, we can associate to each $\overline{y} \in L^\infty((0, L) \times (0, T))$ the function z through (2.8). Recall that $z, z_x \in L^\infty((0, L) \times (0, T))$ and the inequalities (2.9) are satisfied. In view of Theorem 2.4, we can associate to z the null control $\hat{v}$ of minimal norm in $L^2((a'', b'') \times (0, T))$, that is, the solution to (2.22)–(2.23) with a, b and A respectively replaced by a'', b'' and z. Let us denote by $\hat{y}$ the corresponding solution to (2.22).

Then, we can write that $\hat{y} = \theta(t)\hat{u} + \hat{w}$, where $\hat{u}$ and $\hat{w}$ are the unique solutions to the linear systems

$$\begin{cases} \hat{u}_t - \hat{u}_{xx} + z\hat{u}_x = 0 & \text{in} \quad (0, L) \times (0, T), \\ \hat{u}(0, \cdot) = \hat{u}(L, \cdot) = 0 & \text{in} \quad (0, T), \\ \hat{u}(\cdot, 0) = y_0 & \text{in} \quad (0, L) \end{cases} \tag{2.25}$$

and

$$\begin{cases} \hat{w}_t - \hat{w}_{xx} + z\hat{w}_x = \hat{v}1_{(a'',b'')} - \theta_t \hat{u} & \text{in } \ (0,L)\times(0,T), \\ \hat{w}(0,\cdot) = \hat{w}(L,\cdot) = 0 & \text{in } \ (0,T), \\ \hat{w}(\cdot,0) = 0, \ \hat{w}(\cdot,T) = 0 & \text{in } \ (0,L), \end{cases} \tag{2.26}$$

respectively.

If we now set $w := (1-\eta(x))\hat{w}$, then we have that w is the unique solution of the parabolic system

$$\begin{cases} w_t - w_{xx} + zw_x = v - \theta_t \hat{u} & \text{in } \ (0,L)\times(0,T), \\ w(0,\cdot) = w(L,\cdot) = 0 & \text{in } \ (0,T), \\ w(\cdot,0) = 0, \ w(\cdot,T) = 0 & \text{in } \ (0,L), \end{cases} \tag{2.27}$$

where $v := \eta\theta_t\hat{u} - \eta_x z\hat{w} + 2\eta_x\hat{w}_x + \eta_{xx}\hat{w} + (1-\eta(x))\hat{v}1_{(a'',b'')}$.

Notice that $(1-\eta)\hat{v}1_{(a'',b'')} \equiv 0$, since $\eta \equiv 1$ in $[a',b']$. Therefore, one has

$$v = \eta\theta_t\hat{u} - \eta_x z\hat{w} + 2\eta_x\hat{w}_x + \eta_{xx}\hat{w} \tag{2.28}$$

and then $\operatorname{supp}(v) \subset (a,b)$.

Let us prove that $v \in L^\infty((a,b)\times(0,T))$ and

$$\|v\|_\infty \le \widehat{C}\|y_0\|_\infty, \tag{2.29}$$

for some

$$\widehat{C} = e^{C(a,b,L)(1+1/T+(1+T)\|\overline{y}\|_\infty^2)}. \tag{2.30}$$

First, note that $\hat{u} \in L^\infty((0,L)\times(0,T))$ and $\|\hat{u}\|_\infty \le \|y_0\|_\infty$. Defining

$$G = (a,a') \cup (b',b),$$

we see that it suffices to check that $\eta_x z\hat{w}$, $\eta_x\hat{w}_x$ and $\eta_{xx}\hat{w}$ belong to $L^\infty(G\times(0,T))$, with norms in $L^\infty(G\times(0,T))$ bounded by a constant times the L^2-norm of $\hat{v}$ and the L^∞-norm of y_0, since η_x and η_{xx} are identically zero in a neighborhood of $[a',b']$.

From the usual parabolic estimates for (2.26) and the estimate (2.9), we first obtain that

$$\|\hat{w}_t\|_{L^2(L^2)} + \|\hat{w}\|_{L^2(H^2)} + \|\hat{w}\|_{L^\infty(H_0^1)} \le \|\hat{v}1_{(a'',b'')} - \theta_t\hat{u}\|_{L^2(L^2)} e^{C\|\overline{y}\|_\infty^2}. \tag{2.31}$$

In particular, we have $\hat{w} \in L^\infty((a,b)\times(0,T))$, with appropriate estimates.

On the other hand, $\theta_t\hat{u} \in L^\infty((0,L)\times(0,T))$ and, from the equation satisfied by $\hat{w}$, we have

$$\hat{w}_t - \hat{w}_{xx} + z\hat{w}_x = -\theta_t\hat{u} \ \text{ in } \ [(0,a'')\cup(b'',L)]\times(0,T).$$

Hence, from standard (local in space) parabolic estimates, we deduce that $\hat{w}$ belongs to the space $X^p(0,T;G) = \{\, \hat{w} \in L^p(0,T;W^{2,p}(G)) \ : \ \hat{w}_t \in L^p(0,T;L^p(G)) \,\}$ for all

$2 < p < +\infty$.

Then, using Lemma 3.3 (p. 80) of [89], we can take $p > 3$ to get the embedding $X^p(0,T;G) \hookrightarrow C^0([0,T];C^1(\overline{G}))$ and $\hat{w}_x \in C^0(\overline{G}\times[0,T])$. This proves that $\hat{w}_x \in L^\infty(G)$, again with the appropriate estimates.

Therefore, if we define $y := \theta(t)\hat{u} + w$, one has

$$\begin{cases} y_t - y_{xx} + z y_x = v 1_{(a,b)} & \text{in} \quad (0,L)\times(0,T), \\ y(0,\cdot) = y(L,\cdot) = 0 & \text{in} \quad (0,T), \\ y(\cdot,0) = y_0 & \text{in} \quad (0,L) \end{cases} \tag{2.32}$$

and (2.3). Moreover, the control v satisfies (2.29)–(2.30).

Let us set $\Lambda_\alpha(\overline{y}) = y$. In this way, we have been able to introduce a mapping

$$\Lambda_\alpha : L^\infty((0,L)\times(0,T)) \mapsto L^\infty((0,L)\times(0,T))$$

for which the following properties are easy to check:

a) Λ_α is continuous and compact. The compactness can be explained as follows: if $B \subset L^\infty((0,L)\times(0,T))$ is bounded, then $\Lambda_\alpha(B)$ is bounded in the space W in (2.12) and, therefore, it is relatively compact in $L^\infty((0,L)\times(0,T))$, in view of classical results of the Aubin-Lions' kind, see for instance [117]).

b) If $R > 0$ and $\|y_0\|_\infty \le \varepsilon(R)$ (independent of α!), then Λ_α maps the ball $B_R := \{\,\overline{y} \in L^\infty((0,L)\times(0,T)) : \|\overline{y}\|_\infty \le R\,\}$ into itself.

The consequence is that, again, Schauder's Fixed Point Theorem can be applied and there exist controls $v_\alpha \in L^\infty((0,L)\times(0,T))$ such that the corresponding solutions to (2.2) satisfy (2.3). This achieves the proof of Theorem 2.1.

2.4 Large time null controllability of the Burgers-α system

The proof of Theorem 2.2 is similar. It suffices to replace the assumption "y_0 is small" by an assumption imposing that T is large enough. Again, this makes it possible to apply a fixed point argument.

More precisely, let us accept that, if $y_0 \in H_0^1(0,L)$ and $\|y_0\|_\infty < \pi/L$, then the associated uncontrolled solution y_α to (2.2) satisfies

$$\|y_\alpha(\cdot\,,t)\|_{H_0^1} \le C(y_0)e^{-\frac{1}{2}((\pi/L)^2 - \|y_0\|_\infty^2)t} \tag{2.33}$$

where $C(y_0)$ is a constant only depending on $\|y_0\|_\infty$ and $\|y_0\|_{H_0^1}$. Then, if we first take $v \equiv 0$, the state $y_\alpha(\cdot,t)$ becomes small for large t. In a second step, when $\|y_\alpha(\cdot,t)\|_{H_0^1}$ is sufficiently small, we can apply Theorem 2.1 and drive the state exactly to zero.

Let us now see that (2.33) holds. Arguing as in the proof of Proposition 2.1, we see that

$$\frac{\partial}{\partial t}\|y_\alpha\|_2^2 + \|y_{\alpha,x}\|_2^2 \le \|y_0\|_\infty^2 \|y_\alpha\|_2^2 \tag{2.34}$$

and, using *Poincaré's inequality*, we obtain:

$$\frac{\partial}{\partial t}\|y_\alpha\|_2^2 + (\pi/L)^2\|y_\alpha\|_2^2 \le \|y_0\|_\infty^2 \|y_\alpha\|_2^2.$$

Let us introduce $r = \frac{1}{2}((\pi/L)^2 - \|y_0\|_\infty^2)$. It then follows that

$$\|y_\alpha(\cdot,t)\|_2^2 \le \|y_0\|_2^2 e^{-2rt}. \tag{2.35}$$

Hence, by combining (2.34) and (2.35), it is easy to see that

$$\frac{\partial}{\partial t}\left(e^{rt}\|y_\alpha\|_2^2\right) + e^{rt}\|y_{\alpha,x}\|_2^2 \le (r + \|y_0\|_\infty^2)\|y_0\|_2^2\, e^{-rt}.$$

Integrating from 0 to t yields

$$\int_0^t e^{r\sigma}\|y_{\alpha,x}\|_2^2\, d\sigma \le \left(2 + \frac{\|y_0\|_\infty^2}{r}\right)\|y_0\|_2^2. \tag{2.36}$$

Now, we take the L^2-inner product of (2.6) and $-y_{\alpha,xx}$ and get

$$\frac{\partial}{\partial t}\|y_{\alpha,x}\|_2^2 \le \|y_0\|_\infty^2 \|y_{\alpha,x}\|^2.$$

Multiplying this inequality by e^{rt}, we deduce that

$$\frac{\partial}{\partial t}\left(e^{rt}\|y_{\alpha,x}\|_2^2\right) \le (r + \|y_0\|_\infty^2)\, e^{rt}\, \|y_{\alpha,x}\|_2^2$$

and, consequently, we see from (2.36) that

$$\|y_{\alpha,x}(\cdot,t)\|_2^2 \le \left[(r + \|y_0\|_\infty^2)\left(2 + \frac{\|y_0\|_\infty^2}{r}\right)\|y_0\|_2^2 + \|y_0\|_{H_0^1}^2\right] e^{-rt},$$

which implies (2.33).

Remark 2.2. To our knowledge, it is unknown what can be said when the smallness assumption $\|y_0\|_\infty < \pi/L$ is not satisfied. In fact, it is not clear whether or not the solutions to (2.2) with large initial data and $v \equiv 0$ decay as $t \to +\infty$. □

2.5 Controllability in the limit

In this Section, we are going to prove Theorem 2.3.

For the null controls v_α furnished by Theorem 2.1 and the associated solutions (y_α, z_α) to (2.2), we have the uniform estimates (2.29) and (2.7) with $f = v_\alpha 1_{(a,b)}$. Then, there exists $y \in L^2(0,T;K^2(0,L))$, with $y_t \in L^2((0,L)\times(0,T))$, and $v \in L^\infty((a,b)\times(0,T))$ such that, at least for a subsequence, one has:

$$\begin{aligned} &y_\alpha \to y \quad \text{weakly in} \quad L^2(0,T;K^2(0,L)),\\ &y_{\alpha,t} \to y_t \quad \text{weakly in} \quad L^2((0,L)\times(0,T)),\\ &v_\alpha \to v \quad \text{weakly-}\star \text{ in} \quad L^\infty((a,b)\times(0,T)). \end{aligned} \tag{2.37}$$

As before, the *Aubin-Lions' Lemma* implies

$$y_\alpha \to y \quad \text{strongly in} \quad L^2(0,T;H_0^1(0,L)). \tag{2.38}$$

Using the second equation in (2.2), we see that

$$(z_\alpha - y) - \alpha^2 (z_\alpha - y)_{xx} = (y_\alpha - y) + \alpha^2 y_{xx}.$$

Multiplying this equation by $-(z_\alpha - y)_{xx}$ and integrating in $(0,L)\times(0,T)$, we deduce

$$\begin{aligned} &\int_0^T\!\!\int_0^L |(z_\alpha - y)_x|^2\,dx\,dt + \alpha^2 \int_0^T\!\!\int_0^L |(z_\alpha - y)_{xx}|^2\,dx\,dt\\ &= \int_0^T\!\!\int_0^L (y_\alpha - y)_x (z_\alpha - y)_x\,dx\,dt - \alpha^2 \int_0^T\!\!\int_0^L y_{xx}(z_\alpha - y)_{xx}\,dx\,dt. \end{aligned}$$

Whence,

$$\int_0^T\!\!\int_0^L |(z_\alpha - y)_x|^2\,dx\,dt \le \int_0^T\!\!\int_0^L |(y_\alpha - y)_x|^2\,dx\,dt + \alpha^2 \|y_{xx}\|_2^2.$$

This shows that

$$z_\alpha \to y \quad \text{strongly in} \quad L^2(0,T;H_0^1(0,L)) \tag{2.39}$$

and the transport terms in (2.2) satisfy

$$z_\alpha (y_\alpha)_x \to y y_x \quad \text{strongly in} \quad L^1((0,L)\times(0,T)). \tag{2.40}$$

In this way, for each $\psi \in L^\infty(0,T;H_0^1(0,L))$, we obtain

$$\int_0^T\!\!\int_0^L \left((y_\alpha)_t \psi + (y_\alpha)_x \psi_x + z_\alpha (y_\alpha)_x \psi\right)\,dx\,dt = \int_0^T\!\!\int_0^L v_\alpha 1_{(a,b)} \psi\,dx\,dt. \tag{2.41}$$

Using (2.37) and (2.40), we can pass to the limit, as $\alpha \to 0^+$, in all the terms of (2.41) to find

$$\int_0^T\!\!\int_0^L \left(y_t \psi + y_x \psi_x + y y_x \psi\right)\,dx\,dt = \int_0^T\!\!\int_0^L v 1_{(a,b)} \psi\,dx\,dt, \tag{2.42}$$

that is, y is the unique solution of (2.1) and y satisfies (2.3).

2.6 Additional comments and questions

2.6.1 A boundary controllability result

We can use an extension argument to prove local boundary controllability results similar to those above.

For instance, let us see that the analog of Theorem 2.1 remains true. Thus, let us introduce the controlled system

$$\begin{cases} y_t - y_{xx} + zy_x = 0 & \text{in } (0,L)\times(0,T), \\ z - \alpha^2 z_{xx} = y & \text{in } (0,L)\times(0,T), \\ z(0,\cdot) = y(0,\cdot) = 0,\ z(L,\cdot) = y(L,\cdot) = u & \text{in } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,L), \end{cases} \tag{2.43}$$

where $u = u(t)$ stands for the control function and $y_0 \in H_0^1(0,L)$ is given.

Let a, b and $\tilde{L}$ be given, with $L < a < b < \tilde{L}$. Then, let us define $\tilde{y}_0 : [0,\tilde{L}] \mapsto \mathbb{R}$, with $\tilde{y}_0 := y_0 1_{[0,L]}$. Arguing as in Theorem 2.1, it can be proved that there exists $(\tilde{y},\tilde{v})$, with $\tilde{v} \in L^\infty((a,b)\times(0,T))$,

$$\begin{cases} \tilde{y}_t - \tilde{y}_{xx} + z 1_{[0,L]}\tilde{y}_x = \tilde{v} 1_{(a,b)} & \text{in } (0,\tilde{L})\times(0,T), \\ z - \alpha^2 z_{xx} = \tilde{y} & \text{in } (0,L)\times(0,T), \\ \tilde{y}(0,\cdot) = z(0,\cdot) = z(L,\cdot) = \tilde{y}(\tilde{L},\cdot) = 0 & \text{in } (0,T), \\ \tilde{y}(\cdot,0) = \tilde{y}_0 & \text{in } (0,\tilde{L}), \end{cases}$$

and $\tilde{y}(x,T) \equiv 0$. Then, $y := \tilde{y}1_{(0,L)}$, z and $u(t) := \tilde{y}(L,t)$ satisfy (2.43).

Notice that the control that we have obtained satisfies $u \in C^0([0,T])$, since it can be viewed as the lateral trace of a strong solution of the heat equation with a L^∞ right hand side.

2.6.2 No global null controllability?

To our knowledge, it is unknown whether a general global null controllability result holds for (2.2). We can prove global null controllability "for large α".

More precisely, the following holds:

Theorem 2.5. *Let $y_0 \in H_0^1(0,L)$ and $T > 0$ be given. There exists $\alpha_0 = \alpha_0(y_0,T)$ such that (2.2) can be controlled to zero for all $\alpha > \alpha_0$.*

Proof. The main idea is, again, to apply a fixed point argument in $L^\infty(0,T;L^2(0,L))$.

For each $\overline{y} \in L^\infty(0,T;L^2(0,L))$, we introduce the solution z to (2.8). We notice that

z satisfies

$$\|z\|_2^2 + 2\alpha^2\|z_x\|_2^2 \leq \|\overline{y}\|_2^2,$$
$$2\alpha^2\|z_x\|_2^2 + \alpha^4\|z_{xx}\|_2^2 \leq \|\overline{y}\|_2^2.$$

Then, as in the proof of Theorem 2.1, we consider the solution (y, v) to the system

$$\begin{cases} y_t - y_{xx} + zy_x = v1_{(a,b)} & \text{in} \quad (0,L)\times(0,T), \\ y(0,\cdot) = y(L,\cdot) = 0 & \text{in} \quad (0,T), \\ y(\cdot,0) = y_0 & \text{in} \quad (0,L), \end{cases} \tag{2.44}$$

where we assume that y satisfies (2.3) and v satisfies the estimate

$$\|v\|_\infty \leq \hat{C}\|y_0\|_\infty, \tag{2.45}$$

with

$$\hat{C} = e^{C(a,b,L)(1+1/T+(1+T)\|z\|_\infty^2)}.$$

It is then clear that

$$\|y_t\|_2 + \|y\|_{L^2(H^2)} + \|y\|_{L^\infty(H_0^1)} \leq \ C\|y_0\|_{H_0^1} e^{C(a,b,L)(1+1/T+(1+T)\|z\|_\infty^2)}.$$

Since $\|z\|_\infty^2 \leq \frac{C}{\alpha^2}\|\overline{y}\|_2^2$, we have

$$\|y_t\|_2 + \|y\|_{L^2(H^2)} + \|y\|_{L^\infty(H_0^1)} \leq C\|y_0\|_{H_0^1} e^{C(a,b,L)\left(1+\frac{1}{T}+(1+T)\frac{1}{\alpha^2}\|\overline{y}\|_{L^\infty(L^2)}^2\right)}.$$

We can check that there exist R and α_0 such that

$$C\|y_0\|_{H_0^1} e^{C(a,b,L)\left(1+1/T+(1+T)\frac{1}{\alpha^2}R^2\right)} < R,$$

for all $\alpha > \alpha_0$. Therefore, we can apply the fixed point argument in the ball B_R of $L^\infty(0,T;L^2(0,L))$ for these α. This ends the proof. □

Notice that we cannot expect (2.2) to be globally null-controllable with controls bounded independently of α, since the limit problem (2.1) is not globally null-controllable, see [45, 74]. More precisely, let $y_0 \in H_0^1(0,L)$ and $T > 0$ be given and let us denote by $\hat{\alpha}(y_0,T)$ the infimum of all α_0 furnished by Theorem 2.5. Then, either $\hat{\alpha}(y_0,T) > 0$ or the associated cost of null controllability grows to infinity as $\alpha \to 0$, i.e. the null controls of minimal norm v_α satisfy

$$\limsup_{\alpha\to 0^+} \|v_\alpha\|_{L^\infty((a,b)\times(0,T))} = +\infty.$$

2.6.3 The situation in higher spatial dimensions. The Leray-α system

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain ($N = 2,3$) and let $\omega \subset \Omega$ be a (small) open subset. We will use the notation $Q := \Omega \times (0,T)$ and $\Sigma := \partial\Omega \times (0,T)$ and we will use bold symbols for vector-valued functions and spaces of vector-valued functions.

For any $\mathbf{f}$ and any $\mathbf{y}_0$ in appropriate spaces, we will consider the Navier-Stokes system

$$\begin{cases} \mathbf{y}_t - \Delta\mathbf{y} + (\mathbf{y}\cdot\nabla)\mathbf{y} + \nabla p = \mathbf{f} & \text{in} \quad Q, \\ \nabla\cdot\mathbf{y} = 0 & \text{in} \quad Q, \\ \mathbf{y} = \mathbf{0} & \text{on} \quad \Sigma, \\ \mathbf{y}(0) = \mathbf{y}_0 & \text{in} \quad \Omega. \end{cases} \tag{2.46}$$

As before, we will also introduce a smoothing kernel and a related modification of (2.46). More precisely, the following so called *Leray-α* model will be of interest:

$$\begin{cases} \mathbf{y}_t - \Delta\mathbf{y} + (\mathbf{z}\cdot\nabla)\mathbf{y} + \nabla p = \mathbf{f} & \text{in} \quad Q, \\ \nabla\cdot\mathbf{y} = \nabla\cdot\mathbf{z} = 0 & \text{in} \quad Q, \\ \mathbf{z} - \alpha^2\Delta\mathbf{z} + \nabla\pi = \mathbf{y} & \text{in} \quad Q, \\ \mathbf{y} = \mathbf{z} = \mathbf{0} & \text{on} \quad \Sigma, \\ \mathbf{y}(0) = \mathbf{y}_0 & \text{in} \quad \Omega. \end{cases} \tag{2.47}$$

Let us recall the definitions of some function spaces that are frequently used in the analysis of incompressible fluids:

$$\mathbf{H} = \left\{ \boldsymbol{\varphi} \in \mathbf{L}^2(\Omega) : \nabla\cdot\boldsymbol{\varphi} = 0 \text{ in } \Omega,\ \boldsymbol{\varphi}\cdot\mathbf{n} = 0 \text{ on } \partial\Omega \right\},$$
$$\mathbf{V} = \left\{ \boldsymbol{\varphi} \in \mathbf{H}_0^1(\Omega) : \nabla\cdot\boldsymbol{\varphi} = 0 \text{ in } \Omega \right\}.$$

It is not difficult to prove that, for any $\alpha > 0$, under some reasonable conditions on $\mathbf{f}$ and $\mathbf{y}_0$, (2.47) possesses a unique global weak solution. This is stated rigorously in the following proposition, that we present without proof (the arguments are similar to those in [119]; the detailed proof will appear in a forthcoming paper):

Proposition 2.3. *Assume that $\alpha > 0$. Then, for any $\mathbf{f} \in L^2(0,T;\mathbf{H}^{-1}(\Omega))$ and any $\mathbf{y}_0 \in \mathbf{H}$, there exists exactly one solution $(\mathbf{y}_\alpha, p_\alpha, \mathbf{z}_\alpha, \pi_\alpha)$ to (2.47), with*

$$\mathbf{y}_\alpha \in L^2(0,T;\mathbf{V}) \cap C^0([0,T];\mathbf{H}),\ \mathbf{y}_{\alpha,t} \in L^1(0,T;\mathbf{V}'),$$
$$\mathbf{z}_\alpha \in L^2(0,T;\mathbf{H}^2(\Omega)\cap\mathbf{V}) \cap L^\infty(0,T;\mathbf{H}).$$

Furthermore, the following estimates hold:

$$\begin{aligned} \|\mathbf{y}_{\alpha,t}\|_{L^1(\mathbf{V}')} + \|\mathbf{y}_\alpha\|_{L^2(\mathbf{V})} + \|\mathbf{y}_\alpha\|_{L^\infty(\mathbf{H})} &\le C(\|\mathbf{y}_0\|_2 + \|\mathbf{f}\|_{L^2(\mathbf{H}^{-1})}), \\ \|\mathbf{z}_\alpha\|^2_{L^\infty(\mathbf{H})} + 2\alpha^2\|\mathbf{z}_\alpha\|^2_{L^\infty(\mathbf{V})} &\le \|\mathbf{y}_\alpha\|^2_{L^\infty(\mathbf{H})}, \\ 2\alpha^2\|\nabla\mathbf{z}_\alpha\|^2_{L^\infty(\mathbf{H})} + \alpha^4\|\Delta\mathbf{z}_\alpha\|^2_{L^\infty(\mathbf{H})} &\le \|\mathbf{y}_\alpha\|^2_{L^\infty(\mathbf{H})}. \end{aligned} \tag{2.48}$$

In view of the estimates (2.48), there exists $\mathbf{y} \in L^2(0,T;\mathbf{V}))$ with $\mathbf{y}_t \in L^1(0,T;\mathbf{V}')$ such that, at least for a subsequence,

$$\begin{aligned} \mathbf{y}_\alpha \to \mathbf{y} \quad &\text{weakly in} \quad L^2(0,T;\mathbf{V})), \\ \mathbf{y}_{\alpha,t} \to \mathbf{y}_t \quad &\text{weakly-}\star \text{ in} \quad L^1(0,T;\mathbf{V}'). \end{aligned} \tag{2.49}$$

Thanks to the *Aubin-Lions' Lemma*, the Hilbert space

$$\mathbf{W} = \{ \mathbf{w} \in L^2(0,T;\mathbf{V}); \mathbf{w}_t \in L^1(0,T;\mathbf{V}') \}$$

is compactly embedded in $\mathbf{L}^2(Q)$ and we thus have

$$\mathbf{y}_\alpha \to \mathbf{y} \quad \text{strongly in} \quad \mathbf{L}^2(Q). \tag{2.50}$$

Also, using the second equation in (2.47) we see that

$$(\mathbf{z}_\alpha - \mathbf{y}) - \alpha^2 \Delta(\mathbf{z}_\alpha - \mathbf{y}) + \nabla\pi = (\mathbf{y}_\alpha - \mathbf{y}) + \alpha^2 \Delta \mathbf{y}.$$

Therefore, after some computations, we deduce that

$$\mathbf{z}_\alpha \to \mathbf{y} \quad \text{strongly in} \quad \mathbf{L}^2(Q). \tag{2.51}$$

This proves that we can find p such that $(\mathbf{y}, p)$ is solution to (2.46).

In other words, at least for a subsequence, the solutions to the Leray-α system converge (in the sense of (2.49)) towards a solution to the Navier-Stokes system.

Let us now consider the following controlled systems for the Navier-Stokes and Leray-α systems:

$$\begin{cases} \mathbf{y}_t - \Delta\mathbf{y} + (\mathbf{y}\cdot\nabla)\mathbf{y} + \nabla p = \mathbf{v}1_\omega & \text{in} \quad Q, \\ \nabla\cdot\mathbf{y} = 0 & \text{in} \quad Q, \\ \mathbf{y} = \mathbf{0} & \text{on} \quad \Sigma, \\ \mathbf{y}(0) = \mathbf{y}_0 & \text{in} \quad \Omega. \end{cases} \tag{2.52}$$

and

$$\begin{cases} \mathbf{y}_t - \Delta\mathbf{y} + (\mathbf{z}\cdot\nabla)\mathbf{y} + \nabla p = \mathbf{v}1_\omega & \text{in} \quad Q, \\ \nabla\cdot\mathbf{y} = \nabla\cdot\mathbf{z} = 0 & \text{in} \quad Q, \\ \mathbf{z} - \alpha^2\Delta\mathbf{z} + \nabla\pi = \mathbf{y} & \text{in} \quad Q, \\ \mathbf{y} = \mathbf{z} = \mathbf{0} & \text{on} \quad \Sigma, \\ \mathbf{y}(0) = \mathbf{y}_0 & \text{in} \quad \Omega. \end{cases} \tag{2.53}$$

where $\mathbf{v} = \mathbf{v}(\mathbf{x}, t)$ stands for the control function.

With arguments similar to those in [46], it can be proved that, for any $T > 0$, there exists $\varepsilon > 0$ such that, if $\|\mathbf{y}_0\| < \varepsilon$, for each $\alpha > 0$ we can find controls $\mathbf{v}_\alpha \in \mathbf{L}^2(\omega\times(0,T))$

and associate states $(\mathbf{y}_\alpha, p_\alpha, \mathbf{z}_\alpha, \pi_\alpha)$ satisfying

$$\mathbf{y}_\alpha(\mathbf{x}, T) = \mathbf{0} \quad \text{in} \quad \Omega.$$

In a forthcoming paper, we will show that these null controls $\mathbf{v}_\alpha$ can be bounded independently of α and a result similar to Theorem 2.3 holds for (2.53).

Chapter 3

Uniform local null control of the Leray-α model

Uniform local null control of the Leray-α model

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Abstract. This paper deals with the distributed and boundary controllability of the so called Leray-α model. This is a regularized variant of the Navier-Stokes system (α is a small positive parameter) that can also be viewed as a model for turbulent flows. We prove that the Leray-α equations are locally null controllable, with controls bounded independently of α. We also prove that, if the initial data are sufficiently small, the controls converge as $\alpha \to 0^+$ to a null control of the Navier-Stokes equations. We also discuss some other related questions, such as global null controllability, local and global exact controllability to the trajectories, etc.

3.1 Introduction. The main results

Let $\Omega \subset \mathbb{R}^N (N = 2, 3)$ be a bounded domain whose boundary Γ is of class C^2. Let $\omega \subset \Omega$ be a (small) nonempty open set, let $\gamma \subset \Gamma$ be a (small) nonempty open subset of Γ and assume that $T > 0$. We will use the notation $Q = \Omega \times (0, T)$ and $\Sigma = \Gamma \times (0, T)$ and we will denote by $\mathbf{n} = \mathbf{n}(\mathbf{x})$ the outward unit normal to Ω at the points $\mathbf{x} \in \Gamma$; spaces of $\mathbb{R}^N$-valued functions, as well as their elements, are represented by boldface letters.

The Navier-Stokes system for a homogeneous viscous incompressible fluid (with unit density and unit kinematic viscosity) subject to homogeneous Dirichlet boundary conditions is given by

$$\begin{cases} \mathbf{y}_t - \Delta \mathbf{y} + (\mathbf{y} \cdot \nabla)\mathbf{y} + \nabla p = \mathbf{f} & \text{in} \quad Q, \\ \nabla \cdot \mathbf{y} = 0 & \text{in} \quad Q, \\ \mathbf{y} = \mathbf{0} & \text{on} \quad \Sigma, \\ \mathbf{y}(0) = \mathbf{y}_0 & \text{in} \quad \Omega, \end{cases} \tag{3.1}$$

where $\mathbf{y}$ (the velocity field) and p (the pressure) are the unknowns, $\mathbf{f} = \mathbf{f}(\mathbf{x}, t)$ is a forcing term and $\mathbf{y}_0 = \mathbf{y}_0(\mathbf{x})$ is a prescribed initial velocity field.

In order to prove the existence of a solution to the Navier-Stokes system, Leray in [95] had the idea of creating a turbulence *closure* model without enhancing viscous dissipation. Thus, he introduced a "regularized" variant of (3.1) by modifying the nonlinear term as follows:

$$\begin{cases} \mathbf{y}_t - \Delta \mathbf{y} + (\mathbf{z} \cdot \nabla)\mathbf{y} + \nabla p = \mathbf{f} & \text{in} \quad Q, \\ \nabla \cdot \mathbf{y} = 0 & \text{in} \quad Q, \end{cases}$$

where $\mathbf{y}$ and $\mathbf{z}$ are related by

$$\mathbf{z} = \phi_\alpha * \mathbf{y} \tag{3.2}$$

and ϕ_α is a smoothing kernel. At least formally, the Navier-Stokes equations are recovered in the limit as $\alpha \to 0^+$, so that $\mathbf{z} \to \mathbf{y}$.

In this paper, we will consider a special smoothing kernel, associated to the Stokes-like operator $\mathbf{Id} + \alpha^2\mathbf{A}$, where $\mathbf{A}$ is the Stokes operator (see Section 3.2). This leads to the following modification of the Navier-Stokes equations, called the *Leray-α system* (see [19]) :

$$\begin{cases} \mathbf{y}_t - \Delta\mathbf{y} + (\mathbf{z}\cdot\nabla)\mathbf{y} + \nabla p = \mathbf{f} & \text{in} \quad Q, \\ \mathbf{z} - \alpha^2\Delta\mathbf{z} + \nabla\pi = \mathbf{y} & \text{in} \quad Q, \\ \nabla\cdot\mathbf{y} = 0, \ \nabla\cdot\mathbf{z} = 0 & \text{in} \quad Q, \\ \mathbf{y} = \mathbf{z} = \mathbf{0} & \text{on} \quad \Sigma, \\ \mathbf{y}(0) = \mathbf{y}_0 & \text{in} \quad \Omega. \end{cases} \tag{3.3}$$

In almost all previous works found in the literature, Ω is either the N-dimensional torus and the PDE's in (3.3) are completed with periodic boundary conditions or the whole space $\mathbb{R}^N$. Then, $\mathbf{z}$ satisfies an equation of the kind

$$\mathbf{z} - \alpha^2\Delta\mathbf{z} = \mathbf{y} \tag{3.4}$$

and the model is (apparently) slightly different from (3.3). However, since $\nabla\cdot\mathbf{y} = 0$, it is easy to see that (3.4), in these cases, is equivalent to the equation satisfied by $\mathbf{z}$ and π in (3.3).

It has been shown in [19] that, at least for periodic boundary conditions, the numerical solution of the equations in (3.3) matches successfully with empirical data from turbulent channel and pipe flows for a wide range of Reynolds numbers. Accordingly, the Leray-α system has become preferable to other turbulence models, since the associated computational cost is lower and no introduction of *ad hoc* parameters is required.

In [64], the authors have compared the numerical solutions of three different α-models useful in turbulence modeling (in terms of the Reynolds number associated to a Navier-Stokes velocity field). The results improve as one passes from the Navier-Stokes equations to these models and clearly show that the Leray-α system has the best performance. Therefore, it seems quite natural to carry out a theoretical analysis of the solutions to (3.3).

We will be concerned with the following controlled systems

$$\begin{cases} \mathbf{y}_t - \Delta\mathbf{y} + (\mathbf{z}\cdot\nabla)\mathbf{y} + \nabla p = \mathbf{v}1_\omega & \text{in} \quad Q, \\ \mathbf{z} - \alpha^2\Delta\mathbf{z} + \nabla\pi = \mathbf{y} & \text{in} \quad Q, \\ \nabla\cdot\mathbf{y} = 0, \ \nabla\cdot\mathbf{z} = 0 & \text{in} \quad Q, \\ \mathbf{y} = \mathbf{z} = \mathbf{0} & \text{on} \quad \Sigma, \\ \mathbf{y}(0) = \mathbf{y}_0 & \text{in} \quad \Omega \end{cases} \tag{3.5}$$

and

$$\begin{cases} \mathbf{y}_t - \Delta\mathbf{y} + (\mathbf{z}\cdot\nabla)\mathbf{y} + \nabla p = \mathbf{0} & \text{in} \quad Q, \\ \mathbf{z} - \alpha^2\Delta\mathbf{z} + \nabla\pi = \mathbf{y} & \text{in} \quad Q, \\ \nabla\cdot\mathbf{y} = 0, \ \nabla\cdot\mathbf{z} = 0 & \text{in} \quad Q, \\ \mathbf{y} = \mathbf{z} = \mathbf{h}1_\gamma & \text{on} \quad \Sigma, \\ \mathbf{y}(0) = \mathbf{y}_0 & \text{in} \quad \Omega, \end{cases} \tag{3.6}$$

where $\mathbf{v} = \mathbf{v}(\mathbf{x}, t)$ (respectively $\mathbf{h} = \mathbf{h}(\mathbf{x}, t)$) stands for the control, assumed to act only in the (small) set ω (respectively on γ) during the whole time interval $(0, T)$. The symbol 1_ω (respectively 1_γ) stands for the characteristic function of ω (respectively of γ).

In the applications, the *internal control* $\mathbf{v}1_\omega$ can be viewed as a gravitational or electromagnetic field. The *boundary control* $\mathbf{h}1_\gamma$ is the trace of the velocity field on Σ.

Remark 3.1. It is completely natural to suppose that $\mathbf{y}$ and $\mathbf{z}$ satisfy the same boundary conditions on Σ since, in the limit, we should have $\mathbf{z} = \mathbf{y}$. Consequently, we will assume that the boundary control $\mathbf{h}1_\gamma$ acts simultaneously on both variables $\mathbf{y}$ and $\mathbf{z}$. □

In what follows, $(\cdot,\cdot)$ and $\|\cdot\|$ denote the usual L^2 scalar products and norms (in $L^2(\Omega)$, $\mathbf{L}^2(\Omega)$, $L^2(Q)$, etc.) and K, C, C_1, C_2, ... denote various positive constants (usually depending on ω or γ, Ω and T). Let us recall the definitions of some usual spaces in the context of incompressible fluids :

$$\mathbf{H} = \{\, \mathbf{u} \in \mathbf{L}^2(\Omega) : \nabla\cdot\mathbf{u} = 0 \text{ in } \Omega, \ \mathbf{u}\cdot\mathbf{n} = 0 \text{ on } \Gamma \,\},$$
$$\mathbf{V} = \{\, \mathbf{u} \in \mathbf{H}_0^1(\Omega) : \nabla\cdot\mathbf{u} = 0 \text{ in } \Omega \,\}.$$

Note that, for every $\mathbf{y}_0 \in \mathbf{H}$ and every $\mathbf{v} \in \mathbf{L}^2(\omega \times (0, T))$, there exists a unique solution $(\mathbf{y}, p, \mathbf{z}, \pi)$ for (3.5) that satisfies (among other things)

$$\mathbf{y}, \mathbf{z} \in C^0([0, T]; \mathbf{H});$$

see Proposition 3.1 below. This is in contrast with the lack of uniqueness of the Navier-Stokes system when $N = 3$.

The main goals of this paper are to analyze the controllability properties of (3.5) and (3.6) and determine the way they depend on α as $\alpha \to 0^+$.

The null controllability problem for (3.5) at time $T > 0$ is the following :

For any $\mathbf{y}_0 \in \mathbf{H}$, find $\mathbf{v} \in \mathbf{L}^2(\omega \times (0, T))$ such that the corresponding state (the corresponding solution to (3.5)) satisfies

$$\mathbf{y}(T) = \mathbf{0} \quad \textit{in} \quad \Omega. \tag{3.7}$$

The null controllability problem for (3.6) at time $T > 0$ is the following :

For any $\mathbf{y}_0 \in \mathbf{H}$, find $\mathbf{h} \in \mathbf{L}^2(0, T; \mathbf{H}^{-1/2}(\gamma))$ with $\int_\gamma \mathbf{h}\cdot\mathbf{n}\, d\Gamma = 0$ and an *associated state (the corresponding solution to* (3.6)*) satisfying*

$$\mathbf{y}, \mathbf{z} \in C^0([0,T]; \mathbf{L}^2(\Omega))$$

and (3.7).

Recall that, in the context of the Navier-Stokes equations, J.-L. Lions conjectured in [97] the global distributed and boundary approximate controllability; since then, the controllability of these equations has been intensively studied, but for the moment only partial results are known.

Thus, the global approximate controllability of the two-dimensional Navier-Stokes equations with Navier slip boundary conditions was obtained by Coron in [28]. Also, by combining results concerning global and local controllability, the global null controllability for the Navier-Stokes system on a two-dimensional manifold without boundary was established in Coron and Fursikov [31]; see also Guerrero *et al.* [75] for another global controllability result.

The local exact controllability to bounded trajectories has been obtained by Fursikov and Imanuvilov [62, 60], Imanuvilov [81] and Fernández-Cara *et al.* [46] under various circumstances; see Guerrero [73] and González-Burgos *et al.* [72] for similar results related to the Boussinesq system. Let us also mention [12, 32, 33, 47], where analogous results are obtained with a reduced number of scalar controls.

For the (simplified) one-dimensional viscous Burgers model, positive and negative results can be found in [45, 68, 74]; see also [41], where the authors consider the one-dimensional compressible Navier-Stokes system.

Our first main result in this paper is the following :

Theorem 3.1. *There exists* $\epsilon > 0$*(independent of* α*) such that, for each* $\mathbf{y}_0 \in \mathbf{H}$ *with* $\|\mathbf{y}_0\| \leq \epsilon$*, there exist controls* $\mathbf{v}_\alpha \in L^\infty(0,T;\mathbf{L}^2(\omega))$ *such that the associated solutions to* (3.5) *fulfill* (3.7)*. Furthermore, these controls can be found satisfying the estimate*

$$\|\mathbf{v}_\alpha\|_{L^\infty(\mathbf{L}^2)} \leq C, \tag{3.8}$$

where C *is also independent of* α*.*

Our second main result is the analog of Theorem 3.1 in the framework of boundary controllability. It is the following :

Theorem 3.2. *There exists* $\delta > 0$*(independent of* α*) such that, for each* $\mathbf{y}_0 \in \mathbf{H}$ *with* $\|\mathbf{y}_0\| \leq \delta$*, there exist controls* $\mathbf{h}_\alpha \in L^\infty(0,T;\mathbf{H}^{-1/2}(\gamma))$ *with* $\int_\gamma \mathbf{h}_\alpha \cdot \mathbf{n}\, d\Gamma = 0$ *and associated solutions to* (3.6) *that fulfill* (3.7)*. Furthermore, these controls can be found satisfying the estimate*

$$\|\mathbf{h}_\alpha\|_{L^\infty(H^{-1/2})} \leq C, \tag{3.9}$$

where C *is also independent of* α*.*

The proofs rely on suitable fixed-point arguments. The underlying idea has applied to many other nonlinear control problems. However, in the present cases, we find two specific difficulties :

- In order to find spaces and fixed-point mappings appropriate for *Schauder's fixed point Theorem*, the initial state $\mathbf{y}_0$ must be regular enough. Consequently, we have to establish *regularizing properties* for (3.5) and (3.6); see Lemmas 3.1 and 3.4 below.
- For the proof of the uniform estimates (3.8) and (3.9), careful estimates of the null controls and associated states of some particular linear problems are needed.

We will also prove results concerning the controllability in the limit, as $\alpha \to 0^+$. It will be shown that the null-controls for (3.5) can be chosen in such a way that they converge to null-controls for the Navier-Stokes system

$$\begin{cases} \mathbf{y}_t - \Delta \mathbf{y} + (\mathbf{y}\cdot\nabla)\mathbf{y} + \nabla p = \mathbf{v}1_\omega & \text{in} \quad Q, \\ \nabla\cdot\mathbf{y} = 0 & \text{in} \quad Q, \\ \mathbf{y} = \mathbf{0} & \text{on} \quad \Sigma, \\ \mathbf{y}(0) = \mathbf{y}_0 & \text{in} \quad \Omega. \end{cases} \tag{3.10}$$

Also, it will be seen that the null-controls for (3.6) can be chosen such that they converge to boundary null-controls for the Navier-Stokes system

$$\begin{cases} \mathbf{y}_t - \Delta \mathbf{y} + (\mathbf{y}\cdot\nabla)\mathbf{y} + \nabla p = \mathbf{0} & \text{in} \quad Q, \\ \nabla\cdot\mathbf{y} = 0 & \text{in} \quad Q, \\ \mathbf{y} = \mathbf{h}1_\gamma & \text{on} \quad \Sigma, \\ \mathbf{y}(0) = \mathbf{y}_0 & \text{in} \quad \Omega. \end{cases} \tag{3.11}$$

More precisely, our third and fourth main results are the following :

Theorem 3.3. *Let $\epsilon > 0$ be furnished by Theorem 3.1. Assume that $\mathbf{y}_0 \in \mathbf{H}$ and $\|\mathbf{y}_0\| \le \epsilon$, let $\mathbf{v}_\alpha$ be a null control for (3.5) satisfying (3.8) and let $(\mathbf{y}_\alpha, p_\alpha, \mathbf{z}_\alpha, \pi_\alpha)$ be the associated state. Then, at least for a subsequence, one has*

$$\begin{array}{c} \mathbf{v}_\alpha \to \mathbf{v} \quad \textit{weakly-}\star \textit{ in} \quad L^\infty(0,T;\mathbf{L}^2(\omega)), \\ \mathbf{z}_\alpha \to \mathbf{y} \quad \textit{and} \quad \mathbf{y}_\alpha \to \mathbf{y} \quad \textit{strongly in} \quad \mathbf{L}^2(Q), \end{array}$$

as $\alpha \to 0^+$, where $(\mathbf{y}, \mathbf{v})$ is, together with some p, a state-control pair for (3.10) satisfying (3.7).

Theorem 3.4. *Let $\delta > 0$ be furnished by Theorem 3.2. Assume that $\mathbf{y}_0 \in \mathbf{H}$ and $\|\mathbf{y}_0\| \le \delta$, let $\mathbf{h}_\alpha$ be a null control for (3.6) satisfying (3.9) and let $(\mathbf{y}_\alpha, p_\alpha, \mathbf{z}_\alpha, \pi_\alpha)$ be the associated state. Then, at least for a subsequence, one has*

$$\begin{array}{c} \mathbf{h}_\alpha \to \mathbf{h} \quad \textit{weakly-}\star \textit{ in} \quad L^\infty(0,T;H^{-1/2}(\gamma)), \\ \mathbf{z}_\alpha \to \mathbf{y} \quad \textit{and} \quad \mathbf{y}_\alpha \to \mathbf{y} \quad \textit{strongly in} \quad \mathbf{L}^2(Q), \end{array}$$

as $\alpha \to 0^+$*, where* $(\mathbf{y}, \mathbf{h})$ *is, together with some* p*, a state-control pair for* (3.11) *satisfying* (3.7).

The rest of this paper is organized as follows. In Section 3.2, we will recall some properties of the Stokes operator and we will prove some results concerning the existence, uniqueness and regularity of the solution to (3.3). Section 3.3 deals with the proofs of Theorems 3.1 and 3.3. Section 3.4 deals with the proofs of Theorems 3.2 and 3.4. Finally, in Section 3.5, we present some additional comments and open questions.

3.2 Preliminaries

In this Section, we will recall some properties of the Stokes operator. Then, we will prove that the Leray-α system is well-posed. Also, we will recall the Carleman inequalities and null controllability properties of the Oseen system.

3.2.1 The Stokes operator

Let $\mathbf{P} : \mathbf{L}^2(\Omega) \mapsto \mathbf{H}$ be the orthogonal projector, usually known as the *Leray Projector*. Recall that $\mathbf{P}$ maps $\mathbf{H}^s(\Omega)$ into $\mathbf{H}^s(\Omega) \cap \mathbf{H}$ for all $s \geq 0$.

We will denote by $\mathbf{A}$ the *Stokes operator*, i.e. the self-adjoint operator in $\mathbf{H}$ formally given by $\mathbf{A} = -\mathbf{P}\Delta$. For any $\mathbf{u} \in D(\mathbf{A}) := \mathbf{V} \cap \mathbf{H}^2(\Omega)$ and any $\mathbf{w} \in \mathbf{H}$, the identity $\mathbf{A}\mathbf{u} = \mathbf{w}$ holds if and only if

$$(\nabla \mathbf{u}, \nabla \mathbf{v}) = (\mathbf{w}, \mathbf{v}), \quad \forall\, \mathbf{v} \in \mathbf{V}.$$

It is well known that $\mathbf{A} : D(\mathbf{A}) \mapsto \mathbf{H}$ can be inverted and its inverse $\mathbf{A}^{-1}$ is self-adjoint, compact and positive. Consequently, there exists a nondecreasing sequence of positive numbers λ_j and an associated orthonormal basis of $\mathbf{H}$, denoted by $(\mathbf{w}_j)_{j=1}^{+\infty}$, such that

$$\mathbf{A}\mathbf{w}_j = \lambda_j \mathbf{w}_j, \quad \forall j \geq 1.$$

Accordingly we can introduce the real powers of the Stokes operator. Thus, for any $r \in \mathbb{R}$, we set

$$D(\mathbf{A}^r) = \left\{ \mathbf{u} \in \mathbf{H} : \mathbf{u} = \sum_{j=1}^{+\infty} u_j \mathbf{w}_j, \text{ with } \sum_{j=1}^{+\infty} \lambda_j^{2r} |u_j|^2 < +\infty \right\}$$

and

$$\mathbf{A}^r \mathbf{u} = \sum_{j=1}^{+\infty} \lambda_j^r u_j \mathbf{w}_j, \quad \forall\, \mathbf{u} = \sum_{j=1}^{+\infty} u_j \mathbf{w}_j \in D(\mathbf{A}^r).$$

Let us present a result concerning the domains of the powers of the Stokes operator.

Theorem 3.5. *Let $r \in \mathbb{R}$ be given, with $-\frac{1}{2} < r < 1$. Then*

$$D(\mathbf{A}^{r/2}) = \mathbf{H}^r(\Omega) \cap \mathbf{H}, \quad \textit{whenever} \quad -\frac{1}{2} < r < \frac{1}{2},$$
$$D(\mathbf{A}^{r/2}) = \mathbf{H}_0^r(\Omega) \cap \mathbf{H}, \quad \textit{whenever} \quad \frac{1}{2} \leq r \leq 1.$$

Moreover, $\mathbf{u} \mapsto (\mathbf{u}, \mathbf{A}^r\mathbf{u})^{1/2}$ is a Hilbertian norm in $D(\mathbf{A}^{r/2})$, equivalent to the usual Sobolev $\mathbf{H}^r$-norm. In other words, there exist constants $c_1(r), c_2(r) > 0$ such that

$$c_1(r)\|\mathbf{u}\|_{\mathbf{H}^r} \leq (\mathbf{u}, \mathbf{A}^r\mathbf{u})^{1/2} \leq c_2(r)\|\mathbf{u}\|_{\mathbf{H}^r}, \quad \forall \mathbf{u} \in D(\mathbf{A}^{r/2}).$$

The proof of Theorem 3.5 can be found in [59]. Notice that, in view of the interpolation K-method of Lions and Peetre, we have $D(\mathbf{A}^{r/2}) = D((-\Delta)^{r/2}) \cap \mathbf{H}$. Hence, thanks to an explicit description of $D((-\Delta)^{r/2})$, the stated result holds.

Now, we are going to recall an important property of the semigroup of contractions $e^{-t\mathbf{A}}$ generated by $\mathbf{A}$, see [58] :

Theorem 3.6. *For any $r > 0$, there exists $C(r) > 0$ such that*

$$\|\mathbf{A}^r e^{-t\mathbf{A}}\|_{\mathcal{L}(\mathbf{H};\mathbf{H})} \leq C(r)\, t^{-r}, \quad \forall t > 0. \tag{3.12}$$

In order to prove (3.12), it suffices to observe that, for any $\mathbf{u} = \sum_{j=1}^{+\infty} u_j \mathbf{w}_j \in \mathbf{H}$, one has

$$\mathbf{A}^r e^{-t\mathbf{A}}\mathbf{u} = \sum_{j=1}^{+\infty} \lambda_j^r e^{-t\lambda_j} u_j \mathbf{w}_j.$$

Consequently,

$$\|\mathbf{A}^r e^{-t\mathbf{A}}\mathbf{u}\|^2 = \sum_{j=1}^{+\infty} \left|\lambda_j^r e^{-t\lambda_j} u_j\right|^2 \leq \left(\max_{\lambda \in \mathbb{R}} \lambda^r e^{-t\lambda}\right)^2 \|\mathbf{u}\|^2$$

and, since $\max\limits_{\lambda \in \mathbb{R}} \lambda^r e^{-t\lambda} = (r/e)^r\, t^{-r}$, we get easily (3.12).

3.2.2 Well-posedness for the Leray-α system

Let us see that, for any $\alpha > 0$, under some reasonable conditions on $\mathbf{f}$ and $\mathbf{y}_0$, the Leray-α system (3.3) possesses a unique global weak solution. Before this, let us introduce σ_N given by

$$\sigma_N = \begin{cases} 2 & \text{if } N = 2, \\ 4/3 & \text{if } N = 3. \end{cases}$$

Then, we have the following result :

Proposition 3.1. *Assume that $\alpha > 0$ is fixed. Then, for any $\mathbf{f} \in L^2(0,T;\mathbf{H}^{-1}(\Omega))$ and any $\mathbf{y}_0 \in \mathbf{H}$, there exists exactly one solution $(\mathbf{y}_\alpha, p_\alpha, \mathbf{z}_\alpha, \pi_\alpha)$ to (3.3), with*

$$\begin{aligned} \mathbf{y}_\alpha \in L^2(0,T;\mathbf{V}) \cap C^0([0,T];\mathbf{H}),\ \mathbf{y}_{\alpha,t} \in L^2(0,T;\mathbf{V}'), \\ \mathbf{z}_\alpha \in L^2(0,T;D(\mathbf{A}^{3/2})) \cap C^0([0,T];D(\mathbf{A})). \end{aligned} \tag{3.13}$$

Furthermore, the following estimates hold:

$$\begin{aligned} \|\mathbf{y}_\alpha\|_{L^2(\mathbf{V})} + \|\mathbf{y}_\alpha\|_{C^0([0,T];\mathbf{H})} &\le CB_0(\mathbf{y}_0,\mathbf{f}), \\ \|\mathbf{y}_{\alpha,t}\|_{L^{\sigma_N}(\mathbf{V}')} &\le CB_0(\mathbf{y}_0,\mathbf{f})(1+B_0(\mathbf{y}_0,\mathbf{f})), \\ \|\mathbf{z}_\alpha\|^2_{C^0([0,T];\mathbf{H})} + 2\alpha^2\|\mathbf{z}_\alpha\|^2_{C^0([0,T];\mathbf{V})} &\le CB_0(\mathbf{y}_0,\mathbf{f})^2, \\ 2\alpha^2\|\mathbf{z}_\alpha\|^2_{C^0([0,T];\mathbf{V})} + \alpha^4\|\mathbf{z}_\alpha\|^2_{C^0([0,T];D(\mathbf{A}))} &\le CB_0(\mathbf{y}_0,\mathbf{f})^2. \end{aligned} \tag{3.14}$$

Here, C is independent of α and we have introduced the notation

$$B_0(\mathbf{y}_0,\mathbf{f}) := \|\mathbf{y}_0\| + \|\mathbf{f}\|_{L^2(\mathbf{H}^{-1})}.$$

Proof. The proof follows classical and rather well known arguments; see for instance [35, 119]. For completeness, they will be recalled.

• EXISTENCE : We will reduce the proof to the search of a fixed point of an appropriate mapping Λ_α. [1]

Thus, for each $\overline{\mathbf{y}} \in L^2(0,T;\mathbf{H})$, let $(\mathbf{z},\pi)$ be the unique solution to

$$\begin{cases} \mathbf{z} - \alpha^2\Delta\mathbf{z} + \nabla\pi = \overline{\mathbf{y}} & \text{in} \quad Q, \\ \nabla\cdot\mathbf{z} = 0 & \text{in} \quad Q, \\ \mathbf{z} = \mathbf{0} & \text{on} \quad \Sigma. \end{cases}$$

It is clear that $\mathbf{z} \in L^2(0,T;D(\mathbf{A}))$ and then, thanks to the Sobolev embedding, we have $\mathbf{z} \in L^2(0,T;\mathbf{L}^\infty(\Omega))$. Moreover, the following estimates are satisfied :

$$\begin{aligned} \|\mathbf{z}\|^2 + 2\alpha^2\|\mathbf{z}\|^2_{L^2(\mathbf{V})} &\le \|\overline{\mathbf{y}}\|^2, \\ 2\alpha^2\|\mathbf{z}\|^2_{L^2(\mathbf{V})} + \alpha^4\|\mathbf{z}\|^2_{L^2(D(\mathbf{A}))} &\le \|\overline{\mathbf{y}}\|^2. \end{aligned}$$

From this $\mathbf{z}$, we can obtain the unique solution $(\mathbf{y},p)$ to the linear system of the Oseen kind

$$\begin{cases} \mathbf{y}_t - \Delta\mathbf{y} + (\mathbf{z}\cdot\nabla)\mathbf{y} + \nabla p = \mathbf{f} & \text{in} \quad Q, \\ \nabla\cdot\mathbf{y} = 0 & \text{in} \quad Q, \\ \mathbf{y} = \mathbf{0} & \text{on} \quad \Sigma, \\ \mathbf{y}(0) = \mathbf{y}_0 & \text{in} \quad \Omega. \end{cases}$$

[1] Alternatively, we can prove the existence of a solution by introducing adequate Galerkin approximations and applying (classical) compactness arguments.

Since $\mathbf{f} \in L^2(0,T;\mathbf{H}^{-1}(\Omega))$ and $\mathbf{y}_0 \in \mathbf{H}$, it is clear that

$$\mathbf{y} \in L^2(0,T;\mathbf{V}) \cap C^0([0,T];\mathbf{H}), \quad \mathbf{y}_t \in L^2(0,T;\mathbf{V}')$$

and the following estimates hold:

$$\begin{aligned} \|\mathbf{y}\|_{C^0([0,T];\mathbf{H})} + \|\mathbf{y}\|_{L^2(\mathbf{V})} \leq C_1 B_0(\mathbf{y}_0,\mathbf{f}), \\ \|\mathbf{y}_t\|_{L^2(\mathbf{V}')} \leq C_2(1+\|\mathbf{z}\|_{L^2(D(\mathbf{A}))})B_0(\mathbf{y}_0,\mathbf{f}) \leq C_2(1+\alpha^{-2}\|\overline{\mathbf{y}}\|)B_0(\mathbf{y}_0,\mathbf{f}). \end{aligned} \tag{3.15}$$

Now, we introduce the Banach space

$$\mathbf{W} = \{\mathbf{w} \in L^2(0,T;\mathbf{V}) : \mathbf{w}_t \in L^2(0,T;\mathbf{V}')\},$$

the closed ball

$$\mathbf{K} = \{\, \overline{\mathbf{y}} \in L^2(0,T;\mathbf{H}) : \|\overline{\mathbf{y}}\| \leq C_1\sqrt{T}B_0(\mathbf{y}_0,\mathbf{f}) \,\}$$

and the mapping $\widetilde{\Lambda}_\alpha$, with $\widetilde{\Lambda}_\alpha(\overline{\mathbf{y}}) = \mathbf{y}$, for all $\overline{\mathbf{y}} \in L^2(0,T;\mathbf{H})$. Obviously $\widetilde{\Lambda}_\alpha$ is well defined and maps continuously the whole space $L^2(0,T;\mathbf{H})$ into $\mathbf{W} \cap \mathbf{K}$.

Notice that any bounded set of $\mathbf{W}$ is relatively compact in the space $L^2(0,T;\mathbf{H})$, in view of the classical results of the *Aubin-Lions kind*, see for instance [117].

Let us denote by Λ_α the restriction to $\mathbf{K}$ of $\widetilde{\Lambda}_\alpha$. Then, thanks to (3.15), Λ_α maps $\mathbf{K}$ into itself. Moreover, it is clear that $\Lambda_\alpha : \mathbf{K} \mapsto \mathbf{K}$ satisfies the hypotheses of *Schauder's fixed point Theorem*. Consequently, Λ_α possesses at least one fixed point in $\mathbf{K}$.

This immediately achieves the proof of the existence of a solution satisfying (3.13).

The estimates $(3.14)_a$, $(3.14)_c$ and $(3.14)_d$ are obvious. On the other hand,

$$\begin{aligned} \|\mathbf{y}_{\alpha,t}\|_{L^{\sigma_N}(\mathbf{V}')} &\leq C\left(\|\mathbf{f}\|_{L^2(\mathbf{H}^{-1})} + \|\mathbf{y}_\alpha\|_{L^2(\mathbf{V})} + \|(\mathbf{z}_\alpha\cdot\nabla)\mathbf{y}_\alpha\|_{L^{\sigma_N}(\mathbf{H}^{-1})}\right) \\ &\leq C\left(B_0(\mathbf{y}_0,\mathbf{f}) + \|\mathbf{z}_\alpha\|_{L^{s_N}(\mathbf{L}^4)}\|\mathbf{y}_\alpha\|_{L^{s_N}(\mathbf{L}^4)}\right) \\ &\leq C\left[B_0(\mathbf{y}_0,\mathbf{f}) + \left(\|\mathbf{z}_\alpha\|_{L^\infty(\mathbf{H})} + \|\mathbf{z}_\alpha\|_{L^2(\mathbf{V})}\right)\left(\|\mathbf{y}_\alpha\|_{L^\infty(\mathbf{H})} + \|\mathbf{y}_\alpha\|_{L^2(\mathbf{V})}\right)\right] \\ &\leq CB_0(\mathbf{y}_0,\mathbf{f})(1+B_0(\mathbf{y}_0,\mathbf{f})), \end{aligned}$$

where $s_N = 2\,\sigma_N$. Here, the third inequality is a consequence of the continuous embedding

$$L^\infty(0,T;\mathbf{H}) \cap L^2(0,T;\mathbf{V}) \hookrightarrow L^{s_N}(0,T;\mathbf{L}^4(\Omega)).$$

This estimate completes the proof of (3.14).

- UNIQUENESS: Let $(\mathbf{y}_\alpha, p_\alpha, \mathbf{z}_\alpha, \pi_\alpha)$ and $(\mathbf{y}'_\alpha, {p'}_\alpha, \mathbf{z}'_\alpha, {\pi'}_\alpha)$ be two solutions to (3.3) and let us introduce $\mathbf{u} := \mathbf{y}_\alpha - \mathbf{y}'_\alpha$, $q = p_\alpha - p'_\alpha$, $\mathbf{m} := \mathbf{z}_\alpha - \mathbf{z}'_\alpha$ and $h = \pi_\alpha - \pi'_\alpha$. Then

$$\begin{cases} \mathbf{u}_t - \Delta \mathbf{u} + (\mathbf{z}_\alpha \cdot \nabla)\mathbf{u} + \nabla q = -(\mathbf{m} \cdot \nabla)\mathbf{y}'_\alpha & \text{in} \quad Q, \\ \mathbf{m} - \alpha^2 \Delta \mathbf{m} + \nabla h = \mathbf{u} & \text{in} \quad Q, \\ \nabla \cdot \mathbf{u} = 0, \ \nabla \cdot \mathbf{m} = 0 & \text{in} \quad Q, \\ \mathbf{u} = \mathbf{m} = \mathbf{0} & \text{on} \quad \Sigma, \\ \mathbf{u}(0) = \mathbf{0} & \text{in} \quad \Omega. \end{cases}$$

Since $\mathbf{u} \in L^\infty(0,T;\mathbf{H})$, we have $\mathbf{m} \in L^\infty(0,T;D(\mathbf{A}))$ (where the estimate of this norm depends on α). Therefore, we easily deduce from the first equation of the previous system that

$$\frac{1}{2}\frac{\partial}{\partial t}\|\mathbf{u}\|^2 + \|\nabla \mathbf{u}\|^2 \le \|\mathbf{m}\|_\infty \|\nabla \mathbf{y}'_\alpha\| \|\mathbf{u}\|$$

for all t. Since $\|\mathbf{m}\|_\infty \le C\|\mathbf{m}\|_{D(\mathbf{A})} \le C\alpha^{-2}\|\mathbf{u}\|$, we get

$$\frac{1}{2}\frac{\partial}{\partial t}\|\mathbf{u}\|^2 + \|\nabla \mathbf{u}\|^2 \le C\alpha^{-2}\|\nabla \mathbf{y}'_\alpha\| \|\mathbf{u}\|^2.$$

Therefore, in view of *Gronwall's Lemma*, we see that $\mathbf{u} \equiv \mathbf{0}$. Accordingly, we also have $\mathbf{m} \equiv \mathbf{0}$ and uniqueness holds. □

We are now going to present some results concerning the existence and uniqueness of a strong solution. We start with a global result in the two-dimensional case.

Proposition 3.2. *Assume that $N = 2$ and $\alpha > 0$ is fixed. Then, for any $\mathbf{f} \in L^2(0,T;\mathbf{L}^2(\Omega))$ and any $\mathbf{y}_0 \in \mathbf{V}$, there exists exactly one solution $(\mathbf{y}_\alpha, p_\alpha, \mathbf{z}_\alpha, \pi_\alpha)$ to (3.3), with*

$$\begin{gathered} \mathbf{y}_\alpha \in L^2(0,T;D(\mathbf{A})) \cap C^0([0,T];\mathbf{V}), \ \mathbf{y}_{\alpha,t} \in L^2(0,T;\mathbf{H}), \\ \mathbf{z}_\alpha \in L^2(0,T;D(\mathbf{A}^2)) \cap C^0([0,T];D(\mathbf{A}^{3/2})). \end{gathered} \tag{3.16}$$

Furthermore, the following estimates hold :

$$\begin{gathered} \|\mathbf{y}_{\alpha,t}\| + \|\mathbf{y}_\alpha\|_{C^0([0,T];\mathbf{V})} + \|\mathbf{y}_\alpha\|_{L^2(D(\mathbf{A}))} \le B_1(\|\mathbf{y}_0\|_\mathbf{V}, \|\mathbf{f}\|), \\ \|\mathbf{z}_\alpha\|^2_{C^0([0,T];\mathbf{V})} + 2\alpha^2\|\mathbf{z}_\alpha\|^2_{C^0([0,T];D(\mathbf{A}))} \le \|\mathbf{y}_\alpha\|^2_{C^0([0,T];\mathbf{V})}, \end{gathered} \tag{3.17}$$

where we have introduced the notation

$$B_1(r,s) := (r+s)\left[1 + (r+s)^2\right] e^{C(r^2+s^2)^2}.$$

Proof. First, thanks to Proposition 3.1, we see that there exists a unique weak solution $(\mathbf{y}_\alpha, p_\alpha, \mathbf{z}_\alpha, \pi_\alpha)$ satisfying (3.13)–(3.14). In particular, $\mathbf{z}_\alpha \in L^2(0,T;\mathbf{V})$ and we have

$$\|\mathbf{z}_\alpha(t)\| \le \|\mathbf{y}_\alpha(t)\| \quad \text{and} \quad \|\mathbf{z}_\alpha(t)\|_\mathbf{V} \le \|\mathbf{y}_\alpha(t)\|_\mathbf{V}, \quad \forall t \in [0,T].$$

As usual, we will just check that good estimates can be obtained for $\mathbf{y}_\alpha$, $\mathbf{y}_{\alpha,t}$ and $\mathbf{z}_\alpha$. Thus, we assume that it is possible to multiply by $-\Delta\mathbf{y}_\alpha$ the motion equation satisfied by $\mathbf{y}_\alpha$. Taking into account that $N = 2$, we obtain :

$$\begin{aligned}\frac{1}{2}\frac{\partial}{\partial t}\|\nabla\mathbf{y}_\alpha\|^2 + \|\Delta\mathbf{y}_\alpha\|^2 &= -(\mathbf{f}, \Delta\mathbf{y}_\alpha) + ((\mathbf{z}_\alpha\cdot\nabla)\mathbf{y}_\alpha, \Delta\mathbf{y}_\alpha)\\ &\le \|\mathbf{f}\|^2 + \frac{1}{4}\|\Delta\mathbf{y}_\alpha\|^2 + \|\mathbf{z}_\alpha\|^{1/2}\|\mathbf{z}_\alpha\|_{\mathbf{V}}^{1/2}\|\mathbf{y}_\alpha\|_{\mathbf{V}}^{1/2}\|\Delta\mathbf{y}_\alpha\|^{3/2}\\ &\le \|\mathbf{f}\|^2 + \frac{1}{2}\|\Delta\mathbf{y}_\alpha\|^2 + C\|\mathbf{z}_\alpha\|^2\|\mathbf{z}_\alpha\|_{\mathbf{V}}^2\|\mathbf{y}_\alpha\|_{\mathbf{V}}^2.\end{aligned}$$

Therefore,

$$\frac{\partial}{\partial t}\|\nabla\mathbf{y}_\alpha\|^2 + \|\Delta\mathbf{y}_\alpha\|^2 \le C\left[\|\mathbf{f}\|^2 + \|\mathbf{y}_\alpha\|^2\|\mathbf{y}_\alpha\|_{\mathbf{V}}^2\|\nabla\mathbf{y}_\alpha\|^2\right].$$

In view of Gronwall's Lemma and the estimates in Proposition 3.1, we easily deduce (3.16) and (3.17). □

Notice that, in this two-dimensional case, the strong estimates for $\mathbf{y}_\alpha$ in (3.17) are independent of α; obviously, we cannot expect the same when $N = 3$.

In the three-dimensional case, what we obtain is the following :

Proposition 3.3. *Assume that $N = 3$ and $\alpha > 0$ is fixed. Then, for any $\mathbf{f} \in L^2(0,T;\mathbf{L}^2(\Omega))$ and any $\mathbf{y}_0 \in \mathbf{V}$, there exists exactly one solution $(\mathbf{y}_\alpha, p_\alpha, \mathbf{z}_\alpha, \pi_\alpha)$ to (3.3), with*

$$\begin{gathered}\mathbf{y}_\alpha \in L^2(0,T;D(\mathbf{A})) \cap C^0([0,T];\mathbf{V}),\ \ \mathbf{y}_{\alpha,t} \in L^2(0,T;\mathbf{H}),\\ \mathbf{z}_\alpha \in L^2(0,T;D(\mathbf{A}^2)) \cap C^0([0,T];D(\mathbf{A}^{3/2})).\end{gathered}$$

Furthermore, the following estimates hold :

$$\begin{gathered}\|\mathbf{y}_\alpha\|_{C^0([0,T];\mathbf{V})} + \|\mathbf{y}_\alpha\|_{L^2(D(\mathbf{A}))} + \|\mathbf{y}_{\alpha,t}\| \le B_2(\|\mathbf{y}_0\|_{\mathbf{V}}, \|\mathbf{f}\|, \alpha),\\ \|\mathbf{z}_\alpha\|^2_{C^0([0,T];\mathbf{V})} + 2\alpha^2\|\mathbf{z}_\alpha\|^2_{C^0([0,T];D(\mathbf{A}))} \le \|\mathbf{y}_\alpha\|^2_{C^0([0,T];\mathbf{V})},\end{gathered} \tag{3.18}$$

where we have introduced

$$B_2(r,s,\alpha) := C(r+s)e^{C\alpha^{-4}(r+s)^2}.$$

Proof. Thanks to Proposition 3.1, there exists a unique weak solution $(\mathbf{y}_\alpha, p_\alpha, \mathbf{z}_\alpha, \pi_\alpha)$ satisfying (3.13) and (3.14).

In particular, we obtain that $\mathbf{z}_\alpha \in \mathbf{L}^\infty(Q)$, with

$$\|\mathbf{z}_\alpha\|_\infty \le \frac{C}{\alpha^2}\left(\|\mathbf{y}_0\|_{\mathbf{H}} + \|\mathbf{f}\|_{L^2(\mathbf{H}^{-1})}\right).$$

On the other hand, $\mathbf{y}_0 \in \mathbf{V}$. Hence, from the usual (parabolic) regularity results for Oseen systems, the solution to (3.3) is more regular, i.e. $\mathbf{y}_\alpha \in L^2(0,T;D(\mathbf{A})) \cap C^0([0,T];\mathbf{V})$ and $\mathbf{y}_{\alpha,t} \in L^2(0,T;\mathbf{H})$. Moreover, $\mathbf{y}_\alpha$ verifies the first estimate in (3.18). This achieves the proof. □

Let us now provide a result concerning three-dimensional strong solutions corresponding to small data, with estimates independent of α :

Proposition 3.4. *Assume that $N = 3$. There exists $C_0 > 0$ such that, for any $\alpha > 0$, any $\mathbf{f} \in L^\infty(0,T;\mathbf{L}^2(\Omega))$ and any $\mathbf{y}_0 \in \mathbf{V}$ with*

$$M := \max\left\{\|\nabla \mathbf{y}_0\|^2,\ \|\mathbf{f}\|^{2/3}_{L^\infty(\mathbf{L}^2)}\right\} < \frac{1}{\sqrt{2(1+C_0)T}}, \tag{3.19}$$

the Leray-α system (3.3) possesses a unique solution $(\mathbf{y}_\alpha, p_\alpha, \mathbf{z}_\alpha, \pi_\alpha)$ satisfying

$$\begin{gathered}\mathbf{y}_\alpha \in L^2(0,T;D(\mathbf{A})) \cap C^0([0,T];\mathbf{V}),\ \mathbf{y}_{\alpha,t} \in L^2(0,T;\mathbf{H}),\\ \mathbf{z}_\alpha \in L^2(0,T;D(\mathbf{A})) \cap C^0([0,T];\mathbf{V}).\end{gathered}$$

Furthermore, in that case, the following estimates hold :

$$\begin{aligned}\|\mathbf{y}_\alpha\|^2_{C^0([0,T];\mathbf{V})} + \|\mathbf{y}_\alpha\|^2_{L^2(D(\mathbf{A}))} &\le B_3(M,T),\\ \|\mathbf{z}_\alpha\|^2_{C^0([0,T];\mathbf{V})} + 2\alpha^2\|\mathbf{z}_\alpha\|^2_{L^2(D(\mathbf{A}))} &\le \|\mathbf{y}_\alpha\|^2_{L^\infty(\mathbf{V})},\end{aligned} \tag{3.20}$$

where we have introduced

$$B_3(M,T) := 2\left[M^3 + M + C_0T\left(\frac{M}{\sqrt{1-2(1+C_0)M^2T}}\right)^3\right].$$

Proof. The proof is very similar to the proof of the existence of a local in time strong solution to the Navier-Stokes system; see for instance [25, 119].

As before, there exists a unique weak solution $(\mathbf{y}_\alpha, p_\alpha, \mathbf{z}_\alpha, \pi_\alpha)$ and this solution satisfies (3.13) and (3.14).

By multiplying by $\Delta\mathbf{y}_\alpha$ the motion equation satisfied by $\mathbf{y}_\alpha$, we see that

$$\begin{aligned}\frac{1}{2}\frac{\partial}{\partial t}\|\nabla\mathbf{y}_\alpha\|^2 + \|\Delta\mathbf{y}_\alpha\|^2 &= (\mathbf{f},\Delta\mathbf{y}_\alpha) - ((\mathbf{z}_\alpha\cdot\nabla)\mathbf{y}_\alpha,\Delta\mathbf{y}_\alpha)\\ &\le \frac{1}{2}\|\mathbf{f}\|^2 + \frac{1}{2}\|\Delta\mathbf{y}_\alpha\|^2 + \|\mathbf{z}_\alpha\|_{\mathbf{L}^6}\|\nabla\mathbf{y}_\alpha\|_{\mathbf{L}^3}\|\Delta\mathbf{y}_\alpha\|\\ &\le \frac{1}{2}\|\mathbf{f}\|^2 + \frac{1}{2}\|\Delta\mathbf{y}_\alpha\|^2 + C\|\mathbf{z}_\alpha\|_{\mathbf{V}}\|\mathbf{y}_\alpha\|^{1/2}_{\mathbf{V}}\|\Delta\mathbf{y}_\alpha\|^{3/2}.\end{aligned}$$

Then,

$$\frac{\partial}{\partial t}\|\nabla\mathbf{y}_\alpha\|^2 + \frac{1}{2}\|\Delta\mathbf{y}_\alpha\|^2 \le \|\mathbf{f}\|^2 + C_0\|\nabla\mathbf{y}_\alpha\|^6, \tag{3.21}$$

for some $C_0 > 0$.

Let us see that, under the assumption (3.19), we have

$$\|\nabla \mathbf{y}_\alpha\|^2 \leq \frac{M}{\sqrt{1-2(1+C_0)M^2T}}, \quad \forall t \in [0,T]. \tag{3.22}$$

Indeed, let us introduce the real-valued function ψ given by

$$\psi(t) = \max\left\{M, \|\nabla \mathbf{y}_\alpha(t)\|^2\right\}, \quad \forall t \in [0,T].$$

Then, ψ is almost everywhere differentiable and, in view of (3.19) and (3.21), one has

$$\frac{d\psi}{dt} \leq (1+C_0)\psi^3, \quad \psi(0) = M.$$

Therefore,

$$\psi(t) \leq \frac{M}{\sqrt{1-2(1+C_0)M^2t}} \leq \frac{M}{\sqrt{1-2(1+C_0)M^2T}}$$

and, since $\|\nabla \mathbf{y}_\alpha\|^2 \leq \psi$, (3.22) holds. From this estimate, it is very easy to deduce (3.20). □

The following lemma is inspired by a result by Constantin and Foias for the Navier-Stokes equations, see [25]:

Lemma 3.1. *There exists a continuous function* $\phi : \mathbb{R}_+ \mapsto \mathbb{R}_+$, *with* $\phi(s) \to 0$ *as* $s \to 0^+$, *satisfying the following properties:*

a) For $\mathbf{f} = \mathbf{0}$, *any* $\mathbf{y}_0 \in \mathbf{H}$ *and any* $\alpha > 0$, *there exist arbitrarily small times* $t^* \in (0, T/2)$ *such that the corresponding solution to* (3.3) *satisfies* $\|\mathbf{y}_\alpha(t^*)\|^2_{D(\mathbf{A})} \leq \phi(\|\mathbf{y}_0\|)$.

b) The set of these t^* *has positive measure.*

Proof. We are only going to consider the three-dimensional case; the proof in the two-dimensional case is very similar and even easier.

The proof consists of several steps:

- Let us first see that, for any $k > 3/2$ and any $\tau \in (0, T/2]$, the set

$$R_\alpha(k,\tau) := \{\, t \in [0,\tau] : \|\nabla \mathbf{y}_\alpha(t)\|^2 \leq \frac{k}{\tau}\|\mathbf{y}_0\|^2 \,\}$$

  is non-empty and its measure $|R_\alpha(k,\tau)|$ satisfies $|R_\alpha(k,\tau)| \geq \tau/k$.

  Obviously, we can assume that $\mathbf{y}_0 \not\equiv \mathbf{0}$. Now, if we suppose that $|R_\alpha(k,\tau)| < \tau/k$, we have:

$$\int_0^\tau \|\nabla \mathbf{y}_\alpha(t)\|^2\,dt \geq \int_{R_\alpha(k,\tau)^c} \|\nabla \mathbf{y}_\alpha(t)\|^2\,dt \geq \left(\tau - \frac{\tau}{k}\right)\frac{k}{\tau}\|\mathbf{y}_0\|^2$$
$$= (k-1)\|\mathbf{y}_0\|^2 > \frac{1}{2}\|\mathbf{y}_0\|^2.$$

But, since $\mathbf{f} = 0$ in (3.3), we also have the following estimate:

$$\int_0^\tau \|\nabla \mathbf{y}_\alpha(t)\|^2 \, dt \le \frac{1}{2}\|\mathbf{y}_\alpha(\tau)\|^2 + \int_0^\tau \|\nabla \mathbf{y}_\alpha(t)\|^2 \, dt = \frac{1}{2}\|\mathbf{y}_0\|^2.$$

So, we get a contradiction and, necessarily, $|R_\alpha(k,\tau)| \ge \tau/k$.

- Let us choose $\tau \in (0, T/2]$, $k > 3/2$, $t_{0,\alpha} \in R_\alpha(k,\tau)$ and

$$\overline{T}_\alpha \in \left[t_{0,\alpha} + \frac{\tau^2}{4(1+C_0)k^2\|\mathbf{y}_0\|^4}, t_{0,\alpha} + \frac{3\tau^2}{8(1+C_0)k^2\|\mathbf{y}_0\|^4} \right],$$

where C_0 is the constant furnished by Proposition 3.4. Since $\|\nabla \mathbf{y}_\alpha(t_{0,\alpha})\|^2 \le \frac{k}{\tau}\|\mathbf{y}_0\|^2$, there exists exactly one strong solution to (3.3) in $[t_{0,\alpha}, \overline{T}_\alpha]$ starting from $\mathbf{y}_\alpha(t_{0,\alpha})$ at time $t_{0,\alpha}$ and satisfying

$$\|\nabla \mathbf{y}_\alpha(t)\|^2 \le \frac{2k}{\tau}\|\mathbf{y}_0\|^2, \quad \forall t \in [t_{0,\alpha}, \overline{T}_\alpha].$$

Obviously, it can be assumed that $\overline{T}_\alpha < T$.

Let us introduce the set

$$G_\alpha(t_{0,\alpha}, k, \tau) := \left\{ t \in [t_{0,\alpha}, \overline{T}_\alpha] : \|\Delta \mathbf{y}_\alpha(t)\|^2 \le 65(1+C_0)\left(\frac{k}{\tau}\right)^3 \|\mathbf{y}_0\|^6 \right\}.$$

Then, again $G_\alpha(t_{0,\alpha}, k, \tau)$ is non-empty and possesses positive measure. More precisely, one has

$$|G_\alpha(t_{0,\alpha}, k, \tau)| \ge \frac{\tau^2}{8(1+C_0)k^2\|\mathbf{y}_0\|^4}. \tag{3.23}$$

Indeed, otherwise we would get

$$\begin{aligned}
\frac{1}{2}\int_{t_{0,\alpha}}^{\overline{T}_\alpha} \|\Delta \mathbf{y}_\alpha(t)\|^2 \, dt &\ge \frac{1}{2}\int_{G_\alpha(t_{0,\alpha},k,\tau)^c} \|\Delta \mathbf{y}_\alpha(t)\|^2 \, dt \\
&\ge 65\left(\overline{T}_\alpha - t_{0,\alpha} - \frac{\tau^2}{8(1+C_0)k^2\|\mathbf{y}_0\|^4}\right)(1+C_0)\left(\frac{k}{\tau}\right)^3 \|\mathbf{y}_0\|^6 \\
&\ge \frac{65k}{16\tau}\|\mathbf{y}_0\|^2 \\
&> 4\frac{k}{\tau}\|\mathbf{y}_0\|^2.
\end{aligned}$$

However, arguing as in the proof of Proposition 3.4, we also have

$$\begin{aligned}\frac{1}{2}\int_{t_{0,\alpha}}^{\overline{T}_\alpha}\|\Delta\mathbf{y}_\alpha(t)\|^2\,dt \leq \|\nabla\mathbf{y}_\alpha(\overline{T}_\alpha)\|^2 + \frac{1}{2}\int_{t_{0,\alpha}}^{\overline{T}_\alpha}\|\Delta\mathbf{y}_\alpha(t)\|^2\,dt \\ \leq \|\nabla\mathbf{y}_\alpha(t_{0,\alpha})\|^2 + C_0\int_{t_{0,\alpha}}^{\overline{T}_\alpha}\|\nabla\mathbf{y}_\alpha(t)\|^6\,dt \\ \leq \frac{k}{\tau}\|\mathbf{y}_0\|^2 + 8\left(\frac{k}{\tau}\|\mathbf{y}_0\|^2\right)^3(\overline{T}_\alpha - t_{0,\alpha}) \leq 4\frac{k}{\tau}\|\mathbf{y}_0\|^2.\end{aligned}$$

Consequently, we arrive again to a contradiction and this proves (3.23).

- Let us fix $\tau \in (0, T/2]$ and $k > 3/2$. We can now define $\phi : \mathbb{R}_+ \mapsto \mathbb{R}_+$ as follows:

$$\phi(s) := 65(1 + C_0)\frac{k}{\tau}^3 s^6.$$

Then, as a consequence of the previous steps, the set

$$\{\, t^* \in [0, T/2] : \|\mathbf{A}\mathbf{y}_\alpha(t^*)\|^2 \leq \phi(\|\mathbf{y}_0\|) \,\}$$

is non-empty and it measure is bounded from below by a positive quantity independent of α. This ends the proof.

□

We will end this section with some estimates:

Lemma 3.2. *Let $s \in [1, 2]$ be given, and let us assume that $\mathbf{f} \in \mathbf{H}^s(\Omega)$. Then there exist unique functions $\mathbf{u} \in D(\mathbf{A}^{s/2})$ and $\pi \in H^{s-1}$ (π is unique up to a constant) such that*

$$\begin{cases} \mathbf{u} - \alpha^2\Delta\mathbf{u} + \nabla\pi = \alpha^2\Delta\mathbf{f} & \text{in} \quad \Omega, \\ \nabla\cdot\mathbf{u} = 0 & \text{in} \quad \Omega, \\ \mathbf{u} = \mathbf{0} & \text{on} \quad \Gamma \end{cases} \tag{3.24}$$

and there exists a constant $C = C(s, \Omega)$ independent of α such that

$$\|\mathbf{u}\|_{D(\mathbf{A}^{s/2})} \leq C\|\mathbf{f}\|_{\mathbf{H}^s(\Omega)}. \tag{3.25}$$

Moreover, by interpolation arguments, $\mathbf{f} \in \mathbf{H}^s(\Omega)$, $s \in (m, m+1)$ then there exist unique functions $\mathbf{u} \in D(\mathbf{A}^{s/2})$ and $\pi \in H^{s-1}(\Omega)$ (π is unique up to a constant) which are solution of the problem above and there exists a constant $C = C(m, \Omega)$ such that

$$\|\mathbf{u}\|_{D(\mathbf{A}^{s/2})} \leq C\|\mathbf{f}\|_{\mathbf{H}^s(\Omega)}. \tag{3.26}$$

When s is an integer ($s = 1$ or $s = 2$), the proof can be obtained by adapting the proof of Proposition 2.3 in [119]. For other values of s, it suffices to use a classical interpolation argument (see [118]).

3.2.3 Carleman inequalities and null controllability

In this Subsection, we will recall some Carleman inequalities and a null controllability result for the Oseen system

$$\begin{cases} \mathbf{y}_t - \Delta \mathbf{y} + (\mathbf{b} \cdot \nabla)\mathbf{y} + \nabla p = \mathbf{v} 1_\omega & \text{in} \quad Q, \\ \nabla \cdot \mathbf{y} = 0 & \text{in} \quad Q, \\ \mathbf{y} = \mathbf{0} & \text{on} \quad \Sigma, \\ \mathbf{y}(0) = \mathbf{y}_0 & \text{in} \quad \Omega, \end{cases} \tag{3.27}$$

where $\mathbf{b} = \mathbf{b}(\mathbf{x}, t)$ is given.

The null controllability problem for (3.27) at time $T > 0$ is the following :

> *For any $\mathbf{y}_0 \in \mathbf{H}$, find $\mathbf{v} \in \mathbf{L}^2(\omega \times (0, T))$ such that the associated solution to (3.27) satisfies (3.7).*

We have the following result from [46] (see also [81]) :

Theorem 3.7. *Assume that $\mathbf{b} \in \mathbf{L}^\infty(Q)$ and $\nabla \cdot \mathbf{b} = 0$. Then, the linear system (3.27) is null-controllable at any time $T > 0$. More precisely, for each $\mathbf{y}_0 \in \mathbf{H}$ there exists $\mathbf{v} \in L^\infty(0, T; \mathbf{L}^2(\omega))$ such that the corresponding solution to (3.27) satisfies (3.7). Furthermore, the control $\mathbf{v}$ can be chosen satisfying the estimate*

$$\|\mathbf{v}\|_{L^\infty(\mathbf{L}^2(\omega))} \leq e^{K(1+\|\mathbf{h}\|_\infty^2)} \|\mathbf{y}_0\|, \tag{3.28}$$

where K only depends on Ω, ω and T.

The proof is a consequence of an appropriate Carleman inequality for the adjoint system of (3.27).

More precisely, let us consider the backwards in time system

$$\begin{cases} -\boldsymbol{\varphi}_t - \Delta \boldsymbol{\varphi} - (\mathbf{b} \cdot \nabla)\boldsymbol{\varphi} + \nabla q = \mathbf{G} & \text{in} \quad Q, \\ \nabla \cdot \boldsymbol{\varphi} = 0 & \text{in} \quad Q, \\ \boldsymbol{\varphi} = \mathbf{0} & \text{on} \quad \Sigma, \\ \boldsymbol{\varphi}(T) = \boldsymbol{\varphi}_0, & \text{in} \quad \Omega. \end{cases} \tag{3.29}$$

The following result is established in [82]:

Proposition 3.5. *Assume that $\mathbf{b} \in \mathbf{L}^\infty(Q)$ and $\nabla \cdot \mathbf{b} = 0$. There exist positive continuous functions α, α^*, $\hat{\alpha}$, ξ, ξ^* and $\hat{\xi}$ and positive constants $\hat{s}$, $\hat{\lambda}$ and $\widehat{C}$, only depending on ω and*

Ω, such that, for any $\varphi_0 \in \mathbf{H}$ and any $\mathbf{G} \in \mathbf{L}^2(Q)$, the solution to the adjoint system (3.29) satisfies:

$$\begin{aligned}
&\iint_Q e^{-2s\alpha}\left[s^{-1}\xi^{-1}(|\varphi_t|^2+|\Delta\varphi|^2)+s\xi\lambda^2|\nabla\varphi|^2+s^3\xi^3\lambda^4|\varphi|^2\right]d\mathbf{x}\,dt\\
&\quad\le \widehat{C}(1+T^2)\left(s^{15/2}\lambda^{20}\iint_Q e^{-4s\hat{\alpha}+2s\alpha^*}\xi^{*15/2}|\mathbf{G}|^2\,d\mathbf{x}\,dt\right.\\
&\qquad\left.+\,s^{16}\lambda^{40}\iint_{\omega\times(0,T)} e^{-8s\hat{\alpha}+6s\alpha^*}\xi^{*16}|\varphi|^2\,d\mathbf{x}\,dt\right),
\end{aligned} \tag{3.30}$$

for all $s \ge \hat{s}(T^4+T^8)$ and for all $\lambda \ge \hat{\lambda}\left(1+\|\mathbf{b}\|_\infty + e^{\hat{\lambda}T\|\mathbf{b}\|_\infty^2}\right)$.

Now, we are going to construct the a null-control for (3.27) like in [46]. First, let us introduce the auxiliary extremal problem

$$\begin{cases} \text{Minimize } \dfrac{1}{2}\left\{\displaystyle\iint_Q \hat{\rho}^2|\mathbf{y}|^2\,d\mathbf{x}\,dt + \iint_{\omega\times(0,T)}\hat{\rho}_0^2|\mathbf{v}|^2\,d\mathbf{x}\,dt\right\} \\ \text{Subject to } (\mathbf{y},\mathbf{v}) \in \mathcal{M}(\mathbf{y}_0,T), \end{cases} \tag{3.31}$$

where the linear manifold $\mathcal{M}(\mathbf{y}_0,T)$ is given by

$$\mathcal{M}(\mathbf{y}_0,T) = \{\,(\mathbf{y},\mathbf{v}) : \mathbf{v}\in\mathbf{L}^2(\omega\times(0,T)),\ (\mathbf{y},p) \text{ solves } (3.27)\,\}$$

and $\hat{\rho}$ and $\hat{\rho}_0$ are respectively given by

$$\hat{\rho} = s^{-15/4}\lambda^{-10}e^{2s\hat{\alpha}-s\alpha^*}\xi^{*-15/4}, \quad \hat{\rho}_0 = s^{-8}\lambda^{-20}e^{4s\hat{\alpha}-3s\alpha^*}\xi^{*-8}.$$

It can be proved that (3.31) possesses exactly one solution $(\mathbf{y},\mathbf{v})$ satisfying

$$\|\mathbf{v}\|_{L^2(\mathbf{L}^2(\omega))} \le e^{K(1+\|\mathbf{b}\|_\infty^2)}\|\mathbf{y}_0\|,$$

where K only depends on ω, Ω and T.

Moreover, thanks to the *Euler-Lagrange's characterization*, the solution to the extremal problem (4.10) is given by

$$\mathbf{y} = \hat{\rho}^{-2}(-\varphi_t - \Delta\varphi - (\mathbf{b}\cdot\nabla)\varphi + \nabla q) \quad \text{and} \quad \mathbf{v} = -\hat{\rho}_0^{-2}\varphi 1_\omega.$$

From the Carleman inequality (3.30), we can conclude that $\rho_2^{-1}\varphi \in L^\infty(0,T;\mathbf{L}^2(\Omega))$ and

$$\|\rho_2^{-1}\varphi\|_{L^\infty(\mathbf{L}^2)} \le C\|\hat{\rho}_0^{-1}\varphi\|_{L^2(\mathbf{L}^2(\omega))},$$

where $\rho_2 = s^{1/2}\xi^{1/2}e^{s\alpha}$.

Hence,

$$\mathbf{v} = -(\hat{\rho}_0)^{-2}\boldsymbol{\varphi}1_\omega = -(\hat{\rho}_0^{-2}\rho_2)(\rho_2^{-1}\boldsymbol{\varphi}1_\omega) \in L^\infty(0,T;\mathbf{L}^2(\Omega))$$

and, therefore,

$$\|\mathbf{v}\|_{L^\infty(\mathbf{L}^2(\omega))} \leq C\|\mathbf{v}\|_{L^2(\mathbf{L}^2(\omega))} \leq e^{K(1+\|\mathbf{b}\|_\infty^2)}\|\mathbf{y}_0\|.$$

3.3 The distributed case: Theorems 3.1 and 3.3

This section is devoted to prove the local null controllability of (3.5) and the uniform controllability property in Theorem 3.3.

Proof of Theorem 3.1. We will use a fixed point argument. Contrarily to the case of the Navier-Stokes equations, it is not sufficient to work here with controls in $\mathbf{L}^2(\omega\times(0,T))$. Indeed, we need a space $\mathbf{Y}$ for $\mathbf{y}$ that ensures $\mathbf{z}$ in $\mathbf{L}^\infty(Q)$ and a space $\mathbf{X}$ for $\mathbf{v}$ guaranteeing that the solution to (3.27) with $\mathbf{b}=\mathbf{z}$ belongs to a compact set of $\mathbf{Y}$. Furthermore, we want estimates in $\mathbf{Y}$ and $\mathbf{X}$ independent of α.

In view of Lemma 3.1, in order to prove Theorem 3.1, we just need to consider the case in which the initial state $\mathbf{y}_0$ belongs to $D(\mathbf{A})$ and possesses a sufficiently small norm in $D(\mathbf{A})$.

Let us fix σ with $N/4<\sigma<1$. Then, for each $\overline{\mathbf{y}}\in L^\infty(0,T;D(\mathbf{A}^\sigma))$, let $(\mathbf{z},\pi)$ be the unique solution to

$$\begin{cases} \mathbf{z}-\alpha^2\Delta\mathbf{z}+\nabla\pi=\overline{\mathbf{y}} & \text{in} \quad Q, \\ \nabla\cdot\mathbf{z}=0 & \text{in} \quad Q, \\ \mathbf{z}=\mathbf{0} & \text{on} \quad \Sigma. \end{cases}$$

Since $\overline{\mathbf{y}}\in L^\infty(0,T;D(\mathbf{A}^\sigma))$, it is clear that $\mathbf{z}\in L^\infty(0,T;D(\mathbf{A}^\sigma))$. Then, thanks to Theorem 3.5, we have $\mathbf{z}\in\mathbf{L}^\infty(Q)$ and the following is satisfied :

$$\begin{aligned} \|\mathbf{z}\|^2_{L^\infty(0,T;D(\mathbf{A}^\sigma))}+2\alpha^2\|\mathbf{z}\|^2_{L^\infty(D(\mathbf{A}^{1/2+\sigma}))} &\leq \|\overline{\mathbf{y}}\|^2_{L^\infty(0,T;D(\mathbf{A}^\sigma))}, \\ 2\alpha^2\|\mathbf{z}\|^2_{L^\infty(D(\mathbf{A}^{1/2+\sigma}))}+\alpha^4\|\mathbf{z}\|^2_{L^\infty(D(\mathbf{A}^{1+\sigma}))} &\leq \|\overline{\mathbf{y}}\|^2_{L^\infty(0,T;D(\mathbf{A}^\sigma))}. \end{aligned} \tag{3.32}$$

In particular, we have :

$$\|\mathbf{z}\|_{L^\infty(0,T;D(\mathbf{A}^\sigma))} \leq \|\overline{\mathbf{y}}\|_{L^\infty(0,T;D(\mathbf{A}^\sigma))}.$$

Let us consider the system (3.27) with $\mathbf{b}$ replaced by $\mathbf{z}$. In view of Theorem 3.7, we can associate to $\mathbf{z}$ the null control $\mathbf{v}$ of minimal norm in $L^\infty(0,T;\mathbf{L}^2(\omega))$ and the corresponding solution $(\mathbf{y},p)$ to (3.27).

Since $\mathbf{y}_0\in D(\mathbf{A})$, $\mathbf{z}\in\mathbf{L}^\infty(Q)$ and $\mathbf{v}\in L^\infty(0,T;\mathbf{L}^2(\omega))$, we have

$$\mathbf{y}\in L^2(0,T;D(\mathbf{A}))\cap C^0([0,T];\mathbf{V}),\ \mathbf{y}_t\in L^2(0,T;\mathbf{H})$$

and the following estimate holds:

$$\|\mathbf{y}_t\|^2_{L^2(\mathbf{H})} + \|\mathbf{y}\|^2_{L^2(D(\mathbf{A}))} + \|\mathbf{y}\|^2_{L^\infty(\mathbf{V})} \le C(\|\mathbf{y}_0\|^2_{\mathbf{V}} + \|\mathbf{v}\|^2_{L^\infty(\mathbf{L}^2(\omega))})e^{C\|\mathbf{z}\|^2_\infty}. \tag{3.33}$$

We will use the following result:

Lemma 3.3. *One has $\mathbf{y} \in L^\infty(0,T;D(\mathbf{A}^{\sigma'}))$, for all $\sigma' \in (\sigma, 1)$, with*

$$\|\mathbf{y}\|_{L^\infty(D(\mathbf{A}^{\sigma'}))} \le C(\|\mathbf{y}_0\|_{D(\mathbf{A})} + \|\mathbf{v}\|_{L^\infty(\mathbf{L}^2(\omega))})e^{C\|\overline{\mathbf{y}}\|^2_{L^\infty(D(\mathbf{A}^\sigma))}}.$$

Proof. In view of (3.27), $\mathbf{y}$ solves the following abstract initial value problem:

$$\begin{cases} \mathbf{y}_t = -\mathbf{A}\mathbf{y} - \mathbf{P}((\mathbf{z}\cdot\nabla)\mathbf{y}) + \mathbf{P}(\mathbf{v}1_\omega) \quad \text{in} \quad [0,T], \\ \mathbf{y}(0) = \mathbf{y}_0. \end{cases}$$

This system can be rewritten as the nonlinear integral equation

$$\mathbf{y}(t) = e^{-t\mathbf{A}}\mathbf{y}_0 - \int_0^t e^{-(t-s)\mathbf{A}}\mathbf{P}((\mathbf{z}\cdot\nabla)\mathbf{y})(s)\,ds + \int_0^t e^{-(t-s)\mathbf{A}}\mathbf{P}(\mathbf{v}1_\omega)(s)\,ds.$$

Consequently, applying the operator $\mathbf{A}^{\sigma'}$ to both sides, we have

$$\mathbf{A}^{\sigma'}\mathbf{y}(t) = \mathbf{A}^{\sigma'}e^{-t\mathbf{A}}\mathbf{y}_0 + \int_0^t \mathbf{A}^{\sigma'}e^{-(t-s)\mathbf{A}}\left[-P((\mathbf{z}\cdot\nabla)\mathbf{y})(s) + P(\mathbf{v}1_\omega)(s)\right]ds.$$

Taking norms in both sides and using Theorem 3.6, we see that

$$\begin{aligned} \|\mathbf{A}^{\sigma'}\mathbf{y}\|(t) &\le \|\mathbf{y}_0\|_{D(A^{\sigma'})} + \int_0^t (t-s)^{-\sigma'}[\|\mathbf{z}(s)\|_\infty\|\nabla\mathbf{y}(s)\| + \|\mathbf{v}(s)1_\omega\|]\,ds \\ &\le C\|\mathbf{y}_0\|_{D(\mathbf{A})} + (\|\mathbf{z}\|_\infty\|\mathbf{y}\|_{L^\infty(\mathbf{V})} + \|\mathbf{v}\|_{L^\infty(\mathbf{L}^2(\omega))})\int_0^t (t-s)^{-\sigma'}\,ds. \end{aligned}$$

Now, using (3.32) and (3.33) and taking into account that $\sigma' < 1$, we easily obtain that

$$\|\mathbf{A}^{\sigma'}\mathbf{y}\|(t) \le C(\|\mathbf{y}_0\|_{D(\mathbf{A})} + \|\mathbf{v}\|_{L^\infty(\mathbf{L}^2(\omega))})\left[1 + \|\overline{\mathbf{y}}\|_{L^\infty(D(\mathbf{A}^\sigma))}e^{C\|\overline{\mathbf{y}}\|^2_{L^\infty(D(\mathbf{A}^\sigma))}}\right].$$

This ends the proof. □

Now, let us set

$$\mathbf{W} = \{\, \mathbf{w} \in L^\infty(0,T;D(\mathbf{A}^{\sigma'})) : \mathbf{w}_t \in L^2(0,T;\mathbf{H}) \,\}$$

and let us consider the closed ball

$$\mathbf{K} = \{\, \overline{\mathbf{y}} \in L^\infty(0,T;D(\mathbf{A}^\sigma)) : \|\overline{\mathbf{y}}\|_{L^\infty(D(\mathbf{A}^\sigma))} \le 1 \,\}$$

and the mapping $\widetilde{\Lambda}_\alpha$, with $\widetilde{\Lambda}_\alpha(\overline{\mathbf{y}}) = \mathbf{y}$ for all $\overline{\mathbf{y}} \in L^\infty(0,T;D(\mathbf{A}^\sigma))$. Obviously, $\widetilde{\Lambda}_\alpha$ is well defined; furthermore, in view of Lemma 3.3 and (3.33), it maps the whole space $L^\infty(0,T;D(\mathbf{A}^\sigma))$ into $\mathbf{W}$.

Notice that, if $\mathbf{U}$ is bounded set of $\mathbf{W}$ then it is relatively compact in the space $L^\infty(0,T;D(\mathbf{A}^\sigma))$, in view of the classical results of the *Aubin-Lions kind*, see for instance [117].

Let us denote by Λ_α the restriction to $\mathbf{K}$ of $\widetilde{\Lambda}_\alpha$. Then, thanks to Lemma 3.3 and (3.28), if $\|\mathbf{y}_0\|_{D(\mathbf{A})} \le \varepsilon$ (independent of α!) Λ_α maps $\mathbf{K}$ into itself. Moreover, it is clear that $\Lambda_\alpha : \mathbf{K} \mapsto \mathbf{K}$ satisfies the hypotheses of *Schauder's fixed point Theorem*. Indeed, this nonlinear mapping is continuous and compact (the latter is a consequence of the fact that, if $\mathbf{B}$ is bounded in $L^\infty(0,T;D(\mathbf{A}^\sigma))$, then $\tilde{\Lambda}_\alpha(\mathbf{B})$ is bounded in $\mathbf{W}$). Consequently, Λ_α possesses at least one fixed point in $\mathbf{K}$, and this ends the proof of Theorem 3.1. □

Proof of Theorem 3.3. Let $\mathbf{v}_\alpha$ be a null control for (3.5) satisfying (3.8) and let $(\mathbf{y}_\alpha, p_\alpha, \mathbf{z}_\alpha, \pi_\alpha)$ be the state associated to $\mathbf{v}_\alpha$. From (3.8) and the estimates (3.14) for the solutions $\mathbf{y}_\alpha$, there exist $\mathbf{v} \in L^\infty\left(0,T;\mathbf{L}^2(\omega)\right)$ and $\mathbf{y} \in L^\infty(0,T;\mathbf{H}) \cap L^2(0,T;\mathbf{V})$ with $\mathbf{y}_t \in L^{\sigma_N}(0,T;\mathbf{V}')$ such that, at least for a subsequence

$$\begin{aligned} \mathbf{v}_\alpha \to \mathbf{v} \quad &\text{weakly-}\star \text{ in} \quad L^\infty(0,T;\mathbf{L}^2(\omega)),\\ \mathbf{y}_\alpha \to \mathbf{y} \quad \text{weakly-}\star \text{ in} \quad L^\infty(0,T;\mathbf{H}) \quad &\text{and weakly in} \quad L^2(0,T;\mathbf{V}),\\ \mathbf{y}_{\alpha,t} \to \mathbf{y}_t \quad &\text{weakly in} \quad L^{\sigma_N}(0,T;\mathbf{V}'). \end{aligned}$$

Since $\mathbf{W} := \{\mathbf{m} \in L^2\left(0,T;\mathbf{V}\right) : \ \mathbf{m}_t \in L^{\sigma_N}(0,T;\mathbf{V}')\}$ is continuously and compactly embedded in $\mathbf{L}^2(Q)$, we have that

$$\mathbf{y}_\alpha \to \mathbf{y} \quad \text{strongly in} \quad \mathbf{L}^2(Q) \quad \text{and a.e.}$$

This is sufficient to pass to the limit in the equations satisfied by $\mathbf{y}_\alpha$, $\mathbf{v}_\alpha$ and $\mathbf{z}_\alpha$. We conclude that $\mathbf{y}$ is, together with some pressure p, a solution to the Navier-Stokes equations associated to a control $\mathbf{v}$ and satisfies (3.7). □

3.4 The boundary case: Theorems 3.2 and 3.4

This section is devoted to prove the local boundary null controllability of (3.6) and the uniform controllability property in Theorem 3.4.

Proof of Theorem 3.2. Again, we will use a fixed point argument. Contrarily to the case of distributed controllability, we will have to work in a space $\widetilde{\mathbf{Y}}$ of functions defined in an extended domain.

Let $\widetilde{\Omega}$ be given, with $\Omega \subset \widetilde{\Omega}$ and $\partial\widetilde{\Omega} \cap \Gamma = \Gamma \backslash \gamma$ such that $\partial\widetilde{\Omega}$ is of class C^2 (see Fig. 3.1). Let $\omega \subset \widetilde{\Omega} \setminus \overline{\Omega}$ be a non empty open subset and let us introduce $\widetilde{Q} := \widetilde{\Omega} \times (0,T)$ and

$\widetilde{\Sigma} := \partial\widetilde{\Omega} \times (0,T)$. The spaces and operators associate to the domain $\widetilde{\Omega}$ will be denoted by $\widetilde{\mathbf{H}}$, $\widetilde{\mathbf{V}}$, $\widetilde{\mathbf{A}}$, etc.

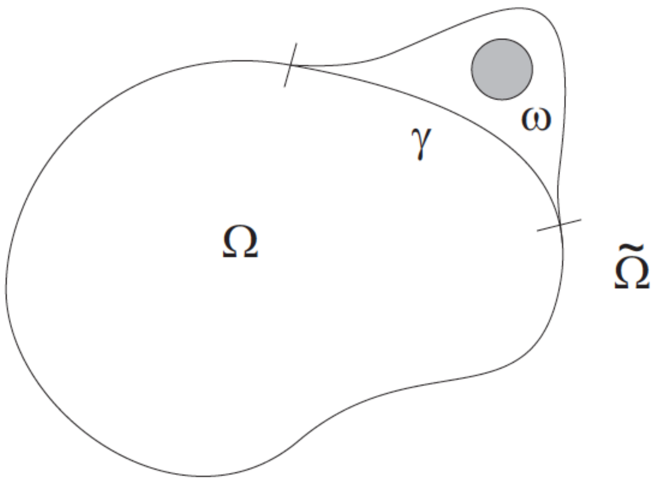

Figure 3.1: The extended domain

Remark 3.2. In view of Lemma 3.1, for the proof of Theorem 3.2 we just need to consider the case in which the initial state $\mathbf{y}_0$ belongs to $\mathbf{V}$ and possesses a sufficiently small norm in $\mathbf{V}$. Indeed, we only have to take initially $\mathbf{h}_\alpha \equiv \mathbf{0}$ and apply Lemma 3.1 to the solution to (3.6). □

Let $\mathbf{y}_0 \in \mathbf{V}$ be given and let us introduce $\tilde{\mathbf{y}}_0$, the extension by zero of $\mathbf{y}_0$. Then $\tilde{\mathbf{y}}_0 \in \widetilde{\mathbf{V}}$.

We will use the following result, similar to Lemma 3.1, whose proof is postponed to the end of the Section:

Lemma 3.4. *There exists a continuous function $\phi : \mathbb{R}_+ \mapsto \mathbb{R}_+$ satisfying $\phi(s) \to 0$ as $s \to 0^+$ with the following property:*

a) For any $\mathbf{y}_0 \in \mathbf{V}$ and any $\alpha > 0$, there exist times $T_0 \in (0,T)$, controls $\mathbf{h}_\alpha \in L^2(0,T_0;\mathbf{H}^{1/2}(\Gamma))$ with $\int_\gamma \mathbf{h}_\alpha \cdot \mathbf{n}\, d\Gamma \equiv 0$, associated solutions $(\mathbf{y}_\alpha, p_\alpha, \mathbf{z}_\alpha, \pi_\alpha)$ to (3.6) in $\Omega \times (0,T_0)$ and arbitrarily small times $t^* \in (0,T/2)$ such that the $\mathbf{y}_\alpha$ can be extended to $\widetilde{\Omega} \times (0,T_0)$ and the extensions satisfy $\|\tilde{\mathbf{y}}_\alpha(t^*)\|^2_{D(\widetilde{\mathbf{A}})} \le \phi(\|\mathbf{y}_0\|_{\mathbf{V}})$.

b) The set of these t^* has positive measure.

c) The controls $\mathbf{h}_\alpha$ are uniformly bounded, i.e.

$$\|\mathbf{h}_\alpha\|_{L^\infty(0,T_0;\mathbf{H}^{1/2}(\gamma))} \le C.$$

In view of Lemma 3.4, for the proof of Theorem 3.2, we just need to consider the case in which the initial state $\mathbf{y}_0$ is such that its extension $\tilde{\mathbf{y}}_0$ to $\widetilde{\Omega}$ belongs to $D(\widetilde{\mathbf{A}})$ and possesses a sufficiently small norm in $D(\widetilde{\mathbf{A}})$.

We will prove that there exists $(\tilde{\mathbf{y}}, \tilde{p}, \mathbf{z}, \pi, \tilde{\mathbf{v}})$, with $\tilde{\mathbf{v}} \in L^\infty(0,T;\mathbf{L}^2(\omega))$, satisfying

$$\begin{cases} \tilde{\mathbf{y}}_t - \Delta\tilde{\mathbf{y}} + (\tilde{\mathbf{z}} \cdot \nabla)\tilde{\mathbf{y}} + \nabla\tilde{p} = \tilde{\mathbf{v}}1_\omega & \text{in} \quad \widetilde{Q}, \\ \mathbf{z} - \alpha^2\Delta\mathbf{z} + \nabla\pi = \tilde{\mathbf{y}} & \text{in} \quad Q, \\ \nabla\cdot\tilde{\mathbf{y}} = 0 & \text{in} \quad \widetilde{Q}, \\ \nabla\cdot\mathbf{z} = 0 & \text{in} \quad Q, \\ \tilde{\mathbf{y}} = \mathbf{0} & \text{on} \quad \widetilde{\Sigma}, \\ \mathbf{z} = \tilde{\mathbf{y}} & \text{on} \quad \Sigma, \\ \tilde{\mathbf{y}}(0) = \tilde{\mathbf{y}}_0 & \text{in} \quad \widetilde{\Omega} \end{cases} \tag{3.34}$$

and $\tilde{\mathbf{y}}(T) = \mathbf{0}$ in $\widetilde{\Omega}$, where $\tilde{\mathbf{z}}$ is the extension by zero of $\mathbf{z}$. Obviously, if this were the case, the restriction of $(\tilde{\mathbf{y}}, \tilde{p})$ to Q, denoted by $(\mathbf{y}, p)$, the couple $(\mathbf{z}, \pi)$ and the lateral trace $\mathbf{h} := \tilde{\mathbf{y}}|_{\gamma\times(0,T)}$ would satisfy (3.6) and (3.7).

Let us fix σ with $N/4 < \sigma < 1$. Then, for each $\overline{\mathbf{y}} \in L^\infty(0,T;D(\widetilde{\mathbf{A}}^\sigma))$, let $\mathbf{w} = \mathbf{w}(\mathbf{x},t)$ and $\pi = \pi(\mathbf{x},t)$ be the unique solution to

$$\begin{cases} \mathbf{w} - \alpha^2\Delta\mathbf{w} + \nabla\pi = \alpha^2\Delta\overline{\mathbf{y}} & \text{in} \quad Q, \\ \nabla\cdot\mathbf{w} = 0 & \text{in} \quad Q, \\ \mathbf{w} = \mathbf{0} & \text{on} \quad \Sigma. \end{cases}$$

Since $\overline{\mathbf{y}} \in L^\infty(0,T;D(\widetilde{\mathbf{A}}^\sigma))$, its restriction to Q belongs to $L^\infty(0,T;\mathbf{H}^{2\sigma}(\Omega))$. Then, Lemma 3.2 implies $\mathbf{w} \in L^\infty(0,T;D(\mathbf{A}^\sigma))$ and, thanks to Theorem 3.5, we also have $\mathbf{w} \in \mathbf{L}^\infty(Q)$ and

$$\|\mathbf{w}\|^2_{L^\infty(0,T;D(\mathbf{A}^\sigma))} \le C\|\overline{\mathbf{y}}\|^2_{L^\infty(0,T;D(\widetilde{\mathbf{A}}^\sigma))},$$

where C is independent of α.

Let $\tilde{\mathbf{w}}$ be the extension by zero of $\mathbf{w}$ and let us set $\tilde{\mathbf{z}} := \overline{\mathbf{y}} + \tilde{\mathbf{w}}$. Let us consider the system (3.27) with Ω replaced by $\widetilde{\Omega}$ and $\mathbf{b}$ replaced by $\tilde{\mathbf{z}}$. In view of Theorem 3.7, we can associate to $\tilde{\mathbf{z}}$ the null control $\tilde{\mathbf{v}}$ of minimal norm in $L^\infty(0,T;\mathbf{L}^2(\omega))$ and the corresponding solution $(\tilde{\mathbf{y}}, \tilde{p})$ to (3.27). Since $\tilde{\mathbf{y}}_0 \in D(\widetilde{\mathbf{A}})$, $\tilde{\mathbf{z}} \in \mathbf{L}^\infty(\widetilde{Q})$ and $\tilde{\mathbf{v}} \in L^\infty(0,T;\mathbf{L}^2(\omega))$, we have

$$\tilde{\mathbf{y}} \in L^2(0,T;D(\widetilde{\mathbf{A}})) \cap C^0([0,T];\widetilde{\mathbf{V}}), \tilde{\mathbf{y}}_t \in L^2(0,T;\widetilde{\mathbf{H}})$$

and the following estimate holds :

$$\|\tilde{\mathbf{y}}_t\|^2_{L^2(\widetilde{\mathbf{H}})} + \|\tilde{\mathbf{y}}\|^2_{L^2(D(\widetilde{\mathbf{A}}))} + \|\tilde{\mathbf{y}}\|^2_{L^\infty(\widetilde{\mathbf{V}})} \le C(\|\tilde{\mathbf{y}}_0\|^2_{\widetilde{\mathbf{V}}} + \|\tilde{\mathbf{v}}\|^2_{L^\infty(\mathbf{L}^2(\omega))})e^{C\|\tilde{\mathbf{z}}\|^2_\infty}. \tag{3.35}$$

Taking $\sigma < \beta < 1$, thanks to Lemma 3.3, one has $\tilde{\mathbf{y}} \in L^\infty(0,T;D(\widetilde{\mathbf{A}}^\beta))$ and

$$\|\tilde{\mathbf{y}}\|_{L^\infty(D(\widetilde{\mathbf{A}}^\beta))} \le C(\|\tilde{\mathbf{y}}_0\|_{D(\widetilde{\mathbf{A}})} + \|\tilde{\mathbf{v}}\|_{L^\infty(\mathbf{L}^2(\omega))})e^{C\|\overline{\mathbf{y}}\|_{L^\infty(0,T;D(\widetilde{\mathbf{A}}^\sigma))}}.$$

Now, let us set

$$\mathbf{W} = \{\, \mathbf{m} \in L^\infty(0,T; D(\widetilde{\mathbf{A}}^\beta)) : \mathbf{m}_t \in L^2(0,T; \widetilde{\mathbf{H}}) \,\},$$

and let us consider the closed ball

$$\mathbf{K} = \{\, \overline{\mathbf{y}} \in L^\infty(0,T; D(\widetilde{\mathbf{A}}^\sigma)) : \|\overline{\mathbf{y}}\|_{L^\infty(D(\widetilde{\mathbf{A}}^\sigma))} \le 1 \,\}$$

and the mapping $\widetilde{\Lambda}_\alpha$, with $\widetilde{\Lambda}_\alpha(\overline{\mathbf{y}}) = \tilde{\mathbf{y}}$ for all $\overline{\mathbf{y}} \in L^\infty(0,T; D(\widetilde{\mathbf{A}}^\sigma))$. Obviously, $\widetilde{\Lambda}_\alpha$ is well defined and maps the whole space $L^\infty(0,T; D(\widetilde{\mathbf{A}}^\sigma))$ into $\mathbf{W}$. Furthermore, any bounded set $\mathbf{U} \subset \mathbf{W}$ then it is relatively compact in $L^\infty(0,T; D(\widetilde{\mathbf{A}}^\sigma))$.

Let us denote by Λ_α the restriction to $\mathbf{K}$ of $\widetilde{\Lambda}_\alpha$. Thanks to Lemma 3.3 and (3.28), there exists $\varepsilon > 0$ (independent of α) such that if $\|\tilde{\mathbf{y}}_0\|_{D(\widetilde{\mathbf{A}})} \le \delta$, Λ_α maps $\mathbf{K}$ into itself and it is clear that $\Lambda_\alpha : \mathbf{K} \mapsto \mathbf{K}$ satisfies the hypotheses of *Schauder's fixed point Theorem*. Consequently, Λ_α possesses at least one fixed point in $\mathbf{K}$ and (3.34) possesses a solution.This ends the proof of Theorem 3.2. □

Proof of Theorem 3.4. The proof is easy, in view of the previous uniform estimates. It suffices to adapt the argument in the proof of Theorem 3.3 and deduce the existence of subsequences that converge (in an appropriate sense) to a solution to (3.11) satisfying (3.7). For brevity, we omit the details. □

Proof of Lemma 3.4. For instance, let us only consider the case $N = 3$. We will reduce the proof to the search of a fixed point of another mapping Φ_α.

For any $\mathbf{y}_0 \in \mathbf{V}$, any $T_0 \in (0,T)$ and any $\overline{\mathbf{y}} \in L^4(0,T_0; \widetilde{\mathbf{V}}))$, let $(\mathbf{w}, \pi)$ be the unique solution to

$$\begin{cases} \mathbf{w} - \alpha^2 \Delta \mathbf{w} + \nabla \pi = \alpha^2 \Delta \overline{\mathbf{y}} & \text{in} \quad \Omega \times (0,T_0), \\ \nabla \cdot \mathbf{w} = 0 & \text{in} \quad \Omega \times (0,T_0), \\ \mathbf{w} = \mathbf{0} & \text{on} \quad \Gamma \times (0,T_0), \end{cases}$$

let $\tilde{\mathbf{w}}$ be the extension by zero of $\mathbf{w}$, let us set $\tilde{\mathbf{z}} := \overline{\mathbf{y}} + \tilde{\mathbf{w}}$ and let us introduce the Oseen system

$$\begin{cases} \tilde{\mathbf{y}}_t - \Delta \tilde{\mathbf{y}} + (\tilde{\mathbf{z}} \cdot \nabla)\tilde{\mathbf{y}} + \nabla \tilde{p} = \mathbf{0} & \text{in} \quad \widetilde{\Omega} \times (0,T_0), \\ \nabla \cdot \tilde{\mathbf{y}} = 0 & \text{in} \quad \widetilde{\Omega} \times (0,T_0), \\ \tilde{\mathbf{y}} = \mathbf{0} & \text{on} \quad \partial\widetilde{\Omega} \times (0,T_0), \\ \tilde{\mathbf{y}}(0) = \tilde{\mathbf{y}}_0 & \text{in} \quad \widetilde{\Omega}. \end{cases}$$

It is clear that the restriction of $\overline{\mathbf{y}}$ to $\Omega \times (0,T_0)$ belongs to $L^4(0,T_0; \mathbf{H}^1(\Omega))$, whence we have from Lemma 3.2 that $\mathbf{w} \in L^4(0,T_0; \mathbf{V})$ and

$$\|\mathbf{w}\|_{L^4(0,T_0;\mathbf{V})} \le C \|\overline{\mathbf{y}}\|_{L^4(0,T_0;\widetilde{\mathbf{V}})}.$$

It is also clear that we can get estimates like those in the proof of Proposition 3.4 for

$\tilde{\mathbf{y}}$. In other words, for any $\mathbf{y}_0 \in \mathbf{V}$, we can find a sufficiently small $T_0 > 0$ such that

$$\tilde{\mathbf{y}} \in L^2(0,T_0;D(\widetilde{\mathbf{A}})) \cap C^0([0,T_0];\widetilde{\mathbf{V}}), \quad \tilde{\mathbf{y}}_t \in L^2(0,T_0;\widetilde{\mathbf{H}})$$

and

$$\|\tilde{\mathbf{y}}\|_{L^2(0,T_0;D(\widetilde{\mathbf{A}}))} + \|\tilde{\mathbf{y}}\|_{C^0([0,T_0];\widetilde{\mathbf{V}})} + \|\tilde{\mathbf{y}}_t\|_{L^2(0,T_0;\widetilde{\mathbf{H}})} \le C\left(T_0, \|\mathbf{y}_0\|_{\mathbf{V}}, \|\overline{\mathbf{y}}\|_{L^4(0,T_0;\widetilde{\mathbf{V}})}\right),$$

where C is nondecreasing with respect to all arguments and goes to zero as $\|\mathbf{y}_0\|_{\mathbf{V}} \to 0$.

Now, let us introduce the mapping $\Phi_\alpha : L^4(0,T_0;\widetilde{\mathbf{V}}) \mapsto L^4(0,T_0;\widetilde{\mathbf{V}})$, with $\Phi_\alpha(\overline{\mathbf{y}}) = \tilde{\mathbf{y}}$ for all $\overline{\mathbf{y}} \in L^4(0,T;\widetilde{\mathbf{V}})$. This is a continuous and compact mapping. Indeed, from well known interpolation results, we have that the embedding

$$L^2(0,T_0;D(\widetilde{\mathbf{A}})) \cap L^\infty(0,T_0;\widetilde{\mathbf{V}}) \hookrightarrow L^4(0,T_0;D(\widetilde{\mathbf{A}}^{3/4}))$$

is continuous and this shows that, if $\tilde{\mathbf{y}}$ is bounded in $L^2(0,T_0;D(\widetilde{\mathbf{A}})) \cap C^0([0,T_0];\widetilde{\mathbf{V}})$ and $\tilde{\mathbf{y}}_t$ is bounded in $L^2(0,T_0;\widetilde{\mathbf{H}})$, then $\tilde{\mathbf{y}}$ belongs to a compact set of $L^4(0,T_0;\widetilde{\mathbf{V}})$.

Then, as in the proofs of Theorems 3.1 and 3.2, we immediately deduce that, whenever $\|\mathbf{y}_0\|_{\mathbf{V}} \le \delta$ (for some δ independent of α), Φ_α possesses at least one fixed point. This shows that the nonlinear system (3.34) is solvable for $\tilde{\mathbf{v}} \equiv \mathbf{0}$ and $\|\mathbf{y}_0\|_{\mathbf{V}} \le \delta$.

Now, the argument in the proof of Lemma 3.1 can be applied in this framework and, as a consequence, we easily deduce Lemma 3.4. □

3.5 Additional comments and questions

3.5.1 Controllability problems for semi-Galerkin approximations

Let $\{\mathbf{w}^1, \mathbf{w}^2, \dots\}$ be a basis of the Hilbert space $\mathbf{V}$. For instance, we can consider the orthogonal base formed by the eigenvectors of the Stokes operator $\mathbf{A}$. Together with (3.5), we can consider the following semi-Galerkin approximated problems:

$$\begin{cases} \mathbf{y}_t - \Delta\mathbf{y} + (\mathbf{z}^m\cdot\nabla)\mathbf{y} + \nabla p = \mathbf{v}1_\omega & \text{in} \quad Q, \\ (\mathbf{z}^m(t) + \alpha^2\nabla\mathbf{z}^m(t) - \mathbf{y}(t), \mathbf{w}) = 0, \ \forall \mathbf{w} \in \mathbf{V}_m, \ t & \text{in} \quad (0,T), \\ \nabla\cdot\mathbf{y} = 0 & \text{in} \quad Q, \\ \mathbf{y} = \mathbf{0} & \text{on} \quad \Sigma, \\ \mathbf{y}(0) = \mathbf{y}_0 & \text{in} \quad \Omega, \end{cases} \tag{3.36}$$

where $\mathbf{z}^m(t) \in \mathbf{V}_m$ and $\mathbf{V}_m$ denotes the space spanned by $\mathbf{w}^1, \dots, \mathbf{w}^m$.

Arguing as in the proof of Theorem 3.1, it is possible to prove a local null controllability result for (3.36). More precisely, for each $m \ge 1$, there exists $\varepsilon_m > 0$ such that, if $\|\mathbf{y}_0\| \le \varepsilon_m$, we can find controls $\mathbf{v}^m$ and associated states $(\mathbf{y}^m, p^m, \mathbf{z}^m)$ satisfying (3.7). Notice that, in view of the equivalence of norms in $\mathbf{V}_m$, the fixed point argument can be applied in this case without any extra regularity assumption on $\mathbf{y}_0$; in other words, Lemma 3.1 is not needed here.

On the other hand, it can also be checked that the maximal ε_m are bounded from below by some positive quantity independent of m and α and the controls $\mathbf{v}^m$ can be found uniformly bounded in $L^\infty(0,T;\mathbf{L}^2(\omega))$. As a consequence, at least for a subsequence, the controls converge weakly-$\star$ in that space to a null control for (3.5).

However, it is unknown whether the problems (3.36) are *globally* null-controllable; see below for other considerations concerning global controllability.

3.5.2 Another strategy: applying an inverse function theorem

There is another way to prove the local null controllability of (3.5) that relies on *Liusternik's Inverse Function Theorem*, see for instance [1]. This strategy has been introduced in [62] and has been applied successfully to the controllability of many semilinear and nonlinear PDE's. In the framework of (3.5), the argument is as follows:

(i) Introduce an appropriate Hilbert space $\mathbf{Y}$ of *state-control pairs* $(\mathbf{y}_\alpha, p_\alpha, \mathbf{z}_\alpha, \pi_\alpha, \mathbf{v}_\alpha)$ satisfying (3.5) and (3.7).

(ii) Introduce a second Hilbert space $\mathbf{Z}$ of right hand sides and initial data and a well-defined mapping $\mathbf{F} : \mathbf{Y} \mapsto \mathbf{Z}$ such that the null controllability of (3.5) with state-controls in $\mathbf{Y}$ is equivalent to the solution of the nonlinear equation

$$\mathbf{F}(\mathbf{y}_\alpha, p_\alpha, \mathbf{z}_\alpha, \pi_\alpha, \mathbf{v}_\alpha) = (\mathbf{0}, \mathbf{y}_0), \quad (\mathbf{y}_\alpha, p_\alpha, \mathbf{z}_\alpha, \pi_\alpha, \mathbf{v}_\alpha) \in \mathbf{Y}. \tag{3.37}$$

(iii) Prove that $\mathbf{F}$ is C^1 in a neighborhood of $(\mathbf{0}, 0, \mathbf{0}, 0, \mathbf{0})$ and $\mathbf{F}'(\mathbf{0}, 0, \mathbf{0}, 0, \mathbf{0})$ is onto.

Arguing as in [46], all this can be accomplished satisfactorily. As a result, (3.37) can be solved for small initial data $\mathbf{y}_0$ and the local null controllability of (3.5) holds.

3.5.3 On global controllability properties

It is unknown whether a general global null controllability result holds for (3.5). This is not surprising, since the same question is also open for the Navier-Stokes system.

What can be proved (as well as for the Navier-Stokes system) is the null controllability for large time: for any given $\mathbf{y}_0 \in \mathbf{H}$, there exists $T_* = T_*(\|\mathbf{y}_0\|)$ such that (3.5) can be driven exactly to zero with controls $\mathbf{v}_\alpha$ uniformly bounded in $L^\infty(0,T_*;\mathbf{L}^2(\omega))$.

Indeed, let ε be the constant furnished by Theorem 3.1 corresponding to the time $T = 1$ (for instance). Let us first take $\mathbf{v}_\alpha \equiv \mathbf{0}$. Then, since the solution to (3.3) with $\mathbf{f} = \mathbf{0}$ satisfies $\|\mathbf{y}_\alpha(t)\| \searrow 0$, there exists T_0 (depending on $\|\mathbf{y}_0\|$ but not on α) such that $\|\mathbf{y}_\alpha(T_0)\| \le \varepsilon$. Therefore, there exist controls $\mathbf{v}'_\alpha \in L^\infty(T_0, T_0+1; \mathbf{L}^2(\omega))$ such that the solution to (3.5) that starts from $\mathbf{y}_\alpha(T_0)$ at time T_0 satisfies $\mathbf{y}_\alpha(T_0+1) = \mathbf{0}$. Hence, the assertion is fulfilled with $T_* = T_0 + 1$ and

$$\mathbf{v}_\alpha = \begin{cases} \mathbf{0} & \text{for} \quad 0 \le t < T_0, \\ \mathbf{v}'_\alpha & \text{for} \quad T_0 \le t \le T_*. \end{cases}$$

A similar argument leads to the null controllability of (3.5) for large α. In other words, it is also true that, for any given $\mathbf{y}_0 \in \mathbf{H}$ and $T > 0$, there exists $\alpha_0 = \alpha_0(\|\mathbf{y}_0\|, T)$ such that, if $\alpha \ge \alpha_0$, then (3.5) can be driven exactly to zero at time T.

3.5.4 The Burgers-α system

There exist similar results for a regularized version of the Burgers equation, more precisely the *Burgers-α system*

$$\begin{cases} y_t - y_{xx} + z y_x = v 1_{(a,b)} & \text{in} \quad (0,L) \times (0,T), \\ z - \alpha^2 z_{xx} = y & \text{in} \quad (0,L) \times (0,T), \\ y(0,t) = y(L,t) = z(0,t) = z(L,t) = 0 & \text{on} \quad (0,T), \\ y(x,0) = y_0(x) & \text{in} \quad (0,L). \end{cases} \tag{3.38}$$

These have been proved in [3].

This system can be viewed as a toy or preliminary model of (3.5). There are, however, several important differences between (3.5) and (3.38):

- The solution to (3.38) satisfies a maximum principle that provides a useful L^∞-estimate.
- There is no apparent energy decay for the uncontrolled solutions. As a consequence, the large time null controllability of (3.38) is unknown.
- It is known that, in the limit $\alpha = 0$, i.e. for the Burgers equation, global null controllability does not hold; consequently, in general, the null controllability of (3.38) with controls bounded independently of α is impossible.

We refer to [3] for further details.

3.5.5 Local exact controllability to the trajectories

It makes sense to consider not only null controllability but also *exact to the trajectories* controllability problems for (3.5). More precisely, let $\overline{\mathbf{y}}_0 \in \mathbf{H}$ be given and let $(\overline{\mathbf{y}}, \overline{p}, \overline{\mathbf{z}}, \overline{\pi})$ a sufficiently regular solution to (3.3) for $\mathbf{f} \equiv \mathbf{0}$ and $\mathbf{y}_0 = \overline{\mathbf{y}}_0$. Then the question is whether, for any given $\mathbf{y}_0 \in \mathbf{H}$, there exist controls $\mathbf{v}$ such that the associated states, i.e. the associated solutions to (3.5), satisfy

$$\mathbf{y}(T) = \overline{\mathbf{y}}(T) \quad \text{in} \quad \Omega.$$

The change of variables

$$\mathbf{y} = \overline{\mathbf{y}} + \mathbf{u}, \quad \mathbf{z} = \overline{\mathbf{z}} + \mathbf{w},$$

allows to rewrite this problem as the null controllability of a system similar, but not identical, to (3.5). It is thus reasonable to expect that a local result holds.

3.5.6 Controlling with few scalar controls

The local null controllability with $N-1$ or even less scalar controls is also an interesting question.

In view of the achievements in [12] and [33] for the Navier-Stokes equations, it is reasonable to expect that results similar to Theorems 3.1 and 3.3 hold with controls $\mathbf{v}$ such that $v_i \equiv 0$ for some i; under some geometrical restrictions, it is also expectable that local exact controllability to the trajectories holds with controls of the same kind, see [47].

3.5.7 Other related controllability problems

There are many other interesting questions concerning the controllability of (3.5) and related systems.

For instance, we can consider questions like those above for the Leray-α equations completed with other boundary conditions : Navier, Fourier or periodic conditions for $\mathbf{y}$ and $\mathbf{z}$, conditions of different kinds on different parts of the boundary, etc. We can also consider Boussinesq-α systems, i.e. systems of the form

$$\begin{cases} \mathbf{y}_t - \Delta\mathbf{y} + (\mathbf{z}\cdot\nabla)\mathbf{y} + \nabla p = \theta\mathbf{k} + \mathbf{v}1_\omega & \text{in} \quad Q, \\ \theta_t - \Delta\theta + \mathbf{z}\cdot\nabla\theta = \mathbf{w}1_\omega & \text{in} \quad Q, \\ \mathbf{z} - \alpha^2\Delta\mathbf{z} + \nabla\pi = \mathbf{y} & \text{in} \quad Q, \\ \nabla\cdot\mathbf{y} = 0, \ \nabla\cdot\mathbf{z} = 0 & \text{in} \quad Q, \\ \mathbf{y} = \mathbf{z} = \mathbf{0}, \ \theta = 0 & \text{on} \quad \Sigma, \\ \mathbf{y}(0) = \mathbf{y}_0, \ \theta(0) = \theta_0 & \text{in} \quad \Omega. \end{cases}$$

Some of these results will be analyzed in a forthcoming paper.

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Chapter 4

On the boundary controllability of incompressible Euler fluids with Boussinesq heat effects

On the boundary controllability of incompressible Euler fluids with Boussinesq heat effects

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Abstract. This paper deals with the boundary controllability of inviscid incompressible fluids for which thermal effects are important. They will be modeled through the so called Boussinesq approximation. In the zero heat diffusion case, by adapting and extending some ideas from J.-M. Coron and O. Glass, we establish the simultaneous global exact controllability of the velocity field and the temperature for 2D and 3D flows. When the heat diffusion coefficient is positive, we present some additional results concerning exact controllability for the velocity field and local null controllability of the temperature.

4.1 Introduction

Let $\Omega \subset \mathbb{R}^N$ be a nonempty, bounded and connected open set whose boundary $\Gamma := \partial\Omega$ is of class C^∞, with $N = 2$ or $N = 3$. Let $\Gamma_0 \subset \Gamma$ be a (small) nonempty open subset of Γ and assume that $T > 0$. For simplicity, we assume that Ω is simply connected.

In the sequel, we will denote by C a generic positive constant; spaces of $\mathbb{R}^N$-valued functions, as well as their elements, are represented by boldfaced letters; we will denote by $\mathbf{n} = \mathbf{n}(\mathbf{x})$ the outward unit normal to Ω at points $\mathbf{x} \in \Gamma$.

In this work, we will be concerned with the boundary controllability of the system:

$$\begin{cases} \mathbf{y}_t + (\mathbf{y}\cdot\nabla)\mathbf{y} = -\nabla p + \vec{\mathbf{k}}\,\theta & \text{in} \quad \Omega\times(0,T), \\ \nabla\cdot\mathbf{y} = 0 & \text{in} \quad \Omega\times(0,T), \\ \theta_t + \mathbf{y}\cdot\nabla\theta = \kappa\,\Delta\theta & \text{in} \quad \Omega\times(0,T), \\ \mathbf{y}\cdot\mathbf{n} = 0 & \text{on} \quad (\Gamma\setminus\Gamma_0)\times(0,T), \\ \mathbf{y}(\mathbf{x},0) = \mathbf{y}_0(\mathbf{x}), \;\; \theta(\mathbf{x},0) = \theta_0(\mathbf{x}) & \text{in} \quad \Omega. \end{cases} \tag{4.1}$$

This system models the behavior of an incompressible homogeneous inviscid fluid with thermal effects. More precisely,

- The field $\mathbf{y}$ and the scalar function p stand for the velocity and the pressure of the fluid in $\Omega\times(0,T)$, respectively.
- The function θ provides the temperature distribution of the fluid.
- The right hand side $\vec{\mathbf{k}}\,\theta$ can be viewed as the *buoyancy force* density ($\vec{\mathbf{k}} \in \mathbb{R}^N$ is a non-zero vector).

- The nonnegative constant $\kappa \geq 0$ is the heat diffusion coefficient.

This system is relevant for the study and description of atmospheric and oceanographic turbulence, as well as other fluid problems where rotation and stratification play dominant roles (see e.g. [107]). In fluid mechanics, (7.4) is used to deal with buoyancy-driven flow; it describes the motion of an incompressible inviscid fluid subject to convective heat transfer under the influence of gravitational forces, see [100].

We will be concerned with the cases $\kappa = 0$ and $\kappa > 0$. When $\kappa = 0$, (7.4) is called the *incompressible inviscid Boussinesq* system.

From now on, we assume that $\alpha \in (0,1)$ and we set

$$\begin{aligned} \mathbf{C}_0^{m,\alpha}(\bar{\Omega};\mathbb{R}^N) &:= \{\, \mathbf{u} \in \mathbf{C}^{m,\alpha}(\bar{\Omega};\mathbb{R}^N) \,:\, \nabla\cdot\mathbf{u} = 0 \text{ in } \bar{\Omega},\ \mathbf{u}\cdot\mathbf{n} = 0 \text{ on } \Gamma \,\}, \\ \mathbf{C}(m,\alpha,\Gamma_0) &:= \{\, \mathbf{u} \in \mathbf{C}^{m,\alpha}(\bar{\Omega};\mathbb{R}^N) \,:\, \nabla\cdot\mathbf{u} = 0 \text{ in } \bar{\Omega},\ \mathbf{u}\cdot\mathbf{n} = 0 \text{ on } \Gamma\backslash\Gamma_0 \,\}, \end{aligned} \tag{4.2}$$

where $\mathbf{C}^{m,\alpha}(\bar{\Omega};\mathbb{R}^N)$ denotes the space of $\mathbb{R}^N$-valued functions whose m-th order derivatives are *Hölder-continuous* in $\bar{\Omega}$ with exponent α. The usual norms in the Banach spaces $\mathbf{C}^0(\bar{\Omega};\mathbb{R}^\ell)$ and $\mathbf{C}^{m,\alpha}(\bar{\Omega};\mathbb{R}^\ell)$ will be respectively denoted by $\|\cdot\|_0$ and $\|\cdot\|_{m,\alpha}$. We will also need to work with the Banach spaces $C^0([0,T];\mathbf{C}^{m,\alpha}(\bar{\Omega};\mathbb{R}^\ell))$, where the usual norms are

$$\|\mathbf{w}\|_{0,m,\alpha} := \max_{[0,T]} \|\mathbf{w}(\cdot\,,t)\|_{m,\alpha}.$$

In particular, $\|\cdot\|_{(0)}$ will stand for $\|\cdot\|_{0,0,0}$.

When $\kappa = 0$, it is appropriate to consider the exact boundary controllability problem for (7.4). In general terms, it can be stated as follows:

> *Given $\mathbf{y}_0$, $\mathbf{y}_1$, θ_0 and θ_1 in appropriate spaces with $\mathbf{y}_0\cdot\mathbf{n} = \mathbf{y}_1\cdot\mathbf{n} = 0$ on $\Gamma\backslash\Gamma_0$, find $(\mathbf{y},p,\theta)$ such that (7.4) holds and, furthermore,*
>
> $$\mathbf{y}(\mathbf{x},T) = \mathbf{y}_1(\mathbf{x}),\ \ \theta(\mathbf{x},T) = \theta_1(\mathbf{x}) \ \ \textit{in}\ \ \Omega. \tag{4.3}$$

If it is always possible to find $\mathbf{y}$, p and θ, it will be said that the incompressible inviscid Boussinesq system is *exactly controllable* for (Ω,Γ_0) at time T.

Notice that, when $\kappa = 0$, in order to determine without ambiguity a unique local in time regular solution to (7.4), it is sufficient to prescribe the normal component of the velocity on the boundary of the flow region and the full field $\mathbf{y}$ and the temperature θ on the inflow section, i.e. only where $\mathbf{y}\cdot\mathbf{n} < 0$, see for instance [99]. Hence, in this case, we can assume that the controls are given as follows:

$$\begin{cases} \mathbf{y}\cdot\mathbf{n} \text{ on } \Gamma_0\times(0,T), \text{ with } \displaystyle\int_{\Gamma_0} \mathbf{y}\cdot\mathbf{n}\,d\Gamma = 0; \\ \mathbf{y} \text{ and } \theta \text{ at any point of } \Gamma_0\times(0,T) \text{ satisfying } \mathbf{y}\cdot\mathbf{n} < 0. \end{cases}$$

The meaning of the exact controllability property is that, when it holds, we can drive the fluid from any initial state $(\mathbf{y}_0, \theta_0)$ exactly to any final state $(\mathbf{y}_1, \theta_1)$, acting only on an arbitrary small part Γ_0 of the boundary during an arbitrary small time interval $(0, T)$.

In the case $\kappa > 0$, the situation is different. Due to the *regularization effect* of the temperature equation, we cannot expect exact controllability, at least for the temperature.

In order to present a suitable boundary controllability problem, let us introduce a nonempty open subset $\gamma \subset \Gamma$. Then, the problem is the following:

> *Given $\mathbf{y}_0$, $\mathbf{y}_1$ and θ_0 in appropriate spaces with $\mathbf{y}_0 \cdot \mathbf{n} = \mathbf{y}_1 \cdot \mathbf{n} = 0$ on $\Gamma \backslash \Gamma_0$ and $\theta_0 = 0$ on $\Gamma \backslash \gamma$, find $(\mathbf{y}, p, \theta)$ with $\theta = 0$ on $(\Gamma \backslash \gamma) \times (0, T)$ such that (7.4) holds and, furthermore,*
>
> $$\mathbf{y}(\mathbf{x}, T) = \mathbf{y}_1(\mathbf{x}), \quad \theta(\mathbf{x}, T) = 0 \;\; in \;\; \Omega. \tag{4.4}$$

If it is always possible to find $\mathbf{y}$, p and θ, it will be said that the incompressible, heat diffusive, inviscid Boussinesq system (7.4) is *exactly-null controllable* for $(\Omega, \Gamma_0, \gamma)$ at time T.

Note that, if $\kappa > 0$ and we fix the same boundary data for $\mathbf{y}$ as before and (for example) Dirichlet data for θ of the form

$$\theta = \theta_* 1_\gamma \;\; \text{on} \;\; \Gamma \times (0, T),$$

there exists at most one solution to (7.4). Therefore, it can be assumed in this case that the controls are the following:

$$\begin{cases} \mathbf{y} \cdot \mathbf{n} \text{ on } \Gamma_0 \times (0, T), \text{ with } \displaystyle\int_{\Gamma_0} \mathbf{y} \cdot \mathbf{n} \, d\Gamma = 0; \\ \mathbf{y} \text{ at any point of } \Gamma_0 \times (0, T) \text{ satisfying } \mathbf{y} \cdot \mathbf{n} < 0; \\ \theta \text{ at any point of } \gamma \times (0, T). \end{cases}$$

Of course, the meaning of the exact-null controllability property is that, when it holds, we can drive the fluid velocity-temperature pair from any initial state $(\mathbf{y}_0, \theta_0)$ exactly to any final state of the form $(\mathbf{y}_1, 0)$, acting only on arbitrary small parts Γ_0 and γ of the boundary during an arbitrary small time interval $(0, T)$.

In the last decades, a lot of researchers has focused attention on the controllability of systems governed by (linear and nonlinear) PDEs. Some related results can be found in [30, 71, 91, 121]. In the context of incompressible ideal fluids, this subject has been mainly investigated by Coron [27, 29] and Glass [65, 66, 67].

In this paper, our first task will be to adapt the techniques and arguments of [29] and [67] to the situations modeled by (7.4). Thus, our first main result is the following:

Theorem 4.1. *If $\kappa = 0$, then the incompressible inviscid Boussinesq system (7.4) is exactly controllable for (Ω, Γ_0) at any time $T > 0$. More precisely, for any $\mathbf{y}_0, \mathbf{y}_1 \in \mathbf{C}(2, \alpha, \Gamma_0)$ and any $\theta_0, \theta_1 \in C^{2,\alpha}(\bar{\Omega})$, there exist $\mathbf{y} \in C^0([0, T]; \mathbf{C}(1, \alpha, \Gamma_0))$, $\theta \in C^0([0, T]; C^{1,\alpha}(\bar{\Omega}))$ and $p \in \mathcal{D}'(\Omega \times (0, T))$ such that one has (7.4) and (4.3).*

The proof of Theorem 4.1 relies on the *extension* and *return* methods. These have been applied in several different contexts to establish controllability; see the seminal works [111] and [26]; see also a long list of applications in [30].

Let us give a sketch of the strategy used in the proof of Theorem 4.1:

- First, we construct a "good" trajectory connecting $(\mathbf{0}, 0)$ to $(\mathbf{0}, 0)$ (see Sections 4.2.1 and 4.2.2).
- Then, we apply the extension method of David L. Russell [111].
- Then, we use a *Fixed-Point Theorem* and we deduce a local exact controllability result.
- Finally, we use an appropriate scaling argument and we obtain the desired global result.

In fact, Theorem 4.1 is a consequence of the following local result:

Proposition 4.1. *Let us assume that* $\kappa = 0$. *There exists* $\delta > 0$ *such that, for any* $\mathbf{y}_0 \in \mathbf{C}(2, \alpha, \Gamma_0)$ *and any* $\theta_0 \in C^{2,\alpha}(\bar{\Omega})$ *with*

$$\max \{\|\mathbf{y}_0\|_{2,\alpha}, \|\theta_0\|_{2,\alpha}\} \leq \delta,$$

there exist $\mathbf{y} \in C^0([0,1]; \mathbf{C}(1, \alpha, \Gamma_0))$, $\theta \in C^0([0,1]; C^{1,\alpha}(\bar{\Omega}))$ *and* $p \in \mathcal{D}'(\Omega \times (0,1))$ *satisfying* (7.4) *in* $\Omega \times (0,1)$ *and the final conditions*

$$\mathbf{y}(\mathbf{x}, 1) = \mathbf{0}, \quad \theta(\mathbf{x}, 1) = 0 \ \ in \ \ \Omega. \tag{4.5}$$

It will be seen later that, in our argument, the $C^{2,\alpha}$-regularity of the initial and final data is needed. However, we can only ensure the existence of a controlled solution that is $C^{1,\alpha}$ in space. It would be interesting to improve this result but, at present, we do not know how.

Our second main result is the following:

Theorem 4.2. *Let* Ω, Γ_0 *and* γ *be given and let us assume that* $\kappa > 0$. *Then* (7.4) *is locally exactly-null controllable. More precisely, for any* $T > 0$ *and any* $\mathbf{y}_0, \mathbf{y}_1 \in \mathbf{C}(2, \alpha, \emptyset)$, *there exists* $\eta > 0$, *depending on* $\mathbf{y}_0$, *such that, for each* $\theta_0 \in C^{2,\alpha}(\bar{\Omega})$ *with*

$$\theta_0 = 0 \ \ on \ \ \Gamma \backslash \gamma, \quad \|\theta_0\|_{2,\alpha} \leq \eta,$$

we can find $\mathbf{y} \in C^0([0,T]; \mathbf{C}^{1,\alpha}(\bar{\Omega}; \mathbb{R}^N)$, $\theta \in C^0([0,T]; C^{1,\alpha}(\bar{\Omega}))$ *with* $\theta = 0$ *on* $(\Gamma \backslash \gamma) \times (0,T)$, *and* $p \in \mathcal{D}'(\Omega \times (0,T))$ *satisfying* (7.4) *and* (4.4).

The proof relies on the following strategy. First, we linearize and control only the temperature θ; this leads the system to a state of the form $(\tilde{\mathbf{y}}_0, 0)$ at (say) time $T/2$. Then, in a second step, we control the velocity field using in part Theorem 4.1. It will be seen that, in order to get good estimates and prove the existence of a fixed point, the initial temperature θ_0 must be small.

To our knowledge, it is unknown whether a global exact-null controllability result holds for (7.4) when $\kappa > 0$. Unfortunately, the cost of controlling θ grows exponentially with the L^∞-norm of the transporting velocity field $\mathbf{y}$ and this is a crucial difficulty to establish estimates independent of the size of the initial data.

The rest of this paper is organized as follows. In Section 4.2, we recall the results needed to prove Theorems 4.1 and 4.2. In Section 4.3, we give the proof of Theorem 4.1. In Section 4.4, we prove Proposition 4.1 in the 2D case; it will be seen that the main ingredients of the proof are the construction of a nontrivial trajectory that starts and ends at $(\mathbf{0}, 0)$ and a Fixed-Point Theorem (the key ideas of the return method). In Section 4.5, we give the proof of Theorem 4.1 in the 3D case. Finally, Section 4.6 contains the proof of Theorem 4.2.

4.2 Preliminary results

In this section, we are going to recall some results used in the proofs of Theorems 4.1 and 4.2. Also, we are going to indicate how to construct a trajectory appropriate to apply the return method.

The following result is an immediate consequence of Banach's Fixed-Point Theorem:

Theorem 4.3. *Let $(B_1, \|\cdot\|_1)$ and $(B_2, \|\cdot\|_2)$ be Banach spaces with B_2 continuously embedded in B_1. Let B be a subset of B_2 and let $G : B \mapsto B$ be a uniformly continuous mapping such that, for some $m \geq 1$ and some $\gamma \in [0, 1)$, one has*

$$\|G^m(u) - G^m(v)\|_1 \leq \gamma \|u - v\|_1 \;\; \forall u, v \in B.$$

Let us denote by $\overline{B}$ the closure of B for the norm $\|\cdot\|_1$. Then, G can be uniquely extended to a continuous mapping $\widetilde{G} : \overline{B} \mapsto \overline{B}$ that possesses a unique fixed-point in $\overline{B}$.

Later, the following lemma will be very important to deduce appropriate estimates. The proof can be found in [4].

Lemma 4.1. *Let m be a nonnegative integer. Assume that $u \in C^0([0, T]; C^{m+1,\alpha}(\overline{\Omega}))$, $g \in C^0([0, T]; C^{m,\alpha}(\overline{\Omega}))$ and $\mathbf{v} \in C^0([0, T]; \mathbf{C}^{m,\alpha}(\overline{\Omega}; \mathbb{R}^N))$ are given, with $\mathbf{v} \cdot \mathbf{n} = 0$ on $\Gamma \times (0, T)$ and*

$$\frac{\partial u}{\partial t} + \mathbf{v} \cdot \nabla u = g \;\text{ in }\; \Omega \times (0, T). \tag{4.6}$$

Then, $u_t \in C^0([0, T]; C^{m,\alpha}(\overline{\Omega}))$ and, for any $m \geq 1$,

$$\frac{d}{dt^+} \|u(\cdot\,, t)\|_{m,\alpha} \leq \|g(\cdot\,, t)\|_{m,\alpha} + K \|\mathbf{v}(\cdot\,, t)\|_{m,\alpha} \|u(\cdot\,, t)\|_{m,\alpha} \quad \text{in} \quad (0, T),$$

where K is a constant only depending on α and m. If $m = 0$, the following holds

$$\frac{d}{dt^+}\|u(\cdot\,,t)\|_{0,\alpha} \leq \|g(\cdot\,,t)\|_{0,\alpha} + \alpha\|\nabla \mathbf{v}(\cdot\,,t)\|_{0,\alpha}\|u(\cdot\,,t)\|_{0,\alpha} \quad \textit{in} \quad (0,T).$$

From Lemma 4.1 and a standard regularization argument, we easily deduce the following:

Lemma 4.2. *Let m be a nonnegative integer. Assume that $u \in C^0([0,T];C^{m,\alpha}(\overline{\Omega}))$, $g \in C^0([0,T];C^{m,\alpha}(\overline{\Omega}))$ and $\mathbf{v} \in C^0([0,T];\mathbf{C}^{m,\alpha}(\overline{\Omega};\mathbb{R}^N))$ are given, with $\mathbf{v}\cdot\mathbf{n} = 0$ on $\Gamma \times (0,T)$ and*

$$\frac{\partial u}{\partial t} + \mathbf{v}\cdot\nabla u = g \ \ \textit{in} \ \ \Omega \times (0,T). \tag{4.7}$$

Then

$$\|u\|_{0,m,\alpha} \leq \left(\int_0^T \|g(\cdot\,,t)\|_{m,\alpha}\,dt + \|u(\cdot\,,0)\|_{m,\alpha}\right)\exp\left(K\int_0^T \|\mathbf{v}(\cdot\,,t)\|_{m,\alpha}\,dt\right),$$

where K is a constant only depending on α and m.

We will also use a technical lemma whose proof can be found in [65]:

Lemma 4.3. *Let us assume that*

$$\begin{array}{rll} \mathbf{w}_0 \in \mathbf{C}^{1,\alpha}(\bar{\Omega};\mathbb{R}^N), \ \ \nabla\cdot\mathbf{w}_0 = 0 & \textit{in} & \Omega, \\ \mathbf{u} \in C^0([0,T];\mathbf{C}^{1,\alpha}(\bar{\Omega};\mathbb{R}^N)), \ \ \mathbf{u}\cdot\mathbf{n} = 0 & \textit{on} & \Gamma\times(0,T), \\ \mathbf{g} \in C^0([0,T];\mathbf{C}^{0,\alpha}(\bar{\Omega},\mathbb{R}^N)), \ \ \nabla\cdot\mathbf{g} = 0 & \textit{in} & \Omega\times(0,T). \end{array}$$

Let $\mathbf{w}$ be a function in $C^0([0,T];\mathbf{C}^{1,\alpha}(\overline{\Omega};\mathbb{R}^N))$ satisfying

$$\begin{cases} \mathbf{w}_t + (\mathbf{u}\cdot\nabla)\mathbf{w} = (\mathbf{w}\cdot\nabla)\mathbf{u} - (\nabla\cdot\mathbf{u})\mathbf{w} + \mathbf{g} & \textit{in} \quad \Omega\times(0,T), \\ \mathbf{w}(\cdot\,,0) = \mathbf{w}_0 & \textit{in} \quad \Omega. \end{cases}$$

Then, $\nabla\cdot\mathbf{w} \equiv 0$. Moreover, there exists $\mathbf{v} \in C^0([0,T];\mathbf{C}^{2,\alpha}(\overline{\Omega};\mathbb{R}^N))$ such that

$$\mathbf{w} = \nabla\times\mathbf{v} \ \ \textit{in} \ \ \Omega\times(0,T).$$

To end this section, we will recall a well known result dealing with the null controllability of general parabolic linear systems of the form

$$\begin{cases} u_t - \kappa\Delta u + \mathbf{w}\cdot\nabla u = v1_\omega & \text{in} \quad \Omega\times(0,T), \\ u = 0 & \text{on} \quad \Gamma\times(0,T), \\ u(\mathbf{x},0) = u_0(\mathbf{x}) & \text{in} \quad \Omega, \end{cases} \tag{4.8}$$

where $\kappa > 0$, $\mathbf{w} \in L^\infty(\Omega\times(0,T))$, $\omega \subset \Omega$ is a non-empty open set and 1_ω is the characteristic function of ω.

It is well known that, for each $u_0 \in L^2(\Omega)$ and each $v \in L^2(\omega \times (0,T))$, there exists exactly one solution u to (4.8), with

$$u \in C^0([0,T]; L^2(\Omega)) \cap L^2(0,T; H_0^1(\Omega)).$$

We also have:

Theorem 4.4. *The linear system (4.8) is null-controllable at any time $T > 0$. In other words, for each $u_0 \in L^2(\Omega)$ there exists $v \in L^2(\omega \times (0,T))$ such that the associated solution to (4.8) satisfies*

$$u(\mathbf{x}, T) = 0 \ \text{ in } \ \Omega. \tag{4.9}$$

Furthermore, the extremal problem

$$\begin{cases} \text{Minimize } \dfrac{1}{2} \displaystyle\iint_{\omega \times (0,T)} |v|^2 \, dx \, dt \\ \text{Subject to: } v \in L^2(\omega \times (0,T)),\ u \text{ satisfies (4.9)} \end{cases} \tag{4.10}$$

possesses exactly one solution $\hat{v}$ satisfying

$$\|\hat{v}\|_2 \leq C_0 \|u_0\|_2, \tag{4.11}$$

where

$$C_0 = \exp\left(C_1 \left(1 + \frac{1}{T} + (1 + T^2) \|\mathbf{w}\|_\infty^2 \right) \right)$$

and C_1 only depends on Ω, ω and κ.

4.2.1 Construction of a trajectory when $N = 2$

We will argue as in [29]. Thus, let $\Omega_1 \subset \mathbb{R}^2$ be a bounded, Lipschitz-contractible open set whose boundary is of class C^∞ and consists of two disjoint closed line segments Γ^- and Γ^+ and two disjoint curves Σ' and Σ'' of class C^∞ such that $\partial\Sigma' \cup \partial\Sigma'' = \partial\Gamma^- \cup \partial\Gamma^+$.

We assume that $\Omega \subset \Omega_1$. We also impose that there is a neighborhood U^- of Γ^- (resp. U^+ of Γ^+) such that $\Omega_1 \cap U^-$ (resp. $\Omega_1 \cap U^+$) coincides with the intersection of U^- (resp. U^+), an open semi-plane limited by the line containing Γ^- (resp. Γ^+) and the band limited by the two straight lines orthogonal to Γ^- (resp. Γ^+) and passing through $\partial\Gamma^-$ (resp. $\partial\Gamma^+$); see Figure 4.1.

Let φ be the solution to

$$\begin{cases} -\Delta\varphi = 0 & \text{in} \quad \Omega_1, \\ \varphi = 1 & \text{on} \quad \Gamma^+, \\ \varphi = -1 & \text{on} \quad \Gamma^-, \\ \dfrac{\partial\varphi}{\partial n} = 0 & \text{on} \quad \Sigma, \end{cases} \tag{4.12}$$

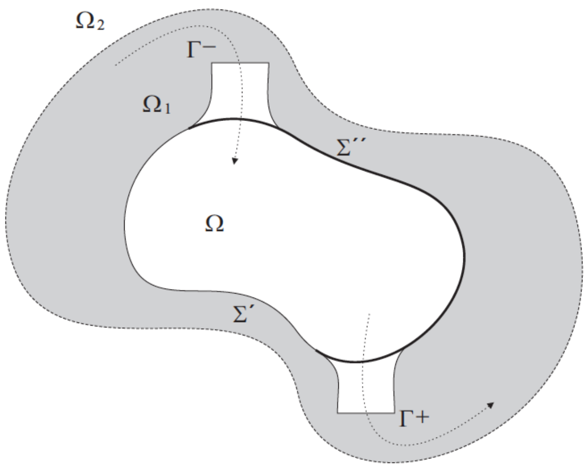

Figure 4.1: The domain Ω_1

where $\Sigma = \Sigma' \cup \Sigma''$. Then, we have the following result from J.-M. Coron [29]:

Lemma 4.4. *One has* $\varphi \in C^\infty(\bar{\Omega}_1)$, $-1 < \varphi(\mathbf{x}) < 1$ *for all* $\mathbf{x} \in \Omega_1$ *and*

$$\nabla\varphi(\mathbf{x}) \neq \mathbf{0} \quad \forall \mathbf{x} \in \bar{\Omega}_1. \tag{4.13}$$

Let $\gamma \in C^\infty([0,1])$ be a non-zero function such that $\operatorname{Supp}\gamma \subset (0,1/2) \cup (1/2,1)$ and the sets $(\operatorname{Supp}\gamma) \cap (0,1/2)$ and $(\operatorname{Supp}\gamma) \cap (1/2,1)$ are non-empty.

Let $M > 0$ be a constant to be chosen below and set

$$\overline{\mathbf{y}}(\mathbf{x},t) := M\gamma(t)\nabla\varphi(\mathbf{x}), \;\; \overline{p}(\mathbf{x},t) := -M\gamma_t(t)\varphi(\mathbf{x}) - \frac{M^2}{2}\gamma(t)^2|\nabla\varphi(\mathbf{x})|^2, \;\; \overline{\theta} \equiv 0.$$

Then (7.4) is satisfied by $(\overline{\mathbf{y}}, \overline{p}, \overline{\theta})$ for $T = 1$, $\mathbf{y}_0 = \mathbf{0}$ and $\theta_0 = 0$. The triplet $(\overline{\mathbf{y}}, \overline{p}, \overline{\theta})$ is thus a nontrivial trajectory of (7.4) that connects the zero state to itself.

Let Ω_3 be a bounded open set of class C^∞ such that $\Omega_1 \subset\subset \Omega_3$. We extend φ to $\bar{\Omega}_3$ as a C^∞ function with compact support in Ω_3 and we still denote this extension by φ. Let us introduce $\mathbf{y}^*(\mathbf{x},t) := M\gamma(t)\nabla\varphi(\mathbf{x})$ (observe that $\overline{\mathbf{y}}$ is the restriction of $\mathbf{y}^*$ to $\bar{\Omega} \times [0,1]$). Also, consider the associated *flux function* $\mathbf{Y}^* : \bar{\Omega}_3 \times [0,1] \times [0,1] \mapsto \bar{\Omega}_3$, defined as follows:

$$\begin{cases} \mathbf{Y}^*_t(\mathbf{x},t,s) = \mathbf{y}^*(\mathbf{Y}^*(\mathbf{x},t,s),t) \\ \mathbf{Y}^*(\mathbf{x},s,s) = \mathbf{x}. \end{cases} \tag{4.14}$$

Obviously, $\mathbf{Y}^*$ contains all the information on the trajectories of the particles transported by the velocity field $\mathbf{y}^*$. The flux $\mathbf{Y}^*$ is of class C^∞ in $\bar{\Omega}_3 \times [0,1] \times [0,1]$. Furthermore, $\mathbf{Y}^*(\cdot\,,t,s)$ is a diffeomorphism of $\bar{\Omega}_3$ onto itself and $(\mathbf{Y}^*(\cdot\,,t,s))^{-1} = \mathbf{Y}^*(\cdot\,,s,t)$ for all $s,t \in [0,1]$.

Remark 4.1. From the definition of $\mathbf{y}^*$ and the boundary conditions on Ω_1 satisfied by

φ, we observe that the particles cannot cross Σ. Since φ is constant on Γ^+, the gradient $\nabla\varphi$ is parallel to the normal vector on Γ^+. Since φ attains a maximum at any point of Γ^+, we have $\nabla\varphi \cdot \mathbf{n} > 0$ on Γ^+, whence $\mathbf{y}^* \cdot \mathbf{n} \geq 0$ on $\Gamma^+ \times [0,1]$. Similarly, $\mathbf{y}^* \cdot \mathbf{n} \leq 0$ on $\Gamma^- \times [0,1]$. Consequently, the particles moving with velocity $\mathbf{y}^*$ can leave Ω_1 only through Γ^+ and can enter Ω_1 only through Γ^-. □

The following lemma shows that the particles that travel with velocity $\mathbf{y}^*$ and are inside $\bar{\Omega}_1$ at time $t = 0$ (resp. $t = 1/2$) will be outside $\bar{\Omega}_1$ at time $t = 1/2$ (resp. $t = 1$).

Lemma 4.5. *There exist $M > 0$ (large enough) and a bounded open set Ω_2 satisfying $\Omega_1 \subset\subset \Omega_2 \subset\subset \Omega_3$ such that*

$$\mathbf{Y}^*(\mathbf{x}, 1/2, 0) \notin \bar{\Omega}_2 \;\; \textit{and} \;\; \mathbf{Y}^*(\mathbf{x}, 1, 1/2) \notin \bar{\Omega}_2 \;\; \forall \mathbf{x} \in \overline{\Omega}_2. \tag{4.15}$$

The proof is given in [29] and relies on the properties of $\mathbf{y}^*$ and, more precisely, on the fact that $t \mapsto \varphi(\mathbf{Y}^*(\mathbf{x}, t, s))$ is nondecreasing.

The next step is to introduce appropriate extension mappings from Ω to Ω_3. We have the following result from [76]:

Lemma 4.6. *For $\ell = 1$ and $\ell = 2$, there exist continuous linear mappings $\pi_\ell : \mathbf{C}^0(\overline{\Omega}; \mathbb{R}^\ell) \mapsto \mathbf{C}^0(\bar{\Omega}_3; \mathbb{R}^\ell)$ such that*

$$\begin{cases} \pi_\ell(\mathbf{f}) = \mathbf{f} \;\; \textit{in} \;\; \Omega \;\; \textit{and} \;\; \textit{Supp}\, \pi_\ell(\mathbf{f}) \subset \Omega_2 \quad \forall \mathbf{f} \in \mathbf{C}^0(\overline{\Omega}; \mathbb{R}^\ell), \\ \pi_\ell \textit{ maps continuously } \mathbf{C}^{m,\lambda}(\overline{\Omega}; \mathbb{R}^\ell) \textit{ into } \mathbf{C}^{m,\lambda}(\overline{\Omega_3}; \mathbb{R}^\ell) \quad \forall m \geq 0, \;\; \forall \lambda \in (0,1). \end{cases}$$

The next lemma asserts that (4.15) holds not only for $\mathbf{y}^*$ but also for any appropriate extension of any flow $\mathbf{z}$ close enough to $\overline{\mathbf{y}}$:

Lemma 4.7. *For each $\mathbf{z} \in C^0(\bar{\Omega} \times [0,1]; \mathbb{R}^2)$, let us set $\mathbf{z}^* = \mathbf{y}^* + \pi_2(\mathbf{z} - \overline{\mathbf{y}})$. There exists $\nu > 0$ such that, if $\|\mathbf{z} - \overline{\mathbf{y}}\|_{(0)} \leq \nu$, then*

$$\mathbf{Z}^*(\mathbf{x}, 1/2, 0) \notin \bar{\Omega}_2 \;\; \textit{and} \;\; \mathbf{Z}^*(\mathbf{x}, 1, 1/2) \notin \bar{\Omega}_2 \;\; \forall \mathbf{x} \in \overline{\Omega}_2, \tag{4.16}$$

where $\mathbf{Z}^*$ is the flux function associated to $\mathbf{z}^*$.

Proof. Let us set

$$\mathbf{A} = \left\{ \mathbf{Y}^*(\mathbf{x}, 1/2, 0) \; : \; \mathbf{x} \in \bar{\Omega}_2 \right\} \cup \left\{ \mathbf{Y}^*(\mathbf{x}, 1, 1/2) \; : \; \mathbf{x} \in \bar{\Omega}_2 \right\}.$$

Both $\mathbf{A}$ and $\bar{\Omega}_2$ are compact subsets of $\mathbb{R}^2$ and, in view of Lemma 4.5, $\mathbf{A} \cap \bar{\Omega}_2 = \emptyset$. Consequently, $d := \text{dist}\,(\mathbf{A},\, \bar{\Omega}_2) > 0$.

Let us introduce $\mathbf{W} := \mathbf{Y}^* - \mathbf{Z}^*$. Then, in view of the *Mean Value Theorem* and the properties of π_2, we have:

$$\begin{aligned}|\mathbf{W}(\mathbf{x},t,s)| &\leq M\int_s^t \gamma(\sigma)|\nabla\varphi(\mathbf{Y}^*(\mathbf{x},\sigma,s)) - \nabla\varphi(\mathbf{Z}^*(\mathbf{x},\sigma,s))|\,d\sigma \\ &\quad + \int_s^t |\pi_2(\mathbf{z}-\overline{\mathbf{y}})(\mathbf{Z}^*(\mathbf{x},\sigma,s),\sigma)|\,d\sigma \\ &\leq M\|\nabla\varphi\|_0 \int_s^t \gamma(\sigma)|\mathbf{W}(\mathbf{x},\sigma,s)|\,d\sigma + \int_s^t \|(\pi_2(\mathbf{z}-\overline{\mathbf{y}}))(\cdot,\sigma)\|_0\,d\sigma \\ &\leq M\|\nabla\varphi\|_0 \int_s^t \gamma(\sigma)|\mathbf{W}(\mathbf{x},\sigma,s)|\,d\sigma + C\int_s^t \|(\mathbf{z}-\overline{\mathbf{y}})(\cdot,\sigma)\|_0\,d\sigma,\end{aligned}$$

where $(\mathbf{x},t,s) \in \bar{\Omega}_3 \times [0,1] \times [0,1]$. Hence, from Gronwall's Lemma, we find that

$$\begin{aligned}|\mathbf{W}(\mathbf{x},t,s)| &\leq C\left(\int_s^t \|\mathbf{z}-\overline{\mathbf{y}}\|_0(\sigma)\,d\sigma\right)\exp\left(M\|\nabla\varphi\|_0\int_s^t \gamma(\sigma)\,d\sigma\right) \\ &\leq Ce^{M\|\nabla\varphi\|_0\|\gamma\|_0}\|\mathbf{z}-\overline{\mathbf{y}}\|_{(0)}\end{aligned}$$

Therefore, there exists $\nu > 0$ such that, if $\|\mathbf{z}-\overline{\mathbf{y}}\|_{(0)} \leq \nu$, one has

$$|\mathbf{W}(\mathbf{x},t,s)| \leq \frac{d}{2} \quad \forall (\mathbf{x},t,s) \in \bar{\Omega}_3 \times [0,1] \times [0,1]. \tag{4.17}$$

Thanks to Lemma 4.5 and (4.17), we necessarily have (4.16) and the proof is achieved. □

4.2.2 Construction of a trajectory when $N = 3$

In this Section, we will follow [67]. As in the two-dimensional case, $\overline{\mathbf{y}}$ will be of the potential form "$\nabla\varphi$", with the property that any particle traveling with velocity $\overline{\mathbf{y}}$ must leave $\overline{\Omega}$ at an appropriate time. The main difference will be that, in this three-dimensional case, "$\nabla\varphi$" is not chosen independent of t.

We first recall a lemma:

Lemma 4.8. *Let $\mathscr{O}$ be a regular bounded open set such that $\Omega \subset\subset \mathscr{O}$. For each $\mathbf{a} \in \bar{\Omega}$, there exists $\phi^{\mathbf{a}} \in C^\infty(\overline{\mathscr{O}} \times [0,1])$ such that $\operatorname{supp}(\phi^{\mathbf{a}}) \subset \mathscr{O} \times (1/4, 3/4)$,*

$$\begin{cases} -\Delta\phi^{\mathbf{a}} = 0 & \text{in} \quad \Omega \times (0,1), \\ \dfrac{\partial\phi^{\mathbf{a}}}{\partial\mathbf{n}} = 0 & \text{on} \quad (\Gamma \setminus \Gamma_0) \times (0,1) \end{cases} \tag{4.18}$$

and

$$\mathbf{\Phi}^{\mathbf{a}}(\mathbf{a},1,0) \in \mathscr{O} \setminus \overline{\Omega},$$

where $\mathbf{\Phi}^{\mathbf{a}} := \mathbf{\Phi}^{\mathbf{a}}(\mathbf{x},t,s)$ is the flux associated to $\nabla\phi^{\mathbf{a}}$, that is, the unique $\mathbb{R}^N-$valued function *in* $\overline{\mathscr{O}} \times [0,1] \times [0,1]$ *satisfying*

$$\begin{cases} \mathbf{\Phi}^{\mathbf{a}}_t(\mathbf{x},t,s) = \nabla\phi^{\mathbf{a}}(\mathbf{\Phi}^{\mathbf{a}}(\mathbf{x},t,s),t), \\ \mathbf{\Phi}^{\mathbf{a}}(\mathbf{x},s,s) = \mathbf{x}. \end{cases} \tag{4.19}$$

The proof is given in [67].

With the help of these $\mathbf{\Phi}^{\mathbf{a}}$, we can construct a vector field $\mathbf{y}^*$ in $\mathscr{O} \times (0,1)$ that makes the particles go from Ω to the outside and then makes them come back.

Indeed, from the continuity of the functions $\mathbf{\Phi}^{\mathbf{a}}$ and the compactness of $\bar{\Omega}$, we can find $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_k$ in $\bar{\Omega}$, real numbers $r_1, \dots, r_k$, smooth functions $\phi^1 := \phi^{\mathbf{a}_1}, \dots, \phi^k := \phi^{\mathbf{a}_k}$ satisfying Lemma 4.8 and a bounded open set $\mathscr{O}_0$ with $\Omega \subset\subset \mathscr{O}_0 \subset\subset \mathscr{O}$, such that

$$\bar{\Omega} \subset \bigcup_{i=1}^{k} B^i \subset\subset \mathscr{O}_0 \ \text{ and } \ \mathbf{\Phi}^i(\overline{B}^i, 1, 0) \subset \mathscr{O} \setminus \overline{\mathscr{O}}_0, \tag{4.20}$$

where $B^i := B(\mathbf{a}_i; r_i)$ and $\mathbf{\Phi}^i := \mathbf{\Phi}^{\mathbf{a}_i}$ for $i = 1, \dots, k$.

As in [67], the definition of $\mathbf{y}^*$ is as follows: let the time t_i be given by

$$\begin{aligned} t_i &= \frac{1}{4} + \frac{i}{4k}, \quad i = 0, \dots, 2k, \\ t_{i+1/2} &= \frac{1}{4} + \left(i + \frac{1}{2}\right)\frac{1}{4k}, \quad i = 0, \dots, 2k-1 \end{aligned} \tag{4.21}$$

and let us set

$$\phi(\mathbf{x},t) = \begin{cases} 0, & (\mathbf{x},t) \in \overline{\mathscr{O}} \times ([0,1/4] \cup [3/4,1]), \\ 8k\phi^j(\mathbf{x}, 8k(t - t_{j-1})), & (\mathbf{x},t) \in \overline{\mathscr{O}} \times \left[t_{j-1}, t_{j-1/2}\right], \\ -8k\phi^j(\mathbf{x}, 8k(t_j - t)), & (\mathbf{x},t) \in \overline{\mathscr{O}} \times \left[t_{j-1/2}, t_j\right] \end{cases} \tag{4.22}$$

for $j = 1, \dots, 2k$, where $\phi^{k+i} := \phi^i$ for $i = 1, \dots, k$; then, we set $\mathbf{y}^* := \nabla\phi$ and $\overline{\mathbf{y}} := \mathbf{y}^*|_{\overline{\Omega} \times [0,1]}$ and we denote by $\mathbf{Y}^*$ the flux associated to $\mathbf{y}^*$.

If we set $\overline{p}(\mathbf{x},t) := -\phi_t(\mathbf{x},t) - \frac{1}{2}|\nabla\phi(\mathbf{x},t)|^2$ and $\overline{\theta} \equiv 0$, then (7.4) and (4.3) are verified by $(\overline{\mathbf{y}}, \overline{p}, \overline{\theta})$ for $T = 1$, $\mathbf{y}_0 = \mathbf{y}_1 = \mathbf{0}$ and $\theta_0 = \theta_1 = 0$.

Thanks to (4.20) and (4.22), one has:

Lemma 4.9. *The following property holds for all* $i = 1, \dots, k$*:*

$$\mathbf{Y}^*(\mathbf{x}, t_{i-1/2}, 0) \in \mathscr{O} \setminus \overline{\mathscr{O}}_0 \quad \textit{and} \quad \mathbf{Y}^*(\mathbf{x}, t_{k+i-1/2}, 1/2) \in \mathscr{O} \setminus \overline{\mathscr{O}}_0 \quad \forall \mathbf{x} \in B^i. \tag{4.23}$$

For the proof, it suffices to notice that, in $\overline{\mathscr{O}} \times [1/4, 3/4] \times [1/4, 3/4]$, $\mathbf{Y}^*(\mathbf{x},t,s)$ is given as follows:

$$\begin{cases} \mathbf{\Phi}^j(\mathbf{x}, 8k(t-t_{j-1}), 8k(s-t_{l-1})) \text{ if } (\mathbf{x},t,s) \in \overline{\mathscr{O}} \times [t_{j-1}, t_{j-1/2}] \times [t_{l-1}, t_{l-1/2}], \\ \mathbf{\Phi}^j(\mathbf{x}, 8k(t-t_{j-1}), 8k(t_l - s)) \text{ if } (\mathbf{x},t,s) \in \overline{\mathscr{O}} \times [t_{j-1}, t_{j-1/2}] \times [t_{l-1/2}, t_l], \\ \mathbf{\Phi}^j(\mathbf{x}, 8k(t_j - t), 8k(s-t_{l-1})) \text{ if } (\mathbf{x},t,s) \in \overline{\mathscr{O}} \times [t_{j-1/2}, t_j] \times [t_{l-1}, t_{l-1/2}], \\ \mathbf{\Phi}^j(\mathbf{x}, 8k(t_j - t), 8k(t_l - s)) \text{ if } (\mathbf{x},t,s) \in \overline{\mathscr{O}} \times [t_{j-1/2}, t_j] \times [t_{l-1/2}, t_l] \end{cases}$$

for all $l, j = 1, \dots, 2k$, where $\mathbf{\Phi}^{k+i}$ the flux associated to $\nabla\phi^{k+i}$ for $i = 1, \dots, k$.

Hence, one has the following for all $i = 1, \dots, k$ and for each $\mathbf{x} \in B^i$:

$$\mathbf{Y}^*(\mathbf{x}, t_{i-1/2}, 0) = \mathbf{Y}^*(\mathbf{x}, t_{i-1/2}, 1/4) = \mathbf{Y}^*(\mathbf{x}, t_{i-1/2}, t_0) = \mathbf{\Phi}^i(\mathbf{x}, 1, 0) \in \mathscr{O} \setminus \overline{\mathscr{O}}_0$$

and

$$\mathbf{Y}^*(\mathbf{x}, t_{k+i-1/2}, 1/2) = \mathbf{Y}^*(\mathbf{x}, t_{k+i-1/2}, t_k) = \mathbf{\Phi}^{k+i}(\mathbf{x}, 1, 0) = \mathbf{\Phi}^i(\mathbf{x}, 1, 0) \in \mathscr{O} \setminus \overline{\mathscr{O}}_0.$$

A result similar to Lemma 4.6 also holds here:

Lemma 4.10. *For $\ell = 1$ and $\ell = 3$, there exist continuous linear mappings $\pi_\ell : \mathbf{C}^0(\overline{\Omega}; \mathbb{R}^\ell) \mapsto \mathbf{C}^0(\overline{\mathscr{O}}; \mathbb{R}^\ell)$ such that*

$$\begin{cases} \pi_\ell(\mathbf{f}) = \mathbf{f} \text{ in } \Omega \text{ and } Supp\, \pi_\ell(\mathbf{f}) \subset \mathscr{O}_0 \quad \forall \mathbf{f} \in \mathbf{C}^0(\overline{\Omega}; \mathbb{R}^\ell), \\ \pi_\ell \text{ maps continuously } \mathbf{C}^{n,\lambda}(\overline{\Omega}; \mathbb{R}^\ell) \text{ into } \mathbf{C}^{n,\lambda}(\overline{\mathscr{O}}; \mathbb{R}^\ell) \quad \forall n \geq 0, \quad \forall \lambda \in (0,1). \end{cases}$$

Finally, we also have that (4.23) holds for the flux corresponding to the of any velocity field close enough to $\overline{\mathbf{y}}$:

Lemma 4.11. *For each $\mathbf{z} \in C^0(\bar{\Omega} \times [0,1]; \mathbb{R}^3)$, let us set $\mathbf{z}^* = \mathbf{y}^* + \pi_3(\mathbf{z} - \overline{\mathbf{y}})$. Then there exists $\nu > 0$ such that, if $\|\mathbf{z} - \overline{\mathbf{y}}\|_{(0)} \leq \nu$ and $i = 1, \dots, k$, one has:*

$$\mathbf{Z}^*(\mathbf{x}, t_{i-1/2}, 0) \in \mathscr{O} \setminus \overline{\mathscr{O}}_0 \quad \text{and} \quad \mathbf{Z}^*(\mathbf{x}, t_{k+i-1/2}, 1/2) \in \mathscr{O} \setminus \overline{\mathscr{O}}_0 \quad \forall \mathbf{x} \in B^i,$$

where $\mathbf{Z}^*$ is the flux associated to $\mathbf{z}^*$.

The proof is very similar to the proof of Lemma 4.7 and will be omitted.

4.3 Proof of Theorem 4.1

This Section is devoted to prove the exact controllability result in Theorem 4.1. We will assume that Proposition 4.1 is satisfied and we will employ a scaling argument.

Let $T > 0$, $\theta_0, \theta_1 \in C^{2,\alpha}(\bar{\Omega})$ and $\mathbf{y}_0, \mathbf{y}_1 \in \mathbf{C}(2, \alpha, \Gamma_0)$ be given. Let us see that, if

$$\|\mathbf{y}_0\|_{2,\alpha} + \|\mathbf{y}_1\|_{2,\alpha} + \|\theta_0\|_{2,\alpha} + \|\theta_1\|_{2,\alpha}$$

is small enough, we can construct a triplet $(\mathbf{y}, p, \theta)$ satisfying (7.4) and (4.3).

If $\varepsilon \in (0, T/2)$ is small enough to have

$$\max\{\varepsilon\|\mathbf{y}_0\|_{2,\alpha}, \varepsilon^2\|\theta_0\|_{2,\alpha}\} \le \delta \ \ (\text{resp. } \max\{\varepsilon\|\mathbf{y}_1\|_{2,\alpha},\ \varepsilon^2\|\theta_1\|_{2,\alpha}\} \le \delta),$$

then, thanks to Proposition 4.1, there exist $(\mathbf{y}^0, \theta^0)$ in $C^0([0,1]; \mathbf{C}^{1,\alpha}(\overline{\Omega}; \mathbb{R}^{N+1}))$ and a pressure p^0 (resp. $(\mathbf{y}^1, \theta^1)$ and p^1) solving (7.4), with $\mathbf{y}^0(\mathbf{x}, 0) \equiv \varepsilon\mathbf{y}_0(\mathbf{x})$ and $\theta^0(\mathbf{x}, 0) \equiv \varepsilon^2\theta_0(\mathbf{x})$ (resp. $\mathbf{y}^1(\mathbf{x}, 0) \equiv -\varepsilon\mathbf{y}_1(\mathbf{x})$ and $\theta^1(\mathbf{x}, 0) = \varepsilon^2\theta_1(\mathbf{x})$) and satisfying (4.5).

Let us choose ε of this form and let us introduce $\mathbf{y} : \bar{\Omega} \times [0,T] \mapsto \mathbb{R}^N$, $p : \bar{\Omega} \times [0,T] \mapsto \mathbb{R}$ and $\theta : \bar{\Omega} \times [0,T] \mapsto \mathbb{R}$ as follows:

$$\begin{cases} \mathbf{y}(\mathbf{x}, t) = \varepsilon^{-1}\mathbf{y}^0(\mathbf{x}, \varepsilon^{-1}t), \\ p(\mathbf{x}, t) = \varepsilon^{-2}p^0(\mathbf{x}, \varepsilon^{-1}t), \\ \theta(\mathbf{x}, t) = \varepsilon^{-2}\theta^0(\mathbf{x}, \varepsilon^{-1}t), \end{cases} \quad \text{for } (\mathbf{x}, t) \in \bar{\Omega} \times [0, \varepsilon],$$

$$\begin{cases} \mathbf{y}(\mathbf{x}, t) = \mathbf{0}, \\ p(\mathbf{x}, t) = 0, \\ \theta(\mathbf{x}, t) = 0, \end{cases} \quad \text{for } (\mathbf{x}, t) \in \bar{\Omega} \times (\varepsilon, T - \varepsilon),$$

$$\begin{cases} \mathbf{y}(\mathbf{x}, t) = -\varepsilon^{-1}\mathbf{y}^1(\mathbf{x}, \varepsilon^{-1}(T - t)), \\ p(\mathbf{x}, t) = \varepsilon^{-2}p^1(\mathbf{x}, \varepsilon^{-1}(T - t)), \\ \theta(\mathbf{x}, t) = \varepsilon^{-2}\theta^1(\mathbf{x}, \varepsilon^{-1}(T - t)), \end{cases} \quad \text{for } (\mathbf{x}, t) \in \bar{\Omega} \times [T - \varepsilon, T].$$

Then, $(\mathbf{y}, \theta) \in C^0([0,T]; \mathbf{C}^{1,\alpha}(\bar{\Omega}; \mathbb{R}^{N+1})$ and the triplet $(\mathbf{y}, p, \theta)$ satisfies (7.4) and (4.3).

4.4 Proof of Proposition 4.1. The 2D case

Let $\mu \in C^\infty([0,1])$ be a function such that $\mu \equiv 1$ in $[0, 1/4]$, $\mu \equiv 0$ in $[1/2, 1]$ and $0 < \mu < 1$. Proposition 4.1 is a consequence of the following result:

Proposition 4.2. *There exists $\delta > 0$ such that, if $\max\{\|\mathbf{y}_0\|_{2,\alpha}, \|\theta_0\|_{2,\alpha}\} \le \delta$, then the coupled system*

$$\begin{cases} \zeta_t + \mathbf{y} \cdot \nabla\zeta = -\vec{\mathbf{k}} \times \nabla\theta & \text{in} \quad \Omega \times (0,1), \\ \theta_t + \mathbf{y} \cdot \nabla\theta = 0 & \text{in} \quad \Omega \times (0,1), \\ \nabla \cdot \mathbf{y} = 0, \ \nabla \times \mathbf{y} = \zeta & \text{in} \quad \Omega \times (0,1), \\ \mathbf{y} \cdot \mathbf{n} = (\overline{\mathbf{y}} + \mu\, \mathbf{y}_0) \cdot \mathbf{n} & \text{on} \quad \Gamma \times (0,1), \\ \zeta(0) = \nabla \times \mathbf{y}_0, \ \theta(0) = \theta_0 & \text{in} \quad \Omega, \end{cases} \tag{4.24}$$

possesses at least one solution $(\zeta, \theta, \mathbf{y})$, *with*

$$(\zeta, \theta, \mathbf{y}) \in C^0([0,1]; C^{0,\alpha}(\overline{\Omega})) \times C^0([0,1]; C^{1,\alpha}(\overline{\Omega})) \times C^0([0,1]; \mathbf{C}^{1,\alpha}(\overline{\Omega}; \mathbb{R}^2)), \tag{4.25}$$

such that

$$\theta(\mathbf{x}, t) = 0 \quad in \quad \Omega \times (1/2, 1) \quad and \quad \zeta(\mathbf{x}, 1) = 0 \quad in \quad \Omega. \tag{4.26}$$

The reminder of this section is devoted to prove Proposition 4.2. We are going to adapt some ideas from Bardos and Frisch [4] and Kato [86], already used in [29] and [65]. Let us give a sketch.

We will start from an arbitrary field $\mathbf{z} := \mathbf{z}(\mathbf{x}, t)$ in a suitable class $\mathbf{S}$ of continuous functions. To this $\mathbf{z}$, we will associate a scalar function θ (a temperature) verifying

$$\begin{cases} \theta_t + \mathbf{z} \cdot \nabla\theta = 0 & \text{in} \quad \Omega \times (0,1), \\ \theta(\mathbf{x}, 0) = \theta_0(\mathbf{x}) & \text{in} \quad \Omega. \end{cases}$$

and

$$\theta(\mathbf{x}, t) = 0 \quad \text{in} \quad \Omega \times (1/2, 1).$$

With the help of θ, we will then construct a function ζ (an associated vorticity) satisfying

$$\begin{cases} \zeta_t + \mathbf{z} \cdot \nabla\zeta = -\vec{\mathbf{k}} \times \nabla\theta & \text{in} \quad \Omega \times (0,1), \\ \zeta(0) = \nabla \times \mathbf{y}_0 & \text{in} \quad \Omega. \end{cases}$$

and

$$\zeta(\mathbf{x}, 1) = 0 \quad \text{in} \quad \Omega.$$

Then, we will construct a field $\mathbf{y} = \mathbf{y}(\mathbf{x}, t)$ such that $\nabla \times \mathbf{y} = \zeta$ and $\nabla \cdot \mathbf{y} = 0$. This way, we will have defined a mapping F with $F(\mathbf{z}) = \mathbf{y}$. We will choose $\mathbf{S}$ such that F maps $\mathbf{S}$ into itself and an appropriate extension of F possesses exactly one fixed-point $\mathbf{y}$. Finally, it will be seen that the triplet $(\zeta, \theta, \mathbf{y})$, where ζ and θ are respectively the vorticity and temperature associated to $\mathbf{y}$, solves (4.24) and satisfies (4.25).

Let us now give the details.

The good definition of $\mathbf{S}$ is as follows. First, let us denote by $\mathbf{S}'$ the set of fields $\mathbf{z} \in C^0([0,1]; \mathbf{C}^{2,\alpha}(\overline{\Omega}; \mathbb{R}^2))$ such that $\nabla \cdot \mathbf{z} = 0$ in $\Omega \times (0,1)$ and $\mathbf{z} \cdot \mathbf{n} = (\overline{\mathbf{y}} + \mu\mathbf{y}_0) \cdot \mathbf{n}$ on $\Gamma \times (0,1)$. Then, for any $\nu > 0$, we set

$$\mathbf{S}_\nu = \{\, \mathbf{z} \in \mathbf{S}' : \|\mathbf{z} - \overline{\mathbf{y}}\|_{0,2,\alpha} \le \nu \,\}.$$

Let $\nu > 0$ be the constant furnished by Lemma 4.7 and let us carry out the previous process with $\mathbf{S} = \mathbf{S}_\nu$. To guarantee that $\mathbf{S}_\nu$ is nonempty, it suffices to assume that the initial data $\mathbf{y}_0$ is sufficiently small in $\mathbf{C}^{2,\alpha}(\bar{\Omega}; \mathbb{R}^2)$. Since, if this is the case, $\overline{\mathbf{y}} + \mu\mathbf{y}_0 \in \mathbf{S}_\nu$.

Let $\mathbf{z} \in \mathbf{S}_\nu$ be given and let us set $\mathbf{z}^* = \mathbf{y}^* + \pi_2(\mathbf{z} - \overline{\mathbf{y}})$. We have the estimate

$$\|\mathbf{z}^*(\cdot\,,t)\|_{2,\alpha} \le \|\mathbf{y}^*(\cdot\,,t)\|_{2,\alpha} + C\|(\mathbf{z} - \overline{\mathbf{y}})(\cdot\,,t)\|_{2,\alpha} \quad \forall t \in [0,1] \tag{4.27}$$

and the following result holds:

Lemma 4.12. *The flux $\mathbf{Z}^*$ associated to $\mathbf{z}^*$ satisfies $\mathbf{Z}^* \in C^1([0,1]\times[0,1];\mathbf{C}^{2,\alpha}(\bar\Omega_3;\mathbb{R}^2))$.*

Recall that $\mathbf{Z}^*$ is, by definition, the unique function satisfying

$$\begin{cases} \mathbf{Z}^*_t(\mathbf{x},t,s) = \mathbf{z}^*(\mathbf{Z}^*(\mathbf{x},t,s),t), \\ \mathbf{Z}^*(\mathbf{x},s,s) = \mathbf{x}, \end{cases} \tag{4.28}$$

and

$$\mathbf{Z}^*(\mathbf{x},t,s) \in \bar\Omega_3 \quad \forall (\mathbf{x},t,s) \in \bar\Omega_3 \times [0,1] \times [0,1].$$

For the proof of Lemma 4.12, it suffices to apply directly the well known (classical) existence, uniqueness and regularity theory of ODEs.

Since $\mathbf{Z}^* \in C^1([0,1]\times[0,1];\mathbf{C}^{2,\alpha}(\bar\Omega_3;\mathbb{R}^2))$, $\theta_0 \in C^{2,\alpha}(\bar\Omega)$ and π_1 maps continuously $C^{2,\alpha}(\bar\Omega)$ into $C^{2,\alpha}(\bar\Omega_3)$, there exists a unique solution $\theta^* \in C^0([0,1/2];C^{2,\alpha}(\bar\Omega_3))$ to the problem

$$\begin{cases} \theta^*_t + \mathbf{z}^* \cdot \nabla\theta^* = 0 & \text{in} \quad \Omega_3 \times (0,1/2), \\ \theta^*(\mathbf{x},0) = \pi_1(\theta_0)(\mathbf{x}) & \text{in} \quad \Omega_3. \end{cases} \tag{4.29}$$

Note that, in (4.29), no boundary condition on θ^* appears. Obviously, this is because $\operatorname{supp}\mathbf{z}^* \subset \Omega_3$.

The solution to (4.29) verifies $(\operatorname{supp}\theta^*(\cdot\,,t)) \subset \mathbf{Z}^*(\Omega_2,t,0)$ for all $t \in [0,1/2]$. In particular, in view of the choice of ν, we get:

$$\operatorname{supp}\theta^*(\cdot\,,1/2) \subset \mathbf{Z}^*(\Omega_2,1/2,0) \subset \Omega_3 \setminus \bar\Omega_2,$$

whence $\theta^*(\mathbf{x},1/2) = 0$ in Ω_2.

Let θ be the following function:

$$\theta(\mathbf{x},t) = \begin{cases} \theta^*(\mathbf{x},t), & (\mathbf{x},t) \in \bar\Omega \times [0,1/2), \\ 0, & (\mathbf{x},t) \in \bar\Omega \times [1/2,1]. \end{cases}$$

Then $\theta \in C^0([0,1];C^{2,\alpha}(\bar\Omega))$ and one has

$$\begin{cases} \theta_t + \mathbf{z} \cdot \nabla\theta = 0 & \text{in} \quad \Omega \times (0,1), \\ \theta(\mathbf{x},0) = \theta_0(\mathbf{x}) & \text{in} \quad \Omega. \end{cases} \tag{4.30}$$

For the construction of ζ, the argument is the following. Firstly, let us introduce $\zeta_0^* := \nabla \times (\pi_2(\mathbf{y}_0))$ and let $\zeta^* \in C^0([0,1/2];C^{1,\alpha}(\bar\Omega_3))$ be the unique solution to the problem

$$\begin{cases} \zeta_t^* + \mathbf{z}^* \cdot \nabla \zeta^* = -\vec{\mathbf{k}} \times \nabla \theta^* & \text{in} \quad \Omega_3 \times (0, 1/2), \\ \zeta^*(\mathbf{x}, 0) = \zeta_0^*(\mathbf{x}) & \text{in} \quad \Omega_3. \end{cases}$$

With this ζ^*, we define $\zeta_{1/2} \in C^{1,\alpha}(\bar{\Omega})$ with

$$\zeta_{1/2}(\mathbf{x}) := \zeta^*(\mathbf{x}, 1/2) \ \text{forall} \ \mathbf{x} \in \bar{\Omega}.$$

Then, let $\zeta^{**} \in C^0([1/2, 1]; C^{1,\alpha}(\bar{\Omega}_3))$ be the unique solution to the problem

$$\begin{cases} \zeta_t^{**} + \mathbf{z}^* \cdot \nabla \zeta^{**} = 0 & \text{in} \quad \Omega_3 \times (1/2, 1), \\ \zeta^{**}(\mathbf{x}, 1/2) = \pi_1(\zeta_{1/2})(\mathbf{x}) & \text{in} \quad \Omega_3. \end{cases}$$

We have $\zeta^{**}(\mathbf{Z}^*(\mathbf{x}, t, 1/2), t) = \pi_1(\zeta_{1/2})(\mathbf{x})$ for all $(\mathbf{x}, t) \in \bar{\Omega}_3 \times [1/2, 1]$ and, again from the choice of ν,

$$\operatorname{supp} \zeta^{**}(\cdot\,, 1) \subset \mathbf{Z}^*(\Omega_2, 1, 1/2) \subset \Omega_3 \setminus \bar{\Omega}_2$$

and $\zeta^{**}(\cdot\,, 1) \equiv 0$ in Ω_2.

Therefore, we can define $\zeta \in C^0([0, 1]; C^{1,\alpha}(\bar{\Omega}))$, with

$$\zeta(\mathbf{x}, t) = \begin{cases} \zeta^*(\mathbf{x}, t), & (\mathbf{x}, t) \in \Omega \times (0, 1/2), \\ \zeta^{**}(\mathbf{x}, t), & (\mathbf{x}, t) \in \Omega \times [1/2, 1). \end{cases}$$

Obviously, ζ is a solution to the initial-value problem

$$\begin{cases} \zeta_t + \mathbf{z} \cdot \nabla \zeta = -\vec{\mathbf{k}} \times \nabla \theta & \text{in} \quad \Omega \times (0, 1), \\ \zeta(\mathbf{x}, 0) = (\nabla \times \mathbf{y}_0)(\mathbf{x}) & \text{in} \quad \Omega. \end{cases} \tag{4.31}$$

With this ζ, we can now get a unique $\mathbf{y} \in C^0([0, 1]; \mathbf{C}^{2,\alpha}(\bar{\Omega}; \mathbb{R}^2))$ such that $\nabla \times \mathbf{y} = \zeta$ in $\Omega \times (0, 1)$, $\nabla \cdot \mathbf{y} = 0$ in $\Omega \times (0, 1)$ and $\mathbf{y} \cdot \mathbf{n} = (\overline{\mathbf{y}} + \mu \mathbf{y}_0) \cdot \mathbf{n}$ on $\Gamma \times [0, 1]$. Indeed, let $\psi \in C^0([0, 1]; C^{3,\alpha}(\bar{\Omega}))$ be the unique solution to the following family of elliptic equations:

$$\begin{cases} -\Delta \psi = \zeta - \mu \nabla \times \mathbf{y}_0 & \text{in } \Omega \times (0, 1), \\ \psi = 0 & \text{on } \Gamma \times (0, 1). \end{cases} \tag{4.32}$$

Then, let us set $\mathbf{y} := \nabla \times \psi + \overline{\mathbf{y}} + \mu \mathbf{y}_0$. Obviously, $\mathbf{y} \in C^0([0, 1]; \mathbf{C}^{2,\alpha}(\bar{\Omega}; \mathbb{R}^2))$ and satisfies the required properties. Since $\mathbf{y}$ is determined by $\mathbf{z}$, we write $\mathbf{y} = F(\mathbf{z})$. Accordingly, $F : \mathbf{S}_\nu \mapsto \mathbf{S}'$ is well defined.

The following result holds:

Lemma 4.13. *There exists $\delta > 0$ such that, if*

$$\max\{\|\mathbf{y}_0\|_{2,\alpha}, \|\theta_0\|_{2,\alpha}\} \leq \delta, \tag{4.33}$$

then $F(\mathbf{S}_\nu) \subset \mathbf{S}_\nu$.

Proof. Let $\mathbf{z} \in \mathbf{S}_\nu$ be given. Then $F(\mathbf{z}) - \overline{\mathbf{y}} = \nabla \times \psi + \mu \mathbf{y}_0$ and we have:

$$\|F(\mathbf{z})(\cdot\,, t) - \overline{\mathbf{y}}(\cdot\,, t)\|_{2,\alpha} \leq C(\|\zeta(\cdot\,, t)\|_{1,\alpha} + \|\mathbf{y}_0\|_{2,\alpha}).$$

Applying Lemma 4.2 to the equations of θ^* and ζ^*, we get

$$\|\theta^*(\cdot\,, t)\|_{2,\alpha} \leq \|\pi_1(\theta_0)\|_{2,\alpha} \exp\left(K \int_0^t \|\mathbf{z}^*(\cdot\,, \tau)\|_{2,\alpha}\, d\tau\right) \tag{4.34}$$

and

$$\|\zeta^*(\cdot\,, t)\|_{1,\alpha} \leq C(\|\pi_2(\mathbf{y}_0)\|_{2,\alpha} + \|\pi_1(\theta_0)\|_{2,\alpha}) \exp\left(K \int_0^t \|\mathbf{z}^*(\cdot\,, \tau)\|_{2,\alpha}\, d\tau\right). \tag{4.35}$$

With similar arguments, we also obtain

$$\|\zeta^{**}(\cdot\,, t)\|_{1,\alpha} \leq C(\|\pi_2(\mathbf{y}_0)\|_{2,\alpha} + \|\pi_1(\theta_0)\|_{2,\alpha}) \exp\left(K \int_0^t \|\mathbf{z}^*(\cdot\,, \tau)\|_{2,\alpha}(\tau) d\tau\right) \tag{4.36}$$

for all $t \in [1/2, 1]$. Thanks to (4.35) and (4.36), we obtain the following for ζ:

$$\|\zeta(\cdot\,, t)\|_{1,\alpha} \leq C(\|\mathbf{y}_0\|_{2,\alpha} + \|\theta_0\|_{2,\alpha}) \exp\left(K \int_0^t \|\mathbf{z}^*(\cdot, \tau)\|_{2,\alpha} d\tau\right). \tag{4.37}$$

Using (4.37), (4.27) and the definition of $\mathbf{S}_\nu$, we see that

$$\begin{aligned}\|F(\mathbf{z})(\cdot\,, t) - \overline{\mathbf{y}}(\cdot\,, t)\|_{2,\alpha} &\leq C_1(\|\mathbf{y}_0\|_{2,\alpha} + \|\theta_0\|_{2,\alpha}) \exp\left(C_2 \int_0^t \|\mathbf{z}(\cdot\,, \tau) - \overline{\mathbf{y}}(\cdot\,, \tau)\|_{2,\alpha}\, d\tau\right)\\ &\leq C_1(\|\mathbf{y}_0\|_{2,\alpha} + \|\theta_0\|_{2,\alpha}) \exp(C_2 \nu).\end{aligned}$$

Let $\delta > 0$ be such that $2C_1 \delta e^{C_2 \nu} \leq \nu$ and let us assume that (4.33) is satisfied. Then

$$\|F(\mathbf{z}) - \overline{\mathbf{y}}\|_{0,2,\alpha} \leq \nu$$

and, consequently, F maps $\mathbf{S}_\nu$ into itself. □

We now prove the existence and uniqueness of a fixed-point of the extension of F in the closure of $\mathbf{S}_\nu$ in $C^0([0,1]; \mathbf{C}^{1,\alpha}(\bar{\Omega}; \mathbb{R}^3))$. For this purpose, we will check that F satisfies the hypotheses of Theorem 4.3.

To this end, we will first establish two important lemmas. The first one is the following:

Lemma 4.14. *There exists $\widetilde{C} > 0$, only depending on $\|\mathbf{y}_0\|_{2,\alpha}$, $\|\theta_0\|_{2,\alpha}$ and ν, such that, for any $\mathbf{z}^1$, $\mathbf{z}^2 \in \mathbf{S}_\nu$, one has:*

$$\|(\zeta^1 - \zeta^2)(\cdot, t)\|_{0,\alpha} \leq \widetilde{C} \int_0^t \|(\mathbf{z}^1 - \mathbf{z}^2)(\cdot, s)\|_{1,\alpha}\, ds \quad \forall t \in [0, 1], \tag{4.38}$$

where ζ^i is the vorticity associated to $\mathbf{z}^i$.

Proof. First of all, let us introduce $\mathbf{w}^* := \mathbf{z}^{*,1} - \mathbf{z}^{*,2}$ and $\Theta^* := \theta^{*,1} - \theta^{*,2}$ (where the notation id self-explaining). Obviously, the estimates (4.27) and (resp. (4.34) and (4.35)) hold for $\mathbf{z}^{*,1}$ and $\mathbf{z}^{*,2}$ (resp. $\theta^{*,1}$ and $\theta^{*,2}$ and $\zeta^{*,1}$ and $\zeta^{*,2}$). Furthermore, it is clear that

$$\Theta^*_t + \mathbf{z}^{*,1} \cdot \nabla\Theta^* = -\mathbf{w}^* \cdot \nabla\theta^{*,2}.$$

Applying Lemma 4.1 to this equation, we have

$$\frac{d}{dt^+}\|\Theta^*(\cdot, t)\|_{1,\alpha} \leq \|\mathbf{w}^*(\cdot, t)\|_{1,\alpha}\|\theta^{*,2}(\cdot, t)\|_{2,\alpha} + K\|\mathbf{z}^{*,1}(\cdot, t)\|_{1,\alpha}\|\Theta^*(\cdot, t)\|_{1,\alpha}. \tag{4.39}$$

In view of Gronwall's Lemma, (4.27) and (4.34), we see that

$$\|\Theta^*(\cdot, t)\|_{1,\alpha} \leq \widetilde{C}_0 \int_0^t \|\mathbf{w}^*(\cdot, s)\|_{1,\alpha}\, ds \quad \forall t \in [0, 1/2]. \tag{4.40}$$

The equations verified by $\Upsilon^* := \zeta^{*,1} - \zeta^{*,2}$ and $\Upsilon^{**} := \zeta^{**,1} - \zeta^{**,2}$ are

$$\Upsilon^*_t + \mathbf{z}^{*,1} \cdot \nabla\Upsilon^* = -\mathbf{w}^* \cdot \nabla\zeta^{*,2} - \vec{\mathbf{k}} \times \nabla\Theta^*$$

and

$$\Upsilon^{**}_t + \mathbf{z}^{*,1} \cdot \nabla\Upsilon^{**} = -\mathbf{w}^* \cdot \nabla\zeta^{**,2},$$

respectively. Consequently, applying Lemma 4.1 to these equations, we get:

$$\frac{d}{dt^+}\|\Upsilon^*(\cdot, t)\|_{0,\alpha} \leq \|(\mathbf{w}^* \cdot \nabla\zeta^{*,2} + \vec{\mathbf{k}} \times \nabla\Theta^*)(\cdot, t)\|_{0,\alpha} + K\|\mathbf{z}^{*,1}(\cdot, t)\|_{1,\alpha}\|\Upsilon^*(\cdot, t)\|_{0,\alpha} \tag{4.41}$$

and

$$\frac{d}{dt^+}\|\Upsilon^{**}(\cdot, t)\|_{0,\alpha} \leq \|(\mathbf{w}^* \cdot \nabla\zeta^{**,2})(\cdot, t)\|_{0,\alpha} + K\|\mathbf{z}^{*,1}(\cdot, t)\|_{1,\alpha}\|\Upsilon^{**}(\cdot, t)\|_{0,\alpha}. \tag{4.42}$$

Applying Gronwall's Lemma, we deduce in view of (4.40) that

$$\|\Upsilon^*(\cdot, t)\|_{0,\alpha} \leq \widetilde{C}_1\|\zeta^{*,2}\|_{0,1,\alpha} \int_0^t \|\mathbf{w}^*(\cdot, s)\|_{1,\alpha}\, ds \quad \forall t \in [0, 1/2]$$

and

$$\|\Upsilon^{**}(\cdot,t)\|_{0,\alpha} \leq \widetilde{C}_2\|\zeta^{*,2}\|_{0,1,\alpha}\int_0^t \|\mathbf{w}^*(\cdot,s)\|_{1,\alpha}\,ds \quad \forall t\in[1/2,1].$$

Finally, we see from these estimates and (4.37) that (4.38) holds. □

Note that $\mathbf{y}^1-\mathbf{y}^2=\nabla\times(\psi^1-\psi^2)$, whence $\nabla\times(\nabla\times(\psi^1-\psi^2))=\zeta^1-\zeta^2$ and $\nabla\times(\psi^1-\psi^2)\cdot\mathbf{n}=0$ on $\Gamma\times[0,1]$.

Let us denote by $\mathbf{M}$ the set of fields $\mathbf{w}\in C^0([0,1];\mathbf{C}^{1,\alpha}(\overline{\Omega};\mathbb{R}^2))$ such that $\nabla\cdot\mathbf{w}=0$ in $\Omega\times(0,1)$ and $\mathbf{w}\cdot\mathbf{n}=0$ on $\Gamma\times(0,1)$. Note that, for any $\mathbf{w}\in\mathbf{M}$, the norms $\|\mathbf{w}\|_{1,\alpha}$ and $\|\nabla\times\mathbf{w}\|_{0,\alpha}$ are equivalent; we will set in the sequel $|||\mathbf{w}|||_{1,\alpha}:=\|\nabla\times\mathbf{w}\|_{0,\alpha}$ for any $\mathbf{w}\in\mathbf{M}$.

Lemma 4.15. *Let $\tilde{C}$ be the constant furnished by Lemma 4.14. For any $\mathbf{z}^1,\mathbf{z}^2\in\mathbf{S}_\nu$, one has*

$$|||(F^m(\mathbf{z}^1)-F^m(\mathbf{z}^2))(\cdot,t)|||_{1,\alpha}\leq\frac{(\tilde{C}t)^m}{m!}\|\mathbf{z}^1-\mathbf{z}^2\|_{0,1,\alpha}\quad\forall m\geq 1. \tag{4.43}$$

Proof. The proof is by induction.

For $m=1$, this is obvious, in view of Lemma 4.14.

Let us assume that (4.43) holds for $m=k$. Applying Lemma 4.14 to $\mathbf{y}^1=F^k(\mathbf{z}^1)$ and $\mathbf{y}^2=F^k(\mathbf{z}^2)$, we have

$$|||(F(\mathbf{y}^1)-F(\mathbf{y}^2))(\cdot,t)|||_{1,\alpha}\leq\tilde{C}\int_0^t\|(\mathbf{y}^1-\mathbf{y}^2)(\cdot,s)\|_{1,\alpha}\,ds\quad\forall t\in[0,1].$$

Therefore, using the induction hypothesis, we obtain:

$$\begin{aligned}|||(F^{k+1}(\mathbf{z}^1)-F^{k+1}(\mathbf{z}^2))(\cdot,t)|||_{1,\alpha}&\leq\tilde{C}\|\mathbf{z}^1-\mathbf{z}^2\|_{0,1,\alpha}\int_0^t\frac{(\tilde{C}s)^k}{k!}\,ds\\&=\frac{(\tilde{C}t)^{k+1}}{(k+1)!}\|\mathbf{z}^1-\mathbf{z}^2\|_{0,1,\alpha}\end{aligned}$$

This ends the proof. □

We deduce that, for some $\widehat{C}>0$, any $m\geq1$ and any $\mathbf{z}^1,\mathbf{z}^2\in\mathbf{S}_\nu$, one has

$$\max_{t\in[0,1]}\|(F^m(\mathbf{z}^1)-F^m(\mathbf{z}^2))(\cdot,t)\|_{1,\alpha}\leq\frac{\widehat{C}\tilde{C}^m}{m!}\left(\max_{\tau\in[0,1]}\|(\mathbf{z}^1-\mathbf{z}^2)(\cdot,\tau)\|_{1,\alpha}\right).$$

Consequently, if m is large enough, $F^m:\mathbf{S}_\nu\mapsto\mathbf{S}_\nu$ is a contraction, that is, there exists $\gamma\in(0,1)$ such that

$$\|F^m(\mathbf{z}^1)-F^m(\mathbf{z}^2)\|_{0,1,\alpha}\leq\gamma\|\mathbf{z}^1-\mathbf{z}^2\|_{0,1,\alpha}\ \ \forall\mathbf{z}^1,\mathbf{z}^2\in\mathbf{S}_\nu. \tag{4.44}$$

Therefore, we can apply Theorem 4.3 with

$$B_1 = C^0([0,1]; \mathbf{C}^{1,\alpha}(\bar{\Omega}; \mathbb{R}^2)), \ \ B_2 = C^0([0,1]; \mathbf{C}^{2,\alpha}(\bar{\Omega}; \mathbb{R}^2)), \ \ B = \mathbf{S}_\nu \ \text{ and } \ G = F,$$

to deduce that F possesses a unique extension $\widetilde{F}$ with a unique fixed-point $\mathbf{y}$ in the closure of $\mathbf{S}_\nu$ in $C^0([0,1]; \mathbf{C}^{1,\alpha}(\bar{\Omega}; \mathbb{R}^2))$. It is easy to check that $\mathbf{y}$ is, together with some ζ and θ, a solution to (4.24) satisfying (4.25) and (4.26).

This ends the proof.

4.5 Proof of Proposition 4.1. The 3D case

In this Section we are going to prove Proposition 4.1 in the three-dimensional case.

To do this, let $\{\rho^i\}$ be a partition of unity associated to the balls B^i introduced in Section 4.2.2 and let us set $\omega_0 = \nabla \times \pi_3(\mathbf{y}_0)$. Proposition 4.1 is a consequence of the following result:

Proposition 4.3. *There exists $\delta > 0$ such that, if $\max\{\|\mathbf{y}_0\|_{2,\alpha}, \|\theta_0\|_{2,\alpha}\} \le \delta$, then the coupled system*

$$\begin{cases} \omega_t + (\mathbf{y}\cdot\nabla)\omega = (\omega\cdot\nabla)\mathbf{y} - \vec{\mathbf{k}} \times \nabla\theta & \text{in} \quad \Omega\times(0,1), \\ \theta_t + \mathbf{y}\cdot\nabla\theta = 0 & \text{in} \quad \Omega\times(0,1), \\ \nabla\cdot\mathbf{y} = 0, \ \nabla\times\mathbf{y} = \omega & \text{in} \quad \Omega\times(0,1), \\ \mathbf{y}\cdot\mathbf{n} = (\bar{\mathbf{y}} + \mu\mathbf{y}_0)\cdot\mathbf{n} & \text{on} \quad \Gamma\times(0,1), \\ \omega(0) = \nabla\times\mathbf{y}_0, \ \theta(0) = \theta_0 & \text{in} \quad \Omega \end{cases} \tag{4.45}$$

possesses at least one solution $(\omega, \theta, \mathbf{y})$, with

$$(\omega,\theta,\mathbf{y}) \in C^0([0,1]; \mathbf{C}^{0,\alpha}(\overline{\Omega}; \mathbb{R}^3)) \times C^0([0,1]; C^{1,\alpha}(\overline{\Omega})) \times C^0([0,1]; \mathbf{C}^{1,\alpha}(\overline{\Omega}; \mathbb{R}^3)), \tag{4.46}$$

such that

$$\theta(\mathbf{x},t) = 0 \quad \text{in} \quad \Omega\times(t_{k-1/2},1) \quad \text{and} \quad \omega(\mathbf{x},t) = 0 \quad \text{in} \quad \Omega\times(t_{2k-1/2},1). \tag{4.47}$$

Let us give the proof of this result. We will repeat the strategy of proof of Proposition 4.2, incorporating some ideas from Bardos and Frisch [4] and Glass [67]; we will use the notation in Section 4.2.2.

First, let us denote by $\mathbf{R}'$ the set of fields $\mathbf{z} \in C^0([0,1]; \mathbf{C}^{2,\alpha}(\overline{\Omega}; \mathbb{R}^3))$ such that $\nabla\cdot\mathbf{z} = 0$ in $\Omega\times(0,1)$ and $\mathbf{z}\cdot\mathbf{n} = (\bar{\mathbf{y}} + \mu\mathbf{y}_0)\cdot\mathbf{n}$ on $\Gamma\times(0,1)$. Then, for any $\nu > 0$, we set

$$\mathbf{R}_\nu = \{\, \mathbf{z} \in \mathbf{R}' : \|\mathbf{z} - \bar{\mathbf{y}}\|_{0,1,\alpha} \le \nu \,\}.$$

Let us fix $\nu > 0$ being the constant furnished by Lemma 4.11. As before, if the initial datum $\mathbf{y}_0$ is sufficiently small in $\mathbf{C}^2(\bar{\Omega}; \mathbb{R}^3)$, then $\mathbf{R}_\nu$ is nonempty.

Now, we are going to construct a mapping $F : \mathbf{R}_\nu \to \mathbf{R}_\nu$.
We start from an arbitrary $\mathbf{z} \in \mathbf{R}_\nu$ and we set $\mathbf{z}^* := \mathbf{y}^* + \pi_3(\mathbf{z} - \overline{\mathbf{y}})$.

First, we denote by θ^* the unique solution to

$$\begin{cases} \theta^*_t + \mathbf{z}^* \cdot \nabla\theta^* = 0 & \text{in} \quad \overline{\mathscr{O}} \times [0, 1/2], \\ \theta^*(\mathbf{x}, 0) = \sum_{i=1}^k \psi^i(\mathbf{x})\, \pi_1(\theta_0)(\mathbf{x}) & \text{in} \quad \overline{\mathscr{O}}. \end{cases}$$

Obviously, $\theta^* = \sum_{i=1}^k \theta^i$, where θ^i is the unique solution to

$$\begin{cases} \theta^i_t + \mathbf{z}^* \cdot \nabla\theta^i = 0 & \text{in} \ \ \overline{\mathscr{O}} \times [0, 1/2], \\ \theta^i(\mathbf{x}, 0) = \psi^i(\mathbf{x})\, \pi_1(\theta_0)(\mathbf{x}) & \text{in} \ \ \overline{\mathscr{O}}. \end{cases} \tag{4.48}$$

The identities

$$\theta^i(\mathbf{Z}^*(\mathbf{x}, t, 0), t) = \psi^i(\mathbf{x})\pi_1(\theta_0)(\mathbf{x}) \quad \forall (\mathbf{x}, t) \in \overline{\mathscr{O}} \times [0, 1/2]$$

imply that

$$\text{Supp } \theta^i(\cdot\,, t) \subset \mathbf{Z}^*(B^i, t, 0) \quad \forall t \in [0, 1/2].$$

Hence, in view of Lemma 4.11, we deduce that

$$\text{Supp } \theta^i(\cdot\,, t_{i-1/2}) \subset \mathbf{Z}^*(B^i, t_{i-1/2}, 0) \subset \mathscr{O} \setminus \overline{\mathscr{O}}_0,$$

whence

$$\theta^i(\cdot\,, t_{i-1/2}) = 0 \quad \text{in} \quad \bar{\Omega}. \tag{4.49}$$

Then, we simply set $\hat{\theta}(\mathbf{x}, t) := \theta^*(\mathbf{x}, t)$ in $\overline{\mathscr{O}} \times [0, t_0]$ and we say that, in $\overline{\mathscr{O}} \times [t_0, 1/2]$, $\hat{\theta}$ is the unique solution to

$$\begin{cases} \hat{\theta}_t + \mathbf{z}^* \cdot \nabla\hat{\theta} = 0 & \text{in} \ \ \overline{\mathscr{O}} \times \left([t_0, 1/2] \setminus \bigcup_{i=1}^k \{t_{i-\frac{1}{2}}\} \right), \\ \hat{\theta}(\mathbf{x}, t_{i-1/2}) = \displaystyle\sum_{l=i}^k \theta^l(\mathbf{x}, t_{i-1/2}) - \theta^i(\mathbf{x}, , t_{i-1/2}) & \text{in} \ \ \overline{\mathscr{O}},\ 1 \le i \le k. \end{cases} \tag{4.50}$$

We notice that $\hat{\theta}(\cdot\,, t_{k-1/2}) \equiv 0$ in $\overline{\mathscr{O}}$. Therefore, $\hat{\theta} \equiv 0$ in $\overline{\mathscr{O}} \times [t_{k-1/2}, 1/2]$. Moreover,

$$\hat{\theta}(\mathbf{x}, t) = \sum_{l=i}^k \theta^l(\mathbf{x}, t) - \theta^i(\mathbf{x}, t) \ \text{ in } \ \overline{\mathscr{O}} \times (t_{i-1/2}, t_{i+1/2}), \ \ 1 \le i \le k - 1. \tag{4.51}$$

We remark that the lateral limits of $\hat{\theta}$ at the points $\{t_{i-1/2}\}_{i=1}^k$ are not necessarily the same in the whole domain $\overline{\mathscr{O}}$.

Let θ be the restriction of $\hat{\theta}$ to $\bar{\Omega}$. Due to (4.49) and (4.50), we see that θ is continuous at the points $\{t_{i-1/2}\}_{i=1}^{k}$ and

$$\begin{cases} \theta_t + \mathbf{z}\cdot\nabla\theta = 0 & \text{in } \Omega\times(0,1/2), \\ \theta(\mathbf{x},0) = \theta_0(\mathbf{x}) & \text{in } \Omega \end{cases} \tag{4.52}$$

and it belongs to $C^0([0,1];C^{1,\alpha}(\overline{\Omega}))$.

In an analogous way as for the temperature, we will define a function $\widehat{\omega}$ in $\overline{\mathscr{O}}\times[0,1]$, whose the restriction to Ω is the function ω satisfying (4.47). The definition of $\widehat{\omega}$ will be made in three parts corresponding, respectively, to the three time intervals $[0,1/2)$, $[1/2,t_{k+1/2})$ and $[t_{k+1/2},1]$.

Let us introduce $\omega_0 := \nabla\times(\pi_3(\mathbf{y}_0))$ and let ω^* be the solution to

$$\begin{cases} \omega^*_t + (\mathbf{z}^*\cdot\nabla)\omega^* = (\omega^*\cdot\nabla)\mathbf{z}^* - (\nabla\cdot\mathbf{z}^*)\omega^* - \overrightarrow{\mathbf{k}}\times\nabla\pi_1(\theta) & \text{in } \mathscr{O}\times(0,1/2), \\ \omega^*(\mathbf{x},0) = \omega_0(\mathbf{x}) & \text{in } \mathscr{O}. \end{cases}$$

With this ω^*, we set $\omega^{**}_{1/2}\in\mathbf{C}^{1,\alpha}(\bar{\Omega})$ with $\omega^{**}_{1/2}(\mathbf{x}) := \omega^*(\mathbf{x}\,,1/2)$ for all $\mathbf{x}\in\bar{\Omega}$. Let us consider ω^{**} the solution to the problem

$$\begin{cases} \omega^{**}_t + (\mathbf{z}^*\cdot\nabla)\omega^{**} = (\omega^{**}\cdot\nabla)\mathbf{z}^* - (\nabla\cdot\mathbf{z}^*)\omega^{**} & \text{in } \mathscr{O}\times(1/2,1), \\ \omega^{**}(\mathbf{x},1/2) = \sum\limits_{i=1}^{k}\psi^i(\mathbf{x})\,\pi_3(\omega^{**}_{1/2})(\mathbf{x}) & \text{in } \mathscr{O}. \end{cases} \tag{4.53}$$

As before, we can decompose ω^{**} as a sum of functions. More precisely, let $\omega^1,\dots,\omega^k$ be the solutions to the problems

$$\begin{cases} \omega^i_t + (\mathbf{z}^*\cdot\nabla)\omega^i = (\omega^i\cdot\nabla)\mathbf{z}^* - (\nabla\cdot\mathbf{z}^*)\omega^i & \text{in } \mathscr{O}\times(1/2,1), \\ \omega^i(\mathbf{x},1/2) = \psi^i(\mathbf{x})\,\pi_3(\omega^{**}_{1/2})(\mathbf{x}) & \text{in } \mathscr{O}. \end{cases} \tag{4.54}$$

Then

$$\omega^{**} = \sum_{i=1}^{k}\omega^i \quad \text{in} \quad \overline{\mathscr{O}}\times[1/2,1].$$

Each ω^i satisfies

$$\omega^i(\mathbf{Z}^*(\mathbf{x},t,1/2),t) = \omega^i(\mathbf{x},1/2) + \int_{1/2}^{t}[(\omega^i\cdot\nabla)\mathbf{z}^* - (\nabla\cdot\mathbf{z}^*)\omega^i](\mathbf{Z}^*(\mathbf{x},\sigma,1/2),\sigma)\,d\sigma.$$

Consequently,

$$|\omega^i(\mathbf{Z}^*(\mathbf{x},t,1/2),t)| \le |\omega^i(\mathbf{x},1/2)| + C\|\mathbf{z}^*\|_{0,1,0}\int_{1/2}^{t}|\omega^i(\mathbf{Z}^*(\mathbf{x},\sigma,1/2),\sigma)|\,d\sigma.$$

Notice that, if $\mathbf{x} \notin B^i$ we then have

$$|\omega^i(\mathbf{Z}^*(\mathbf{x}, t, 1/2), t)| \le C\|\mathbf{z}^*\|_{0,1,0} \int_{1/2}^{t} |\omega^i(\mathbf{Z}^*(\mathbf{x}, \sigma, 1/2), \sigma)|\, d\sigma$$

and, from Gronwall's Lemma, we see that

$$\omega^i(\mathbf{Z}^*(\mathbf{x}, t, 1/2), t) = 0 \quad \forall (\mathbf{x}, t) \in (\overline{\mathscr{O}} \setminus B^i) \times [1/2, 1].$$

A consequence is that $(\text{supp } \omega^i(\cdot\,, t)) \subset \mathbf{Z}^*(B^i, t, 1/2)$, whence we get

$$\omega^i(\mathbf{x}, t_{k+i-1/2}) = 0 \quad \text{for all} \quad \mathbf{x} \in \bar{\Omega}.$$

Then, we simply set $\widehat{\omega}(\mathbf{x}, t) := \omega^*(\mathbf{x}, t)$ in $\overline{\mathscr{O}} \times [0, 1/2]$ and $\widehat{\omega}(\mathbf{x}, t) := \omega^{**}(\mathbf{x}, t)$ in $\overline{\mathscr{O}} \times [1/2, t_{k+1/2}]$ and we say that, in $\mathscr{O} \times [t_{k+1/2}, 1]$, $\widehat{\omega}$ is the unique solution to

$$\begin{cases} \widehat{\omega}_t + (\mathbf{z}^* \cdot \nabla)\widehat{\omega} = (\widehat{\omega} \cdot \nabla)\mathbf{z}^* - (\nabla \cdot \mathbf{z}^*)\widehat{\omega} & \text{in } \ \overline{\mathscr{O}} \times \left([t_{k+\frac{1}{2}}, 1] \setminus \bigcup_{i=1}^{k} \{t_{k+i-\frac{1}{2}}\} \right) \\ \widehat{\omega}(\mathbf{x}, t_{k+i-\frac{1}{2}}) = \displaystyle\sum_{l=i}^{k} \widehat{\omega}^l(\mathbf{x}, t_{k+i-\frac{1}{2}}) - \widehat{\omega}^i(\mathbf{x}, t_{k+i-\frac{1}{2}}) & \text{in } \ \overline{\mathscr{O}}, \ 1 \le i \le k. \end{cases} \tag{4.55}$$

We notice that $\widehat{\omega}(\cdot\,, t_{2k-1/2}) \equiv 0$ in $\overline{\mathscr{O}}$. Therefore, $\widehat{\omega} \equiv 0$ in $\overline{\mathscr{O}} \times [t_{2k-1/2}, 1]$. Moreover,

$$\widehat{\omega}(\mathbf{x}, t) = \sum_{l=i}^{k} \omega^l(\mathbf{x}, t) - \omega^i(\mathbf{x}, t) \ \text{ in } \ \overline{\mathscr{O}} \times (t_{k+i-1/2}, t_{k+i+1/2}), \ \ 1 \le i \le k-1. \tag{4.56}$$

We define ω to be the restriction of $\widehat{\omega}$ to $\bar{\Omega} \times [0, 1]$. It belongs to $C^0([0, 1]; \mathbf{C}^{1,\alpha}(\overline{\Omega}; \mathbb{R}^3))$ and together with the temperature θ, satisfies:

$$\begin{cases} \omega_t + (\mathbf{z} \cdot \nabla)\omega = (\omega \cdot \nabla)\mathbf{z} - \overrightarrow{\mathbf{k}} \times \nabla\theta & \text{in} \quad \Omega \times [0, 1] \\ \omega(\mathbf{x}, 0) = (\nabla \times \mathbf{y}_0)(\mathbf{x}) & \text{in} \quad \Omega \end{cases}$$

and, moreover, $\omega \equiv 0$ in $\bar{\Omega} \times [t_{2k-1/2}, 1]$.

Thanks to Lemma 4.3, ω is divergence-free in $\Omega \times (0, 1)$. Consequently, from classical results, we know that there exists exactly one $\mathbf{y}$ in $C^0([0, 1]; \mathbf{C}^{2,\alpha}(\overline{\Omega}; \mathbb{R}^3))$ such that

$$\begin{cases} \nabla \times \mathbf{y} = \omega, \ \ \nabla \cdot \mathbf{y} = 0 & \text{in} \quad \overline{\Omega} \times (0, 1), \\ \mathbf{y} \cdot \mathbf{n} = (\mu \mathbf{y}_0 + \overline{\mathbf{y}}) \cdot \mathbf{n} & \text{on} \quad \Gamma \times (0, 1). \end{cases} \tag{4.57}$$

Since $\mathbf{y}$ is uniquely determined by $\mathbf{z}$, we write $F(\mathbf{z}) = \mathbf{y}$. The mapping $F : \mathbf{R}_\nu \mapsto \mathbf{R}'$ is thus well defined.

In view of some estimates similar to the $2D$ case, we can take the initial data small enough to have $F(\mathbf{R}_\nu) \subset \mathbf{R}_\nu$. More precisely, one has:

Lemma 4.16. *There exists $\delta > 0$ such that, if $\{\|\mathbf{y}_0\|_{2,\alpha}, \|\theta_0\|_{2,\alpha}\} \leq \delta$, one has $F(\mathbf{z}) \in \mathbf{R}_\nu$ for all $\mathbf{z} \in \mathbf{R}_\nu$.*

The end of the proof of Proposition 4.3 is very similar to the final part of Section 4.4.

Essentially, what we have to prove is that, for some $m \geq 1$, F^m is a contraction for the usual norm in $C^0([0,1]; \mathbf{C}^{1,\alpha}(\overline{\Omega}; \mathbb{R}^3))$. Indeed, after this we can apply Theorem 4.3 with $B_1 = C^0([0,1]; \mathbf{C}^{1,\alpha}(\overline{\Omega}; \mathbb{R}^3))$, $B_2 = C^0([0,1]; \mathbf{C}^{2,\alpha}(\overline{\Omega}; \mathbb{R}^3))$, $B = \mathbf{R}_\nu$ and $G = F$ and deduce the existence of a fixed-point of the extension $\widetilde{F}$ in the closure of $\mathbf{R}_\nu$ in $C^0([0,1]; C^{1,\alpha}(\overline{\Omega}; \mathbb{R}^3))$.

But this can be done easily, arguing as in the proof of Lemma 4.15. For brevity, we omit the details.

4.6 Proof of Theorem 4.2

Theorem 4.2 is an easy consequence of the following result:

Proposition 4.4. *For each $\mathbf{y}_0 \in \mathbf{C}(2, \alpha, \emptyset)$ there exist $T^* \in (0, T)$ and $\eta > 0$ such that, if $\theta_0 \in C^{2,\alpha}(\bar{\Omega})$, $\theta_0 = 0$ on $\Gamma \backslash \gamma$ and $\|\theta_0\|_{2,\alpha} \leq \eta$, then the system*

$$\begin{cases} \mathbf{y}_t + (\mathbf{y} \cdot \nabla)\mathbf{y} = -\nabla p + \vec{\mathbf{k}}\, \theta & \text{in} \quad \Omega \times (0, T^*), \\ \nabla \cdot \mathbf{y} = 0 & \text{in} \quad \Omega \times (0, T^*), \\ \theta_t + \mathbf{y} \cdot \nabla \theta = \kappa\, \Delta \theta & \text{in} \quad \Omega \times (0, T^*), \\ \mathbf{y} \cdot \mathbf{n} = 0 & \text{on} \quad \Gamma \times (0, T^*), \\ \theta = 0 & \text{on} \quad (\Gamma \backslash \gamma) \times (0, T^*), \\ \mathbf{y}(\mathbf{x}, 0) = \mathbf{y}_0(\mathbf{x}),\ \theta(\mathbf{x}, 0) = \theta_0(\mathbf{x}) & \text{in} \quad \Omega, \end{cases} \tag{4.58}$$

possesses at least one solution $\mathbf{y} \in C^0([0, T^*]; \mathbf{C}^{2,\alpha}(\bar{\Omega}; \mathbb{R}^N))$, $\theta \in C^0([0, T^*]; C^{2,\alpha}(\bar{\Omega}))$ and $p \in \mathcal{D}'(\Omega \times (0, T^*))$ such that

$$\theta(\mathbf{x}, T^*) = 0 \quad \text{in} \quad \Omega. \tag{4.59}$$

Indeed, if Proposition 4.4 holds, we can consider (7.4) and control first the temperature θ exactly to zero at time T^*. To do this, we need initial data as above, that is, $\mathbf{y}_0 \in \mathbf{C}(2, \alpha, \emptyset)$ and $\theta_0 \in C^{2,\alpha}(\bar{\Omega})$ such that $\theta_0 = 0$ on $\Gamma \backslash \gamma$ and $\|\theta_0\|_{2,\alpha} \leq \delta$. Then, in a second step, we can apply the results in [29] and [67] to the Euler system in $\Omega \times (T^*, T)$, with initial data $\mathbf{y}(\cdot\,, T^*)$. In other words, we can find new controls in (T^*, T) that drive the velocity field exactly to any final state $\mathbf{y}_1$.

Proof of Proposition 4.4: For simplicity, we will consider only the case $N = 2$. We will apply a fixed-point argument that guarantees the existence of a solution to (4.58)-(4.59).

We start from an arbitrary $\overline{\theta} \in C^0([0,T/2];C^{1,\alpha}(\overline{\Omega}))$. To this $\overline{\theta}$, arguing as in Section 4.3, we can associate a field $\mathbf{y} \in C^0([0,T/2];\mathbf{C}^{2,\alpha}(\bar{\Omega};\mathbb{R}^N))$ verifying

$$\begin{cases} \mathbf{y}_t + (\mathbf{y}\cdot\nabla)\mathbf{y} = -\nabla p + \vec{\mathbf{k}}\,\overline{\theta} & \text{in } \Omega\times(0,T/2), \\ \nabla\cdot\mathbf{y} = 0 & \text{in } \Omega\times(0,T/2), \\ \mathbf{y}\cdot\mathbf{n} = 0 & \text{on } \Gamma\times(0,T/2), \\ \mathbf{y}(\mathbf{x},0) = \mathbf{y}_0(\mathbf{x}) & \text{in } \Omega \end{cases}$$

and

$$\|\mathbf{y}\|_{0,2,\alpha} \le C(\|\mathbf{y}_0\|_{2,\alpha} + \|\overline{\theta}\|_{0,2,\alpha}).$$

Let $\tilde{\Omega} \subset \mathbb{R}^2$ be a connected open set with boundary $\tilde{\Gamma} = \partial\tilde{\Omega}$ of class C^2 such that $\Omega \subset \tilde{\Omega}$ and $\tilde{\Gamma}\cap\Gamma = \Gamma\setminus\gamma$ (see Fig. 4.2). Let $\omega \subset \tilde{\Omega}\setminus\overline{\Omega}$ be a non-empty open subset.

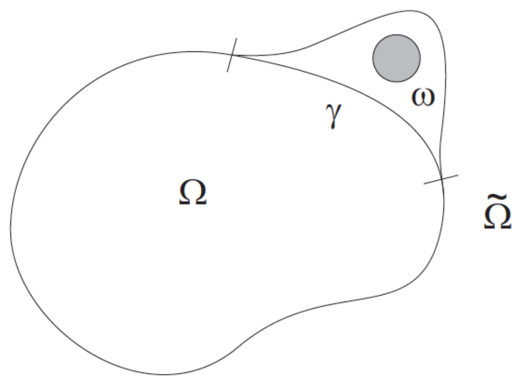

Figure 4.2: The domain $\tilde{\Omega}$ and the subdomain ω.

Then, as in Theorem 4.4, we associate to $\mathbf{y}$ a pair $(\tilde{\theta},\tilde{v})$ satisfying

$$\begin{cases} \tilde{\theta}_t + \pi(\mathbf{y})\cdot\nabla\tilde{\theta} = \kappa\Delta\tilde{\theta} + \tilde{v}1_\omega & \text{in } \tilde{\Omega}\times(0,T/2), \\ \tilde{\theta} = 0 & \text{on } \tilde{\Gamma}\times(0,T/2), \\ \tilde{\theta}(\mathbf{x},0) = \tilde{\pi}(\theta_0)(\mathbf{x}),\ \ \tilde{\theta}(\mathbf{x},T/2) = 0 & \text{in } \tilde{\Omega}, \end{cases}$$

where π and $\tilde{\pi}$ are extension operators from Ω into $\tilde{\Omega}$ that preserve regularity. Let θ be the restriction of $\tilde{\theta}$ to $\overline{\Omega}\times[0,T/2]$. Then, θ satisfies:

$$\begin{cases} \theta_t + \mathbf{y}\cdot\nabla\theta = \kappa\Delta\theta & \text{in } \Omega\times(0,T/2), \\ \theta = \tilde{\theta}1_\gamma & \text{on } \Gamma\times(0,T/2), \\ \theta(\mathbf{x},0) = \theta_0(\mathbf{x}),\ \ \theta(\mathbf{x},T/2) = 0 & \text{in } \Omega. \end{cases}$$

Moreover, from parabolic regularity, it is not difficult to check that the following inequalities hold:

$$\|\theta_t\|_{0,0,\alpha} + \|\theta\|_{0,2,\alpha} \leq C\|\theta_0\|_{2,\alpha}^2 \, e^{C\|\mathbf{y}\|_{0,2,\alpha}} \leq C\|\theta_0\|_{2,\alpha} \, e^{C(\|\mathbf{y}_0\|_{2,\alpha}+\|\overline{\theta}\|_{0,2,\alpha})}.$$

Now, let us introduce the Banach space

$$W = \{\, \theta \in C^0([0,T/2]; C^{2,\alpha}(\overline{\Omega})) : \theta_t \in C^0([0,T/2]; C^{0,\alpha}(\overline{\Omega})) \,\}$$

and let us consider the closed ball

$$B := \{\, \overline{\theta} \in C^0([0,T/2]; C^{1,\alpha}(\overline{\Omega})) : \|\overline{\theta}\|_{0,1,\alpha} \leq 1 \,\}$$

and the mapping Λ, with

$$\Lambda(\overline{\theta}) = \theta \quad \forall \overline{\theta} \in C^0([0,T/2]; C^{1,\alpha}(\overline{\Omega})).$$

Obviously, Λ is well defined. Furthermore, in view of the previous inequalities, it maps continuously the whole space $C^0([0,T/2]; C^{1,\alpha}(\overline{\Omega}))$ into W, that is compactly embedded in $C^0([0,T/2]; C^{1,\alpha}(\overline{\Omega}))$, in view of the classical results of the Aubin-Lions kind, see for instance [117].

On the other hand, if $\eta > 0$ is sufficiently small (depending on $\|\mathbf{y}_0\|_{2,\alpha}$) and $\|\theta_0\|_{2,\alpha} \leq \eta$, Λ maps B into itself. Consequently, the hypotheses of Schauder's Theorem are satisfied and Λ possesses at least one fixed-point in B.

This ends the proof. □

Part II

Numerical results on the control of several equations and parabolic systems

Chapter 5

A mixed formulation for the direct approximation of L^2-weighted controls for the linear heat equation

A mixed formulation for the direct approximation of L^2-weighted controls for the linear heat equation

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Abstract. This paper deals with the numerical computation of null controls for the linear heat equation. The goal is to compute approximations of controls that drive the solution from a prescribed initial state to zero at a given positive time. In [Fernandez-Cara & Münch, Strong convergence approximations of null controls for the 1D heat equation, 2013], a so-called primal method is described leading to a strongly convergent approximation of distributed control: the controls minimize quadratic weighted functionals involving both the control and the state and are obtained by solving the corresponding optimality conditions. In this work, we adapt the method to approximate the control of minimal square integrable-weighted norm. The optimality conditions of the problem are reformulated as a mixed formulation involving both the state and its adjoint. We prove the well-posedness of the mixed formulation (in particular the inf-sup condition) then discuss several numerical experiments. The approach covers both the boundary and the inner situation and is valid in any dimension.

5.1 Introduction. The null controllability problem

Let $\Omega \subset \mathbb{R}^N$ be a bounded connected open set whose boundary $\partial\Omega$ is regular enough (for instance of class C^2). Let $\omega \subset \Omega$ be a (small) nonempty open subset and assume that $T > 0$. In the sequel, for any $\tau > 0$ we denote by Q_τ, q_τ and Σ_τ the sets $\Omega \times (0, \tau)$, $\omega \times (0, \tau)$ and $\partial\Omega \times (0, \tau)$, respectively.

This work is concerned with the null controllability problem for the heat equation

$$\begin{cases} y_t - \nabla \cdot (c(x)\nabla y) + d(x,t)y = v\,1_\omega, & \text{in} \quad Q_T, \\ y = 0, & \text{on} \quad \Sigma_T, \\ y(x,0) = y_0(x), & \text{in} \quad \Omega. \end{cases} \tag{5.1}$$

Here, we assume that $c := (c_{i,j}) \in C^1(\overline{\Omega}; \mathcal{M}_N(\mathbb{R}))$ with $(c(x)\xi, \xi) \geq c_0|\xi|^2$ in $\overline{\Omega}$ ($c_0 > 0$), $d \in L^\infty(Q_T)$ and $y_0 \in L^2(\Omega)$; $v = v(x,t)$ is the *control* (a function in $L^2(q_T)$) and $y = y(x,t)$ is the associated state. Moreover, 1_ω is the characteristic function associated to the set ω.

In the sequel, we shall use the following notation :

$$L\,y := y_t - \nabla \cdot (c(x)\nabla y) + d(x,t)y, \qquad L^\star\varphi := -\varphi_t - \nabla \cdot (c(x)\nabla\varphi) + d(x,t)\varphi.$$

For any $y_0 \in L^2(\Omega)$ and $v \in L^2(q_T)$, there exists exactly one solution y to (5.1), with the regularity $y \in C^0([0,T];L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$ (see [98, 15]). Accordingly, for any final time $T > 0$, the associated null controllability problem at time T is the following : for each $y_0 \in L^2(\Omega)$, find $v \in L^2(q_T)$ such that the corresponding solution to (5.1) satisfies

$$y(\cdot\,,T) = 0 \quad \text{in } \Omega. \tag{5.2}$$

The controllability of PDEs is an important area of research and has been the subject of many papers in recent years. Some relevant references are [90, 96, 112] and [30]. In particular, we refer to [62] and [92] where the null controllability of (5.1) is proved.

The numerical approximation is also a fundamental issue, since it is not in general possible to get explicit expression of controls. Due to the strong regularization property of the heat kernel, numerical approximation of controls is a rather delicate issue. The same holds in inverse problems theory when parabolic equations and systems are involved (see [40]). This has been exhibited numerically in [13] who made use of a duality argument and focused on the control of minimal square integrable norm: the problem reads

$$\begin{cases} \text{Minimize } \; J_1(y,v) := \dfrac{1}{2}\displaystyle\iint_{q_T} |v(x,t)|^2\,dx\,dt \\ \text{Subject to } \; (y,v) \in \mathcal{C}(y_0,T) \end{cases} \tag{5.3}$$

where $\mathcal{C}(y_0;T)$ denotes the linear manifold

$$\mathcal{C}(y_0;T) := \{\,(y,v) : v \in L^2(q_T),\; y \text{ solves (5.1) and satisfies (5.2)}\,\}.$$

The earlier contribution is due to Glowinski and Lions in [70] (updated in [71]) and relies on duality arguments. Duality allows to replace the original constrained minimization problem by an unconstrained and *a priori* easier minimization (dual) problem. The dual problem associated with (5.3) is :

$$\min_{\varphi_T \in \mathcal{H}} J_1^\star(\varphi_T) := \frac{1}{2}\iint_{q_T} |\varphi(x,t)|^2\,dxdt + \int_\Omega y_0(x)\varphi(x,0)dx \tag{5.4}$$

where the variable φ solves the backward heat equation :

$$L^\star\varphi = 0 \quad \text{in} \quad Q_T, \qquad \varphi = 0 \quad \text{on} \quad \Sigma_T; \quad \varphi(\cdot,T) = \varphi_T \quad \text{in} \quad \Omega, \tag{5.5}$$

and the Hilbert space $\mathcal{H}$ is defined as the completion of $\mathcal{D}(\Omega)$ with respect to the norm $\|\varphi_T\|_{\mathcal{H}} := \|\varphi\|_{L^2(q_T)}$. In view of the unique continuation property to (5.5), the mapping $\varphi_T \mapsto \|\varphi_T\|_{\mathcal{H}}$ is a Hilbertian norm in $\mathcal{D}(\Omega)$. Hence, we can certainly consider the completion of $\mathcal{D}(\Omega)$ for this norm. The coercivity of the functional $J_1^\star$ in $\mathcal{H}$ is a consequence of the so-called *observability inequality*

$$\|\varphi(\cdot,0)\|^2_{L^2(\Omega)} \leq C \iint_{q_T} |\varphi(x,t)|^2\,dx\,dt \quad \forall \varphi_T \in \mathcal{H}, \tag{5.6}$$

where φ solves (5.5). This inequality holds for some constant $C = C(\omega, T)$ and, in turn, is a consequence of some appropriate global Carleman inequalities; see [62]. The minimization of $J_1^\star$ is numerically ill-posed, essentially because of the hugeness of the completed space $\mathcal{H}$. The control of minimal square integrable norm highly oscillates near the final time T, property which is hard to capture numerically. We refer to [6, 87, 101, 105] where this phenomenon is highlighted under several perspectives.

Moreover, at the level of the approximation, the minimization of $J_1^\star$ requires to find a finite dimensional and conformal approximation of $\mathcal{H}$ such that the corresponding discrete adjoint solution satisfies (5.5), which is in general impossible for polynomial piecewise approximations. In practice, the trick initially described in [70], consists first to introduce a discrete and consistent approximation of (5.1) and then to minimize the corresponding discrete conjugate functional. However, this requires to get some uniform discrete observability inequalities which is a delicate issue, strongly depend on the approximations used (we refer to [10, 42, 123] and the references therein) and is still open in the general case of the heat equation with non constant coefficients. This fact and the hugeness of $\mathcal{H}$ has raised many authors to relax the controllability problem: precisely, the constraint (5.2). We mention the references [10, 13, 123] and notably [9, 50, 88] for some numerical realizations.

In [49] (see also [48] in a semi-linear case), a different - so-called primal approach - allowing more general results has been used and consists to solve directly optimality conditions : specifically, the following general extremal problem (initially introduced by Fursikov and Imanuvilov in [62]) is considered :

$$\begin{cases} \text{Minimize } \; J(y,v) := \dfrac{1}{2}\displaystyle\iint_{Q_T} \rho^2 |y|^2\,dx\,dt + \dfrac{1}{2}\iint_{q_T} \rho_0^2 |v|^2\,dx\,dt \\ \text{Subject to } \; (y,v) \in \mathcal{C}(y_0,T). \end{cases} \tag{5.7}$$

The weights $\rho = \rho(x,t)$ and $\rho_0 = \rho_0(x,t)$ are continuous, uniformly positive and are assumed to belong to $L^\infty(Q_{T-\delta})$ for any $\delta > 0$ (hence, they can blow up as $t \to T^-$). Under those conditions, the extremal problem (5.7) is well-posed (see [49]).

Moreover, the explicit occurrence of the term y in the functional allow to solve directly the optimality conditions associated with (5.7): defining the Hilbert space P as the completion of the linear space $P_0 = \{q \in C^\infty(\overline{Q_T}) : q = 0 \text{ on } \Sigma_T\}$ with respect to the scalar product

$$(p,q)_P := \iint_{Q_T} \rho^{-2} L^\star p\, L^\star q\,dx\,dt + \iint_{q_T} \rho_0^{-2}\, p\, q\,dx\,dt, \tag{5.8}$$

the optimal pair (y, v) for J is characterized as follows

$$y = \rho^{-2} L^\star p \quad \text{in} \quad Q_T, \qquad v = -\rho_0^{-2} p \, 1_\omega \quad \text{in} \quad Q_T \tag{5.9}$$

in term of an additional variable $p \in P$ unique solution to the following variational equality :

$$(p, q)_P = \int_\Omega y_0(x)\, q(x, 0)\, dx, \quad \forall q \in P. \tag{5.10}$$

The well-posedeness of this formulation is ensured as soon as the weights ρ_0, ρ are of Carleman type (in particular ρ and ρ_0 blow up exponentially as $t \to T^-$); this specific behavior near T reinforces the null controllability requirement and prevents the control of any oscillations near the final time.

The search of a control v in the manifold $\mathcal{C}(y_0, T)$ is reduced to solve the (elliptic) variational formulation (5.10). In [49], the approximation of (5.10) is performed in the framework of the finite element theory through a discretization of the space-time domain Q_T. In practice, an approximation p_h of p is obtained in a direct way by inverting a symmetric positive definite matrix, in contrast with the iterative (and possibly divergent) methods used within dual methods. Moreover, a major advantage of this approach is that a conformal approximation, say P_h of P, leads to the strong convergence of p_h toward p in P, and consequently from (5.9), to a strong convergence in $L^2(q_T)$ of $v_h := -\rho_0^{-2} p_h 1_\omega$ toward v, a null control for (5.1). It is worth to mention that, for any $h > 0$, v_h is not *a priori* an exact control for any finite dimensional system (which is not necessary at all in practice) but an approximation for the L^2-norm of the control v.

The variational formulation (5.10) derived from the optimality conditions (5.9) is obtained assuming that the weights ρ and ρ_0 are both strictly positive in Q_T and q_T respectively. In particular, this approach does not apply for the control of minimal L^2-norm, for which simply $\rho := 0$ and $\rho_0 := 1$. The main reason of the present work is to adapt this approach to cover the case $\rho := 0$ and therefore obtain directly an approximation v_h of the control of some minimal weighted L^2-norm. To do so, we adapt the idea developed in [24] devoted to the wave equation. We also mention [103] where a different space-time variational approach (based on Least-squares principles) is used to approximate null controls for the heat equation.

The paper is organized as follows. In Section 5.2, we associate to the dual problem (5.4) an equivalent mixed formulation which relies on the optimality conditions associated to the problem (5.7) with $\rho = 0$. In Section 5.2.1, we first address the penalization case and write the constraint $L^\star\varphi = 0$ as an equality in $L^2(Q_T)$. We then show the well-posedness of this mixed formulation, in particular we check the inf-sup condition (Theorem 5.1). The mixed formulation allows to approximate simultaneously the dual variable and the primal one, controlled solution of (5.1). Interestingly, we also derive an equivalent extremal problem in the primal variable y only (see Prop 5.2, Section 5.2.1). In Section 5.2.2, we reproduce the analysis relaxing the condition $L^\star\varphi = 0$ in the weaker space $L^2(0,T,H^{-1}(\Omega))$. Then, in Section 5.2.3, by using the Global Carleman estimate (5.34), we show that a well-posed mixed formulation is also available for the limit and singular case for which $\varepsilon = 0$ leading to Theorem 5.3. Section 5.3 is devoted to the numerical approximation of the mixed formulation (5.16) in the case $\varepsilon > 0$ (Section 5.3.1) and of the mixed formulation (5.32) in the case $\varepsilon = 0$ (section 5.3.2). Conformal approximations based on space-time finite elements are employed. In Section 5.3.3, we numerically check that the approximations used lead to discrete inf-sup properties, uniformly w.r.t. the discretization parameter h. Then the remaining of Section 5.3 is devoted to some experiments which emphasize the remarkable robustness of the method. Section 5.4 concludes with some perspectives.

5.2 Control of minimal weighted L^2-norm : mixed reformulations

In order to avoid the minimization of the conjugate functional $J^\star$ with respect to the final state φ_T by an iterative process, we now present a direct way to approximate the control of minimal square integrable norm in the spirit of the primal approach recalled in the introduction and developed in [49]. We adapt the case of the wave equation studied in [24].

5.2.1 The penalized case: Mixed formulation I

Let $\rho_\star \in \mathbb{R}^+_\star$ and let $\rho_0 \in \mathcal{R}$ defined by

$$\mathcal{R} := \{w : w \in C(Q_T); w \geq \rho_\star > 0 \text{ in } Q_T; w \in L^\infty(Q_{T-\delta})\ \forall \delta > 0\} \tag{5.11}$$

so that in particular, the weight ρ_0 may blow up as $t \to T^-$. We first consider the approximate controllability case. For any $\varepsilon > 0$, the problem reads as follows:

$$\begin{cases} \text{Minimize } \ J_\varepsilon(y,v) := \dfrac{1}{2}\displaystyle\iint_{q_T} \rho_0^2 |v|^2\, dt + \frac{1}{2\varepsilon}\|y(\cdot,T)\|^2_{L^2(\Omega)} \\ \text{Subject to } \ (y,v) \in \mathcal{A}(y_0;T) \end{cases} \tag{5.12}$$

where ε denotes a penalty parameter (see [9, 13, 50]) and where $\mathcal{A}(y_0;T)$ denotes the linear manifold $\mathcal{A}(y_0;T) := \{\, (y,v) : v \in L^2(q_T),\ y \text{ solves } (5.1)\,\}$. The corresponding conjugate and well-posed problem is given by

$$\begin{cases} \text{Minimize } \ J^\star_\varepsilon(\varphi_T) := \dfrac{1}{2}\displaystyle\iint_{q_T} \rho_0^{-2}|\varphi(x,t)|^2 dx\, dt + \frac{\varepsilon}{2}\|\varphi_T\|^2_{L^2(\Omega)} + (y_0, \varphi(\cdot,0))_{L^2(\Omega)} \\ \text{Subject to } \ \varphi_T \in L^2(\Omega) \end{cases} \tag{5.13}$$

where φ solves (5.5).

We recall that the penalized problem (5.12) is a consistent approximation of the original null controllability problem, in the sense that its unique solution converges to the solution of (5.7) with $\rho = 0$ as $\varepsilon \to 0$. We refer for instance to [50], Prop. 3.3 for a proof of the following result, consequence of the null controllability for the heat equation.

Proposition 5.1. *Let $(y_\varepsilon, v_\varepsilon)$ be the solution of Problem (5.12) and let (y, v) be the solution of Problem (5.7) with $\rho = 0$. Then, one has*

$$y_\varepsilon \to y \ \textit{strongly in} \ \ L^2(Q_T), \quad v_\varepsilon \to v \ \textit{strongly in} \ \ L^2(q_T)$$

as $\varepsilon \to 0^+$.

Mixed formulation

Since the variable φ, solution of (5.5), is completely and uniquely determined by the data φ_T, the main idea of the reformulation is to keep φ as main variable.

We introduce the linear space $\Phi^0 := \{\varphi \in C^2(\overline{Q_T}),\ \varphi = 0 \text{ on } \Sigma_T\}$. For any $\eta > 0$, we define the bilinear form

$$(\varphi, \overline{\varphi})_{\Phi^0} := \iint_{q_T} \rho_0^{-2} \varphi\, \overline{\varphi}\, dx\, dt + \varepsilon(\varphi(\cdot, T), \overline{\varphi}(\cdot, T))_{L^2(\Omega)} + \eta \iint_{Q_T} L^\star \varphi\, L^\star \overline{\varphi}\, dx\, dt, \ \ \forall \varphi, \overline{\varphi} \in \Phi^0.$$

From the unique continuation property for the heat equation, this bilinear form defines for any $\varepsilon \geq 0$ a scalar product. For any $\varepsilon > 0$, let Φ_ε be the completion of Φ^0 for this scalar product. We denote the norm over Φ_ε by $\|\cdot\|_{\Phi_\varepsilon}$ such that

$$\|\varphi\|^2_{\Phi_\varepsilon} := \|\rho_0^{-1}\varphi\|^2_{L^2(q_T)} + \varepsilon\|\varphi(\cdot, T)\|^2_{L^2(\Omega)} + \eta\|L^\star \varphi\|^2_{L^2(Q_T)}, \quad \forall \varphi \in \Phi_\varepsilon. \tag{5.14}$$

Finally, we defined the closed subset W_ε of Φ_ε by

$$W_\varepsilon = \{\varphi \in \Phi_\varepsilon : L^\star \varphi = 0 \text{ in } L^2(Q_T)\}$$

and we endow W_ε with the same norm than Φ_ε.

Then, we define the following extremal problem :

$$\min_{\varphi \in W_\varepsilon} \hat{J}^\star_\varepsilon(\varphi) := \frac{1}{2} \iint_{q_T} \rho_0^{-2} |\varphi(x,t)|^2 dx\, dt + \frac{\varepsilon}{2}\|\varphi(\cdot, T)\|^2_{L^2(\Omega)} + (y_0, \varphi(\cdot, 0))_{L^2(\Omega)}. \tag{5.15}$$

Standard energy estimates for the heat equation imply that, for any $\varphi \in W_\varepsilon$, $\varphi(\cdot, 0) \in L^2(\Omega)$ so that the functional $\hat{J}^\star_\varepsilon$ is well-defined over W_ε. Moreover, since for any $\varphi \in W_\varepsilon$, $\varphi(\cdot, T)$ belongs to $L^2(\Omega)$, Problem (5.15) is equivalent to the minimization problem (5.13). As announced, the main variable is now φ submitted to the constraint equality (in $L^2(Q_T)$) $L^\star \varphi = 0$. This constraint equality is addressed by introducing a Lagrangian multiplier.

We consider the following mixed formulation : find $(\varphi_\varepsilon, \lambda_\varepsilon) \in \Phi_\varepsilon \times L^2(Q_T)$ solution of

$$\begin{cases} a_\varepsilon(\varphi_\varepsilon, \overline{\varphi}) + b(\overline{\varphi}, \lambda_\varepsilon) &= l(\overline{\varphi}), \qquad \forall \overline{\varphi} \in \Phi_\varepsilon \\ b(\varphi_\varepsilon, \overline{\lambda}) &= 0, \qquad \forall \overline{\lambda} \in L^2(Q_T), \end{cases} \tag{5.16}$$

where

$$a_\varepsilon : \Phi_\varepsilon \times \Phi_\varepsilon \to \mathbb{R}, \quad a_\varepsilon(\varphi, \overline{\varphi}) := \iint_{q_T} \rho_0^{-2} \varphi\, \overline{\varphi}\, dx\, dt + \varepsilon(\varphi(\cdot, T), \overline{\varphi}(\cdot, T))_{L^2(\Omega)}$$

$$b : \Phi_\varepsilon \times L^2(Q_T) \to \mathbb{R}, \quad b(\varphi, \lambda) := -\iint_{Q_T} L^\star \varphi\, \lambda\, dx\, dt$$

$$l : \Phi_\varepsilon \to \mathbb{R}, \quad l(\varphi) := -(y_0, \varphi(\cdot, 0))_{L^2(\Omega)}.$$

We have the following result :

Theorem 5.1. *(i) The mixed formulation (5.16) is well-posed.*

(ii) The unique solution $(\varphi_\varepsilon, \lambda_\varepsilon) \in \Phi_\varepsilon \times L^2(Q_T)$ is the unique saddle-point of the Lagrangian $\mathcal{L}_\varepsilon : \Phi_\varepsilon \times L^2(Q_T) \to \mathbb{R}$ defined by

$$\mathcal{L}_\varepsilon(\varphi, \lambda) := \frac{1}{2} a_\varepsilon(\varphi, \varphi) + b(\varphi, \lambda) - l(\varphi). \tag{5.17}$$

(iii) The optimal function φ_ε is the minimizer of $\hat{J}^\star_\varepsilon$ over W_ε while the optimal multiplier $\lambda_\varepsilon \in L^2(Q_T)$ is the state of the heat equation (5.1) in the weak sense.

Proof. We easily check that the bilinear form a_ε is continuous over $\Phi_\varepsilon \times \Phi_\varepsilon$, symmetric and positive and that the bilinear form b_ε is continuous over $\Phi_\varepsilon \times L^2(Q_T)$. Furthermore, for any fixed ε, the continuity of the linear form l over Φ_ε can be viewed from the energy estimate :

$$\|\varphi(\cdot, 0)\|^2_{L^2(\Omega)} \le C \left(\iint_{Q_T} |L^\star \varphi|^2 dx\, dt + \|\varphi(\cdot, T)\|^2_{L^2(\Omega)} \right), \quad \forall \varphi \in \Phi_\varepsilon,$$

for some $C > 0$, so that $\|\varphi(\cdot, 0)\|^2_{L^2(\Omega)} \le C \max(\eta^{-1}, \varepsilon^{-1}) \|\varphi\|^2_{\Phi_\varepsilon}$.

Therefore, the well-posedness of the mixed formulation is a consequence of the following two properties (see [11]):

- a_ε is coercive on $\mathcal{N}(b)$, where $\mathcal{N}(b)$ denotes the kernel of b :

$$\mathcal{N}(b) := \{\varphi \in \Phi_\varepsilon \ : \ b(\varphi, \lambda) = 0 \text{ for every } \lambda \in L^2(Q_T)\};$$

- b satisfies the usual "inf-sup" condition over $\Phi_\varepsilon \times L^2(Q_T)$: there exists $\delta > 0$ such that

$$\inf_{\lambda \in L^2(Q_T)} \sup_{\varphi \in \Phi_\varepsilon} \frac{b(\varphi, \lambda)}{\|\varphi\|_{\Phi_\varepsilon} \|\lambda\|_{L^2(Q_T)}} \ge \delta. \tag{5.18}$$

From the definition of a_ε, the first point is clear : for all $\varphi \in \mathcal{N}(b) = W_\varepsilon$, $a_\varepsilon(\varphi,\varphi) = \|\varphi\|^2_{\Phi_\varepsilon}$. Let us check the inf-sup condition. For any fixed $\lambda^0 \in L^2(Q_T)$, we define the (unique) element φ^0 of

$$L^\star\varphi^0 = -\lambda^0 \quad \text{in} \quad Q_T, \qquad \varphi^0 = 0 \quad \text{on} \quad \Sigma_T; \quad \varphi^0(\cdot,T) = 0 \quad \text{in} \quad \Omega,$$

so that φ^0 solves the backward heat equation with source term $-\lambda^0 \in L^2(Q_T)$, null Dirichlet boundary condition and zero initial state. Since $-\lambda^0 \in L^2(Q_T)$, then using energy estimates, there exists a constant $C_{\Omega,T} > 0$ such that the solution φ^0 of the backward heat equation with source term λ^0 satisfies the inequality

$$\iint_{q_T} \rho_0^{-2}|\varphi^0|^2 dx\,dt \leq \rho_\star^{-2}\iint_{q_T} |\varphi^0|^2 dx\,dt \leq \rho_\star^{-2}\, C_{\Omega,T}\,\|\lambda^0\|^2_{L^2(Q_T)}.$$

Consequently, $\varphi^0 \in \Phi_\varepsilon$. In particular, we have $b(\varphi^0,\lambda^0) = \|\lambda^0\|^2_{L^2(Q_T)}$ and

$$\sup_{\varphi\in\Phi_\varepsilon}\frac{b(\varphi,\lambda^0)}{\|\varphi\|_{\Phi_\varepsilon}\|\lambda^0\|_{L^2(Q_T)}} \geq \frac{b(\varphi^0,\lambda^0)}{\|\varphi^0\|_{\Phi_\varepsilon}\|\lambda^0\|_{L^2(Q_T)}} = \frac{\|\lambda^0\|^2_{L^2(Q_T)}}{(\|\rho_0^{-1}\varphi^0\|^2_{L^2(q_T)}+\eta\|\lambda_0\|^2_{L^2(Q_T)})^{\frac12}\,\|\lambda_0\|_{L^2(Q_T)}}.$$

Combining the above two inequalities, we obtain

$$\sup_{\varphi_0\in\Phi_\varepsilon}\frac{b(\varphi_0,\lambda_0)}{\|\varphi_0\|_{\Phi_\varepsilon}\|\lambda_0\|_{L^2(Q_T)}} \geq \frac{1}{\sqrt{\rho_\star^2\, C_{\Omega,T}+\eta}}$$

and, hence, (5.18) holds with $\delta = \left(\rho_\star^2\, C_{\Omega,T}+\eta\right)^{-1/2}$.

The point (ii) is due to the symmetry and to the positivity of the bilinear form a_ε. (iii) Concerning the third point, the equality $b(\varphi_\varepsilon,\overline{\lambda}) = 0$ for all $\overline{\lambda} \in L^2(Q_T)$ implies that $L^\star\varphi_\varepsilon = 0$ as an $L^2(Q_T)$ function, so that if $(\varphi_\varepsilon,\lambda_\varepsilon) \in \Phi_\varepsilon \times L^2(Q_T)$ solves the mixed formulation, then $\varphi_\varepsilon \in W_\varepsilon$ and $\mathcal{L}_\varepsilon(\varphi_\varepsilon,\lambda_\varepsilon) = \hat{J}^\star_\varepsilon(\varphi_\varepsilon)$. Finally, the first equation of the mixed formulation reads as follows:

$$\iint_{q_T} \rho_0^{-2}\varphi_\varepsilon\,\overline{\varphi}\,dx\,dt + \varepsilon(\varphi_\varepsilon(\cdot,T),\overline{\varphi}(\cdot,T)) - \iint_{Q_T} L^\star\overline{\varphi}(x,t)\,\lambda_\varepsilon(x,t)\,dx\,dt = l(\overline{\varphi}), \quad \forall\overline{\varphi}\in\Phi_\varepsilon,$$

or equivalently, since the control is given by $v_\varepsilon := \rho_0^{-2}\varphi_\varepsilon\,1_\omega$,

$$\iint_{q_T} v_\varepsilon\,\overline{\varphi}\,dx\,dt + (\varepsilon\varphi_\varepsilon(\cdot,T),\overline{\varphi}(\cdot,T)) - \iint_{Q_T} L^\star\overline{\varphi}(x,t)\,\lambda_\varepsilon(x,t)\,dx\,dt = l(\overline{\varphi}), \quad \forall\overline{\varphi}\in\Phi_\varepsilon.$$

But this means that $\lambda_\varepsilon \in L^2(Q_T)$ is solution of the heat equation in the transposition sense. Since $y_0 \in L^2(\Omega)$ and $v_\varepsilon \in L^2(q_T)$, λ_ε must coincide with the unique weak solution to (5.1) ($y_\varepsilon = \lambda_\varepsilon$) such that $\lambda_\varepsilon(\cdot,T) = -\varepsilon\varphi_\varepsilon(\cdot,T)$. As a conclusion, the optimal pair $(y_\varepsilon,v_\varepsilon)$ to (5.12) is characterized in term of the adjoint variable φ_ε solution of (5.16) by $v_\varepsilon = \rho_0^{-2}\varphi_\varepsilon\,1_\omega$ and $y_\varepsilon(\cdot,T) = -\varepsilon\varphi_\varepsilon(\cdot,T)$. □

Theorem 5.1 reduces the search of the approximated control to the resolution of the mixed formulation (5.16), or equivalently the search of the saddle point for $\mathcal{L}_\varepsilon$. In general, it is convenient to "augment" the Lagrangian (see [57]), and consider instead the Lagrangian $\mathcal{L}_{\varepsilon,r}$ defined for any $r > 0$ by

$$\begin{cases} \mathcal{L}_{\varepsilon,r}(\varphi,\lambda) := \dfrac{1}{2}a_{\varepsilon,r}(\varphi,\varphi) + b(\varphi,\lambda) - l(\varphi), \\ a_{\varepsilon,r}(\varphi,\varphi) := a_\varepsilon(\varphi,\varphi) + r\displaystyle\iint_{Q_T} |L^\star\varphi|^2\,dx\,dt. \end{cases}$$

Since $a_\varepsilon(\varphi,\varphi) = a_{\varepsilon,r}(\varphi,\varphi)$ on W_ε and since the function φ_ε such that $(\varphi_\varepsilon,\lambda_\varepsilon)$ is the saddle point of $\mathcal{L}_\varepsilon$ verifies $\varphi_\varepsilon \in W_\varepsilon$, the lagrangian $\mathcal{L}_\varepsilon$ and $\mathcal{L}_{\varepsilon,r}$ share the same saddle-point.

Dual problem of the extremal problem (5.15)

The mixed formulation allows to solve simultaneously the dual variable φ_ε, argument of the conjugate functional (5.15), and the Lagrange multiplier λ_ε. Since λ_ε turns out to be the (approximate) controlled state of (5.1), we may qualify λ_ε as the primal variable of the problem. We derive in this section the corresponding extremal problem involving only that variable λ_ε.

For any $r > 0$, let us define the linear operator $\mathcal{A}_{\varepsilon,r}$ from $L^2(Q_T)$ into $L^2(Q_T)$ by

$$\mathcal{A}_{\varepsilon,r}\lambda := L^\star\varphi, \quad \forall\lambda \in L^2(Q_T)$$

where $\varphi \in \Phi_\varepsilon$ is the unique solution to

$$a_{\varepsilon,r}(\varphi,\overline{\varphi}) = -b(\overline{\varphi},\lambda), \quad \forall\overline{\varphi} \in \Phi_\varepsilon. \tag{5.19}$$

Note that the assumption $r > 0$ is necessary here in order to guarantee the well-posedness of (5.19). Precisely, for any $r > 0$, the form $a_{\varepsilon,r}$ defines a norm equivalent to the norm on Φ_ε (see (5.14)).

We have the following crucial lemma :

Lemma 5.1. *For any $r > 0$, the operator $\mathcal{A}_{\varepsilon,r}$ is a strongly elliptic, symmetric isomorphism from $L^2(Q_T)$ into $L^2(Q_T)$.*

Proof. From the definition of $a_{\varepsilon,r}$, we easily get that $\|\mathcal{A}_{\varepsilon,r}\lambda\|_{L^2(Q_T)} \leq r^{-1}\|\lambda\|_{L^2(Q_T)}$ and the continuity of $\mathcal{A}_{\varepsilon,r}$. Next, consider any $\lambda' \in L^2(Q_T)$ and denote by φ' the corresponding unique solution of (5.19) so that $\mathcal{A}_{\varepsilon,r}\lambda' := L^\star\varphi'$. Relation (5.19) with $\overline{\varphi} = \varphi'$ then implies that

$$\iint_{Q_T} (\mathcal{A}_{\varepsilon,r}\lambda')\lambda\,dx\,dt = a_{\varepsilon,r}(\varphi,\varphi') \tag{5.20}$$

and therefore the symmetry and positivity of $\mathcal{A}_{\varepsilon,r}$. The last relation with $\lambda' = \lambda$ implies that $\mathcal{A}_{\varepsilon,r}$ is also positive definite.

Finally, let us check the strong ellipticity of $\mathcal{A}_{\varepsilon,r}$, equivalently that the bilinear functional

$$(\lambda, \lambda') \mapsto \iint_{Q_T} (\mathcal{A}_{\varepsilon,r}\lambda)\lambda' \, dx \, dt$$

is $L^2(Q_T)$-elliptic. Thus we want to show that

$$\iint_{Q_T} (\mathcal{A}_{\varepsilon,r}\lambda)\lambda \, dx \, dt \geq C\|\lambda\|^2_{L^2(Q_T)}, \quad \forall \lambda \in L^2(Q_T) \tag{5.21}$$

for some positive constant C. Suppose that (5.21) does not hold; there exists then a sequence $\{\lambda_n\}_{n\geq 0}$ of $L^2(Q_T)$ such that

$$\|\lambda_n\|_{L^2(Q_T)} = 1, \quad \forall n \geq 0, \qquad \lim_{n\to\infty} \iint_{Q_T} (\mathcal{A}_{\varepsilon,r}\lambda_n)\lambda_n \, dx \, dt = 0.$$

Let us denote by φ_n the solution of (5.19) corresponding to λ_n. From (5.20), we then obtain that

$$\lim_{n\to\infty} \|L^\star \varphi_n\|_{L^2(Q_T)} = 0, \qquad \lim_{n\to\infty} \|\rho_0^{-1}\varphi_n\|_{L^2(q_T)} = 0, \qquad \lim_{n\to\infty} \|\varphi_n(\cdot, T)\|_{L^2(\Omega)} = 0. \tag{5.22}$$

From (5.19) with $\lambda = \lambda_n$ and $\varphi = \varphi_n$, we have

$$\iint_{q_T} \rho_0^{-2}\varphi_n \, \overline{\varphi} \, dx \, dt + \varepsilon \int_0^1 \varphi_n(\cdot, T)\overline{\varphi}(\cdot, T) dx + \iint_{Q_T} (rL^\star\varphi_n - \lambda_n) L^\star \overline{\varphi} \, dx \, dt = 0, \quad \forall \overline{\varphi} \in \Phi_\varepsilon. \tag{5.23}$$

We define the sequence $\{\overline{\varphi}_n\}_{n\geq 0}$ as follows :

$$L^\star \overline{\varphi}_n = r\, L^\star \varphi_n - \lambda_n \quad \text{in} \quad Q_T, \qquad \overline{\varphi}_n = 0 \quad \text{on} \quad \Sigma_T; \quad \overline{\varphi}_n(\cdot, T) = 0 \quad \text{in} \quad \Omega,$$

so that, for all $n \geq 0$, $\overline{\varphi}_n$ is the solution of the backward heat equation with zero initial datum and source term $r\, L^\star\varphi_n - \lambda_n$ in $L^2(Q_T)$. Using again energy type estimates, we get

$$\|\rho_0^{-1}\overline{\varphi}_n\|_{L^2(q_T)} \leq \rho_\star^{-1}\|\overline{\varphi}_n\|_{L^2(q_T)} \leq \rho_\star^{-1} C_{\Omega,T}\|rL^\star\varphi_n - \lambda_n\|_{L^2(Q_T)},$$

so that $\overline{\varphi}_n \in \Phi_\varepsilon$. Then, using (5.23) with $\overline{\varphi} = \overline{\varphi}_n$, we get

$$\|rL^\star\varphi_n - \lambda_n\|_{L^2(Q_T)} \leq \rho_\star^{-1} C_{\Omega,T}\|\rho_0^{-1}\varphi_n\|_{L^2(q_T)}.$$

Then, from (5.22), we conclude that $\lim_{n\to+\infty} \|\lambda_n\|_{L^2(Q_T)} = 0$ leading to a contradiction and to the strong ellipticity of the operator $\mathcal{A}_{\varepsilon,r}$. □

The introduction of the operator $\mathcal{A}_{\varepsilon,r}$ is motivated by the following proposition:

Proposition 5.2. *For any $r > 0$, let $\varphi^0 \in \Phi_\varepsilon$ be the unique solution of*

$$a_{\varepsilon,r}(\varphi^0, \overline{\varphi}) = l(\overline{\varphi}), \quad \forall \overline{\varphi} \in \Phi_\varepsilon$$

and let $J^{\star\star}_{\varepsilon,r} : L^2(Q_T) \to L^2(Q_T)$ be the functional defined by

$$J^{\star\star}_{\varepsilon,r}(\lambda) := \frac{1}{2} \iint_{Q_T} (\mathcal{A}_{\varepsilon,r}\lambda)\, \lambda \, dx\, dt - b(\varphi^0, \lambda).$$

The following equality holds :

$$\sup_{\lambda \in L^2(Q_T)} \inf_{\varphi \in \Phi_\varepsilon} \mathcal{L}_{\varepsilon,r}(\varphi, \lambda) = - \inf_{\lambda \in L^2(Q_T)} J^{\star\star}_{\varepsilon,r}(\lambda) \;\; + \;\; \mathcal{L}_{\varepsilon,r}(\varphi^0, 0).$$

Proof. For any $\lambda \in L^2(Q_T)$, let us denote by $\varphi_\lambda \in \Phi_\varepsilon$ the minimizer of $\varphi \mapsto \mathcal{L}_{\varepsilon,r}(\varphi, \lambda)$; φ_λ satisfies the equation

$$a_{\varepsilon,r}(\varphi_\lambda, \overline{\varphi}) + b(\overline{\varphi}, \lambda) = l(\overline{\varphi}), \quad \forall \overline{\varphi} \in \Phi_\varepsilon$$

and can be decomposed as follows : $\varphi_\lambda = \psi_\lambda + \varphi^0$ where $\psi_\lambda \in \Phi_\varepsilon$ solves

$$a_{\varepsilon,r}(\psi_\lambda, \overline{\varphi}) + b(\overline{\varphi}, \lambda) = 0, \quad \forall \overline{\varphi} \in \Phi_\varepsilon.$$

We then have

$$\begin{aligned}
\inf_{\varphi \in \Phi_\varepsilon} \mathcal{L}_{\varepsilon,r}(\varphi, \lambda) &= \mathcal{L}_{\varepsilon,r}(\varphi_\lambda, \lambda) = \mathcal{L}_{\varepsilon,r}(\psi_\lambda + \varphi^0, \lambda) \\
&= \frac{1}{2} a_{\varepsilon,r}(\psi_\lambda + \varphi^0, \psi_\lambda + \varphi^0) + b(\psi_\lambda + \varphi^0, \lambda) - l(\psi_\lambda + \varphi^0) \\
&:= X_1 + X_2 + X_3
\end{aligned}$$

with

$$\begin{cases} X_1 := \frac{1}{2} a_{\varepsilon,r}(\psi_\lambda, \psi_\lambda) + b(\psi_\lambda, \lambda) + b(\varphi^0, \lambda) \\ X_2 := a_{\varepsilon,r}(\varphi^0, \psi_\lambda) - l(\psi_\lambda), \quad X_3 := \frac{1}{2} a_{\varepsilon,r}(\varphi^0, \varphi^0) - l(\varphi^0). \end{cases}$$

From the definition of φ^0, $X_2 = 0$ while $X_3 = \mathcal{L}_{\varepsilon,r}(\varphi^0, 0)$. Eventually, from the definition of ψ_λ,

$$X_1 = \frac{1}{2} b(\psi_\lambda, \lambda) + b(\varphi^0, \lambda) = -\frac{1}{2} \iint_{Q_T} (\mathcal{A}_{\varepsilon,r}\lambda)\, \lambda \, dx\, dt + b(\varphi^0, \lambda) = -J^{\star\star}_{\varepsilon,r}(\lambda)$$

and the result follows. □

From the ellipticity of the operator $\mathcal{A}_{\varepsilon,r}$, the minimization of the functional $J^{\star\star}_{\varepsilon,r}$ over $L^2(Q_T)$ is well-posed. It is interesting to note that with this extremal problem involving only λ, we are coming to the primal variable, controlled solution of (5.1) (see Theorem

5.1, (iii)). This argument allows notably to avoid the direct minimization of J_ε (introduced in Problem (5.12)) with respect to the state y (ill-conditioned due to the term ε^{-1} for ε small). Here, any constraint equality is assigned to the variable λ.

5.2.2 The penalized case : Mixed formulation II (relaxing the condition $L^\star\varphi_\varepsilon = 0$ in $L^2(Q_T)$)

The previous mixed formulation amounts to find a backward solution φ_ε satisfying the condition $L^\star\varphi_\varepsilon = 0$ in $L^2(Q_T)$. For numerical purposes, it may be interesting to relax this condition, which typically leads to the use of C^1 type approximations in the space variable (see Section 5.3). In order to circumvent this difficulty, we introduce and analyze in this section a second penalized mixed formulation where the condition on φ_ε is relaxed, namely we impose the constraint $L^\star\varphi_\varepsilon = 0$ in $L^2(0,T;H^{-1}(\Omega))$.

Considering as before the full adjoint variable φ as the main variable, we associated to (5.13) the following extremal problem :

$$\min_{\varphi\in\widehat{W}_\varepsilon} \hat{J}^\star_\varepsilon(\varphi) = \frac{1}{2}\iint_{Q_T} \rho_0^{-2}|\varphi(x,t)|^2 dx\,dt + \frac{\varepsilon}{2}\|\varphi(\cdot,T)\|^2_{L^2(\Omega)} + \int_\Omega y_0(x)\varphi(x,0)dx, \quad (5.24)$$

over the space $\widehat{W}_\varepsilon = \left\{\varphi \in \widehat{\Phi}_\varepsilon : L^\star\varphi = 0 \text{ in } L^2(0,T;H^{-1}(\Omega))\right\}$. The space $\widehat{\Phi}_\varepsilon$ is again defined as the completion of Φ^0 with respect to the inner product

$$(\varphi,\overline{\varphi})_{\widehat{\Phi}_\varepsilon} := \iint_{q_T} \rho_0^{-2}\varphi\,\overline{\varphi}\,dx\,dt + \varepsilon(\varphi(\cdot,T),\overline{\varphi}(\cdot,T)) + \eta\left(\iint_{Q_T}\nabla\varphi\,\nabla\overline{\varphi}\,dx\,dt + \int_0^T(\varphi_t,\overline{\varphi}_t)_{H^{-1}}dt\right),$$

defined over Φ^0. We denote by $\|\cdot\|_{\widehat{\Phi}_\varepsilon}$ the associated norm such that

$$\|\varphi\|^2_{\widehat{\Phi}_\varepsilon} := \|\rho_0^{-1}\varphi\|^2_{L^2(q_T)} + \varepsilon\|\varphi(\cdot,T)\|^2_{L^2(\Omega)} + \eta(\|\nabla\varphi\|^2_{L^2(Q_T)} + \|\varphi_t\|^2_{L^2(0,T;H^{-1})}), \quad \forall\varphi\in\widehat{\Phi}_\varepsilon. \quad (5.25)$$

Lemma 5.2. *The equality $\widehat{W}_\varepsilon = W_\varepsilon$ holds. Therefore, the minimization problem (5.24) is equivalent to the minimization (5.15).*

Proof. First, let us see that $W_\varepsilon \subset \widehat{W}_\varepsilon$. To do this, it is enough see that $\Phi_\varepsilon \subset \widehat{\Phi}_\varepsilon$. In fact, if $\varphi \in \Phi_\varepsilon$ then there exists a sequence $(\varphi^n)_{n=1}^\infty$ in Φ_0 such that $\varphi^n \to \varphi$ in Φ_ε. So, we can conclude that $\varphi^n \to \varphi$ in $L^2(0,T;H_0^1(\Omega))$ and $\varphi^n_t \to \varphi_t$ in $L^2(0,T;H^{-1}(\Omega))$. Hence, $\varphi^n \to \varphi$ in $\widehat{\Phi}_\varepsilon$.

Secondly, let us see that $\widehat{W}_\varepsilon \subset W_\varepsilon$. Indeed, if $\widehat{\varphi} \in \widehat{W}_\varepsilon$ then $\widehat{\varphi} \in \widehat{\Phi}_\varepsilon$ and $L^\star\widehat{\varphi} = 0$. Let us denote $\widehat{\varphi}_T := \widehat{\varphi}(\cdot,T)$, so there exists a sequence $(\varphi^n_T)_{n=1}^\infty$ in $C_0^\infty(\Omega)$ such that $\varphi^n_T \to \widehat{\varphi}_T$ in $L^2(\Omega)$. Now, if $(\varphi^n)_{n=1}^\infty$ is a sequence such that $L^\star\varphi^n = 0$, $\varphi^n = 0$ on Σ_T and $\varphi^n(\cdot,T) := \varphi^n_T$ then this sequence belongs to Φ^0. Hence, $\varphi^n \to \widehat{\varphi}$ in $\widehat{\Phi}_\varepsilon$ and $\varphi^n \to \widehat{\varphi}$ in Φ_ε. Therefore, $\widehat{\varphi}$ belongs to W_ε. □

The main variable is now φ submitted to the constraint equality $L^\star\varphi = 0 \in L^2(0,T;H^{-1})$. As before, this constraint is addressed by introducing a mixed formulation given as follows : find $(\varphi_\varepsilon, \lambda_\varepsilon) \in \widehat{\Phi}_\varepsilon \times \widehat{\Lambda}_\varepsilon$ solution of

$$\left\{ \begin{array}{rcll} \hat{a}_\varepsilon(\varphi_\varepsilon, \overline{\varphi}) + \hat{b}(\overline{\varphi}, \lambda_\varepsilon) & = & \hat{l}(\overline{\varphi}), & \forall \overline{\varphi} \in \widehat{\Phi}_\varepsilon \\ \hat{b}(\varphi_\varepsilon, \overline{\lambda}) & = & 0, & \forall \overline{\lambda} \in \widehat{\Lambda}_\varepsilon, \end{array} \right. \tag{5.26}$$

where $\widehat{\Lambda}_\varepsilon := L^2(0,T;H_0^1(\Omega))$ and

$$\hat{a}_\varepsilon : \widehat{\Phi}_\varepsilon \times \widehat{\Phi}_\varepsilon \to \mathbb{R}, \quad \hat{a}_\varepsilon(\varphi, \overline{\varphi}) := \iint_{q_T} \rho_0^{-2} \varphi\, \overline{\varphi}\, dx\, dt + \varepsilon(\varphi(\cdot,T), \overline{\varphi}(\cdot,T))_{L^2(\Omega)}$$

$$\hat{b} : \widehat{\Phi}_\varepsilon \times \widehat{\Lambda}_\varepsilon \to \mathbb{R},$$

$$\begin{aligned} \hat{b}(\varphi, \lambda) &:= -\int_0^T < L^\star\varphi, \lambda >_{H^{-1}(\Omega), H_0^1(\Omega)} dt \\ &= \int_0^T \langle \varphi_t(t), \lambda(t) \rangle_{H^{-1}, H_0^1} dt - \iint_{Q_T} \Big((c(x)\nabla\varphi, \nabla\lambda) + d(x,t)\varphi\lambda \Big) dx\, dt \end{aligned}$$

$$\hat{l} : \widehat{\Phi}_\varepsilon \to \mathbb{R}, \quad \hat{l}(\varphi) := -\int_\Omega y_0(x)\varphi(x,0)dx.$$

Similarly to Theorem 5.1, the following holds :

Theorem 5.2. *(i) The mixed formulation (5.26) is well-posed.*

(ii) The unique solution $(\varphi_\varepsilon, \lambda_\varepsilon) \in \widehat{\Phi}_\varepsilon \times \widehat{\Lambda}_\varepsilon$ is the unique saddle-point of the Lagrangian operator $\widehat{\mathcal{L}}_\varepsilon : \widehat{\Phi}_\varepsilon \times \widehat{\Lambda}_\varepsilon \to \mathbb{R}$ defined by

$$\widehat{\mathcal{L}}_\varepsilon(\varphi, \lambda) := \frac{1}{2}\hat{a}_\varepsilon(\varphi, \varphi) + \hat{b}(\varphi, \lambda) - \hat{l}(\varphi). \tag{5.27}$$

(iii) The optimal function φ_ε is the minimizer of $\hat{J}_\varepsilon^\star$ over $\widehat{W}_\varepsilon$ while the optimal multiplier $\lambda_\varepsilon \in \hat{\Lambda}_\varepsilon$ is the weak solution of the heat equation (5.1).

Proof. We easily check that the bilinear form $\hat{a}_\varepsilon$ is continuous over $\widehat{\Phi}_\varepsilon \times \widehat{\Phi}_\varepsilon$, symmetric and positive and that the bilinear form $\hat{b}$ is continuous over $\widehat{\Phi}_\varepsilon \times \widehat{\Lambda}_\varepsilon$. Furthermore, the continuity of the linear form $\hat{l}$ over $\widehat{\Phi}_\varepsilon$ is a direct by the continuous embedding $\widehat{\Phi}_\varepsilon \hookrightarrow C^0([0,T];L^2(\Omega))$. Therefore, the well-posedness of the mixed formulation is a consequence of the following two properties (see [11]):

- $\hat{a}_\varepsilon$ is coercive on $\mathcal{N}(\hat{b})$, where $\mathcal{N}(\hat{b})$ denotes the kernel of $\hat{b}$:

$$\mathcal{N}(\hat{b}) = \left\{ \varphi \in \widehat{\Phi}_\varepsilon \text{ such that } \hat{b}(\varphi, \lambda) = 0 \text{ for every } \lambda \in \widehat{\Lambda}_\varepsilon \right\}.$$

- $\hat{b}$ satisfies the usual "inf-sup" condition over $\widehat{\Phi}_\varepsilon \times \widehat{\Lambda}_\varepsilon$: there exists $\delta > 0$ such that

$$\inf_{\lambda \in \widehat{\Lambda}_\varepsilon} \sup_{\varphi \in \widehat{\Phi}_\varepsilon} \frac{\hat{b}(\varphi, \lambda)}{\|\varphi\|_{\widehat{\Phi}_\varepsilon} \|\lambda\|_{\widehat{\Lambda}_\varepsilon}} \geq \delta. \tag{5.28}$$

From the definition of $\hat{a}_\varepsilon$, the first point is clear : for all $\varphi \in \mathcal{N}(\hat{b}_\varepsilon) = \widehat{W}_\varepsilon$, thanks to classical energy estimates, we have

$$\begin{aligned}
\hat{a}_\varepsilon(\varphi, \varphi) &= \|\rho_0^{-1}\varphi\|^2_{L^2(q_T)} + \frac{\varepsilon}{2}\|\varphi(\cdot, T)\|^2_{L^2(\Omega)} + \frac{\varepsilon}{2}\|\varphi(\cdot, T)\|^2_{L^2(\Omega)} \\
&\geq \|\rho_0^{-1}\varphi\|^2_{L^2(q_T)} + \frac{\varepsilon}{2}\|\varphi(\cdot, T)\|^2_{L^2(\Omega)} + \varepsilon\, C(\|\nabla\varphi\|^2_{L^2(Q_T)} + \|\varphi_t\|^2_{L^2(0,T;H^{-1})}) \\
&\geq C_{\varepsilon,\eta}\|\varphi\|^2_{\widehat{\Phi}_\varepsilon},
\end{aligned}$$

where $C = C(T, c_0, \|d\|_\infty) > 0$ and $C_{\varepsilon,\eta} := \min(2^{-1}, C\,\varepsilon\,\eta^{-1})$.

Let us check the inf-sup condition. For any fixed $\lambda^0 \in \widehat{\Lambda}_\varepsilon$, we define the (unique) element φ^0 of

$$L^\star\varphi^0 = \Delta\lambda^0 \quad \text{in} \quad Q_T, \qquad \varphi^0 = 0 \quad \text{on} \quad \Sigma_T; \quad \varphi^0(\cdot, T) = 0 \quad \text{in} \quad \Omega,$$

so that φ^0 solves the backward heat equation with source term $\Delta\lambda^0$, null Dirichlet boundary condition and zero initial state. Since $\Delta\lambda^0 \in L^2(0,T;H^{-1})$, then $\varphi^0 \in \widehat{\Phi}_\varepsilon$: precisely, using energy estimates, there exists a constant $C > 0$ such that φ^0 satisfies the inequalities

$$\|\nabla\varphi^0\|^2_{L^2(Q_T)} \leq C\|\nabla\lambda^0\|^2_{L^2(Q_T)}$$

and

$$\begin{aligned}
\|\varphi^0\|^2_{\widehat{\Phi}_\varepsilon} &= \|\rho_0^{-1}\varphi^0\|^2_{L^2(q_T)} + \varepsilon\|\varphi^0(\cdot, T)\|^2_{L^2(\Omega)} + \eta(\|\nabla\varphi^0\|^2_{L^2(Q_T)} + \|\varphi^0_t\|^2_{L^2(0,T;H^{-1})}) \\
&= \|\rho_0^{-1}\varphi^0\|^2_{L^2(q_T)} + \eta(\|\nabla\varphi^0\|^2_{L^2(Q_T)} + \|\varphi^0_t\|^2_{L^2(0,T;H^{-1})}) \\
&\leq C_\eta\|\nabla\lambda^0\|^2_{L^2(Q_T)}.
\end{aligned}$$

where $C = C(T, \|c\|_\infty, \|d\|_\infty) > 0$ and $C_\eta := C(1 + \eta)$.

Consequently, $\varphi^0 \in \widehat{\Phi}_\varepsilon$. In particular, we have $\hat{b}(\varphi^0, \lambda^0) = \|\nabla\lambda^0\|^2_{L^2(Q_T)}$ and

$$\sup_{\varphi \in \widehat{\Phi}_\varepsilon} \frac{\hat{b}(\varphi, \lambda^0)}{\|\varphi\|_{\widehat{\Phi}_\varepsilon}\|\lambda^0\|_{\widehat{\Lambda}_\varepsilon}} \geq \frac{\hat{b}(\varphi^0, \lambda^0)}{\|\varphi^0\|_{\widehat{\Phi}_\varepsilon}\|\lambda^0\|_{\widehat{\Lambda}_\varepsilon}} \geq \frac{\|\nabla\lambda^0\|^2_{L^2(Q_T)}}{C_\eta^{1/2}\|\nabla\lambda_0\|_{L^2(Q_T)}\|\nabla\lambda_0\|_{L^2(Q_T)}}.$$

Combining the above two inequalities, we obtain

$$\sup_{\varphi \in \widehat{\Phi}_\varepsilon} \frac{\hat{b}(\varphi, \lambda^0)}{\|\varphi\|_{\widehat{\Phi}_\varepsilon}\|\lambda^0\|_{\widehat{\Lambda}_\varepsilon}} \geq \frac{1}{\sqrt{C_\eta}}$$

and, hence, (5.28) holds with $\delta = C_\eta^{-\frac{1}{2}}$.

The point (ii) is due to the symmetry and to the positivity of the bilinear form $\hat{a}_\varepsilon$.

Concerning the third assertion, the equality $b(\hat{\varphi}_\epsilon, \overline{\lambda}) = 0$ for all $\overline{\lambda} \in \widehat{\Lambda}_\varepsilon$ implies that $L^\star \varphi_\epsilon = 0$ as an $L^2(0,T;H^{-1})$ function, so that if $(\varphi_\varepsilon, \lambda_\varepsilon) \in \widehat{\Phi}_\varepsilon \times \widehat{\Lambda}_\varepsilon$ solves the mixed formulation (5.26), then $\varphi_\varepsilon \in \widehat{W}_\varepsilon$ and $\widehat{\mathcal{L}}_\varepsilon(\varphi_\varepsilon, \lambda_\varepsilon) = \hat{J}^\star_\varepsilon(\varphi_\varepsilon)$. This implies that φ_ε of the two mixed formulations coincide.

Assuming $y_0 \in L^2(\Omega)$ and $v \in L^2(q_T)$, it is said here that $y \in L^2(0,T;H^1_0(\Omega))$ is the (unique) solution by transposition of the heat equation (5.1) if and only if, for every $g \in L^2(0,T;H^{-1})$, we have

$$\int_0^T \langle g, y\rangle_{H^{-1},H^1_0} dt = \iint_{q_T} v\,\overline{\varphi}\,dx\,dt + (\overline{\varphi}(\cdot,0), y_0)_{L^2(\Omega)},$$

where $\overline{\varphi}$ solves

$$L^\star \overline{\varphi} = g \quad \text{in} \quad Q_T, \qquad \overline{\varphi} = 0 \quad \text{on} \quad \Sigma_T, \quad \overline{\varphi}(\cdot,T) = 0 \quad \text{in} \quad \Omega.$$

As $g \mapsto (v, \overline{\varphi})_{L^2(q_T)} + (\overline{\varphi}(\cdot,0), y_0)_{L^2(\Omega)}$ is linear and continuous on $L^2(0,T;H^{-1})$ the Riesz representation theorem guarantees that this definition makes sense.

Finally, the first equation of the mixed formulation (5.26) reads as follows:

$$\begin{aligned}&\iint_{q_T} \rho_0^{-2}\varphi_\varepsilon\,\overline{\varphi}\,dx\,dt + \varepsilon(\varphi_\varepsilon(\cdot,T), \overline{\varphi}(\cdot,T)) + \int_0^T \langle \overline{\varphi}_t, \lambda_\varepsilon\rangle_{H^{-1},H^1_0}\\ &- \iint_{Q_T} (c(x)\nabla\overline{\varphi}, \nabla\lambda_\varepsilon) + d(x,t)\overline{\varphi}\lambda_\varepsilon\,dx\,dt = \hat{l}(\overline{\varphi}), \quad \forall \overline{\varphi} \in \widehat{\Phi}_\varepsilon,\end{aligned}$$

or equivalently, since the control is given by $v_\varepsilon = \rho_0^{-2}\varphi_\varepsilon$ (recall that the formulations (5.15) and (5.24) are equivalent),

$$\begin{aligned}&\iint_{q_T} v_\varepsilon\,\overline{\varphi}\,dx\,dt + (\varepsilon\varphi_\varepsilon(\cdot,T), \overline{\varphi}(\cdot,T)) + \int_0^T \langle \overline{\varphi}_t, \lambda_\varepsilon\rangle_{H^{-1},H^1_0}\,dt\\ &- \iint_{Q_T} (c(x)\nabla\overline{\varphi}, \nabla\lambda_\varepsilon) + d(x,t)\overline{\varphi}\lambda_\varepsilon\,dx\,dt = \hat{l}(\overline{\varphi}), \quad \forall \overline{\varphi} \in \widehat{\Phi}_\varepsilon.\end{aligned}$$

But this means that $\lambda_\varepsilon \in \widehat{\Lambda}_\varepsilon$ is solution of the heat equation in the transposition sense. Since $y_0 \in L^2(\Omega)$ and $v_\varepsilon \in L^2(q_T)$, λ_ε must coincide with the unique weak solution to (5.1) $(y_\varepsilon = \lambda_\varepsilon)$ and, in particular, we can conclude that $y_\varepsilon(\cdot,T) = -\varepsilon\varphi_\varepsilon(\cdot,T)$. So from the unique of the weak solution, the solution $(\varphi_\varepsilon, \lambda_\varepsilon)$ of the two mixed formulation coincides. □

The equivalence of the mixed formulation (5.26) with the mixed formulation (5.16) is related to the regularizing property of the heat kernel. At the numerical level, the advantage is that this formulation leads naturally to continuous spaces of approximation both in time and space.

5.2.3 Third mixed formulation of the controllability problem : the limit case $\varepsilon = 0$

We consider in this section the limit case of Section 5.2.1 corresponding to $\varepsilon = 0$, i.e. to the null controllability. The conjugate functional $J^\star$ corresponding to this case is given in the introduction, see (5.4), with a weight ρ_0^{-2} (recall that $\rho_0 \in \mathcal{R}$ defined by (5.11)) in the first term, precisely

$$\min_{\varphi_T \in \mathcal{H}} J^\star(\varphi_T) := \frac{1}{2}\iint_{q_T} \rho_0^{-2}(x,t)|\varphi(x,t)|^2\,dxdt + (y_0, \varphi(\cdot,0))_{L^2(\Omega)} \tag{5.29}$$

where the variable φ solves the backward heat equation (5.5) and $\mathcal{H}$ is again defined as the completion of the $L^2(\Omega)$ space with respect to the norm $\|\varphi_T\|_{\mathcal{H}} := \|\rho_0^{-1}\varphi\|_{L^2(q_T)}$. As explained in the introduction, the limit case is much more singular due to the hugeness of the space $\mathcal{H}$. At the limit $\varepsilon = 0$, the control of the terminal state $\varphi(\cdot,T)$ is lost in $L^2(\Omega)$.

Let $\rho \in \mathcal{R}$. Proceeding as before, we consider again the space $\widetilde{\Phi}_0 = \{\varphi \in C^2(\overline{Q_T}) : \varphi = 0 \text{ on } \Sigma_T\}$ and then, for any $\eta > 0$, we define the bilinear form

$$(\varphi,\overline{\varphi})_{\widetilde{\Phi}_{\rho_0,\rho}} := \iint_{q_T} \rho_0^{-2}\varphi\,\overline{\varphi}\,dx\,dt + \eta\iint_{Q_T} \rho^{-2}L^\star\varphi\,L^\star\overline{\varphi}\,dx\,dt, \quad \forall \varphi,\overline{\varphi} \in \widetilde{\Phi}_0.$$

The introduction of the weight ρ, which does not appear in the original problem (5.29) will be motivated at the end of this Section. From the unique continuation property for the heat equation, this bilinear form defines for any $\eta > 0$ a scalar product. Let then $\widetilde{\Phi}_{\rho_0,\rho}$ be the completion of $\widetilde{\Phi}_0$ for this scalar product. We denote the norm over $\widetilde{\Phi}_{\rho_0,\rho}$ by $\|\cdot\|_{\widetilde{\Phi}_{\rho_0,\rho}}$ such that

$$\|\varphi\|^2_{\widetilde{\Phi}_{\rho_0,\rho}} := \|\rho_0^{-1}\varphi\|^2_{L^2(q_T)} + \eta\|\rho^{-1}L^\star\varphi\|^2_{L^2(Q_T)}, \quad \forall \varphi \in \widetilde{\Phi}_{\rho_0,\rho}. \tag{5.30}$$

Finally, we defined the closed subset $\widetilde{W}_{\rho_0,\rho}$ of $\widetilde{\Phi}_{\rho_0,\rho}$ by

$$\widetilde{W}_{\rho_0,\rho} = \{\varphi \in \widetilde{\Phi}_{\rho_0,\rho} : \rho^{-1}L^\star\varphi = 0 \text{ in } L^2(Q_T)\}$$

and we endow $\widetilde{W}_{\rho_0,\rho}$ with the same norm than $\widetilde{\Phi}_{\rho_0,\rho}$.

We then define the following extremal problem :

$$\min_{\varphi \in \widetilde{W}_{\rho_0,\rho}} \hat{J}^\star(\varphi) = \frac{1}{2}\iint_{q_T} \rho_0^{-2}|\varphi(x,t)|^2 dx\,dt + (y_0,\, \varphi(\cdot,0))_{L^2(\Omega)}. \tag{5.31}$$

For any $\varphi \in \widetilde{W}_{\rho_0,\rho}$, $L^\star\varphi = 0$ a.e. in Q_T and $\|\varphi\|_{\widetilde{W}_{\rho_0,\rho}} = \|\rho_0^{-1}\varphi\|_{L^2(q_T)}$ so that $\varphi(\cdot,T)$ belongs by definition to the abstract space $\mathcal{H}$: consequently, extremal problems (5.31) and (5.29) are equivalent. In particular, from the regularizing property of the heat kernel, $\varphi(\cdot,0)$ belongs to $L^2(\Omega)$ and the linear term in φ in $\hat{J}^\star$ is well defined.

Then, we consider the following mixed formulation : find $(\varphi,\lambda) \in \widetilde{\Phi}_{\rho_0,\rho} \times L^2(Q_T)$ solution of

$$\begin{cases} \tilde{a}(\varphi,\overline{\varphi}) + \tilde{b}(\overline{\varphi},\lambda) & = & \tilde{l}(\overline{\varphi}), & \forall \overline{\varphi} \in \widetilde{\Phi}_{\rho_0,\rho} \\ \tilde{b}(\varphi,\overline{\lambda}) & = & 0, & \forall \overline{\lambda} \in L^2(Q_T), \end{cases} \tag{5.32}$$

where

$$\tilde{a} : \widetilde{\Phi}_{\rho_0,\rho} \times \widetilde{\Phi}_{\rho_0,\rho} \to \mathbb{R}, \quad \tilde{a}(\varphi,\overline{\varphi}) = \iint_{q_T} \rho_0^{-2} \varphi\, \overline{\varphi}\, dx\, dt$$

$$\tilde{b} : \widetilde{\Phi}_{\rho_0,\rho} \times L^2(Q_T) \to \mathbb{R}, \quad \tilde{b}(\varphi,\lambda) = -\iint_{Q_T} \rho^{-1} L^\star \varphi\, \lambda\, dx\, dt$$

$$\tilde{l} : \widetilde{\Phi}_{\rho_0,\rho} \to \mathbb{R}, \quad \tilde{l}(\varphi) = -(y_0, \varphi(\cdot,0))_{L^2(\Omega)}.$$

Before studying this mixed formulation, let us do the following comment. The continuity of $\tilde{l}$ over the space $\widetilde{\Phi}_{\rho_0,\rho}$ holds true for a precise choice of the weights which appear in Carleman type estimates for parabolic equations (see [62]): we recall the following important result.

Proposition 5.3 ([62]). *Let the weights $\rho^c, \rho_0^c \in \mathcal{R}$ (see (5.11)) be defined as follows :*

$$\rho^c(x,t) := \exp\left(\frac{\beta(x)}{T-t}\right), \quad \beta(x) := K_1\left(e^{K_2} - e^{\beta_0(x)}\right),$$
$$\rho_0^c(x,t) := (T-t)^{3/2}\rho^c(x,t), \tag{5.33}$$

where the K_i are sufficiently large positive constants (depending on T, c_0 and $\|c\|_{C^1(\overline{\Omega})}$) such that

$$\beta_0 \in C^\infty(\overline{\Omega}),\ \beta > 0 \quad in \quad \Omega,\ \beta = 0 \quad on \quad \partial\Omega, \quad Supp(\nabla\beta) \subset \overline{\Omega} \setminus \omega.$$

Then, there exists a constant $C > 0$, depending only on ω, T, such that

$$\|\varphi(\cdot,0)\|_{L^2(\Omega)} \le C\|\varphi\|_{\widetilde{\Phi}_{\rho_0^c,\rho^c}}, \quad \forall \varphi \in \widetilde{\Phi}_{\rho_0^c,\rho^c}. \tag{5.34}$$

The estimate (5.34) is a consequence of the celebrated global Carleman inequality satisfied by the solution of (5.5), introduced and popularized in [62]. It allows to obtain the following existence and uniqueness result :

Theorem 5.3. *Let $\rho_0 \in \mathcal{R}$ and $\rho \in \mathcal{R} \cap L^\infty(Q_T)$ and assume that there exists a positive constant K such that*

$$\rho_0 \le K\rho_0^c, \quad \rho \le K\rho^c \quad in \quad Q_T. \tag{5.35}$$

Then, we have :

(i) The mixed formulation (5.32) defined over $\widetilde{\Phi}_{\rho_0,\rho} \times L^2(Q_T)$ is well-posed.

(ii) The unique solution $(\varphi,\lambda)\in\widetilde{\Phi}_{\rho_0,\rho}\times L^2(Q_T)$ *is the unique saddle-point of the Lagrangian* $\widetilde{\mathcal{L}}:\widetilde{\Phi}_{\rho_0,\rho}\times L^2(Q_T)\to\mathbb{R}$ *defined by*

$$\widetilde{\mathcal{L}}(\varphi,\lambda)=\frac{1}{2}\tilde{a}(\varphi,\varphi)+\tilde{b}(\varphi,\lambda)-\tilde{l}(\varphi). \tag{5.36}$$

(iii) The optimal function φ *is the minimizer of* $\hat{J}^\star$ *over* $\widetilde{\Phi}_{\rho_0,\rho}$ *while* $\rho^{-1}\lambda\in L^2(Q_T)$ *is the state of the heat equation (5.1) in the weak sense.*

Proof. The proof is similar to the proof of Theorem 5.1. From the definition, the bilinear form $\tilde{a}$ is continuous over $\widetilde{\Phi}_{\rho_0,\rho}\times\widetilde{\Phi}_{\rho_0,\rho}$, symmetric and positive and the bilinear form $\tilde{b}$ is continuous over $\widetilde{\Phi}_{\rho_0,\rho}\times L^2(Q_T)$. Furthermore, the continuity of the linear form $\tilde{l}$ over $\widetilde{\Phi}_{\rho_0,\rho}$ is the consequence of the estimate (5.34): precisely, from the assumptions (5.35), the inclusion $\widetilde{\Phi}_{\rho_0,\rho}\subset\widetilde{\Phi}_{\rho_0^c,\rho^c}$ hold true. Therefore, estimate (5.34) implies

$$\|\varphi(\cdot,0)\|_{L^2(\Omega)}\le C\|\varphi\|_{\widetilde{\Phi}_{\rho_0^c,\rho^c}}\le CK^{-1}\|\varphi\|_{\widetilde{\Phi}_{\rho_0,\rho}},\quad\forall\varphi\in\widetilde{\Phi}_{\rho_0,\rho}. \tag{5.37}$$

Therefore, the well-posedeness of the formulation (5.32) is the consequence of two properties: first, the coercivity of the form $\tilde{a}$ on the kernel $\mathcal{N}(\tilde{b}):=\{\varphi\in\widetilde{\Phi}_{\rho_0,\rho}:\tilde{b}(\varphi,\lambda)=0\,\forall\lambda\in L^2(Q_T)\}$: again, this holds true since the kernel coincides with the space $\widetilde{W}_{\rho_0,\rho}$. Second, the inf-sup property which reads as :

$$\inf_{\lambda\in L^2(Q_T)}\sup_{\varphi\in\widetilde{\Phi}_{\rho_0,\rho}}\frac{\tilde{b}(\varphi,\lambda)}{\|\varphi\|_{\widetilde{\Phi}_{\rho_0,\rho}}\|\lambda\|_{L^2(Q_T)}}\ge\delta \tag{5.38}$$

for some $\delta>0$. For any fixed $\lambda^0\in L^2(Q_T)$, we define the unique element φ^0 solution of

$$\rho^{-1}L^\star\varphi=-\lambda^0\text{ in }Q_T,\quad\varphi=0\text{ on }\Sigma_T,\quad\varphi(\cdot,T)=0\text{ in }\Omega.$$

Using energy estimates, we have

$$\|\rho_0^{-1}\varphi^0\|_{L^2(q_T)}\le\rho_\star^{-1}\|\varphi^0\|_{L^2(Q_T)}\le\rho_\star^{-1}\|\rho\lambda^0\|_{L^2(Q_T)}\le\rho_\star^{-1}\|\rho\|_{L^\infty(Q_T)}\|\lambda^0\|_{L^2(Q_T)} \tag{5.39}$$

which proves that $\varphi^0\in\widetilde{\Phi}_{\rho_0,\rho}$ and that

$$\sup_{\varphi\in\widetilde{\Phi}_{\rho_0,\rho}}\frac{\tilde{b}(\varphi,\lambda^0)}{\|\varphi\|_{\widetilde{\Phi}_{\rho_0,\rho}}\|\lambda^0\|_{L^2(Q_T)}}\ge\frac{\tilde{b}(\varphi^0,\lambda^0)}{\|\varphi^0\|_{\widetilde{\Phi}_{\rho_0,\rho}}\|\lambda^0\|_{L^2(Q_T)}}=\frac{\|\lambda^0\|_{L^2(Q_T)}}{\left(\|\rho_0^{-1}\varphi^0\|^2_{L^2(q_T)}+\eta\|\lambda_0\|^2_{L^2(Q_T)}\right)^{\frac{1}{2}}}.$$

Combining the above two inequalities, we obtain

$$\sup_{\varphi_0\in\widetilde{\Phi}_{\rho_0,\rho}}\frac{b(\varphi_0,\lambda_0)}{\|\varphi_0\|_{\widetilde{\Phi}_{\rho_0,\rho}}\|\lambda_0\|_{L^2(Q_T)}}\ge\frac{1}{\sqrt{\rho_\star^{-2}\|\rho\|^2_{L^\infty(Q_T)}+\eta}}$$

and, hence, (5.38) holds with $\delta = \left(\rho_\star^{-2}\|\rho\|^2_{L^\infty(Q_T)} + \eta\right)^{-1/2}$.

The point (ii) is again due to the positivity and symmetry of the form $\tilde{a}$.

Concerning the last point of the Theorem, the equality $\tilde{b}(\varphi, \overline{\lambda}) = 0$ for all $\overline{\lambda} \in L^2(Q_T)$ implies that $\rho^{-1}L^\star\varphi = 0$ as an $L^2(Q_T)$ function, so that if $(\varphi, \lambda) \in \widetilde{\Phi}_{\rho,\rho_0} \times L^2(Q_T)$ solves the mixed formulation (5.32), then $\varphi \in \widetilde{W}_{\rho,\rho_0}$ and $\widetilde{\mathcal{L}}(\varphi, \lambda) = \hat{J}^\star(\varphi)$. Finally, the first equation of the mixed formulation (5.32) reads as follows:

$$\iint_{q_T} \rho_0^{-2}\varphi\,\overline{\varphi}\,dx\,dt - \iint_{Q_T} \rho^{-1}L^\star\overline{\varphi}\lambda\,dx\,dt = \tilde{l}(\overline{\varphi}), \quad \forall\overline{\varphi} \in \widetilde{\Phi},$$

or equivalently, since the control is given by $v := \rho_0^{-2}\varphi\,1_\omega$,

$$\iint_{q_T} v\,\overline{\varphi}\,dx\,dt - \iint_{Q_T} L^\star\overline{\varphi}(\rho^{-1}\lambda)\,dx\,dt = \tilde{l}(\overline{\varphi}), \quad \forall\overline{\varphi} \in \widetilde{\Phi},$$

This means that $\rho^{-1}\lambda \in L^2(Q_T)$ is solution of the heat equation with source term $v\,1_\omega$ in the transposition sense and such that $(\rho^{-1}\lambda)(\cdot, T) = 0$. Since $y_0 \in L^2(\Omega)$ and $v \in L^2(q_T)$, $\rho^{-1}\lambda$ must coincide with the unique weak solution to (5.1) ($y = \rho^{-1}\lambda$) and, in particular, $y(\cdot, T) = 0$. □

Remark 5.1. The well-posedness of the mixed formulation (5.32), precisely the inf-sup property (5.38), is open in the case where the weight ρ is simply in $\mathcal{R}$ (instead of $\mathcal{R} \cap L^\infty(Q_T)$): in this case, the weight ρ may blow up at $t \to T^-$. In order to get (5.38), it suffices to prove that the function $\psi := \rho_0^{-1}\varphi$ solution of the boundary value problem

$$\rho^{-1}L^\star(\rho_0\psi) = -\lambda^0 \text{ in } Q_T, \quad \psi = 0 \text{ on } \Sigma_T, \quad \psi(\cdot, T) = 0 \text{ in } \Omega$$

for any $\lambda_0 \in L^2(Q_T)$ satisfies the following estimate for some positive constant C

$$\|\psi\|_{L^2(q_T)} \le C\|\rho^{-1}L^\star(\rho_0\psi)\|_{L^2(Q_T)}.$$

In the cases of interest for which the weights ρ_0 and ρ blow up at $t \to T^-$ (for instance given by ρ_0^c and ρ^c), this estimates is open and does not seem to be a consequence of the estimate (5.34).

Let us now comment the introduction of the weight ρ. The solution φ of the mixed formulation (5.32) belongs to $\widetilde{W}_{\rho_0,\rho}$ and therefore does not depend on the weight ρ (recall that ρ is strictly positive); this is in agreement with the fact that ρ does not appear in the original formulation formulation (5.29). Therefore, this weight may be seen as a parameter to improve some specific properties of the mixed formulation, specifically at the numerical level. Precisely, in the limit case $\varepsilon = 0$, we recall that the trace $\varphi|_{t=T}$ of the solution does not belong to $L^2(\Omega)$ but to a much larger and singular space. Very likely, a similar behavior occurs for the function $L^\star\varphi$ near $\Omega \times \{T\}$ so that the constraint $L^\star\varphi = 0$ in $L^2(Q_T)$ introduced in Section 5.2.1 is too "strong" and must be replaced at the limit in ε by the relaxed one $\rho^{-1}L^\star\varphi = 0$ in $L^2(Q_T)$ with ρ^{-1} "small" near $\Omega \times \{T\}$. Remark that this is actually the effect and the role of the Carleman type weights ρ^c defined by (5.33) and initially introduced in [62].

As in Section 5.2.1, it is convenient to "augment" the Lagrangian and consider instead the Lagrangian $\mathcal{L}_r$ defined for any $r > 0$ by

$$\begin{cases} \mathcal{L}_r(\varphi,\lambda) := \dfrac{1}{2}\tilde{a}_r(\varphi,\varphi) + \tilde{b}(\varphi,\lambda) - \tilde{l}(\varphi), \\ \tilde{a}_r(\varphi,\varphi) := \tilde{a}(\varphi,\varphi) + r \displaystyle\iint_{Q_T} |\rho^{-1}L^\star\varphi|^2\,dx\,dt. \end{cases}$$

Finally, similarly to Lemma 5.1 and Proposition 5.2, we have the following result.

Let $\rho_0 \in \mathcal{R}$ and $\rho \in \mathcal{R} \cap L^\infty(Q_T)$

Proposition 5.4. *For any $r > 0$, let $\rho_0 \in \mathcal{R}$ and $\rho \in \mathcal{R} \cap L^\infty(Q_T)$ verifying (5.35). Let us define the linear operator $\mathcal{A}_r$ from $L^2(Q_T)$ into $L^2(Q_T)$ by*

$$\mathcal{A}_r\lambda := \rho^{-1}L^\star\varphi, \quad \forall\lambda \in L^2(Q_T),$$

where $\varphi \in \widetilde{\Phi}_{\rho_0,\rho}$ is the unique solution to

$$a_r(\varphi,\overline{\varphi}) = -b(\overline{\varphi},\lambda), \quad \forall\overline{\varphi} \in \widetilde{\Phi}_{\rho_0,\rho}.$$

$\mathcal{A}_r$ is a strongly elliptic, symmetric isomorphism from $L^2(Q_T)$ into $L^2(Q_T)$. Let $\hat{J}_r^{\star\star}$ be the functional defined by

$$\hat{J}_r^{\star\star} : L^2(Q_T) \mapsto L^2(Q_T), \quad \hat{J}_r^{\star\star}(\lambda) := \frac{1}{2}\iint_{Q_T} (\mathcal{A}_r\lambda)\,\lambda\,dx\,dt - \tilde{b}(\varphi^0,\lambda).$$

where $\varphi^0 \in \widetilde{\Phi}_{\rho_0,\rho}$ is the unique solution of

$$\tilde{a}_r(\varphi^0,\overline{\varphi}) = \tilde{l}(\overline{\varphi}), \quad \forall\overline{\varphi} \in \widetilde{\Phi}_{\rho_0,\rho}.$$

The following equality holds :

$$\sup_{\lambda\in L^2(Q_T)} \inf_{\varphi\in\widetilde{\Phi}_{\rho_0,\rho}} \mathcal{L}_r(\varphi,\lambda) = -\inf_{\lambda\in L^2(Q_T)} \hat{J}_r^{\star\star}(\lambda) \;+\; \mathcal{L}_r(\varphi^0,0).$$

5.3 Numerical approximation and experiments

5.3.1 Discretization of the mixed formulation (5.16)

We now turn to the discretization of the mixed formulation (5.16) assuming $r > 0$. Let then $\Phi_{\varepsilon,h}$ and $M_{\varepsilon,h}$ be two finite dimensional spaces parametrized by the variable h such that, for any $\varepsilon > 0$,

$$\Phi_{\varepsilon,h} \subset \Phi_\varepsilon, \quad M_{\varepsilon,h} \subset L^2(Q_T), \quad \forall h > 0.$$

Then, we can introduce the following approximated problems : find $(\varphi_h, \lambda_h) \in \Phi_{\varepsilon,h} \times M_{\varepsilon,h}$ solution of

$$\left\{ \begin{array}{rcll} a_{\varepsilon,r}(\varphi_h, \overline{\varphi}_h) + b(\overline{\varphi}_h, \lambda_h) & = & l(\overline{\varphi}_h), & \forall \overline{\varphi}_h \in \Phi_{\varepsilon,h} \\ b(\varphi_h, \overline{\lambda}_h) & = & 0, & \forall \overline{\lambda}_h \in M_{\varepsilon,h}. \end{array} \right. \tag{5.40}$$

The well-posedness of this mixed formulation is again a consequence of two properties : the coercivity of the bilinear form $a_{\varepsilon,r}$ on the subset $\mathcal{N}_h(b) = \{\varphi_h \in \Phi_{\varepsilon,h}; b(\varphi_h, \lambda_h) = 0 \quad \forall \lambda_h \in M_{\varepsilon,h}\}$. Actually, from the relation

$$a_{\varepsilon,r}(\varphi, \varphi) \geq C_{r,\eta} \|\varphi\|^2_{\Phi_\varepsilon}, \quad \forall \varphi \in \Phi_\varepsilon,$$

where $C_{r,\eta} = \min\{1, r/\eta\}$, the form $a_{\varepsilon,r}$ is coercive on the full space Φ_ε, and so *a fortiori* on $\mathcal{N}_h(b) \subset \Phi_{\varepsilon,h} \subset \Phi_\varepsilon$. The second property is a discrete inf-sup condition : there exists $\delta_h > 0$ such that

$$\inf_{\lambda_h \in M_{\varepsilon,h}} \sup_{\varphi_h \in \Phi_{\varepsilon,h}} \frac{b(\varphi_h, \lambda_h)}{\|\varphi_h\|_{\Phi_{\varepsilon,h}} \|\lambda_h\|_{M_{\varepsilon,h}}} \geq \delta_h. \tag{5.41}$$

For any fixed h, the spaces $M_{\varepsilon,h}$ and $\Phi_{\varepsilon,h}$ are of finite dimension so that the infimum and supremum in (5.41) are reached: moreover, from the property of the bilinear form $a_{\varepsilon,r}$, it is standard to prove that δ_h is strictly positive (see Section 5.3.3). Consequently, for any fixed $h > 0$, there exists a unique couple (φ_h, λ_h) solution of (5.40). On the other hand, the property $\inf_h \delta_h > 0$ is in general difficult to prove and depends strongly on the choice made for the approximated spaces $M_{\varepsilon,h}$ and $\Phi_{\varepsilon,h}$. We shall analyze numerically this property in Section 5.3.3.

Remark 5.2. For $r = 0$, the discrete mixed formulation (5.40) is not well-posed over $\Phi_{\varepsilon,h} \times M_{\varepsilon,h}$ because the bilinear form $a_{\varepsilon,r=0}$ is not coercive over the discrete kernel of b: the equality $b(\lambda_h, \varphi_h) = 0$ for all $\lambda_h \in M_{\varepsilon,h}$ does not imply that $L^\star \varphi_h$ vanishes. Therefore, the term $r\|L^\star \varphi_h\|^2_{L^2(Q_T)}$ may be understood as a numerical stabilization term: for any $h > 0$, it ensures the uniform coercivity of the form $a_{\varepsilon,r}$ (and so the well-posedness) and vanishes at the limit in h. We also emphasize that this term is not a regularization term as it does not add any regularity to the solution φ_h.

As in [23], the finite dimensional and conformal space $\Phi_{\varepsilon,h}$ must be chosen such that $L^\star \varphi_h$ belongs to $L^2(Q_T)$ for any $\varphi_h \in \Phi_{\varepsilon,h}$. This is guaranteed as soon as φ_h possesses second-order derivatives in $L^2_{loc}(Q_T)$. Any conformal approximation based on standard triangulation of Q_T achieves this sufficient property as soon as it is generated by spaces of functions continuously differentiable with respect to the variable x and spaces of continuous functions with respect to the variable t.

We introduce a triangulation $\mathcal{T}_h$ such that $\overline{Q_T} = \cup_{K \in \mathcal{T}_h} K$ and we assume that $\{\mathcal{T}_h\}_{h>0}$ is a regular family. Then, we introduce the space $\Phi_{\varepsilon,h}$ as follows :

$$\Phi_{\varepsilon,h} = \{\varphi_h \in C^1(\overline{Q_T}) : \varphi_h|_K \in \mathbb{P}(K) \quad \forall K \in \mathcal{T}_h, \ \varphi_h = 0 \text{ on } \Sigma_T\}$$

where $\mathbb{P}(K)$ denotes an appropriate space of polynomial functions in x and t. In this work, we consider for $\mathbb{P}(K)$ the so-called *Bogner-Fox-Schmit* (BFS for short) C^1-element defined for rectangles.

In the one dimensional setting considered in the sequel, it involves 16 degrees of freedom, namely the values of $\varphi_h, \varphi_{h,x}, \varphi_{h,t}, \varphi_{h,xt}$ on the four vertices of each rectangle K. Therefore $\mathbb{P}(K) = \mathbb{P}_{3,x} \otimes \mathbb{P}_{3,t}$ where $\mathbb{P}_{r,\xi}$ is by definition the space of polynomial functions of order r in the variable ξ. We refer to [21] page 76.

We also define the finite dimensional space

$$M_{\varepsilon,h} = \{\lambda_h \in C^0(\overline{Q_T}) : \lambda_h|_K \in \mathbb{Q}(K) \quad \forall K \in \mathcal{T}_h\},$$

where $\mathbb{Q}(K)$ denotes the space of affine functions both in x and t on the element K.

Again, in the one dimensional setting, for rectangle, we simply have $\mathbb{Q}(K) = \mathbb{P}_{1,x} \otimes \mathbb{P}_{1,t}$.

We also mention that the approximation is conformal : for any $h > 0$, we have $\Phi_{\varepsilon,h} \subset \Phi_\varepsilon$ and $M_{\varepsilon,h} \subset L^2(Q_T)$.

Let $n_h = \dim \Phi_{\varepsilon,h}, m_h = \dim M_{\varepsilon,h}$ and let the real matrices $A_{\varepsilon,r,h} \in \mathbb{R}^{n_h,n_h}$, $B_h \in \mathbb{R}^{m_h,n_h}$, $J_h \in \mathbb{R}^{m_h,m_h}$ and $L_h \in \mathbb{R}^{n_h}$ be defined by

$$\begin{cases} a_{\varepsilon,r}(\varphi_h, \overline{\varphi_h}) =< A_{\varepsilon,r,h}\{\varphi_h\}, \{\overline{\varphi_h}\} >_{\mathbb{R}^{n_h},\mathbb{R}^{n_h}}, \quad \forall \varphi_h, \overline{\varphi_h} \in \Phi_{\varepsilon,h}, \\ b(\varphi_h, \lambda_h) =< B_h\{\varphi_h\}, \{\lambda_h\} >_{\mathbb{R}^{m_h},\mathbb{R}^{m_h}}, \quad \forall \varphi_h \in \Phi_{\varepsilon,h}, \forall \lambda_h \in M_{\varepsilon,h}, \\ \displaystyle\iint_{Q_T} \lambda_h \overline{\lambda_h}\, dx\, dt =< J_h\{\lambda_h\}, \{\overline{\lambda_h}\} >_{\mathbb{R}^{m_h},\mathbb{R}^{m_h}}, \quad \forall \lambda_h, \overline{\lambda_h} \in M_{\varepsilon,h}, \\ l(\varphi_h) =< L_h, \{\varphi_h\} >, \quad \forall \varphi_h \in \Phi_{\varepsilon,h} \end{cases}$$

where $\{\varphi_h\} \in \mathbb{R}^{n_h}$ denotes the vector associated to φ_h and $< \cdot, \cdot >_{\mathbb{R}^{n_h},\mathbb{R}^{n_h}}$ the usual scalar product over $\mathbb{R}^{n_h}$. With these notations, Problem (5.40) reads as follows : find $\{\varphi_h\} \in \mathbb{R}^{n_h}$ and $\{\lambda_h\} \in \mathbb{R}^{m_h}$ such that

$$\begin{pmatrix} A_{\varepsilon,r,h} & B_h^T \\ B_h & 0 \end{pmatrix}_{\mathbb{R}^{n_h+m_h,n_h+m_h}} \begin{pmatrix} \{\varphi_h\} \\ \{\lambda_h\} \end{pmatrix}_{\mathbb{R}^{n_h+m_h}} = \begin{pmatrix} L_h \\ 0 \end{pmatrix}_{\mathbb{R}^{n_h+m_h}}. \tag{5.42}$$

The matrix $A_{\varepsilon,r,h}$ as well as the mass matrix J_h are symmetric and positive definite for any $h > 0$ and any $r > 0$. On the other hand, the matrix of order $m_h + n_h$ in (6.29) is symmetric but not positive definite. We use exact integration methods developed in [39] for the evaluation of the coefficients of the matrices. The system (6.29) is solved using the direct LU decomposition method.

Let us also mention that for $r = 0$, although the formulation (5.16) is well-posed, numerically, the corresponding matrix $A_{\varepsilon,0,h}$ is not invertible in agreement with Remark 5.2. In the sequel, we shall consider strictly positive values for r.

Once an approximation φ_h is obtained, an approximation $v_{\varepsilon,h}$ of the control v_ε is given by $v_{\varepsilon,h} = \rho_0^{-2}\varphi_{\varepsilon,h}\, 1_\omega$. The corresponding controlled state $y_{\varepsilon,h}$ may be obtained by solving (5.1) with standard forward approximation (we refer to [23], Section 4 where this is detailed). Here, since the controlled state is directly given by the multiplier λ, we simply use λ_h as an approximation of y and we do not report here the computation of y_h.

In the sequel, we only report numerical experiments in the one dimensional setting. We use uniform rectangular meshes. Each element is a rectangle of lengths Δx and Δt; $\Delta x > 0$ and $\Delta t > 0$ denote as usual the discretization parameters in space and time, respectively. We note

$$h := \max\{\text{diam}(K), K \in \mathcal{T}_h\}$$

where $\text{diam}(K)$ denotes the diameter of K.

5.3.2 Normalization and discretization of the mixed formulation (5.32)

The same approximation may be used for the mixed formulation (5.32). In particular, we easily check that the finite dimensional spaces $M_{\varepsilon,h}$ and $\Phi_{\varepsilon,h}$ (which actually do not depend on ε) are conformal approximation of $L^2(Q_T)$ and $\widetilde{\Phi}_{\rho_0,\rho}$ respectively. However, in the limit case $\varepsilon = 0$, a normalization of the variable φ, which is singular and takes arbitrarily large amplitude in the neighborhood of $\Omega \times \{T\}$ is very convenient and suitable in practice. Following [49], we introduce the variable $\psi := \rho_0^{-1}\varphi \in \rho_0^{-1}\widetilde{\Phi}_{\rho_0,\rho}$ and replace the mixed formulation (5.32) by the equivalent one: find $(\psi, \lambda) \in \rho_0^{-1}\widetilde{\Phi}_{\rho_0,\rho} \times L^2(Q_T)$ solution of

$$\begin{cases} \hat{a}(\psi, \overline{\psi}) + \hat{b}(\overline{\psi}, \lambda) & = & \hat{l}(\overline{\psi}), \qquad \forall \overline{\psi} \in \rho_0^{-1}\widetilde{\Phi}_{\rho_0,\rho} \\ \hat{b}(\psi, \overline{\lambda}) & = & 0, \qquad \forall \overline{\lambda} \in L^2(Q_T), \end{cases} \tag{5.43}$$

where

$$\hat{a} : \rho_0^{-1}\widetilde{\Phi}_{\rho_0,\rho} \times \rho_0^{-1}\widetilde{\Phi}_{\rho_0,\rho} \to \mathbb{R}, \quad \hat{a}(\psi, \overline{\psi}) = \iint_{q_T} \psi\, \overline{\psi}\, dx\, dt$$

$$\hat{b} : \rho_0^{-1}\widetilde{\Phi}_{\rho_0,\rho} \times L^2(Q_T) \to \mathbb{R}, \quad \hat{b}(\psi, \lambda) = -\iint_{Q_T} \rho^{-1} L^\star(\rho_0\, \psi)\, \lambda\, dx\, dt$$

$$\hat{l} : \rho_0^{-1}\widetilde{\Phi}_{\rho_0,\rho} \to \mathbb{R}, \quad \hat{l}(\varphi) = -(y_0, \rho_0(\cdot, 0)\psi(\cdot, 0))_{L^2(\Omega)}.$$

Well-posedness of this formulation is the consequence of Theorem 5.3. Moreover, the optimal controlled state is still given by $\rho^{-1}\lambda$ while the optimal control is expressed in term of the variable ψ as $v = \rho_0^{-1}\psi\, 1_\omega$.

The corresponding discretization approximation (augmented with the term $r\|\rho^{-1}L^\star(\rho_0\psi)\|_{L^2(Q_T)}$) reads as follows: find $(\psi_h, \lambda_h) \in \Phi_h \times M_h$ solution of

$$\begin{cases} \hat{a}_r(\psi_h, \overline{\psi}_h) + \hat{b}(\overline{\psi}_h, \lambda_h) & = & \hat{l}(\overline{\varphi}_h), \qquad \forall \overline{\psi}_h \in \Phi_h \\ \hat{b}(\psi_h, \overline{\lambda}_h) & = & 0, \qquad \forall \overline{\lambda}_h \in M_h. \end{cases} \tag{5.44}$$

with

$$\begin{aligned} a_r(\psi_h,\overline{\psi}_h) :=& a(\psi_h,\overline{\psi}_h) + r(\rho^{-1}L^\star(\rho_0\psi_h),\rho^{-1}L^\star(\rho_0\overline{\psi}_h))_{L^2(Q_T)} \\ =&(\psi_h,\overline{\psi}_h)_{L^2(q_T)} + r(\rho^{-1}L^\star(\rho_0\psi_h),\rho^{-1}L^\star(\rho_0\overline{\psi}_h))_{L^2(Q_T)}. \end{aligned}$$

for any $r>0$.

Remark 5.3. When the weights ρ_0 and ρ are chosen in such a way that they are compensated each other in the term $\rho^{-1}L^\star(\rho_0\psi)$, the change of variable has the effect to reduced the amplitude (with respect to the time variable) of the coefficients in the integrals of $\hat{a}_r$ and $\hat{b}$, and therefore, at the discrete level, to improve significantly the condition number of square matrix $\hat{A}_{r,h}$ so that $\hat{a}_r(\psi_h,\overline{\psi}_h) =< \hat{A}_{r,h}\{\psi_h\},\{\overline{\psi}_h\} >_{\mathbb{R}^{n_h},\mathbb{R}^{n_h}}$. In this respect, the change of variable, can be seen as a preconditioner for the mixed formulation (5.32).

Similarly to (6.29), we note the matrix form of (5.44) as follows :

$$\begin{pmatrix} \hat{A}_{r,h} & \hat{B}_h^T \\ \hat{B}_h & 0 \end{pmatrix}_{\mathbb{R}^{n_h+m_h,n_h+m_h}} \begin{pmatrix} \{\psi_h\} \\ \{\lambda_h\} \end{pmatrix}_{\mathbb{R}^{n_h+m_h}} = \begin{pmatrix} \hat{L}_h \\ 0 \end{pmatrix}_{\mathbb{R}^{n_h+m_h}}, \tag{5.45}$$

where $\hat{B}_h$ is the matrix so that $\hat{b}(\psi_h,\lambda_h) =< \hat{B}_h\{\psi_h\},\{\lambda_h\} >_{\mathbb{R}^{m_h},\mathbb{R}^{m_h}}$ and $\hat{L}_h$ is the matrix so that $\hat{l}(\psi_h) =< \hat{L}_h,\{\psi_h\} >$.

5.3.3 The discrete inf-sup test

Before giving and discussing some numerical experiments, we first test numerically the discrete inf-sup condition (5.41). Taking $\eta = r > 0$ so that $a_{\varepsilon,r}(\varphi,\overline{\varphi}) = (\varphi,\overline{\varphi})_{\Phi_\varepsilon}$ exactly for all $\varphi,\overline{\varphi}\in\Phi_\varepsilon$, it is readily seen (see for instance [16]) that the discrete inf-sup constant satisfies

$$\delta_{\varepsilon,r,h} = \inf\left\{\sqrt{\delta} : B_hA_{\varepsilon,r,h}^{-1}B_h^T\{\lambda_h\} = \delta\, J_h\{\lambda_h\}, \quad \forall\,\{\lambda_h\}\in\mathbb{R}^{m_h}\setminus\{0\}\right\}. \tag{5.46}$$

The matrix $B_hA_{\varepsilon,r,h}^{-1}B_h^T$ enjoys the same properties than the matrix $A_{\varepsilon,r,h}$: it is symmetric and positive definite so that the scalar $\delta_{\varepsilon,h}$ defined in term of the (generalized) eigenvalue problem (5.46) is strictly positive. This eigenvalue problem is solved using the power iteration algorithm (assuming that the lowest eigenvalue is simple): for any $\{v_h^0\}\in\mathbb{R}^{m_h}$ such that $\|\{v_h^0\}\|_2=1$, compute for any $n\geq 0$, $\{\varphi_h^n\}\in\mathbb{R}^{n_h}$, $\{\lambda_h^n\}\in\mathbb{R}^{m_h}$ and $\{v_h^{n+1}\}\in\mathbb{R}^{m_h}$ iteratively as follows :

$$\begin{cases} A_{\varepsilon,r,h}\{\varphi_h^n\} + B_h^T\{\lambda_h^n\} = 0 \\ B_h\{\varphi_h^n\} = -J_h\{v_h^n\} \end{cases}, \quad \{v_h^{n+1}\} = \frac{\{\lambda_h^n\}}{\|\{\lambda_h^n\}\|_2}.$$

The scalar $\delta_{\varepsilon,r,h}$ defined by (5.46) is then given by : $\delta_{\varepsilon,r,h} = \lim_{n\to\infty}(\|\{\lambda_h^n\}\|_2)^{-1/2}$.

We now give some numerical values of $\delta_{\varepsilon,r,h}$ with respect to h for the C^1-finite element introduced in Section 5.3.1.

We consider the one dimensional case for which $\Omega = (0,1)$ and take for simplicity $c := 1/10$ and $d := 0$. Values of the diffusion c and of the potential d do not affect qualitatively the results.

In the spirit of the previous work [49], we consider the following choice for the weight $\rho_0 \in \mathcal{R}$:

$$\rho_0(x,t) := (T-t)^{3/2} \exp\left(\frac{K_1}{(T-t)}\right), \quad (x,t) \in Q_T, \qquad K_1 := \frac{3}{4} \tag{5.47}$$

so that ρ_0 blows exponentially as $t \to T^-$. This allows a smooth behavior of the corresponding control $v := \rho_0^{-2}\varphi\, 1_\omega$. Let us insist however that the mixed formulation is well-posed for any weight $\rho_0 \in \mathcal{R}$, in particular $\rho_0 := 1$ (leading to the control of minimal L^2-norm and for which we refer to [105]). ρ_0 is independent of the variable x for simplicity.

We consider the following data $\omega = (0.2, 0.5)$, $T = 1/2$, and $\Omega = (0,1)$. Tables 5.1, 5.2 and 5.3 provides the values of $\delta_{\varepsilon,r,h}$ with respect to h and ε for $r = 10^{-2}, 1$ and $r = 10^2$, respectively. In view of the definition, we check that $\delta_{\varepsilon,r,h}$ increases as $r \to 0$ and $\varepsilon \to 0$. We also observe, that for r large enough (see Tables 5.2 and 5.3), the value of the inf-sup constant is almost constant with respect to ε and behaves like

$$\delta_{\varepsilon,r,h} \approx C_{\varepsilon,r,h} \times r^{-1/2} \tag{5.48}$$

for some constant $C_{\varepsilon,r,h} \in (0,1)$. More importantly, we observe that for any r and ε, the value of $\delta_{\varepsilon,r,h}$ is bounded by below uniformly with respect to the discretization parameter h. The same behavior is observed for other discretizations such that $\Delta t \neq \Delta x$, other supports ω and other choices for the weight ρ_0 (in particular $\rho_0 := 1$).

Consequently, we may conclude that the finite approximation we have used do "pass" the discrete inf-sup test. It is interesting to note that this is in contrast with the situation for the wave equation for which the parameter r have to be adjusted carefully with respect to h; we refer to [24]. Moreover, as it is usual in mixed finite element theory, such a property together with the uniform coercivity of form $a_{\varepsilon,r}$ then implies the convergence of the approximation sequence (φ_h, λ_h) solution of (5.40).

h	7.07×10^{-2}	3.53×10^{-2}	1.76×10^{-2}	8.83×10^{-3}
$\varepsilon = 10^{-2}$	8.358	8.373	8.381	8.386
$\varepsilon = 10^{-4}$	9.183	9.213	9.229	9.237
$\varepsilon = 10^{-8}$	9.263	9.318	9.354	9.383

Table 5.1: $\delta_{\varepsilon,r,h}$ w.r.t. ε and h ; $r = 10^{-2}$; $\Omega = (0,1)$, $\omega = (0.2, 0.5)$, $T = 1/2$.

Similarly, Table 5.4 displays the discrete inf-sup constant corresponding to the limit

h	7.07×10^{-2}	3.53×10^{-2}	1.76×10^{-2}	8.83×10^{-3}
$\varepsilon=10^{-2}$	9.933×10^{-1}	9.938×10^{-1}	9.940×10^{-1}	9.941×10^{-1}
$\varepsilon=10^{-4}$	9.933×10^{-1}	9.938×10^{-1}	9.941×10^{-1}	9.942×10^{-1}
$\varepsilon=10^{-8}$	9.933×10^{-1}	9.938×10^{-1}	9.941×10^{-1}	9.942×10^{-1}

Table 5.2: $\delta_{\varepsilon,r,h}$ w.r.t. ε and h ; $r=1$; $\Omega=(0,1)$, $\omega=(0.2,0.5)$, $T=1/2$.

h	7.07×10^{-2}	3.53×10^{-2}	1.76×10^{-2}	8.83×10^{-3}
$\varepsilon=10^{-2}$	9.933×10^{-2}	9.939×10^{-2}	9.940×10^{-2}	9.941×10^{-2}
$\varepsilon=10^{-4}$	9.933×10^{-2}	9.939×10^{-2}	9.941×10^{-2}	9.942×10^{-2}
$\varepsilon=10^{-8}$	9.933×10^{-2}	9.939×10^{-2}	9.941×10^{-2}	9.942×10^{-2}

Table 5.3: $\delta_{\varepsilon,r,h}$ w.r.t. ε and h ; $r=10^2$; $\Omega=(0,1)$, $\omega=(0.2,0.5)$, $T=1/2$.

case of the mixed formulation (5.43):

$$\begin{aligned}&\inf_{\lambda_h\in\widehat{M}_h}\sup_{\psi_h\in\widehat{\Phi}_h}\frac{\hat{b}(\psi_h,\lambda_h)}{\|\lambda_h\|_{L^2(Q_T)}\|\psi_h\|_{\rho_0^{-1}\widetilde{\Phi}_{\rho_0,\rho}}}\\ &=\inf_{\lambda_h\in\widehat{M}_h}\sup_{\psi_h\in\widehat{\Phi}_h}\frac{\displaystyle\iint_{q_T}\lambda_h\,\rho^{-1}L^\star(\rho_0\psi_h)\,dx\,dt}{\|\lambda_h\|_{L^2(Q_T)}(\|\psi_h\|^2_{L^2(q_T)}+r\|\rho^{-1}L^\star(\rho_0\,\psi_h)\|^2_{L^2(Q_T)})^{1/2}}.\end{aligned}$$

We take here a weight ρ independent of the variable x given by

$$\rho(x,t):=\exp\Big(\frac{K_1}{(T-t)}\Big),\quad (x,t)\in Q_T,\quad K_1:=\frac{3}{4}. \tag{5.49}$$

Again, for the limit case, the value given in the Table suggest a similar behavior observed for $\varepsilon>0$: the constant is uniformly bounded by below with respect to the parameter h and behaves like $r^{-1/2}$ for r large enough (up to 1). Remark that, due to the introduction of the weight $\rho\neq 1$, the inf-sup constants given by Table 5.4 are not the limit (as $\varepsilon\to 0$) of the previous Tables.

h	7.07×10^{-2}	3.53×10^{-2}	1.76×10^{-2}	8.83×10^{-3}	4.41×10^{-3}
$r=10^2$	6.9×10^{-2}	6.91×10^{-2}	7.06×10^{-2}	8.08×10^{-2}	9.52×10^{-2}
$r=1$	6.89×10^{-1}	6.91×10^{-1}	6.96×10^{-1}	7.94×10^{-1}	8.66×10^{-1}
$r=10^{-2}$	1.944	1.922	1.845	1.775	1.731

Table 5.4: $\varepsilon=0$; $\delta_{r,h}$ w.r.t. r and h; $\Omega=(0,1)$, $\omega=(0.2,0.5)$, $T=1/2$.

5.3.4 Numerical experiments for the mixed formulation (5.16)

We report in this section experiments for the mixed formulation (5.16) and for simplicity we consider only the one dimensional case: $\Omega = (0,1)$ and $T = 1/2$.

Let us first remark that in general explicit solutions $(\varphi_\varepsilon, \lambda_\varepsilon)$ of (5.16) are not available. However, when the coefficient c and d are constant, we may obtain a semi-explicit representation (using Fourier decomposition) of the minimizer $\varphi_{\varepsilon,T}$ of the conjugate functional $J^\star_\varepsilon$ (see (5.13)), and consequently of the corresponding adjoint variable φ_ε, the control of weighted minimal square integrable norm $v = \rho_0^{-2}\varphi_\varepsilon 1_\omega$ and finally the controlled state y_ε solution of (5.1). In practice, the obtention of the Fourier representation amounts to solve a symmetric linear system. We refer to the Appendix for the details.

Such representation allows to evaluate precisely the distance of the exact solution $(\varphi_\varepsilon, \lambda_\varepsilon)$ from the approximation (φ_h, λ_h) with respect to h and validate the convergence of the approximation with respect to h.

As for the initial data, we first simply take the first mode of the Laplacian, that is, $y_0(x) = \sin(\pi x)$, $x \in (0,1)$. In view of the regularization property of the heat equation, the regularity of the initial data has a very restricted effect on the optimal control and the robustness of the method.

We take $c(x) := 10^{-1}$, $d(x,t) := 0$ and recall that in the uncontrolled case ($\omega = \emptyset$), these data leads to $\|y(\cdot,T)\|_{L^2(0,1)} = \sqrt{1/2}e^{-\pi^2 cT} \approx 4.31 \times 10^{-1}$. Finally, we take $\omega = (0.2, 0.5)$.

For $r = 1$, Tables 5.5, 5.6 and 5.7 report some norms with respect to h for $\varepsilon = 10^{-2}$, 10^{-4} and $\varepsilon = 10^{-8}$ respectively. The cases $r = 10^2$ and $r = 10^{-2}$ are reported in the Appendix, in Tables 5.17, 5.18, 5.19 and 5.20, 5.21, 5.22 respectively. In the Tables, φ_ε and y_ε denotes the unique solution of (5.16) given by (5.51) and (5.53). In the Tables, $\kappa_{\varepsilon,h}$ denotes the condition number associated to (6.29), independent of the initial data y_0 [1].

We first check that the L^2-norm $\|\lambda_{\varepsilon,h}(\cdot,T)\|_{L^2(0,1)}$ of the final state is of the order of $\sqrt{\varepsilon}$ and that the condition number $\kappa_{\varepsilon,h}$ behave polynomially with respect to h; on the other hand, we observe a low variation of $\kappa_{\varepsilon,h}$ with respect to ε; $\kappa_{\varepsilon,h} \approx O(h^{5.9})$ for $\varepsilon = 10^{-2}$ and $\kappa_{\varepsilon,h} \approx O(h^{7.3})$ for $\varepsilon = 10^{-8}$.

Then, we check the convergence as h tends to zero of the approximations $(v_{\varepsilon,h}, \lambda_{\varepsilon,h})$ toward the optimal pair $(v_\varepsilon, y_\varepsilon)$ in $L^2(q_T) \times L^2(Q_T)$ for any values of ε and r.

More precisely, for large enough value of ε (here $\varepsilon = 10^{-2}$), we observe a quasi linear rate of convergence for both $\frac{\|\rho_0(v_\varepsilon - v_{\varepsilon,h})\|_{L^2(q_T)}}{\|\rho_0 v_\varepsilon\|_{L^2(q_T)}}$ and $\frac{\|y_\varepsilon - \lambda_{\varepsilon,h}\|_{L^2(Q_T)}}{\|y_\varepsilon\|_{L^2(Q_T)}}$ with respect to h, independent of the value of the parameter r. We refer to Figure 5.1. For small values of ε, we observe a reduced convergence both for the control and the state (see Figure 5.2 for $\varepsilon = 10^{-4}$ and Figure 5.3 for $\varepsilon = 10^{-8}$). We recall that as ε tends to zero, the space Φ_ε

[1] The condition number $\kappa(\mathcal{M}_h)$ of any square matrix $\mathcal{M}_h$ is defined by $\kappa(\mathcal{M}_h) = |||\mathcal{M}_h|||_2 |||\mathcal{M}_h^{-1}|||_2$ where the norm $|||\mathcal{M}_h|||_2$ stands for the largest singular value of $\mathcal{M}_h$.

degenerates into a much larger space and φ_ε highly oscillates near T. Remark also that for $\varepsilon = 10^{-8}$, the constraint $L^\star\varphi_\varepsilon = 0$ as an $L^2(Q_T)$ function is badly represented: this is due to the loss of regularity on the variable φ_ε (in the neighborhood of T) as $\varepsilon \to 0^+$. This does not prevent the convergence of the variable $\varphi_{\varepsilon,h}$ for the norm Φ_ε, in particular the control $v_{\varepsilon,h} = \rho^{-2}\varphi_{\varepsilon,h} 1_\omega$, and of the variable $\lambda_{\varepsilon,h}$. We will come back to this situation in detail in the section devoted to the limit case $\varepsilon = 0$. Moreover, for small value of ε, the parameter r does have an influence; precisely, a low value of r (here $r = 10^{-2}$) leads to better relative errors : this is in agreement with the behavior of the inf-sup constant $\delta_{\varepsilon,r,h}$ which increases with $r^{-1/2}$.

h	1.41×10^{-1}	7.07×10^{-2}	3.53×10^{-2}	1.76×10^{-2}	8.83×10^{-3}
$m_h + n_h$	330	1 155	4 305	16 605	65 205
$\Vert L^\star\varphi_{\varepsilon,h}\Vert_{L^2(Q_T)}$	1.32×10^{-1}	3.75×10^{-2}	9.66×10^{-3}	2.42×10^{-3}	7.82×10^{-4}
$\frac{\Vert\rho_0(v_\varepsilon - v_{\varepsilon,h})\Vert_{L^2(q_T)}}{\Vert\rho_0 v_\varepsilon\Vert_{L^2(q_T)}}$	1.10×10^{-1}	6.21×10^{-2}	3.29×10^{-2}	1.68×10^{-2}	8.57×10^{-3}
$\frac{\Vert y_\varepsilon - \lambda_{\varepsilon,h}\Vert_{L^2(Q_T)}}{\Vert y_\varepsilon\Vert_{L^2(Q_T)}}$	5.13×10^{-2}	2.84×10^{-2}	1.48×10^{-2}	7.60×10^{-3}	3.89×10^{-3}
$\Vert\lambda_{\varepsilon,h}(\cdot, T)\Vert_{L^2(0,1)}$	1.54×10^{-1}	1.61×10^{-1}	1.65×10^{-1}	1.67×10^{-1}	1.68×10^{-1}
$\kappa_{\varepsilon,h}$	1.52×10^{9}	1.10×10^{11}	6.80×10^{12}	3.83×10^{14}	1.96×10^{16}

Table 5.5: Mixed formulation (5.16) - $r = 1$ and $\varepsilon = 10^{-2}$ with $\omega = (0.2, 0.5)$.

h	1.41×10^{-1}	7.07×10^{-2}	3.53×10^{-2}	1.76×10^{-2}	8.83×10^{-3}
$\Vert L^\star\varphi_{\varepsilon,h}\Vert_{L^2(Q_T)}$	1.383	1.471	9.05×10^{-1}	2.56×10^{-1}	6.54×10^{-2}
$\frac{\Vert\rho_0(v_\varepsilon - v_{\varepsilon,h})\Vert_{L^2(q_T)}}{\Vert\rho_0 v_\varepsilon\Vert_{L^2(q_T)}}$	6.72×10^{-1}	3.22×10^{-1}	1.15×10^{-1}	5.49×10^{-2}	2.74×10^{-2}
$\frac{\Vert y_\varepsilon - \lambda_{\varepsilon,h}\Vert_{L^2(Q_T)}}{\Vert y_\varepsilon\Vert_{L^2(Q_T)}}$	2.73×10^{-1}	1.86×10^{-1}	5.89×10^{-2}	2.51×10^{-2}	1.26×10^{-2}
$\Vert\lambda_{\varepsilon,h}(\cdot, T)\Vert_{L^2(0,1)}$	8.50×10^{-2}	5.74×10^{-2}	3.39×10^{-2}	3.11×10^{-2}	3.13×10^{-2}
$\kappa_{\varepsilon,h}$	3.02×10^{9}	3.91×10^{11}	3.86×10^{13}	3.25×10^{15}	2.46×10^{17}

Table 5.6: Mixed formulation (5.16) - $r = 1$ and $\varepsilon = 10^{-4}$ with $\omega = (0.2, 0.5)$.

h	1.41×10^{-1}	7.07×10^{-2}	3.53×10^{-2}	1.76×10^{-2}	8.83×10^{-3}
$\Vert L^\star\varphi_{\varepsilon,h}\Vert_{L^2(Q_T)}$	1.48	2.03	2.50	2.52	2.61
$\frac{\Vert\rho_0(v_\varepsilon - v_{\varepsilon,h})\Vert_{L^2(q_T)}}{\Vert\rho_0 v_\varepsilon\Vert_{L^2(q_T)}}$	1.44	1.01	7.92×10^{-1}	6.65×10^{-1}	4.89×10^{-1}
$\frac{\Vert y_\varepsilon - \lambda_{\varepsilon,h}\Vert_{L^2(Q_T)}}{\Vert y_\varepsilon\Vert_{L^2(Q_T)}}$	8.42×10^{-1}	8.27×10^{-1}	5.73×10^{-1}	4.35×10^{-1}	2.89×10^{-1}
$\Vert\lambda_{\varepsilon,h}(\cdot, T)\Vert_{L^2(0,1)}$	8.63×10^{-2}	6.65×10^{-2}	2.39×10^{-2}	1.23×10^{-2}	4.43×10^{-3}
$\kappa_{\varepsilon,h}$	3.12×10^{9}	4.30×10^{11}	6.05×10^{13}	1.13×10^{16}	1.90×10^{18}

Table 5.7: Mixed formulation (5.16) - $r = 1$ and $\varepsilon = 10^{-8}$ with $\omega = (0.2, 0.5)$.

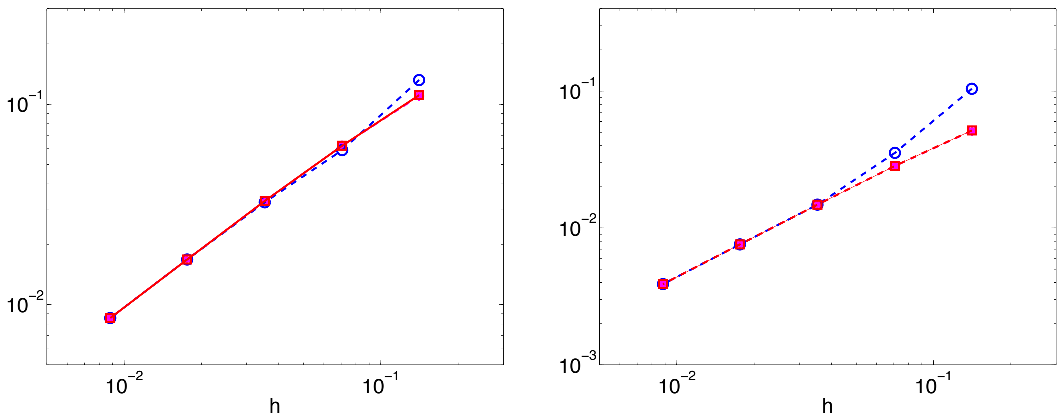

Figure 5.1: $\omega = (0.2, 0.5)$; $y_0(x) = \sin(\pi x)$: $\varepsilon = 10^{-2}$. ; $\frac{\|\rho_0(v_\varepsilon - v_{\varepsilon,h})\|_{L^2(q_T)}}{\|\rho_0 v_\varepsilon\|_{L^2(q_T)}}$ (**Left**) and $\frac{\|y_\varepsilon - \lambda_{\varepsilon,h}\|_{L^2(Q_T)}}{\|y_\varepsilon\|_{L^2(Q_T)}}$ (**Right**) vs. h for $r = 10^2$ ($\circ$), $r = 1.$ ($\star$) and $r = 10^{-2}$ ($\Box$).

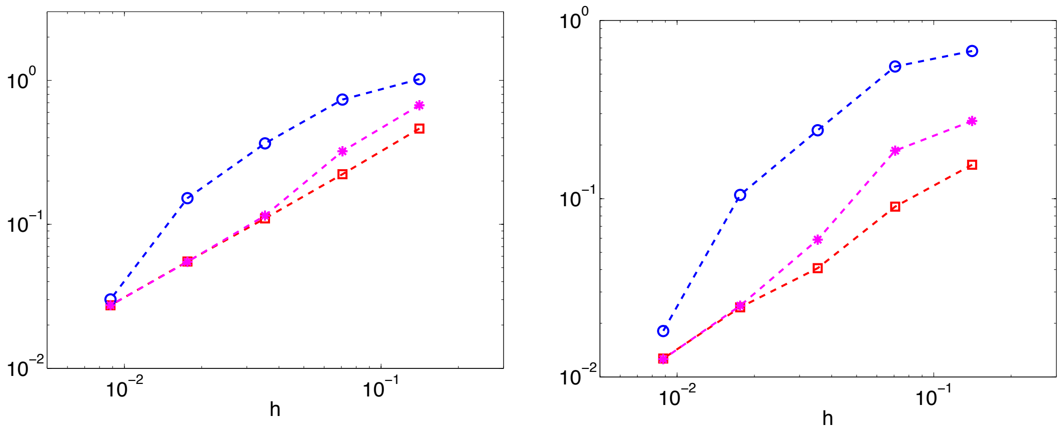

Figure 5.2: $\omega = (0.2, 0.5)$; $y_0(x) = \sin(\pi x)$: $\varepsilon = 10^{-4}$. ; $\frac{\|\rho_0(v_\varepsilon - v_{\varepsilon,h})\|_{L^2(q_T)}}{\|\rho_0 v_\varepsilon\|_{L^2(q_T)}}$ (**Left**) and $\frac{\|y_\varepsilon - \lambda_{\varepsilon,h}\|_{L^2(Q_T)}}{\|y_\varepsilon\|_{L^2(Q_T)}}$ (**Right**) vs. h for $r = 10^2$ ($\circ$), $r = 1.$ ($\star$) and $r = 10^{-2}$ ($\Box$).

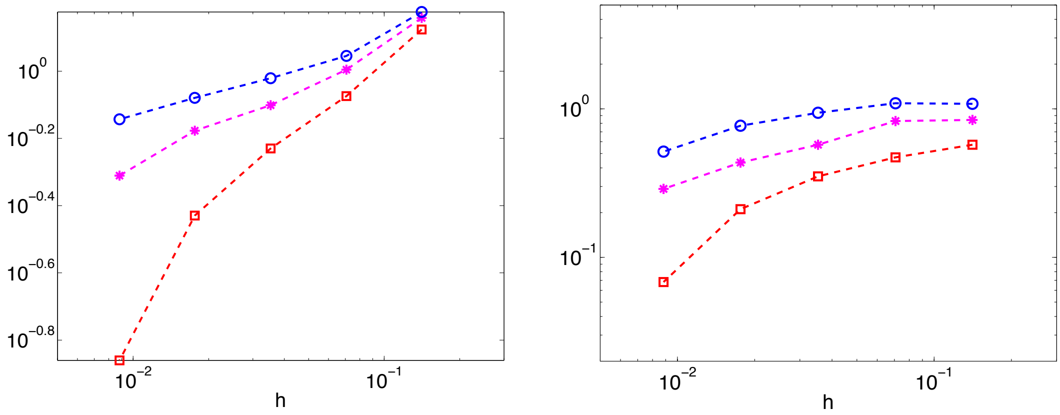

Figure 5.3: $\omega = (0.2, 0.5)$; $y_0(x) = \sin(\pi x)$: $\varepsilon = 10^{-8}$. ; $\frac{\|\rho_0(v_\varepsilon - v_{\varepsilon,h})\|_{L^2(q_T)}}{\|\rho_0 v_\varepsilon\|_{L^2(q_T)}}$ **(Left)** and $\frac{\|y_\varepsilon - \lambda_{\varepsilon,h}\|_{L^2(Q_T)}}{\|y_\varepsilon\|_{L^2(Q_T)}}$ **(Right)** vs. h for $r = 10^2$ ($\circ$), $r = 1.$ ($\star$) and $r = 10^{-2}$ ($\square$).

Remarkably, we highlight that the variational approach developed here allows, for any ε, a direct and robust approximation of one control for the heat equation. As discussed at length in [50, 105], the minimization of the conjugate functional $J_\varepsilon^\star$ using conjugate gradient algorithm requires a great number of iterates for small ε (typically $\varepsilon = 10^{-8}$ and $\omega = (0.2, 0.5)$) and diverge for small values of h.

Eventually, we present one experiment for the mixed formulation (5.26) introduced in Section 5.2.2, which require only the use of continuous finite element approximation. Precisely, we use here $\mathbb{P}_1$ finite elements for both the states φ_ε and λ_ε. With the same data as before, Table 5.8 reports some norms with respect to h for $r = 1$ and $\varepsilon = 10^{-4}$. We still observe the convergence when h tends to zero, but as expected (comparing with Table 5.6 corresponding to the mixed formulation (5.16)) with lower convergence rates: for instance, we obtain that $\|y_\varepsilon - \lambda_{\varepsilon,h}\|_{L^2(Q_T)} = \mathcal{O}(h^{0.91})$ while, from Table 5.6, we obtain $\|y_\varepsilon - \lambda_{\varepsilon,h}\|_{L^2(Q_T)} = \mathcal{O}(h^{1.17})$. We also refer to Section 4.1 of [49] where a different mixed formulation is discussed in this context.

h	1.41×10^{-1}	7.07×10^{-2}	3.53×10^{-2}	1.76×10^{-2}	8.83×10^{-3}
$m_h + n_h$	132	462	1 732	6 642	26 082
$\|L^\star \varphi_{\varepsilon,h}\|_{L^2(0,T;H^{-1}(0,1))}$	7.90×10^{-1}	4.42×10^{-1}	2.47×10^{-1}	1.37×10^{-1}	7.72×10^{-2}
$\frac{\|\rho_0(v_\varepsilon - v_{\varepsilon,h})\|_{L^2(q_T)}}{\|\rho_0 v_\varepsilon\|_{L^2(q_T)}}$	6.55×10^{-1}	3.50×10^{-1}	1.86×10^{-1}	9.87×10^{-2}	5.27×10^{-2}
$\frac{\|y_\varepsilon - \lambda_{\varepsilon,h}\|_{L^2(Q_T)}}{\|y_\varepsilon\|_{L^2(Q_T)}}$	3.86×10^{-1}	2.06×10^{-1}	1.09×10^{-1}	5.82×10^{-2}	3.11×10^{-2}

Table 5.8: Mixed formulation (5.26) - $r = 1$ and $\varepsilon = 10^{-4}$ with $\omega = (0.2, 0.5)$.

5.3.5 Conjugate gradient for $J^{\star\star}_{\varepsilon,r}$

We illustrate here the Section 5.2.1 and minimize the functional $J^{\star\star}_{\varepsilon,r} : L^2(Q_T) \to \mathbb{R}$ defined in Proposition 5.2 with respect to the variable λ_ε. From the ellipticity of the operator $\mathcal{A}_{\varepsilon,r}$, we use a conjugate gradient method which in the present case reads as follows :

(i) *Initialization*
Let $\lambda^0_\varepsilon \in L^2(Q_T)$ be a given function;
Solve

$$\begin{cases} \overline{\varphi}^0_\varepsilon \in \Phi_\varepsilon \\ a_{\varepsilon,r}(\overline{\varphi}^0_\varepsilon, \overline{\varphi}) + b_\varepsilon(\overline{\varphi}, \lambda^0_\varepsilon) = l_\varepsilon(\overline{\varphi}), \quad \forall \overline{\varphi} \in \Phi_\varepsilon \end{cases}$$

and set $g^0_\varepsilon = L^\star \overline{\varphi}^0_\varepsilon$ and set $w^0_\varepsilon = g^0_\varepsilon$.

For $n \geq 0$, assuming that $\lambda^n_\varepsilon, g^n_\varepsilon$ and w^n_ε are known with $g^n_\varepsilon \neq 0$ and $w^n_\varepsilon \neq 0$, compute $\lambda^{n+1}_\varepsilon, g^{n+1}_\varepsilon$ and w^{n+1}_ε as follows

(ii) *Steepest descent*
Compute $\overline{\varphi}^n \in \Phi_\varepsilon$ solution to

$$a_{\varepsilon,r}(\overline{\varphi}^n_\varepsilon, \overline{\varphi}) = -b_\varepsilon(\overline{\varphi}, w^n_\varepsilon), \quad \forall \overline{\varphi} \in \Phi_\varepsilon$$

and $\overline{w}^n_\varepsilon = L^\star \overline{\varphi}^n_\varepsilon$ and then compute

$$\rho_n = \|g^n_\varepsilon\|^2_{L^2(Q_T)} / (\overline{w}^n_\varepsilon, w^n_\varepsilon)_{L^2(Q_T)}.$$

and set

$$\lambda^{n+1}_\varepsilon = \lambda^n_\varepsilon - \rho_n w^n_\varepsilon.$$

(iii) *Testing the convergence and construction of the new descent direction*
Update g^n_ε by

$$g^{n+1}_\varepsilon = g^n_\varepsilon - \rho^n \overline{w}^n_\varepsilon.$$

If $\|g^{n+1}_\varepsilon\|_{L^2(Q_T)} / \|g^0_\varepsilon\|_{L^2(Q_T)} \leq \gamma$, take $\lambda_\varepsilon = \lambda^{n+1}_\varepsilon$. Else, compute

$$\gamma_n = \|g^{n+1}_\varepsilon\|^2_{L^2(Q_T)} / \|g^n_\varepsilon\|^2_{L^2(Q_T)}$$

and update w^n_ε via

$$w^{n+1}_\varepsilon = g^{n+1}_\varepsilon + \gamma_n w^n_\varepsilon.$$

Do $n = n + 1$ and return to step (ii).

As mentioned in [69] where this approach is discussed at length for Stokes and Navier-Stokes systems, this algorithm can be viewed as a sophisticated version of Uzawa type algorithm to solve the mixed formulation (5.16). Concerning the speed of convergence of this algorithm, it follows, for instance, from [34] that

$$\|\lambda_\varepsilon^n - \lambda_\varepsilon\|_{L^2(Q_T)} \leq 2\sqrt{\nu(\mathcal{A}_{\varepsilon,r})}\left(\frac{\sqrt{\nu(\mathcal{A}_{\varepsilon,r})}-1}{\sqrt{\nu(\mathcal{A}_{\varepsilon,r})}+1}\right)^n \|\lambda_\varepsilon^0 - \lambda_\varepsilon\|_{L^2(Q_T)}, \quad \forall n \geq 1$$

where λ_ε minimizes $J^{\star\star}_{\varepsilon,r}$. $\nu(\mathcal{A}_{\varepsilon,r}) = \|\mathcal{A}_{\varepsilon,r}\|\|\mathcal{A}_{\varepsilon,r}^{-1}\|$ denotes the condition number of the operator $\mathcal{A}_{\varepsilon,r}$.

Eventually, once the above algorithm has converged we can compute $\varphi_\varepsilon \in \Phi_\varepsilon$ as solution of

$$a_{\varepsilon,r}(\varphi_\varepsilon, \overline{\varphi}) + b_\varepsilon(\overline{\varphi}, \lambda_\varepsilon) = l_\varepsilon(\overline{\varphi}), \quad \forall \overline{\varphi} \in \Phi_\varepsilon.$$

We use the same spaces $\Phi_{\varepsilon,h}$ and $M_{\varepsilon,h}$ as described in Section 5.3.1. In practice, each iteration amounts to solve a linear system involving the matrix $A_{\varepsilon,r,h}$ of size $n_h = 4m_h$ (see (6.29)) which is sparse, symmetric and positive definite. We use the Cholesky method.

From the previous estimate, the performances of the algorithm are related to the condition number of the operator $\mathcal{A}_{\varepsilon,r}$ restricted to $M_{\varepsilon,h} \subset L^2(Q_T)$, which coincides here (see [11]) with the condition number of the symmetric and positive definite matrix $B_h A^{-1}_{\varepsilon,r,h} B_h^T$ introduced in (5.46). Using again the power iteration algorithm, we obtain that, for any h, the largest eigenvalue of $B_h A^{-1}_{\varepsilon,r,h} B_h^T$ is very closed to r^{-1} (and bounded by r^{-1}). This is in agreement with the estimate $\|\mathcal{A}_{\varepsilon,r}\lambda\|_{L^2(Q_T)} \leq r^{-1}\|\lambda\|_{L^2(Q_T)}$ for all $\lambda \in L^2(Q_T)$. Consequently, the condition number is expressed in term of r and of the discrete inf-sup constant $\delta_{\varepsilon,h}$ as follows :

$$\nu(B_h A^{-1}_{\varepsilon,r,h} B_h^T) \approx r^{-1}\delta^{-2}_{\varepsilon,r,h}.$$

Since, from our observation in Section 5.3.3, the discrete inf-sup constant $\delta_{\varepsilon,r,h}$ is uniformly bounded by below with respect to h, we deduce that the condition number is uniformly bounded by above with respect to the discretization parameter. This implies that the convergence of the sequence $\{\lambda^n_{\varepsilon,h}\}_{(n>0)}$, minimizing for $J^{\star\star}_{\varepsilon,r}$ over $M_{\varepsilon,h}$ is independent of h. This is exactly what we observe from our numerical experiments. Morever, from (5.48), we get that the number $\nu(B_h A^{-1}_{\varepsilon,r,h} B_h^T) \approx C^{-2}_{\varepsilon,r,h}$ is very closed to one. We refer to Tables 5.9 and 5.10 for the values.

h	7.07×10^{-2}	3.53×10^{-2}	1.76×10^{-2}	8.83×10^{-3}
$\varepsilon = 10^{-2}$	1.431	1.426	1.423	1.423
$\varepsilon = 10^{-4}$	1.185	1.177	1.173	1.171
$\varepsilon = 10^{-8}$	1.165	1.151	1.142	1.135

Table 5.9: $r^{-1}\delta^{-2}_{\varepsilon,r,h}$ w.r.t. ε and h ; $r = 10^{-2}$; $\Omega = (0,1)$, $\omega = (0.2, 0.5)$, $T = 1/2$.

h	7.07×10^{-2}	3.53×10^{-2}	1.76×10^{-2}	8.83×10^{-3}
$\varepsilon = 10^{-2}$	1.013	1.012	1.012	1.011
$\varepsilon = 10^{-4}$	1.013	1.012	1.011	1.011
$\varepsilon = 10^{-8}$	1.013	1.012	1.011	1.011

Table 5.10: $r^{-1}\delta_{\varepsilon,r,h}^{-2}$ w.r.t. ε and h ; $r = 1$ and $r = 10^2$; $\Omega = (0, 1)$, $\omega = (0.2, 0.5)$, $T = 1/2$.

We consider the same data as in Section 5.3.4, that is, $\omega = (0.2, 0.5)$, $y_0(x) = \sin(\pi x)$ and $T = 1/2$. We take $\gamma = 10^{-10}$ as a stopping threshold for the algorithm (that is the algorithm is stopped as soon as the norm of the residue g^n at the iterate n satisfies $\|g_\varepsilon^n\|_{L^2(Q_T)} \leq 10^{-10}\|g_\varepsilon^0\|_{L^2(Q_T)}$). The algorithm is initiated with $\lambda_{\varepsilon,h}^0 = 0$ in Q_T.

We check that the method provides, for the same value of r, ε and h, exactly the same approximation $\lambda_{\varepsilon,h}$ than the previous direct method (see Tables 5.5, etc). Table 5.11, 5.12 and 5.13, we simply give the number of iterates of the conjugate gradient algorithm for $r = 10^2$, $r = 1$ and $r = 10^{-2}$ with respect to h and ε respectively. For each case, the convergence is reached in very few iterates, independent of h. Once again, this is in contrast with the behavior of the conjugate gradient algorithm when this latter is used to minimize $J_\varepsilon^\star$ with respect to φ_T defined by (5.13). The number of convergence is also almost independent of ε and r. Since the gradient of $J_{\varepsilon,r}^{\star\star}$ is given by $\nabla J_{\varepsilon,r}^{\star\star}(\lambda') = \mathcal{A}_{\varepsilon,r}(\lambda') - L^\star\varphi_0$ for all $\lambda' \in L^2(Q_T)$, in particular $\nabla J_{\varepsilon,r}^{\star\star}(\lambda_\varepsilon) = L^\star\varphi_\varepsilon$, a larger value of the augmentation parameter r reduces (slightly here) the number of iterates.

According to this very low number of iterates, it seems more advantageous not only in term of memory resource but also in term of time execution to solve the extremal problem in the variable λ_ε than the (equivalent) mixed formulation (5.40). The matrix $A_{\varepsilon,r,h}$ of order n_h is very sparse, symmetric, positive definite, diagonal bloc (for which the Cholesky method is very efficient) while the matrix defined by (6.29), of order $m_h + n_h = 5/4n_h$ requires the use of for instance the Gauss decomposition method. Note however that the condition number of the matrix $A_{\varepsilon,r,h}$ is not independent of h but behaves polynomially (see Table 5.11 where the value is reported for $r = 1$.). On the other hand, the condition number slightly decreases with r (recall that the norm over Φ_ε contains the term $r\|L^\star\varphi\|_{L^2(Q_T)}$): consequently, for very stiff situation (typically ω very small), there may be a balance between large values of r leading to a better numerical robustness and low values of r leading to smaller relative errors on $v_{\varepsilon,h}$ and $\lambda_{\varepsilon,h}$.

For very small values of both h (leading to very fine meshes) of the order $h = 10^{-3}$ and ε, we observe some instabilities on the approximation $\lambda_{\varepsilon,h}$ (very likely due to the condition number of the matrix $A_{\varepsilon,r,h}$ which exceeds 10^{25} in this case). A preconditioning technique introduced in the next section is needed in these cases.

We do not describe experiments for the mixed formulation introduced in 5.2.2, which require the use of continuous finite element approximation. We refer to [49] in a closed

h	1.41×10^{-1}	7.07×10^{-2}	3.53×10^{-2}	1.76×10^{-2}	8.83×10^{-3}
$m_h = card(\{\lambda_{\varepsilon,h}\})$	66	231	861	3 321	13 041
$\sharp$ iterates - $\varepsilon = 10^{-2}$	5	5	5	5	5
$\sharp$ iterates - $\varepsilon = 10^{-4}$	5	5	5	4	4
$\sharp$ iterates - $\varepsilon = 10^{-8}$	5	5	5	5	5
$\kappa(A_{\varepsilon,r,h})$ - $\varepsilon = 10^{-2}$	1.51×10^{9}	1.10×10^{11}	6.81×10^{12}	3.83×10^{14}	1.91×10^{16}

Table 5.11: Mixed formulation (5.16) - $r = 1$ - $\omega = (0.2, 0.5)$; Conjugate gradient algorithm.

h	1.41×10^{-1}	7.07×10^{-2}	3.53×10^{-2}	1.76×10^{-2}	8.83×10^{-3}
$\sharp$ iterates - $\varepsilon = 10^{-2}$	5	5	4	4	4
$\sharp$ iterates - $\varepsilon = 10^{-4}$	5	5	5	4	4
$\sharp$ iterates - $\varepsilon = 10^{-8}$	5	5	5	5	4

Table 5.12: Mixed formulation (5.16) - $r = 10^2$ - $\omega = (0.2, 0.5)$; Conjugate gradient algorithm.

context.

5.3.6 Numerical experiments for the mixed formulation (5.32) - limit case $\varepsilon = 0$.

We now report in this section some experiments corresponding to the limit case, that is $\varepsilon = 0$, of the mixed formulation (5.32). We consider again the first mode : $y_0(x) = \sin(\pi x)$, take $\omega = (0.2, 0.5)$, $T = 1/2$ and the exponential type weights ρ_0 and ρ given by (5.47) and (5.49) respectively.

This particular choice of the weights allows to rewrite the quantity $\rho^{-1} L^\star \varphi$ in term of the new variable ψ as follow

$$\begin{aligned} \rho^{-1} L^\star(\rho_0 \psi) &= \rho^{-1} \rho_0 L^\star \psi - \rho^{-1} \rho_{0t} \psi \\ &= (T-t)^{3/2} L^\star \psi + \left(-\frac{3}{2}(T-t)^{1/2} + K_1 (T-t)^{-1/2} \right) \psi \end{aligned} \tag{5.50}$$

and thus eliminate the exponential singularity near T^{-1}. Only a much weaker polynomial singularity, precisely $(T-t)^{-1/2}$ remains.

Moreover, we define as "exact" solution (y, v) the solution obtained with a very fine mesh corresponding to $h \approx 1.1 \times 10^{-3}$, a number of element equal to $819\,200$ and a number of degrees of freedom equal to $m_h + n_h = 3\,284\,484$. With these values, we get the following norms :

$$\|\rho^{-1} \lambda_{h=1.1 \times 10^{-3}}\|_{L^2(Q_T)} \approx 3.592 \times 10^{-1}, \quad \|\rho_0 v_{h=1.1 \times 10^{-3}}\|_{L^2(q_T)} \approx 18.6634.$$

We do not use the Fourier expansion approach described in the Appendix, since the optimality equation (5.52) is ill-posed for $\varepsilon = 0$ and leads to instability as the number of

h	1.41×10^{-1}	7.07×10^{-2}	3.53×10^{-2}	1.76×10^{-2}	8.83×10^{-3}
$\sharp$ iterates - $\varepsilon = 10^{-2}$	9	9	8	8	8
$\sharp$ iterates - $\varepsilon = 10^{-4}$	8	8	8	8	8
$\sharp$ iterates - $\varepsilon = 10^{-8}$	8	8	7	7	7

Table 5.13: Mixed formulation (5.16) - $r = 10^{-2}$ - $\omega = (0.2, 0.5)$; Conjugate gradient algorithm.

modes used in the sum increases. On the contrary, the minimization of $J_r^{\star\star}$- equivalent to the resolution of the mixed formulation (5.32) exhibits a remarkable robustness as $h \to 0$. Eventually, we mention that the mesh used is so fine that the corresponding result is (almost) independent of the parameter r.

Tables 5.14, 5.15 and 5.16 reports some norms with respect to h for $r = 10^{-2}$, $r = 1$ and $r = 10^2$, respectively. Let us first mention that we again obtain exactly the same approximations from the direct resolution of the system (5.45) and from the minimization of $J_r^{\star\star}$.

As in the case $\varepsilon > 0$, we observe the convergence of $\rho^{-1}\lambda_h$ and $\rho_0 v_h$ in $L^2(Q_T)$ and $L^2(q_T)$ respectively as $h \to 0^+$. For instance, for $r = 1$, we obtain

$$\frac{\|\rho_0(v - v_h)\|_{L^2(q_T)}}{\|\rho_0 v\|_{L^2(q_T)}} \approx e^{1.04} h^{0.429}, \quad \frac{\|y - \rho^{-1}\lambda_h\|_{L^2(Q_T)}}{\|y\|_{L^2(Q_T)}} \approx e^{1.27} h^{0.704}.$$

Figure 5.4 depicts the evolution of these relatives errors with respect to h for $r = 10^{-2}, 1$ and $r = 10^2$. Again, in view of the values of the inf-sup constant of Table 5.4, we check that the lower value $r = 10^{-2}$ provides a faster convergence of the approximation. It is also interesting to remark that low errors for the state $\rho^{-1}\lambda_h$ and the control v_h are obtained with a relatively large value of the norm $\|\rho^{-1}L^\star\varphi_h\|_{L^2(Q_T)}$. This suggests that the constraint equality $L^\star\varphi = 0$ in $L^2(Q_T)$ may be replaced by a weaker one as discussed in Section 5.2.2. We do not present experiments for the weaker formulation (5.26) and refer to Section 4 of [49] in a closed context. Tables 5.14, 5.15 and 5.16 also report some results from the minimization of the functional $J_r^{\star\star}$ using the conjugate gradient algorithm. For $r = 1$ and $r = 10^2$, the quantity $r^{-1}\delta_{r,h}^{-2}$ - bounded by above of the condition number of $\hat{B}_h \hat{A}_{r,h}^{-1} \hat{B}_h^T$ - slightly decreases with h; the convergence of the algorithm is reached in few iterations independent of h. The value $r = 10^{-2}$ requires about 50 iterations for all the discretization considered.

Remarkably, the change of variable performed in the limit case allows to reduce very significantly the condition number $\kappa(A_{r,h})$ of the matrix $A_{r,h}$ (almost independent of r): see Table 5.14. This allows to consider very small values of the parameter h without producing any instabilities.

This high robustness of the approximation is definitively in contrast classical dual methods discussed in [105] and the references therein: We recall that for $\varepsilon = 0$, the minimization of $J_{\varepsilon=0}^\star$ defined by (5.13) fails as soon as h is small enough.

Figure 5.5 and Figure 5.6 depict over Q_T the approximation $y_h := \rho^{-1}\lambda_h$ and $v_h := \rho_0^{-1}\psi_h 1_{q_T}$ for $h = 8.83 \times 10^{-3}$. In particular, the smallness of both the diffusion coefficient and the size of the support ω leads to a large amplitude of the control at the initial time. This is in contrast with the boundary control situation where one acts directly on the state (or its first derivative).

Eventually, in order to validate one more time our computations, we have approximated by a standard time-marching algorithm the solution of (5.1) with $v = v_h$. Specifically, we have used a C^1-approximation with $\mathbb{P}_{3,x}(0,1)$ elements in space and the second-step implicit Gear scheme (of order two) for the time discretization. Tables 5.14, 5.15 and 5.16 report the L^2-norm of the state at the final time, i.e. $\|y_h(\cdot,T)\|_{L^2(0,1)}$. For each value of r, the L^2-norm decreases linearly to 0 with h. For any h, the non-zero value of $\|y_h(\cdot,T)\|_{L^2(0,1)}$ is, first due to the fact that v_h is not an exact null-control for any discrete system, and second to the consistency error of the approximation used.

h	3.53×10^{-2}	1.76×10^{-2}	8.83×10^{-3}	4.41×10^{-3}	2.2×10^{-3}
$\|\rho^{-1}L^\star(\rho_0\psi_h)\|_{L^2(Q_T)}$	29.76	24.86	21.12	17.92	15.42
$\frac{\|\rho_0(v-v_h)\|_{L^2(q_T)}}{\|\rho_0 v\|_{L^2(q_T)}}$	5.35×10^{-1}	3.34×10^{-1}	2.42×10^{-1}	1.63×10^{-1}	8.45×10^{-2}
$\|\rho_0 v_h\|_{L^2(q_T)}$	15.20	16.642	17.52	18.07	18.43
$\|\rho^{-1}\lambda_h\|_{L^2(Q_T)}$	3.15×10^{-1}	3.34×10^{-1}	3.46×10^{-1}	3.52×10^{-1}	3.56×10^{-1}
$\frac{\|y-\rho^{-1}\lambda_h\|_{L^2(Q_T)}}{\|y\|_{L^2(Q_T)}}$	1.96×10^{-1}	1.20×10^{-1}	6.97×10^{-2}	3.67×10^{-2}	1.49×10^{-2}
$\sharp$ CG iterates	52	55	56	56	55
$r^{-1}\delta_{r,h}^{-2}$	27.04	29.37	31.73	33.37	−
$\kappa(A_{r,h})$	9.5×10^4	1.4×10^7	3.03×10^9	1.1×10^{12}	−
n_h=size($A_{r,h}$)	3 444	13 284	52 264	206 724	823 044
$\|y_h(\cdot,T)\|_{L^2(0,1)}$	1.52×10^{-1}	6.109×10^{-2}	2.59×10^{-2}	1.162×10^{-2}	5.41×10^{-3}

Table 5.14: Mixed formulation (5.32) - $r = 10^{-2}$ and $\varepsilon = 0$ with $\omega = (0.2, 0.5)$.

h	3.53×10^{-2}	1.76×10^{-2}	8.83×10^{-3}	4.41×10^{-3}	2.2×10^{-3}
$\|\rho^{-1}L^\star(\rho_0\psi_h)\|_{L^2(Q_T)}$	3.659	3.276	2.808	2.377	2.002
$\frac{\|\rho_0(v-v_h)\|_{L^2(q_T)}}{\|\rho_0 v\|_{L^2(q_T)}}$	6.97×10^{-1}	4.82×10^{-1}	3.69×10^{-1}	2.81×10^{-1}	2.06×10^{-1}
$\|\rho_0 v_h\|_{L^2(q_T)}$	13.37	15.33	16.62	17.45	17.99
$\|\rho^{-1}\lambda_h\|_{L^2(Q_T)}$	3.35×10^{-1}	3.40×10^{-1}	3.41×10^{-1}	3.42×10^{-1}	3.52×10^{-1}
$\frac{\|y-\rho^{-1}\lambda_h\|_{L^2(Q_T)}}{\|y\|_{L^2(Q_T)}}$	3.28×10^{-1}	2.13×10^{-1}	1.33×10^{-1}	8.09×10^{-2}	4.63×10^{-2}
$\sharp$ CG iterates	12	11	10	9	9
$r^{-1}\delta_{r,h}^{-2}$	2.092	2.062	1.585	1.333	−
$\|y_h(\cdot,T)\|_{L^2(0,1)}$	1.19×10^{-1}	5.39×10^{-2}	2.42×10^{-2}	1.12×10^{-2}	5.29×10^{-3}

Table 5.15: Mixed formulation (5.32) - $r = 1$ and $\varepsilon = 0$ with $\omega = (0.2, 0.5)$.

h	3.53×10^{-2}	1.76×10^{-2}	8.83×10^{-3}	4.41×10^{-3}	2.2×10^{-3}
$\Vert\rho^{-1}L^\star(\rho_0\psi_h)\Vert_{L^2(Q_T)}$	0.428	0.426	0.380	0.321	0.215
$\frac{\Vert\rho_0(v-v_h)\Vert_{L^2(q_T)}}{\Vert\rho_0 v\Vert_{L^2(q_T)}}$	8.83×10^{-1}	6.80×10^{-1}	5.24×10^{-1}	4.16×10^{-1}	3.25×10^{-1}
$\Vert\rho_0 v_h\Vert_{L^2(q_T)}$	9.880	12.706	14.82	16.256	17.338
$\Vert\rho^{-1}\lambda_h\Vert_{L^2(Q_T)}$	0.2546	0.2926	0.3189	0.3352	0.3477
$\frac{\Vert y-\rho^{-1}\lambda_h\Vert_{L^2(Q_T)}}{\Vert y\Vert_{L^2(Q_T)}}$	5.86×10^{-1}	4.04×10^{-1}	2.63×10^{-1}	1.66×10^{-1}	9.88×10^{-2}
$\sharp$ CG iterates	10	8	7	5	5
$r^{-1}\delta_{r,h}^{-2}$	2.092	2.007	1.53	1.103	$-$
$\Vert y_h(\cdot,T)\Vert_{L^2(0,1)}$	8.26×10^{-2}	4.24×10^{-2}	2.11×10^{-2}	1.03×10^{-2}	5.12×10^{-3}

Table 5.16: Mixed formulation (5.32) - $r = 10^2$ and $\varepsilon = 0$ with $\omega = (0.2, 0.5)$.

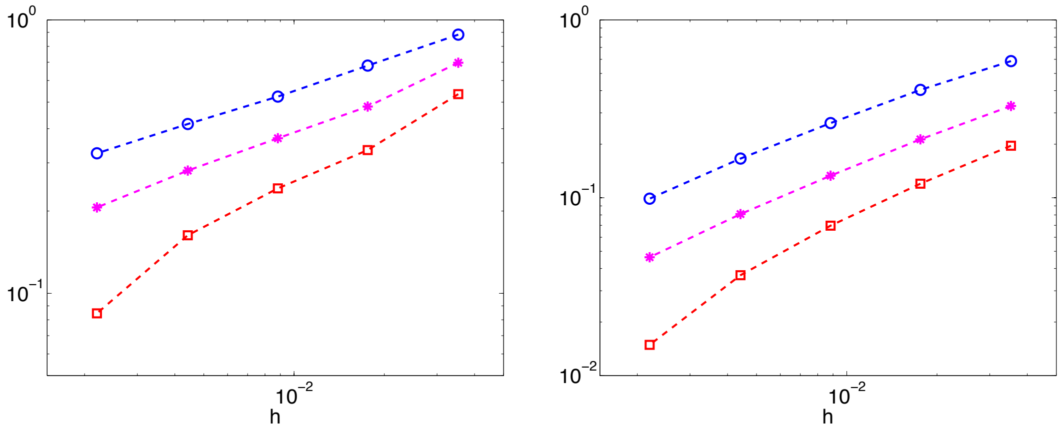

Figure 5.4: $\omega = (0.2, 0.5)$; $y_0(x) = \sin(\pi x)$: $\varepsilon = 0.$; $\frac{\Vert\rho_0(v-v_h)\Vert_{L^2(q_T)}}{\Vert\rho_0 v\Vert_{L^2(q_T)}}$ **(Left)** and $\frac{\Vert y-\rho^{-1}\lambda_h\Vert_{L^2(Q_T)}}{\Vert y\Vert_{L^2(Q_T)}}$ **(Right)** vs. h for $r = 10^2$ ($\circ$), $r = 1.$ ($\star$) and $r = 10^{-2}$ ($\square$).

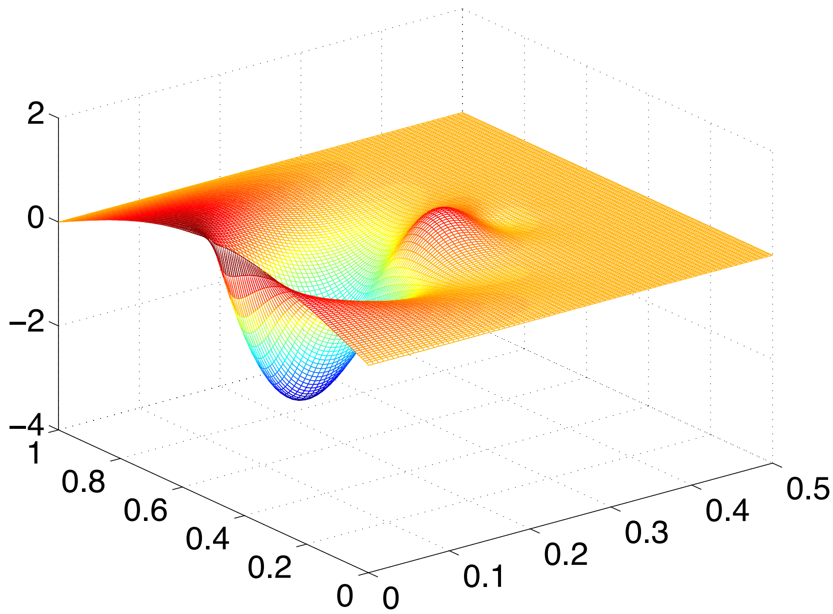

Figure 5.5: $\omega = (0.2, 0.5)$; Approximation $\rho^{-1}\lambda_h$ of the controlled state y over Q_T - $r = 1$ and $h = 8.83 \times 10^{-3}$.

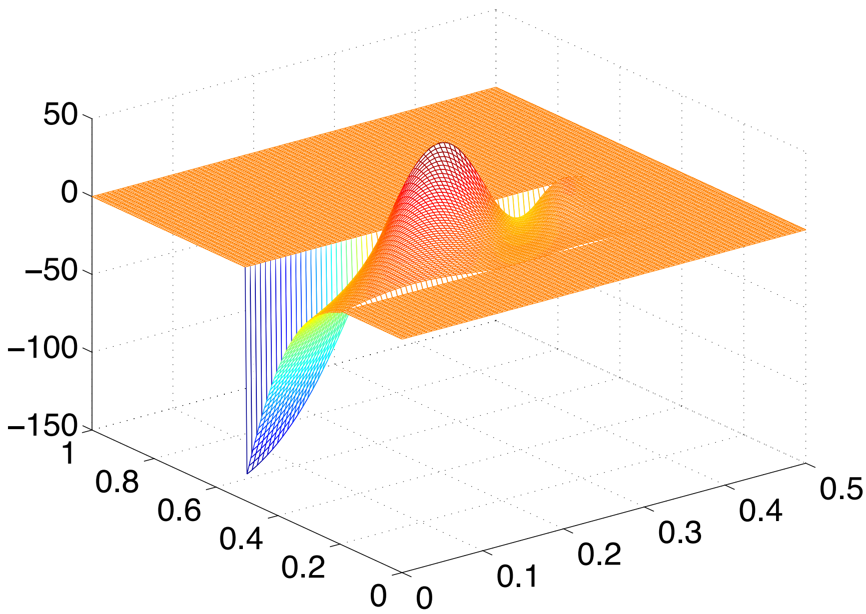

Figure 5.6: $\omega = (0.2, 0.5)$; Approximation $v_h = \rho_0^{-1}\psi_h$ of the null control v over Q_T - $r = 1$ and $h = 8.83 \times 10^{-3}$.

The experiments reported here - in the limit case $\varepsilon = 0$ - are obtained for a specific choice of the weights ρ_0 and ρ. Precisely, the weight ρ_0 is such that the approximation $v_h := \rho_0^{-2}\varphi_h 1_\omega$ vanishes exponentially as $t \to T^-$. This allows in particular to avoid the high oscillatory behavior of the control of minimal L^2-norm, that is when $\rho_0 := 1$ in q_T. The exponential behavior of the control implies a similar behavior of the corresponding controlled stated $\rho^{-1}\lambda$, so that the choice of the parameter ρ made here, is also natural. Remark that ρ is not bounded and therefore does not strictly satisfied the hypothesis of Theorem 5.3. Seemingly, this has no influence at the numerical level. This specific choice of the parameter ρ allows to perform a change of variable and therefore reduce significantly the condition number of the discrete problem. We also point out that, if the mixed formulation (5.32) is well-posed for any ρ, ρ_0 satisfying the hypothesis of Theorem 5.3, the constant of continuity of the linear form $\tilde{l}$ depends strongly - in view of the Carleman estimate (5.34) - of ρ and ρ_0. This affects the convergence and the robustness of the method. Thus, for ρ_0 as before and $\rho := 1$, the condition number is too large for small values of h (typically $h \approx 10^{-3}$) and leads to wrong results. Remark that for $\rho := 1$, the exponential decreases of ρ_0^{-1} cannot be compensated by ρ (see (5.34)) so that the change of variable is inefficient.

5.4 Concluding remarks and Perspectives

The mixed formulation we have introduced here in order to address the null controllability of the heat equation seems original and adapted the work [24] devoted to the wave equation. This formulation is nothing else than the Euler system associated to the conjugate functional and depends on both the dual adjoint variable and a Lagrange multiplier, which turns out to be the primal state of the heat equation to be controlled. The approach, recently used in a different way in [49], leads to a variational problem defined over time-space functional Hilbert spaces, without distinction between the time and the space variable. The main ingredients allowing to prove the well-posedness of the mixed formulation are an observability inequality and a direct inequality (usually deduced from energy estimates). For these reasons, the mixed reformulation may also be employed to any other controllable systems for which such inequalities are available. In particular, we may consider the Stokes system as in [102].

At the practical level, the discrete mixed time-space formulation is solved in a systematic way in the framework of the finite element theory: in contrast to the classical approach initially developed in [70], there is no need to take care of the time discretization nor of the stability of the resulting scheme, which is often a delicate issue. The resolution amounts to solve a sparse symmetric linear system : the corresponding matrix can be preconditioned if necessary, and may be computed once for all as it does not depend on the initial data to be controlled. Eventually, as discussed in [24], Section 4.3 (but not employed here), the space-time discretization of the domain allows an adaptation of the mesh so as to reduce the computational cost and capture the main features of the solutions. We also emphasize that the higher dimensional case is very similar as it requires C^1 approximation in space.

The numerical experiments reported in this work suggest a very good behavior of the approach: the strong convergence of the sequences $\{v_h\}_{h>0}$, approximation of the controls of minimal weighted square integrable norm, are clearly observed as the discretization parameter h tends to zero (as the consequence of the uniform inf-sup discrete property). It is worth to mention that, within this mixed formulation approach, the strong convergence of the approximations (as obtained within a closed but different approach in [49] assuming that the weights ρ_0 and ρ coincide with the Carleman weight) is still to be done. From the uniform coercivity of the bilinear form in the primal variable, a strong convergence is guaranteed by a uniform discrete inf-sup property. In view of the complicated and unusual constraint $L^\star\varphi = 0$ and of the C^1 nature of the approximation, the proof of such uniform property is probably very hard to get. However, it seems possible to bypass this property by adding to the Lagrangian the stabilization terms (for instance in the limit case $\varepsilon = 0$)

$$-\|L(\rho^{-1}\lambda_h) - \rho_0^{-2}\varphi_h\, 1_\omega\|^2_{L^2(Q_T)}, \quad -\|\lambda_h(\cdot,0) - y_0\|^2_{L^2(0,1)}$$

which vanish at the continuous level (writing $Ly = v\,1_\omega$ with $y = \rho^{-1}\lambda$ and $v = \rho^{-2}\varphi 1_\omega$, see Theorem 5.3) and give coercivity property for the variable λ_h. This will be examined in a future work.

The approach may also be extended to the boundary case. We also mention that the variational approach developed here based on a space-time formulation is also very well-adapted to the case where the support of the inner control evolves in time and takes the form

$$q_T := \{(x,t) \in Q_T; \quad a(t) < x < b(t) \quad t \in (0,T)\}$$

with any a, b in $C^0([0,T],]0,1[)$. We refer to [14] which examines this case for the wave equation.

Eventually, we also mention that this approach which consists in solving directly the optimality conditions of a controllability problem may be employed to solve inverse problems where, for instance, the solution of the heat equation has to be recovered from a partial observation, typically localized on a sub-domain q_T of the working domain: actually, the optimality conditions associated to a least-square type functional can be expressed as a mixed formulation very closed to (5.3). This issue will be analyzed in a future work.

5.5 Appendix

5.5.1 Appendix : Fourier expansion of the control of minimal $L^2(\rho_0, q_T)$ norm.

We expand in term of Fourier series the control of minimal $L^2(\rho_0, q_T)$ norm v for the (5.1) and the corresponding controlled solution y. We use these expansions in Section 5.3.4 to evaluate with respect to h the error $\|y_\varepsilon - \lambda_{\varepsilon,h}\|_{L^2(Q_T)}$ and $\|\rho_0(v_\varepsilon - v_{\varepsilon,h})\|_{L^2(q_T)}$ where the sequence $(\varphi_{\varepsilon,h}, \lambda_{\varepsilon,h})$ solves the discrete mixed formulation (5.40). We use the characterization of the couple $(y_\varepsilon, v_\varepsilon)$ in term of the adjoint solution φ_ε (see (5.5)), unique minimizer in $L^2(\Omega)$ of $J^\star_\varepsilon$ defined by (5.13).

We first note $(a_{\varepsilon,p})_{(p>0)}$ the Fourier coefficients in $l^2(\mathbb{N})$ of the minimizer $\varphi_{T,\varepsilon} \in L^2(0,1)$ of (5.13) such that

$$\varphi_{\varepsilon,T}(x) = \sum_{p>0} a_{\varepsilon,p} \sin(p\pi x), \quad x \in (0,1). \tag{5.51}$$

The adjoint state takes the form $\varphi_\varepsilon(x,t) = \sum_{p\geq 1} a_{\varepsilon,p} e^{c\pi^2 p^2 (t-T)} \sin(p\pi x)$ in Q_T.

The optimality equation associated to the functional $J^\star_\varepsilon$ then reads,

$$DJ^\star_\varepsilon(\varphi_{\varepsilon,T}) \cdot \overline{\varphi_T} = \iint_{q_T} \rho_0^{-2} \varphi_\varepsilon \overline{\varphi}\, dx\, dt + \varepsilon \int_0^1 \varphi_{\varepsilon,T} \overline{\varphi_T} + (y_0, \overline{\varphi}(\cdot,0)) = 0, \forall \overline{\varphi_T} \in L^2(0,1)$$

and can be rewritten in terms of the $(a_{\varepsilon,p})_{p>0}$ as follows :

$$< \{\overline{a_p}\}_{p>0}, \mathcal{M}_{q_T,\varepsilon}\{a_p\}_{p>0} >=< \{\overline{a_p}\}_{p>0}, \mathcal{F}_{y_0} > \quad \forall \overline{a}_{\varepsilon,p} \in l^2(\mathbb{N}) \tag{5.52}$$

where $\mathcal{M}_{q_T,\varepsilon}$ denotes a symmetric positive definite matrix and $\mathcal{F}_{y_0}$ a vector obtained from the expansion (5.51). The resolution of the infinite dimensional system (reduced to a finite dimension one by truncation of the sums) allows an approximation of the minimizer $\varphi_{T,\varepsilon}$ of $J^\star_\varepsilon$.

Finally, we use that the control of minimal $L^2(\rho_0, q_T)$ norm is given by $v_\varepsilon = \rho_0^{-2}\varphi_\varepsilon 1_\omega$ and find that the corresponding controlled solution may be expanded as follows

$$y_\varepsilon(x,t) = \sum_{q>0}\Big(e^{-c\pi^2 q^2 t} b_q^0 + \sum_{p\geq 1} a_{\varepsilon,p} c_{q,p}(\omega) d_{q,p}(t)]\Big)\sin(p\pi x), \quad (x,t)\in Q_T \tag{5.53}$$

with

$$c_{p,q}(\omega) := 2\int_\omega \sin(p\pi x)\sin(q\pi x)\,dx;\;\; d_{p,q}(t) := \int_0^t \rho_0^{-2}(s) e^{c\pi^2(p^2(s-T)+q^2(s-t))}\,ds,\;\; t\in(0,T).$$

$(b_q^0)_{q>0}$ denotes the Fourier coefficients of the initial data $y_0 \in L^2(0,1)$.

5.5.2 Appendix: Tables

h	1.41×10^{-1}	7.07×10^{-2}	3.53×10^{-2}	1.76×10^{-2}	8.83×10^{-3}
$\Vert L^\star\varphi_{\varepsilon,h}\Vert_{L^2(Q_T)}$	3.84×10^{-2}	2.90×10^{-2}	9.27×10^{-3}	2.41×10^{-3}	7.78×10^{-4}
$\frac{\Vert\rho_0(v_\varepsilon-v_{\varepsilon,h})\Vert_{L^2(q_T)}}{\Vert\rho_0 v_\varepsilon\Vert_{L^2(q_T)}}$	1.32×10^{-1}	5.90×10^{-2}	3.24×10^{-2}	1.68×10^{-2}	8.57×10^{-3}
$\frac{\Vert y_\varepsilon-\lambda_{\varepsilon,h}\Vert_{L^2(Q_T)}}{\Vert y_\varepsilon\Vert_{L^2(Q_T)}}$	1.04×10^{-1}	3.54×10^{-2}	1.48×10^{-2}	7.59×10^{-3}	3.89×10^{-3}
$\Vert\lambda_{\varepsilon,h}(\cdot,T)\Vert_{L^2(0,1)}$	2.02×10^{-1}	1.68×10^{-1}	1.65×10^{-1}	1.67×10^{-1}	1.68×10^{-1}
κ_ε	4.44×10^{9}	4.20×10^{11}	3.84×10^{13}	3.25×10^{15}	5.72×10^{16}

Table 5.17: Mixed formulation (5.16) - $r = 10^2$ and $\varepsilon = 10^{-2}$ with $\omega = (0.2, 0.5)$.

h	1.41×10^{-1}	7.07×10^{-2}	3.53×10^{-2}	1.76×10^{-2}	8.83×10^{-3}
$\Vert L^\star\varphi_{\varepsilon,h}\Vert_{L^2(Q_T)}$	6.19×10^{-2}	1.57×10^{-1}	1.56×10^{-1}	1.50×10^{-1}	6.21×10^{-2}
$\frac{\Vert\rho_0(v_\varepsilon-v_{\varepsilon,h})\Vert_{L^2(q_T)}}{\Vert\rho_0 v_\varepsilon\Vert_{L^2(q_T)}}$	1.02	7.36×10^{-1}	3.65×10^{-1}	1.52×10^{-1}	3.01×10^{-2}
$\frac{\Vert y_\varepsilon-\lambda_{\varepsilon,h}\Vert_{L^2(Q_T)}}{\Vert y_\varepsilon\Vert_{L^2(Q_T)}}$	6.74×10^{-1}	5.51×10^{-1}	2.42×10^{-1}	1.05×10^{-1}	1.81×10^{-2}
$\Vert\lambda_{\varepsilon,h}(\cdot,T)\Vert_{L^2(0,1)}$	2.23×10^{-1}	1.76×10^{-1}	7.86×10^{-2}	4.87×10^{-2}	3.28×10^{-2}
κ_ε	5.31×10^{9}	8.31×10^{11}	9.64×10^{13}	1.47×10^{16}	1.50×10^{18}

Table 5.18: Mixed formulation (5.16) - $r = 10^2$ and $\varepsilon = 10^{-4}$ with $\omega = (0.2, 0.5)$.

h	1.41×10^{-1}	7.07×10^{-2}	3.53×10^{-2}	1.76×10^{-2}	8.83×10^{-3}
$\Vert L^\star \varphi_{\varepsilon,h}\Vert_{L^2(Q_T)}$	6.23×10^{-2}	1.63×10^{-1}	1.77×10^{-1}	2.66×10^{-1}	2.24×10^{-1}
$\frac{\Vert\rho_0(v_\varepsilon - v_{\varepsilon,h})\Vert_{L^2(q_T)}}{\Vert\rho_0 v_\varepsilon\Vert_{L^2(q_T)}}$	1.50	1.11	9.53×10^{-1}	8.33×10^{-1}	7.19×10^{-1}
$\frac{\Vert y_\varepsilon - \lambda_{\varepsilon,h}\Vert_{L^2(Q_T)}}{\Vert y_\varepsilon\Vert_{L^2(Q_T)}}$	1.08	1.09	9.4×10^{-1}	7.69×10^{-1}	5.15×10^{-1}
$\Vert\lambda_{\varepsilon,h}(\cdot,T)\Vert_{L^2(0,1)}$	2.24×10^{-1}	1.79×10^{-1}	8.10×10^{-2}	5.67×10^{-2}	1.71×10^{-2}
κ_ε	5.32×10^{9}	8.59×10^{11}	9.86×10^{13}	1.84×10^{16}	3.07×10^{18}

Table 5.19: Mixed formulation (5.16) - $r = 10^2$ and $\varepsilon = 10^{-8}$ with $\omega = (0.2, 0.5)$.

h	1.41×10^{-1}	7.07×10^{-2}	3.53×10^{-2}	1.76×10^{-2}	8.83×10^{-3}
$\Vert L^\star \varphi_{\varepsilon,h}\Vert_{L^2(Q_T)}$	2.86×10^{-1}	7.15×10^{-2}	1.84×10^{-2}	4.86×10^{-3}	1.40×10^{-3}
$\frac{\Vert\rho_0(v_\varepsilon - v_{\varepsilon,h})\Vert_{L^2(q_T)}}{\Vert\rho_0 v_\varepsilon\Vert_{L^2(q_T)}}$	1.11×10^{-1}	6.21×10^{-2}	3.29×10^{-2}	1.68×10^{-2}	8.57×10^{-3}
$\frac{\Vert y_\varepsilon - \lambda_{\varepsilon,h}\Vert_{L^2(Q_T)}}{\Vert y_\varepsilon\Vert_{L^2(Q_T)}}$	5.16×10^{-2}	2.84×10^{-2}	1.48×10^{-2}	7.59×10^{-3}	3.89×10^{-3}
$\Vert\lambda_{\varepsilon,h}(\cdot,T)\Vert_{L^2(0,1)}$	1.53×10^{-1}	1.61×10^{-1}	1.65×10^{-1}	1.67×10^{-1}	1.68×10^{-1}
κ_ε	9.15×10^{8}	2.07×10^{10}	8.05×10^{11}	3.25×10^{13}	1.45×10^{15}

Table 5.20: Mixed formulation (5.16) - $r = 10^{-2}$ and $\varepsilon = 10^{-2}$ with $\omega = (0.2, 0.5)$.

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h	1.41×10^{-1}	7.07×10^{-2}	3.53×10^{-2}	1.76×10^{-2}	8.83×10^{-3}
$\Vert L^\star \varphi_{\varepsilon,h}\Vert_{L^2(Q_T)}$	10.77	3.821	1.018	2.59×10^{-1}	6.56×10^{-2}
$\frac{\Vert\rho_0(v_\varepsilon - v_{\varepsilon,h})\Vert_{L^2(q_T)}}{\Vert\rho_0 v_\varepsilon\Vert_{L^2(q_T)}}$	4.63×10^{-1}	2.23×10^{-1}	1.10×10^{-1}	5.52×10^{-2}	2.74×10^{-2}
$\frac{\Vert y_\varepsilon - \lambda_{\varepsilon,h}\Vert_{L^2(Q_T)}}{\Vert y_\varepsilon\Vert_{L^2(Q_T)}}$	1.55×10^{-1}	9.03×10^{-2}	4.08×10^{-2}	2.46×10^{-2}	1.27×10^{-2}
$\Vert\lambda_{\varepsilon,h}(\cdot,T)\Vert_{L^2(0,1)}$	3.22×10^{-2}	2.85×10^{-2}	2.99×10^{-2}	3.08×10^{-2}	3.12×10^{-2}
κ_ε	3.04×10^{9}	1.33×10^{11}	7.55×10^{12}	3.88×10^{14}	1.96×10^{16}

Table 5.21: Mixed formulation (5.16) - $r = 10^{-2}$ and $\varepsilon = 10^{-4}$ with $\omega = (0.2, 0.5)$.

h	1.41×10^{-1}	7.07×10^{-2}	3.53×10^{-2}	1.76×10^{-2}	8.83×10^{-3}
$\Vert L^\star \varphi_{\varepsilon,h}\Vert_{L^2(Q_T)}$	21.872	19.388	26.098	28.310	21.249
$\Vert\rho_0(v_\varepsilon - v_{\varepsilon,h})\Vert_{L^2(q_T)}$	14.989	9.459	6.606	4.175	1.556
$\frac{\Vert\rho_0(v_\varepsilon - v_{\varepsilon,h})\Vert_{L^2(q_T)}}{\Vert\rho_0 v_\varepsilon\Vert_{L^2(q_T)}}$	1.33	8.43×10^{-1}	5.89×10^{-1}	3.72×10^{-1}	1.38×10^{-1}
$\frac{\Vert y_\varepsilon - \lambda_{\varepsilon,h}\Vert_{L^2(Q_T)}}{\Vert y_\varepsilon\Vert_{L^2(Q_T)}}$	5.73×10^{-1}	$4.71 \times 10^{-}$	3.51×10^{-1}	2.11×10^{-1}	6.82×10^{-2}
$\Vert\lambda_{\varepsilon,h}(\cdot,T)\Vert_{L^2(0,1)}$	3.31×10^{-2}	1.31×10^{-2}	5.99×10^{-3}	2.83×10^{-3}	8.26×10^{-4}
κ_ε	4.08×10^{9}	3.04×10^{11}	4.54×10^{13}	6.79×10^{15}	1.30×10^{18}

Table 5.22: Mixed formulation (5.16) - $r = 10^{-2}$ and $\varepsilon = 10^{-8}$ with $\omega = (0.2, 0.5)$.

Chapter 6

On the numerical controllability of the two-dimensional heat, Stokes and Navier-Stokes equations

On the numerical controllability of the two-dimensional heat, Stokes and Navier-Stokes equations

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Abstract. The aim of this work is to present strategies to solve numerically some controllability problems for the two-dimensional heat equation, the Stokes equations and the Navier-Stokes equations with Dirichlet boundary conditions. The main idea is to adapt the Fursikov-Imanuvilov's formulation, see [A.V. Fursikov, O.Yu. Imanuvilov: *Controllability of Evolutions Equations*, Lectures Notes Series, Vol. 34, Seoul National University, 1996]; this approach has been followed recently for the one-dimensional heat equation by the first two authors. More precisely, we minimize over the class of admissible null controls a functional that involves weighted integrals of the state and the control, with weights that blow up near the final time. The associated optimality conditions can be viewed as a differential system in the three variables x_1, x_2 and t that is second-order in time and fourth-order in space, completed with appropriate boundary conditions. We present several mixed formulations of the problems and, then, associated mixed finite element Lagrangian approximations that are relatively easy to handle. Finally, we exhibit some numerical experiments.

6.1 Introduction. The controllability problems

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain whose boundary $\Gamma := \partial\Omega$ is regular enough. Let $\omega \subset \Omega$ be a (small) nonempty open subset and assume that $T > 0$. We will use the notation $Q_\tau = \Omega \times (0,\tau)$, $\Sigma_\tau = \Gamma \times (0,\tau)$, $q_\tau = \omega \times (0,\tau)$ and $\mathbf{n} = \mathbf{n}(\mathbf{x})$ will denote the outward unit normal to Ω at any point $\mathbf{x} \in \Gamma$.

Throughout this paper, C will denote a generic positive constant (usually depending on Ω, ω and T) and bold letters and symbols will stand for vector-valued functions and spaces; for instance, $\mathbf{L}^2(\Omega)$ is the Hilbert space of the functions $\mathbf{u} = (u_1, u_2)$ with $u_1,\, u_2 \in L^2(\Omega)$.

This paper is concerned with the global null controllability problems for the heat equation

$$\begin{cases} y_t - \nu\Delta y + G(\mathbf{x},t)\, y = v1_\omega & \text{in} \quad Q_T, \\ y = 0 & \text{on} \quad \Sigma_T, \\ y(\cdot,0) = y_0 & \text{in} \quad \Omega \end{cases} \tag{6.1}$$

and the Stokes equations

$$\begin{cases} \mathbf{y}_t - \nu\Delta\mathbf{y} + \nabla\pi = \mathbf{v}1_\omega & \text{in} \quad Q_T, \\ \nabla\cdot\mathbf{y} = 0 & \text{in} \quad Q_T, \\ \mathbf{y} = \mathbf{0} & \text{on} \quad \Sigma_T, \\ \mathbf{y}(\cdot,0) = \mathbf{y}_0 & \text{in} \quad \Omega \end{cases} \tag{6.2}$$

and the local exact controllability to the trajectories for the Navier-Stokes equations

$$\begin{cases} \mathbf{y}_t - \nu\Delta\mathbf{y} + (\mathbf{y}\cdot\nabla)\mathbf{y} + \nabla\pi = \mathbf{v}1_\omega & \text{in} \quad Q_T, \\ \nabla\cdot\mathbf{y} = 0 & \text{in} \quad Q_T, \\ \mathbf{y} = \mathbf{0} & \text{on} \quad \Sigma_T, \\ \mathbf{y}(\cdot,0) = \mathbf{y}_0 & \text{in} \quad \Omega. \end{cases} \tag{6.3}$$

Here, $v = v(\mathbf{x},t)$ and $\mathbf{v} = \mathbf{v}(\mathbf{x},t)$ stand for the controls (they are assumed to act on ω during the whole time interval $(0,T)$; the symbol 1_ω stands for the characteristic function of ω). Moreover, $\nu > 0$ and we assume that $G \in L^\infty(Q_T)$.

Let us first consider the system (6.1). It is well known that, for any $y_0 \in L^2(\Omega)$, $T > 0$ and $v \in L^2(q_T)$, there exists exactly one solution y to (6.1), with

$$y \in C^0([0,T];L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega)).$$

The *null controllability* problem for (6.1) at time T is the following:

> *For any $y_0 \in L^2(\Omega)$, find a control $v \in L^2(q_T)$ such that the associated solution to* (6.1) *satisfies*

$$y(\mathbf{x},T) = 0 \quad \textit{in} \quad \Omega. \tag{6.4}$$

The following result is also well known, see [62]:

Theorem 6.1. *The heat equation* (6.1) *is null-controllable at any time $T > 0$.*

Let us now consider the systems (6.2) and (6.3). Let us recall the definitions of some usual spaces in the context of incompressible fluids:

$$\begin{aligned} \mathbf{H} &:= \left\{ \boldsymbol{\varphi} \in \mathbf{L}^2(\Omega) : \nabla\cdot\boldsymbol{\varphi} = 0 \text{ in } \Omega, \ \boldsymbol{\varphi}\cdot\mathbf{n} = 0 \text{ on } \Gamma \right\}, \\ \mathbf{V} &:= \left\{ \boldsymbol{\varphi} \in \mathbf{H}_0^1(\Omega) : \nabla\cdot\boldsymbol{\varphi} = 0 \text{ in } \Omega \right\}, \\ U &:= \left\{ \psi \in H^1(\Omega) : \int_\Omega \psi(\mathbf{x})\,d\mathbf{x} = 0 \right\}. \end{aligned}$$

For any $\mathbf{y}_0 \in \mathbf{H}$, $T > 0$ and $\mathbf{v} \in \mathbf{L}^2(q_T)$, there exists exactly one solution $(\mathbf{y},\pi)$ to the Stokes equations (6.2) and also (since we are in the $2D$ case), one solution $(\mathbf{y},\pi)$ to the

Navier-Stokes equations (6.3). In both cases

$$\mathbf{y} \in C^0\left([0,T];\mathbf{H}\right) \cap L^2\left(0,T;\mathbf{V}\right), \quad \pi \in L^2_{\rm loc}(0,T;U).$$

In the context of the Stokes system (6.2), the *null controllability* problem at time T is the following:

> *For any $\mathbf{y}_0 \in \mathbf{H}$, find a control $\mathbf{v} \in \mathbf{L}^2(q_T)$ such that the associated solution to (6.2) satisfies*
>
> $$\mathbf{y}(\mathbf{x},T) = \mathbf{0} \quad in \quad \Omega. \tag{6.5}$$

Again, the following result is well known, see again[62]:

Theorem 6.2. *The Stokes system (6.2) is null-controllable at any time $T > 0$.*

Let us introduce the concept of *exact controllability to the trajectories* for the Navier-Stokes equations. The idea is that, even if we cannot reach every element of the state space exactly, we can try to reach (in finite time T) any state on any trajectory.

Thus, let $(\overline{\mathbf{y}},\overline{\pi})$ be a solution to the uncontrolled Navier-Stokes equations:

$$\begin{cases} \overline{\mathbf{y}}_t - \nu\Delta\overline{\mathbf{y}} + (\overline{\mathbf{y}}\cdot\nabla)\overline{\mathbf{y}} + \nabla\overline{\pi} = \mathbf{0} & \text{in} \quad Q_T, \\ \nabla\cdot\overline{\mathbf{y}} = 0 & \text{in} \quad Q_T, \\ \overline{\mathbf{y}} = \mathbf{0} & \text{on} \quad \Sigma_T, \\ \overline{\mathbf{y}}(\cdot,0) = \overline{\mathbf{y}}_0 & \text{in} \quad \Omega. \end{cases} \tag{6.6}$$

We will look for controls $\mathbf{v} \in \mathbf{L}^2(q_T)$ such that the associated solutions to (6.3) satisfy

$$\mathbf{y}(\mathbf{x},T) = \overline{\mathbf{y}}(\mathbf{x},T) \quad \text{in} \quad \Omega. \tag{6.7}$$

The problem of exact controllability to the trajectories for (6.3) is the following:

> *For any $\mathbf{y}_0 \in \mathbf{H}$ and any trajectory $(\overline{\mathbf{y}},\overline{\pi})$, find a control $\mathbf{v} \in \mathbf{L}^2(q_T)$ such that the associated solution to (6.3) satisfies (6.7).*

The following result shows that this problem can be solved at least locally when $\overline{\mathbf{y}}$ is bounded; a proof can be found in [46, 81, 82]:

Theorem 6.3. *The Navier-Stokes equations (6.3) are locally exact controllable to the trajectories $(\overline{\mathbf{y}},\overline{\pi})$ with*

$$\overline{\mathbf{y}} \in \mathbf{L}^\infty(Q_T), \ \ \overline{\mathbf{y}}(\cdot,0) \in \mathbf{V}. \tag{6.8}$$

In other words, for any $T > 0$ and any solution to (6.6) satisfying (6.8), there exists $\varepsilon > 0$ with the following property: if $\mathbf{y}_0 \in \mathbf{V}$ and $\|\mathbf{y}_0 - \overline{\mathbf{y}}(\cdot,0)\|_{\mathbf{V}} \le \varepsilon$, one can find a control $\mathbf{v} \in \mathbf{L}^2(q_T)$ such that the associated solution to (6.3) satisfies (6.7).

The aim of this paper is to present efficient strategies for the numerical solution of the previous controllability problems.

The paper is organized as follows. In Section 6.2, we present a method that furnishes numerical approximations of null controls of the heat equation. We present a mixed formulation that can be approximated with the help of Lagrangian (C^0 in space and time) finite elements. In Sections 6.3 and 6.4, we present similar numerical strategies to solve numerically the previous controllability problems for the Stokes and the Navier-Stokes equations. These methods are illustrated with several numerical experiments.

6.2 A strategy for the computation of null controls for the heat equation

In this Section, we will start from a formulation of the null controllability problem for (6.1) introduced and extensively used by Fursikov and Imanuvilov, see [62]. Let us fix the notation

$$Ly := y_t - \nu\Delta y + G(\mathbf{x},t)y, \quad L^*p := -p_t - \nu\Delta p + G(\mathbf{x},t)p$$

and let the weights ρ, β and ρ_i be given by

$$\rho(\mathbf{x},t) := e^{\beta(\mathbf{x})/(T-t)}, \ \beta(\mathbf{x}) := K_1\left(e^{K_2} - e^{\beta_0(\mathbf{x})}\right), \ \rho_i(\mathbf{x},t) := (T-t)^{3/2-i}\rho(\mathbf{x},t), \quad (6.9)$$

where $i = 0,1,2$, K_1 and K_2 are sufficiently large positive constants (depending on T) and $\beta_0 = \beta_0(\mathbf{x})$ is a regular bounded function that is positive in Ω, vanishes on Γ and satisfies

$$|\nabla\beta_0| > 0 \text{ in } \overline{\Omega}\setminus\omega;$$

for a justification of the existence of β_0, see [62].

The main idea relies on considering the extremal problem

$$\begin{cases} \text{Minimize } \ J(y,v) = \dfrac{1}{2}\left(\displaystyle\iint_{Q_T} \rho^2|y|^2\,d\mathbf{x}\,dt + \iint_{q_T} \rho_0^2|v|^2\,d\mathbf{x}\,dt\right) \\ \text{Subject to } \ (y,v) \in \mathcal{H}(y_0,T). \end{cases} \quad (6.10)$$

Here, for any $y_0 \in L^2(\Omega)$ and any $T > 0$, the linear manifold $\mathcal{H}(y_0,T)$ is given by

$$\mathcal{H}(y_0,T) := \{(y,v) : v \in L^2(q_T), \ (y,v) \text{ satisfies } (6.1) \text{ and } (6.4)\}.$$

We have the following result:

Theorem 6.4. *For any $y_0 \in L^2(\Omega)$ and any $T > 0$, there exists exactly one solution to (6.10).*

This result is a consequence of an appropriate *Carleman inequality*.

More precisely, let us introduce the space

$$P_0 := \{\, p \in C^2(\overline{Q}_T) : p = 0 \text{ on } \Sigma_T \,\}. \tag{6.11}$$

Then, one has the following result from [62]:

Proposition 6.1. *There exists C_0, only depending on Ω, ω and T, such that the following estimate holds for all $p \in P_0$:*

$$\begin{aligned}\iint_{Q_T} \left[\rho_2^{-2}(|p_t|^2 + |\Delta p|^2) + \rho_1^{-2}|\nabla p|^2 + \rho_0^{-2}|p|^2\right] d\mathbf{x}\, dt \\ \le C_0 \iint_{Q_T} (\rho^{-2}|L^* p|^2 + \rho_0^{-2}|p|^2 1_\omega)\, d\mathbf{x}\, dt.\end{aligned} \tag{6.12}$$

Let us introduce the bilinear form $k(\cdot,\cdot)$, with

$$k(p, p') := \iint_{Q_T} \left(\rho^{-2} L^* p\, L^* p' + 1_\omega \rho_0^{-2} p\, p'\right) d\mathbf{x}\, dt \quad \forall p, p' \in P_0. \tag{6.13}$$

In view of the unique continuation property of the heat equation, $k(\cdot\,,\cdot)$ is a scalar product in P_0. Indeed, if $p \in P_0$, $L^* p = 0$ in Q_T, $p = 0$ on Σ_T and $p = 0$ in q_T, then we necessarily have $p \equiv 0$.

Let P be the completion of P_0 with respect to this scalar product. Then P is a Hilbert space, the functions $p \in P$ satisfy

$$\iint_{Q_T} \rho^{-2}|L^* p|^2\, d\mathbf{x}\, dt + \iint_{q_T} \rho_0^{-2}|p|^2\, d\mathbf{x}\, dt < +\infty \tag{6.14}$$

and, from Proposition 6.1 and a standard density argument, we also have (6.12) for all $p \in P$.

Another consequence of Proposition 6.1 is that we can characterize the space P as follows:

$$P = \left\{\, p : p, p_t, \partial_{x_i} p, \partial_{x_i x_j} p \in L^2(0, T - \delta; L^2(\Omega))\ \forall \delta > 0,\ (6.14) \text{ holds},\ p = 0 \text{ on } \Sigma_T \,\right\}.$$

In particular, we see that any $p \in P$ satisfies $p \in C^0([0, T - \delta]; H_0^1(\Omega))$ for all $\delta > 0$ and, moreover,

$$\|p(\cdot\,, 0)\|_{H_0^1(\Omega)} \le C\, k(p, p)^{1/2} \quad \forall p \in P. \tag{6.15}$$

The following result holds:

Theorem 6.5. *Let the weights ρ and ρ_0 be chosen as in Proposition 6.1. Let (y, v) be the unique solution to (6.10). Then, one has*

$$y = \rho^{-2} L^* p, \quad v = -\rho_0^{-2} p\big|_{q_T}, \tag{6.16}$$

where p is the unique solution to the following variational equality in the Hilbert space P:

$$\begin{cases} \displaystyle\iint_{Q_T} \left(\rho^{-2}L^*p\, L^*p' + 1_\omega \rho_0^{-2} p\, p'\right)\, d\mathbf{x}\, dt = \int_\Omega y_0(\mathbf{x})\, p'(\mathbf{x},0)\, d\mathbf{x} \\ \forall\, p' \in P;\ p \in P. \end{cases} \tag{6.17}$$

We can interpret (6.17) as the weak formulation of a boundary-value problem for a PDE that is fourth-order in $\mathbf{x}$ and second-order in t. Indeed, taking "test functions" $p' \in P$ first with $p' \in C_0^\infty(Q_T)$, then $p' \in C^2(\overline{\Omega} \times (0,T))$ and finally $p' \in C^2(\overline{Q}_T)$, we see easily that p must necessarily satisfy:

$$\begin{cases} L(\rho^{-2}L^*p) + 1_\omega \rho_0^{-2} p = 0 & \text{in} \quad Q_T, \\ p = 0, \ \ \rho^{-2}L^*p = 0 & \text{on} \quad \Sigma_T, \\ \rho^{-2}L^*p\big|_{t=0} = y_0, \ \ \rho^{-2}L^*p\big|_{t=T} = 0 & \text{in} \quad \Omega. \end{cases} \tag{6.18}$$

By introducing the linear form ℓ_0, with

$$\langle \ell_0, p\rangle := \int_\Omega y_0(\mathbf{x})\, p(\mathbf{x},0)\, d\mathbf{x} \quad \forall p \in P, \tag{6.19}$$

we see from (6.15) that ℓ_0 is continuous and (6.17) can be rewritten in the form

$$k(p,p') = \langle \ell_0, p'\rangle \quad \forall p' \in P;\ p \in P. \tag{6.20}$$

Let P_h denote a finite dimensional subspace of P. A natural approximation of (6.20) is the following:

$$k(p_h,p_h') = \langle \ell_0, p_h'\rangle \quad \forall p_h' \in P_h;\ p_h \in P_h. \tag{6.21}$$

Thus, to solve numerically the variational equality (6.20), it suffices to construct explicitly finite dimensional spaces $P_h \subset P$.

Notice however that this is possible but not simple from the numerical viewpoint. The reason is that, if $p \in P$, then $\rho^{-1}L^*p$ must belong to $L^2(Q_T)$ and $\rho_0^{-1}p\big|_{q_T}$ must belong to $L^2(q_T)$. From the Carleman inequality (6.12), we also see that p must possess first-order time derivatives and up to second-order spatial derivatives in $L^2_{\text{loc}}(Q_T)$. Therefore, an approximation based on a standard triangulation of Q_T requires spaces P_h of functions that must be C^0 in $(\mathbf{x},t)$ and C^1 in $\mathbf{x}$ and this can be complex and too expensive.

Spaces of this kind are constructed for instance in [21]. For example, good behavior is observed for the so called *Argyris*, *Bell* or *Bogner-Fox-Schmit* finite elements; the reader is referred to [49, 104] for numerical approximations of this kind in the framework of one spatial dimension.

In spite of its complexity, the direct approximation of (6.21) has an advantage: it is possible to adapt the standard finite element theory to this framework and deduce strong convergence results for the numerical controls and states.

6.2.1 First mixed formulation with modified variables

Let us introduce the new variable
$$z := L^*p \tag{6.22}$$
and let us set $Z := L^2(\rho^{-1}; Q_T)$. Then $z \in Z$ and $L^*p - z = 0$ (an equality in Z).

Notice that this identity can also be written in the form

$$\iint_{Q_T} (z - L^*p)\, \psi \; d\mathbf{x}\, dt = 0 \quad \forall \psi \in C_0^\infty(Q_T);$$

Accordingly, we introduce the following reformulation of (6.20):

$$\begin{cases} \displaystyle\iint_{Q_T} (\rho^{-2} z\, z' + \rho_0^{-2} p\, p' 1_\omega)\, d\mathbf{x}\, dt + \iint_{Q_T} (z' - L^*p')\, \lambda\, d\mathbf{x}\, dt = \int_\Omega y_0(\mathbf{x})\, p'(\mathbf{x},0)\, d\mathbf{x}, \\ \displaystyle\iint_{Q_T} (z - L^*p)\, \lambda'\, d\mathbf{x}\, dt = 0, \\ \qquad \forall ((z',p'),\lambda') \in W;\; ((z,p),\lambda) \in W, \end{cases} \tag{6.23}$$

where $W := X \times Y$, $X := Z \times P$ and $Y := L^2(\rho; Q_T)$.

Notice that the definitions of Z, P and Y are the appropriates to keep all the terms in (6.23) meaningful.

Let us introduce the bilinear forms $\alpha(\cdot\,,\cdot) : X \times X \mapsto \mathbb{R}$ and $\beta(\cdot\,,\cdot) : X \times Y \mapsto \mathbb{R}$, with
$$\alpha((z,p),(z',p')) := \iint_{Q_T} \left(\rho^{-2} z\, z' + \rho_0^{-2} p\, p' 1_\omega\right)\, d\mathbf{x}\, dt$$

and
$$\beta((z,p),\lambda) := \iint_{Q_T} (z - L^*p)\, \lambda\, d\mathbf{x}\, dt$$

and the linear form $\ell : X \mapsto \mathbb{R}$, with

$$\langle \ell, (z,p)\rangle := \int_\Omega y_0(\mathbf{x})\, p(\mathbf{x},0)\, d\mathbf{x}.$$

Then, $\alpha(\cdot\,,\cdot)$, $\beta(\cdot\,,\cdot)$ and ℓ are well-defined and continuous and (6.23) reads:

$$\begin{cases} \alpha((z,q),(z',p')) + \beta((z',p'),\lambda) = \langle \ell, (z',p')\rangle, \\ \beta((z,p),\lambda') = 0, \\ \qquad \forall ((z',p'),\lambda') \in W;\; ((z,p),\lambda) \in W. \end{cases} \tag{6.24}$$

This is a mixed formulation of the variational problem (6.10). The following result holds:

Proposition 6.2. *There exists exactly one solution to (6.24). Furthermore, (6.20) and (6.24) are equivalent problems in the following sense:*

(i) If $((z,p),\lambda)$ solves (6.24), then p solves (6.20).

(ii) Conversely, if p solves (6.20), there exists $\lambda \in Y$ such that the triplet $((z,p),\lambda)$ with $z := L^*p$ solves (6.24).

Proof. Let us introduce the space

$$V := \{\, (z,p) \in X : \beta((z,p),\lambda) = 0 \quad \forall \lambda \in Y \,\}.$$

We will check that

- $\alpha(\cdot\,,\cdot)$ is coercive in V;
- $\beta(\cdot\,,\cdot)$ satisfies the usual "inf-sup" condition with respect to X and Y.

This will be sufficient to guarantee the existence and uniqueness of a solution to (6.24); see for instance [11, 110].

The proofs of the assertions above are straightforward. Indeed, we first notice that, for any $(z,p) \in V$, $z = L^*p$ and thus

$$\begin{aligned}
\alpha((z,p),(z,p)) &= \iint_{Q_T} \left(\rho^{-2}|z|^2 + \rho_0^{-2}|p|^2 1_\omega\right)\, d\mathbf{x}\, dt \\
&= \frac{1}{2}\iint_{Q_T} \rho^{-2}|z|^2\, d\mathbf{x}\, dt + \frac{1}{2}\iint_{Q_T} \rho^{-2}|L^*p|^2\, d\mathbf{x}\, dt + \iint_{Q_T} \rho_0^{-2}|p|^2 1_\omega\, d\mathbf{x}\, dt \\
&= \frac{1}{2}\|(z,p)\|_X^2 + \frac{1}{2}\iint_{Q_T} \rho_0^{-2}|p|^2 1_\omega\, d\mathbf{x}\, dt \\
&\geq \frac{1}{2}\|(z,p)\|_X^2.
\end{aligned}$$

This proves that $\alpha(\cdot\,,\cdot)$ is coercive in V.

On the other hand, for any $\lambda \in Y$ there exists $(z^0,p^0) \in X$ such that

$$\beta((z^0,p^0),\lambda) = \|\lambda\|_Y^2 \quad \text{and} \quad \|(z^0,p^0)\|_X \leq C\|\lambda\|_Y.$$

Indeed, we can take for instance $(z^0,p^0) = (\rho^2\lambda, 0)$. Then $\|(z^0,p^0)\|_X = \|\lambda\|_Y$ and

$$\sup_{(z,p)\in X} \frac{\beta((z,p),\lambda)}{\|(z,p)\|_X\|\lambda\|_Y} \geq \frac{\beta((z^0,p^0),\lambda)}{\|(z^0,p^0)\|_X\|\lambda\|_Y} \geq 1.$$

Hence, $\beta(\cdot\,,\cdot)$ certainly satisfies the "inf-sup" condition in $X \times Y$. □

An advantage of (6.24) with respect to the previous formulation (6.20) is that the solution $((z,p),\lambda)$ furnishes directly the state-control couple that solves (6.10). Indeed, it suffices to take

$$y = \rho^{-2}z, \quad v = -\rho_0^{-2}p\big|_{q_T}.$$

However, we still find spatial second-order derivatives in the integrals in (6.24) and, consequently, a finite element approximation of (6.24) still needs C^1 in space functions.

6.2.2 Second mixed formulation with modified variables

Let us introduce the spaces

$$\begin{aligned}
\widetilde{Z} &:= L^2(\rho^{-1}; Q_T),\\
\widetilde{Y} &:= \left\{ \lambda : \iint_{Q_T} \left(\rho_2^2|\lambda|^2 + \rho_1^2|\nabla\lambda|^2\right) d\mathbf{x}\,dt < +\infty,\ \lambda\big|_{\Sigma_T} = 0 \right\},\\
\widetilde{R} &:= \left\{ p : \iint_{Q_T} \left[\rho_2^{-2}|p_t|^2 + \rho_1^{-2}|\nabla p|^2 + \rho_0^{-2}|p|^2\right] d\mathbf{x}\,dt < +\infty,\ p\big|_{\Sigma_T} = 0 \right\},\\
\widetilde{X} &:= \widetilde{Z} \times \widetilde{R},\ \widetilde{W} := \widetilde{X} \times \widetilde{Y}
\end{aligned}$$

the bilinear forms $\widetilde{\alpha}(\cdot\,,\cdot) : \widetilde{X} \times \widetilde{X} \mapsto \mathbb{R}$ and $\widetilde{\beta}(\cdot\,,\cdot) : \widetilde{X} \times \widetilde{Y} \mapsto \mathbb{R}$, with

$$\widetilde{\alpha}((z,p),(z',p')) := \iint_{Q_T} \left(\rho^{-2} z\, z' + \rho_0^{-2} p\, p' 1_\omega\right)\, d\mathbf{x}\,dt$$

and

$$\widetilde{\beta}((z,p),\lambda) := \iint_{Q_T} \left[(z + p_t - G(\mathbf{x},t)\,p)\,\lambda - \nu\nabla p\cdot\nabla\lambda\right]\, d\mathbf{x}\,dt$$

and the linear form $\tilde{\ell} : \widetilde{R} \mapsto \mathbb{R}$, with

$$\langle \tilde{\ell}, (z,p)\rangle := \int_\Omega y_0(\mathbf{x})\, p(\mathbf{x},0)\, d\mathbf{x}.$$

Then $\widetilde{\alpha}(\cdot\,,\cdot)$, $\widetilde{\beta}(\cdot\,,\cdot)$ and $\tilde{\ell}$ are well-defined and continuous. Let us consider the mixed formulation

$$\left\{\begin{array}{l}
\widetilde{\alpha}((z,p),(z',p')) + \widetilde{\beta}((z',p'),\lambda) = \langle\tilde{\ell},(z',p')\rangle,\\
\widetilde{\beta}((z,p),\lambda') = 0,\\
\qquad \forall (z',p',\lambda') \in \widetilde{W};\ (z,p,\lambda) \in \widetilde{W}.
\end{array}\right. \tag{6.25}$$

Notice again that $\widetilde{Z}$, $\widetilde{R}$ and $\widetilde{Y}$ have been defined in such a way that all the terms in (6.25) remain meaningful.

It is easy to see that any possible solution to (6.25) also solves (6.24). Consequently, there exists at most one solution to (6.25). However, unfortunately, a rigorous proof of the existence of a solution to (6.25) is, to our knowledge, unknown. In practice, what we should be able to prove is that the following inf-sup condition holds:

$$\inf_{\lambda\in\widetilde{Y}} \sup_{(z,p)\in\widetilde{X}} \frac{\widetilde{\beta}((z,p),\lambda)}{\|(z,p)\|_{\widetilde{X}} \|\lambda\|_{\widetilde{Y}}} > 0.$$

But whether or not this holds, it is an open question.

Nevertheless, although we cannot prove that (6.25) is an equivalent reformulation of (6.24), we will use in the following Sections this system to compute numerical approximations of the control and the state. It will be seen in Section 6.2.6 that this approach works well in practice.

6.2.3 A reformulation of (6.25)

It is very convenient from the numerical viewpoint to introduce the following new variables:

$$\hat{z} := \rho^{-1}L^{*}p, \quad \hat{p} := \rho_0^{-1}p. \tag{6.26}$$

This will serve to improve the conditioning of the approximations given below.

The mixed problem (6.25) can be rewritten in the new variables as follows:

$$\begin{cases} \widehat{\alpha}((\hat{z},\hat{p}),(\hat{z}',\hat{p}')) + \widehat{\beta}((\hat{z}',\hat{p}'),\hat{\lambda}) = \langle \hat{\ell},(\hat{z}',\hat{p}')\rangle, \\ \widehat{\beta}((\hat{z},\hat{p}),\hat{\lambda}') = 0, \\ \qquad \forall (\hat{z}',\hat{p}',\hat{\lambda}') \in \widehat{W};\ (\hat{z},\hat{p},\hat{\lambda}) \in \widehat{W}, \end{cases} \tag{6.27}$$

where

$$\widehat{W} := \widehat{X} \times \widehat{Y}, \quad \widehat{X} := \widehat{Z} \times \widehat{R}, \quad \widehat{Z} := L^2(Q_T),$$

$$\widehat{R} := \left\{ \hat{p} : \iint_{Q_T} \left[(T-t)^4|\hat{p}_t|^2 + (T-t)^2|\nabla\hat{p}|^2 + |\hat{p}|^2\right] d\mathbf{x}\,dt < +\infty,\ \hat{p}|_{\Sigma_T} = 0 \right\},$$

$$\widehat{Y} := \left\{ \hat{\lambda} : \iint_{Q_T} \left[(T-t)^{-1}|\hat{\lambda}|^2 + (T-t)|\nabla\hat{\lambda}|^2\right] d\mathbf{x}\,dt < +\infty,\ \hat{\lambda}|_{\Sigma_T} = 0 \right\}$$

and the bilinear forms $\widehat{\alpha}(\cdot\,,\cdot) : \widehat{X} \times \widehat{X} \mapsto \mathbb{R}$ and $\widehat{\beta}(\cdot\,,\cdot) : \widehat{X} \times \widehat{Y} \mapsto \mathbb{R}$ are given by

$$\widehat{\alpha}((\hat{z},\hat{p}),(\hat{z}',\hat{p}')) := \iint_{Q_T} \left(\hat{z}\,\hat{z}' + \hat{p}\,\hat{p}' 1_{\omega}\right)\, d\mathbf{x}\,dt$$

and

$$\begin{aligned} \widehat{\beta}((\hat{z},\hat{p}),\hat{\lambda}) := &\iint_{Q_T} (T-t)^{3/2}\left(\hat{p}_t\hat{\lambda} - \nu\nabla\hat{p}\cdot\nabla\hat{\lambda} - G\,\hat{p}\,\hat{\lambda}\right)\, d\mathbf{x}\,dt \\ &+ \iint_{Q_T} \left[\hat{z} + 2\nu(T-t)^{1/2}\nabla\beta\cdot\nabla\hat{p}\right]\hat{\lambda}\, d\mathbf{x}\,dt \\ &+ \iint_{Q_T} \left[(T-t)^{1/2}\left(-3/2+\nu\Delta\beta\right) + (T-t)^{-1/2}\left(\beta+\nu|\nabla\beta|^2\right)\right]\hat{p}\,\hat{\lambda}\, d\mathbf{x}\,dt \end{aligned}$$

and the linear form $\hat{\ell} : \widehat{R} \mapsto \mathbb{R}$ is given by

$$\langle \hat{\ell}, (\hat{z}, \hat{p}) \rangle := \int_{\Omega} \rho_0(\mathbf{x}, 0) y_0(\mathbf{x})\, \hat{p}(\mathbf{x}, 0)\, d\mathbf{x}.$$

6.2.4 A numerical approximation based on Lagrangian finite elements

For simplicity, we will assume that ω is a polygonal subset of Ω. Let $\{\Omega_\kappa\}$ be a family of polygonal domains with $\omega \subset \Omega_\kappa \subset \Omega$ and meas $(\Omega \setminus \Omega_\kappa) \to 0$ as $\kappa \to 0$. Let $\mathcal{T}_\kappa$ be a classical 2-simplex triangulation of $\overline{\Omega}_\kappa$ such that $\overline{\omega} = \bigcup_{F \in \mathcal{T}_\kappa, F \subset \omega} F$ and let $\mathcal{P}_\tau$ denote a partition of the time interval $[0, T]$. Here, it can be understood that κ and τ respectively denote space and time mesh size parameters. We will use the notation $h := (\kappa, \tau)$ and we will denote by $\mathcal{Q}_h$ the family of all sets of the form

$$K = F \times [t_1, t_2], \ \text{ with } F \in \mathcal{T}_\kappa, \ [t_1, t_2] \in \mathcal{P}_\tau$$

and by $\mathcal{R}_h$ the family of all sets of the form

$$K = F \times [t_1, t_2], \ \text{ with } F \in \mathcal{T}_\kappa, \ F \subset \omega, \ [t_1, t_2] \in \mathcal{P}_\tau.$$

Let us introduce $Q_{\kappa,T} := \Omega_\kappa \times (0, T)$. We have

$$\overline{Q}_{\kappa,T} = \bigcup_{K \in \mathcal{Q}_h} K \quad \text{and} \quad \overline{q}_T = \bigcup_{K \in \mathcal{R}_h} K.$$

For any couple of integers $m, n \geq 1$, we will set

$$\widehat{Z}_h(m, n) = \{\, \hat{z}_h \in C^0(\overline{Q}_{\kappa,T}) : \hat{z}_h|_K \in (\mathbb{P}_{m,\mathbf{x}} \otimes \mathbb{P}_{n,t})(K) \ \ \forall K \in \mathcal{Q}_h \,\}$$

and

$$\widehat{V}_h(m, n) = \{\, \hat{z}_h \in \widehat{Z}_h(m, n) : \hat{z}_h = 0 \ \text{ on } \ \partial\Omega_\kappa \times (0, T) \,\}.$$

Here, $\mathbb{P}_{\eta,\xi}$ denotes the space of polynomial functions of order η in the variable ξ.

Then, $\widehat{Z}_h(m, n)$ and $\widehat{V}_h(m, n)$ are finite dimensional subspaces of the Hilbert space $H^1(Q_{\kappa,T})$. Moreover, $\widehat{V}_h(m, n) \subset \widehat{Y}$ and $\widehat{V}_h(m, n) \subset \widehat{R}$. Therefore, for any m, n, m', n', m'', $n'' \geq 1$, we can define $\widehat{X}_h(m, n, m', n') := \widehat{Z}_h(m, n) \times \widehat{V}_h(m', n')$, the product space

$$\widehat{W}_h = \widehat{W}_h(m, n, m', n', m'', n'') := \widehat{X}_h(m, n, m', n') \times \widehat{V}_h(m'', n'')$$

is a finite dimensional subspace of W and the following mixed approximation to (6.27) makes sense:

$$\begin{cases} \widehat{\alpha}((\hat{z}_h, \hat{p}_h), (\hat{z}'_h, \hat{p}'_h)) + \widehat{\beta}((\hat{z}'_h, \hat{p}'_h), \hat{\lambda}_h) = \langle \hat{\ell}, (\hat{z}'_h, \hat{p}'_h) \rangle, \\ \widehat{\beta}((\hat{z}_h, \hat{p}_h), \hat{\lambda}'_h) = 0, \\ \qquad \forall (\hat{z}'_h, \hat{p}'_h, \hat{\lambda}'_h) \in \widehat{W}_h; \ (\hat{z}_h, \hat{p}_h, \hat{\lambda}_h) \in \widehat{W}_h. \end{cases} \tag{6.28}$$

Let $n_h = \dim \widehat{X}_h(m,n,m',n'), m_h = \dim \widehat{V}_h(m'',n'')$ and let the real matrices $\widehat{A}_h \in \mathbb{R}^{n_h,n_h}$, $\widehat{B}_h \in \mathbb{R}^{m_h,n_h}$ and $\widehat{L}_h \in \mathbb{R}^{n_h}$ be defined by

$$\begin{cases} \widehat{\alpha}((\hat{z}_h,\hat{p}_h),(\hat{z}'_h,\hat{p}'_h)) = \langle \widehat{A}_h\{(\hat{z}_h,\hat{p}_h)\},\{(\hat{z}'_h,\hat{p}'_h)\}\rangle_{\mathbb{R}^{n_h},\mathbb{R}^{n_h}} \quad \forall(\hat{z}_h,\hat{p}_h),(\hat{z}'_h,\hat{p}'_h) \in \widehat{X}_h, \\ \widehat{\beta}((\hat{z}_h,\hat{p}_h),\hat{\lambda}_h) = \langle \widehat{B}_h\{(\hat{z}_h,\hat{p}_h)\},\{\hat{\lambda}_h\}\rangle_{\mathbb{R}^{m_h},\mathbb{R}^{m_h}} \quad \forall(\hat{z}_h,\hat{p}_h) \in \widehat{X}_h, \forall\hat{\lambda}_h \in \widehat{V}_h, \\ \hat{\ell}(\hat{z}_h,\hat{p}_h) = \langle \widehat{L}_h,\{\hat{z}_h,\hat{p}_h\}\rangle \quad \forall(\hat{z}_h,\hat{p}_h) \in \widehat{X}_h, \end{cases}$$

where $\{\hat{\lambda}_h\} \in \mathbb{R}^{m_h}$ (resp. $\{\hat{\lambda}'_h\}$) denotes the vector associated to $\hat{\lambda}_h$ (resp. $\hat{\lambda}'_h$), $\{\hat{z}_h,\hat{p}_h\} \in \mathbb{R}^{n_h}$ (resp. $\{\hat{z}_h,\hat{p}_h\}$) denotes the vector associated to $(\hat{z}_h,\hat{p}_h)$ (resp. $(\hat{z}_h,\hat{p}_h)$) and $\langle\cdot,\cdot\rangle_{\mathbb{R}^{n_h},\mathbb{R}^{n_h}}$ (resp. $\langle\cdot,\cdot\rangle_{\mathbb{R}^{m_h},\mathbb{R}^{m_h}}$) the usual scalar product over $\mathbb{R}^{n_h}$ (resp. $\mathbb{R}^{m_h}$). With these notations, the problem (6.28) reads as follows: find $\{(\hat{z}_h,\hat{p}_h)\} \in \mathbb{R}^{n_h}$ and $\{\hat{\lambda}_h\} \in \mathbb{R}^{m_h}$ such that

$$\begin{pmatrix} \widehat{A}_h & \widehat{B}_h^T \\ \widehat{B}_h & 0 \end{pmatrix}_{\mathbb{R}^{n_h+m_h,n_h+m_h}} \begin{pmatrix} \{(\hat{z}_h,\hat{p}_h)\} \\ \{\hat{\lambda}_h\} \end{pmatrix}_{\mathbb{R}^{n_h+m_h}} = \begin{pmatrix} \widehat{L}_h \\ 0 \end{pmatrix}_{\mathbb{R}^{n_h+m_h}}. \tag{6.29}$$

The system (6.29) can be solved either by a direct solver method or using an iterative algorithm like for instance the *Arrow-Hurwicz method* (for completeness, we will describe this method in the following Section).

6.2.5 The Arrow-Hurwicz algorithm

If n_h and m_h are large, the coefficient matrix in (6.29) can be ill-conditioned. Furthermore, $\widehat{A}_h$ is only semidefinite positive and not definite positive (notice that $\widehat{\alpha}(\cdot\,,\cdot)$ is not coercive in $\widehat{X}_h$). For this reason, it is convenient to solve (6.29) using an iterative method not relying on the inversion of $\widehat{A}_h$. Among other possibilities, we have checked that a good choice is the so called *Arrow-Hurwicz algorithm*. It is the following:

ALG 1 (Arrow-Hurwicz):

(i) *Initialize*
Fix $r, s > 0$. Let $(\hat{z}_h^{(0)},\hat{p}_h^{(0)})$ and $\hat{\lambda}_h^{(0)}$ be arbitrarily chosen elements of $\widehat{X}_h$ and $\widehat{V}_h$, respectively. Take, for instance, $(\hat{z}_h^{(0)},\hat{p}_h^{(0)}) = (0,0)$ and $\hat{\lambda}_h^{(0)} = 0$.

For $k \geq 0$, assume that $(\hat{z}_h^{(k)},\hat{p}_h^{(k)})$ and $\hat{\lambda}_h^{(k)}$ are known. Then we do the following:

(ii) *Advance in* $(\hat{z}_h,\hat{p}_h)$: Let $(\hat{z}_h^{(k+1)},\hat{p}_h^{(k+1)})$ be defined by

$$\{\hat{z}_h^{(k+1)},\hat{p}_h^{(k+1)}\} = \{\hat{z}_h^{(k)},\hat{p}_h^{(k)}\} - r\left[\widehat{A}_h\{\hat{z}_h^{(k)},\hat{p}_h^{(k)}\} - \widehat{L}_h + \widehat{B}_h^T\{\hat{\lambda}_h^{(k)}\}\right].$$

(iii) *Advance in* $\hat{\lambda}_h$: Let $\hat{\lambda}_h^{(k+1)}$ be defined by

$$\{\hat{\lambda}_h^{(k+1)}\} = \{\hat{\lambda}_h^{(k)}\} + rs\widehat{B}_h\{\hat{z}_h^{(k+1)},\hat{p}_h^{(k+1)}\}.$$

Check convergence. If the stopping test is not satisfied, replace k by $k+1$ and return to step (ii).

Remark 6.1. The best choice of the parameters r and s is determined by the smallest and greatest eigenvalues associated to some operators involving the matrix $\widehat{A}_h$ and $\widehat{B}_h$; see for example [22, 108, 110]. The main advantage of **ALG 1** with respect to other (iterative or not) algorithms is that we do not have to invert in practice any matrix. The drawback is that we have to find good values of r and s and, obviously, this needs some extra work. □

6.2.6 A numerical experiment

We present now some numerical results. From $(\hat{z}_h, \hat{p}_h)$, we obtain an approximation of the control by setting $v_h = -\rho_0^{-1}\hat{p}_h 1_\omega$. The corresponding controlled state y_h can be computed by solving the equation (6.1) with standard techniques, for instance using the *Cranck-Nicolson method*. Since the state is directly given by $\rho^{-1}\hat{z}$, we simply take $y_h = \rho^{-1}\hat{z}_h$.

We present in this Section an experiment concerning the numerical solution of (6.27). The computations have been performed with *Freefem++*, see [77]. We have used P_2-Lagrange finite elements in $(\mathbf{x}, t)$ for all the variables $\hat{p}$, $\hat{z}$ and $\hat{\lambda}$. We have taken $\Omega = (0, L_1) \times (0, L_2)$, with $L_1 = L_2 = 1$ and $\nu = 1$. For any $(a, b) \in \Omega$, we have considered the function $\beta_0^{(a,b)}$, where

$$\beta_0^{(a,b)}(x_1, x_2) = \frac{x_1(L_1 - x_1)x_2(L_2 - x_2)e^{-[(x_1-c_a)^2+(x_2-c_b)^2]}}{a(L_1 - a)b(L_2 - b)e^{-[(a-c_a)^2+(b-c_b)^2]}},$$

$$c_a = a - \frac{L_1 - 2a}{2a(L_1 - a)}, \quad c_b = b - \frac{L_2 - 2b}{2b(L_2 - b)}.$$

Then, if (a, b) belongs to ω, the function $\beta_0^{(a,b)}$ satisfies the conditions in (6.9). We have taken $T = 1$, $\omega = (0.2, 0.6) \times (0.2, 0.6)$, $G(\mathbf{x}, t) :\equiv 1$, $K_1 = 1$, $K_2 = 2$, $(a, b) = (0.5, 0.5)$ and $y_0(\mathbf{x}) \equiv 1000$. In view of the regularizing effect of the heat equation, the lack of regularity of the initial-boundary data does not have serious consequences.

The computational domain and the mesh are shown in Fig. 6.1. With these data, the behavior of the Arrow-Hurwicz algorithm is depicted in Table 6.1, where the first and second relative errors are respectively given by

$$\frac{\|(\hat{z}_h^{(k+1)}, \hat{p}_h^{(k+1)}) - (\hat{z}_h^{(k)}, \hat{p}_h^{(k)})\|_{L^2(Q_T)}}{\|(\hat{z}_h^{(k+1)}, \hat{p}_h^{(k+1)})\|_{L^2(Q_T)}}$$

and

$$\frac{\|\hat{\lambda}_h^{(k+1)} - \hat{\lambda}_h^{(k)}\|_{L^2(Q_T)}}{\|\hat{\lambda}_h^{(k+1)}\|_{L^2(Q_T)}}.$$

Some illustrative views of the numerical approximations of the control and the state can be found in Fig. 6.2-6.3.

Iterate	Rel. error 1	Rel. error 2
1	0.225333	1.000000
10	0.019314	0.236549
20	0.010068	0.065812
30	0.005633	0.046212
40	0.000358	0.003397
50	0.000117	0.001357

Table 6.1: The behavior of ALG 1 for (6.27).

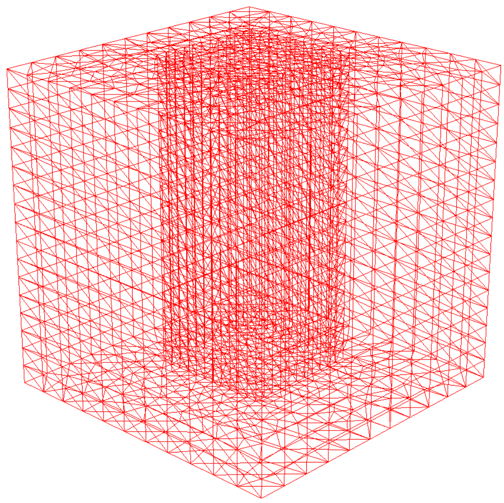

Figure 6.1: The domain and the mesh. Number of vertices: 2800. Number of elements (tetrahedra): 14094. Total number of variables: 20539.

6.3 A strategy for the computation of null controls for the Stokes equations

In this Section, we will present a formulation of the null controllability problem for (6.2) inspired by the same ideas (again, Fursikov-Imanunuvilov's formulation). Specifically,

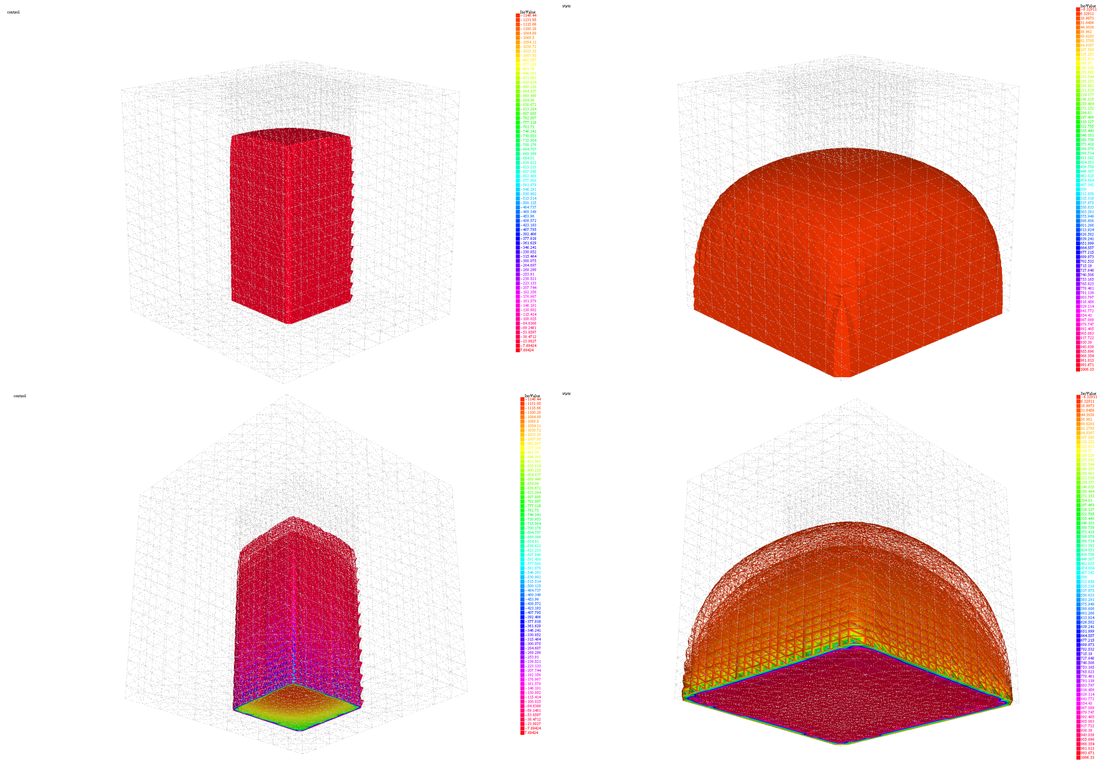

Figure 6.2: $\omega = (0.2, 0.6)$; $y_0(\mathbf{x}) = 1000$. Visualizations of the sets $\{(\mathbf{x}, t) : v_h(\mathbf{x}, t) = 0\}$ (**Left**) and $\{(\mathbf{x}, t) : y_h(\mathbf{x}, t) = 0\}$ (**Right**). Minimal (maximal) values of v_h and y_h: 7.69 and -6.32 (resp. -1146.44 and 1006.33).

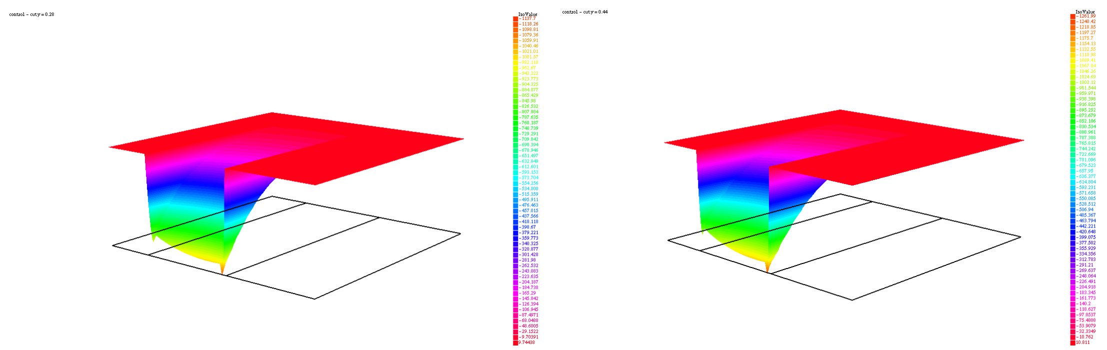

Figure 6.3: Cuts of the control v_h at $x_1 = 0.28$ (**Left**) and $x_1 = 0.44$ (**Right**).

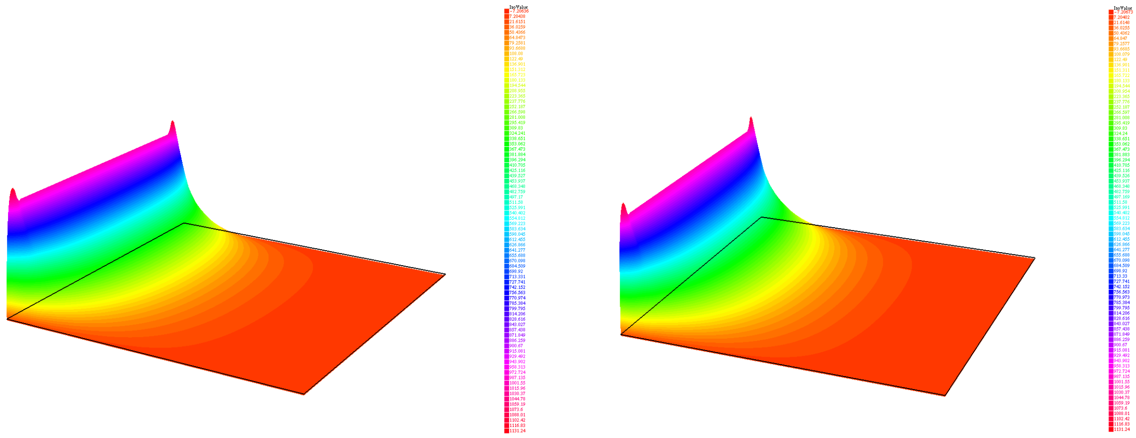

Figure 6.4: Cuts of the state y_h at $x_1 = 0.28$ (**Left**) and $x_1 = 0.44$ (**Right**).

we will try to solve numerically the problem

$$\left\{ \begin{array}{l} \text{Minimize } \; J(\mathbf{y}, \mathbf{v}) = \dfrac{1}{2} \displaystyle\iint_{Q_T} \rho^2 |\mathbf{y}|^2 \, d\mathbf{x} \, dt + \dfrac{1}{2} \iint_{q_T} \rho_0^2 |\mathbf{v}|^2 \, d\mathbf{x} \, dt \\ \text{Subject to } \; (\mathbf{y}, \mathbf{v}) \in \mathcal{S}(\mathbf{y}_0, T), \end{array} \right. \tag{6.30}$$

where $\mathbf{y}_0 \in \mathbf{H}$, $T > 0$, the linear manifold $\mathcal{S}(\mathbf{y}_0, T)$ is given by

$$\mathcal{S}(\mathbf{y}_0, T) = \{ \, (\mathbf{y}, \mathbf{v}) : \mathbf{v} \in \mathbf{L}^2(q_T), \; (\mathbf{y}, \mathbf{v}) \text{ satisfies (6.2) for some } \pi \text{ and fulfills (6.5)} \, \}$$

and it is again assumed that the weights ρ and ρ_0 satisfy (6.9).

We have:

Theorem 6.6. *For any $\mathbf{y}_0 \in \mathbf{H}$ and $T > 0$, there exists exactly one solution to* (6.30).

Again, this result can be viewed as a consequence of a *Carleman inequality*. Thus, let us set

$$\mathbf{L}\mathbf{y} := \mathbf{y}_t - \nu \Delta \mathbf{y}, \quad \mathbf{L}^* \mathbf{p} := -\mathbf{p}_t - \nu \Delta \mathbf{p}$$

and let us introduce the space

$$\mathbf{\Phi}_0 = \left\{ (\mathbf{p}, \sigma) : p_i, \sigma \in C^2(\overline{Q}_T), \; \nabla \cdot \mathbf{p} \equiv 0, \; p_i = 0 \text{ on } \Sigma_T, \; \int_\Omega \sigma(\mathbf{x}, t) d\mathbf{x} = 0 \;\; \forall t \right\}. \tag{6.31}$$

Then, one has the following (see [62, 81]):

Proposition 6.3. *The function β_0 and the associated weights ρ, ρ_0, ρ_1 and ρ_2 furnished by Proposition 6.1 can be chosen such that there exists C_1, only depending on Ω, ω and T, with the* *following property:*

$$\begin{aligned}&\iint_{Q_T} \left[\rho_2^{-2}(|\mathbf{p}_t|^2 + |\Delta\mathbf{p}|^2) + \rho_1^{-2}|\nabla\mathbf{p}|^2 + \rho_0^{-2}|\mathbf{p}|^2 + \rho^{-2}|\nabla\sigma|^2\right] d\mathbf{x}\, dt \\ &\qquad \le C_1 \left(\iint_{Q_T} \rho^{-2}|\mathbf{L}^*\mathbf{p} + \nabla\sigma|^2 \, d\mathbf{x}\, dt + \iint_{q_T} \rho_0^{-2}|\mathbf{p}|^2 \, d\mathbf{x}\, dt \right)\end{aligned} \tag{6.32}$$

for all $(\mathbf{p}, \sigma) \in \mathbf{\Phi}_0$.

Let us introduce the bilinear form

$$m((\mathbf{p},\sigma),(\mathbf{p}',\sigma')) := \iint_{Q_T} \left[\rho^{-2}(\mathbf{L}^*\mathbf{p} + \nabla\sigma)\cdot(\mathbf{L}^*\mathbf{p}' + \nabla\sigma') + 1_\omega \rho_0^{-2}\mathbf{p}\cdot\mathbf{p}'\right] d\mathbf{x}\, dt. \tag{6.33}$$

In view of the unique continuation property of the Stokes system, $m(\cdot\,,\cdot)$ is a scalar product in $\mathbf{\Phi}_0$: if $(\mathbf{p},\sigma) \in \mathbf{\Phi}_0$, $\mathbf{L}^*\mathbf{p} + \nabla\sigma = \mathbf{0}$ in Q_T and $\mathbf{p} = \mathbf{0}$ in q_T, then we have $\mathbf{p} \equiv \mathbf{0}$ and $\sigma \equiv 0$ (notice that, in fact, under these circumstances, $\mathbf{p}(\cdot\,,t)$ and $\sigma(\cdot\,,t)$ are analytic for all t).

Let $\mathbf{\Phi}$ be the completion of $\mathbf{\Phi}_0$ with respect to this scalar product. As before, $\mathbf{\Phi}$ is a Hilbert space, the functions $(\mathbf{p},\sigma) \in \mathbf{\Phi}$ satisfy

$$\iint_{Q_T} \rho^{-2}|\mathbf{L}^*\mathbf{p} + \nabla\sigma|^2 \, d\mathbf{x}\, dt + \iint_{q_T} \rho_0^{-2}|\mathbf{p}|^2 \, d\mathbf{x}\, dt < +\infty \tag{6.34}$$

and, from Proposition 6.3 and a density argument, we also have (6.32) for all $(\mathbf{p},\sigma) \in \mathbf{\Phi}$.

We also see from Proposition 6.3 that

$$\begin{aligned}\mathbf{\Phi} = \Big\{ &(\mathbf{p},\sigma) : p_i, \sigma, \partial_t p_i, \partial_{x_j} p_i, \partial_{x_j}\sigma, \partial_{x_j x_k} p_i \in L^2(0, T-\delta; L^2(\Omega)) \ \forall \delta > 0, \\ &(6.34) \text{ holds}, \ \nabla\cdot\mathbf{p} \equiv 0 \text{ in } Q_T, \ p_i = 0 \text{ on } \Sigma_T, \ \int_\Omega \sigma(\mathbf{x},t)\, d\mathbf{x} = 0 \ \forall t \Big\}\end{aligned} \tag{6.35}$$

and, in particular, any $(\mathbf{p},\sigma) \in \mathbf{\Phi}$ satisfies $\mathbf{p} \in C^0([0, T-\delta]; \mathbf{V})$ for all $\delta > 0$ and

$$\|\mathbf{p}(\cdot\,,0)\|_{\mathbf{V}} \le C\, m((\mathbf{p},\sigma),(\mathbf{p},\sigma))^{1/2} \quad \forall (\mathbf{p},\sigma) \in \mathbf{\Phi}. \tag{6.36}$$

The following result is proved in [62]:

Theorem 6.7. *Let the weights* ρ *and* ρ_0 *be chosen as in Proposition 6.3. Let* $(\mathbf{y}, \mathbf{v})$ *be the unique solution to (6.30). Then one has*

$$\mathbf{y} = \rho^{-2}(L^*\mathbf{p} + \nabla\sigma), \quad \mathbf{v} = -\rho_0^{-2}\mathbf{p}\big|_{\omega\times(0,T)}, \tag{6.37}$$

where $(\mathbf{p}, \sigma)$ *is the solution to the following variational equality in* $\mathbf{\Phi}$*:*

$$\begin{cases} \displaystyle\iint_{Q_T} (\rho^{-2}(\mathbf{L}^*\mathbf{p} + \nabla\sigma) \cdot (\mathbf{L}^*\mathbf{p}' + \nabla\sigma') + \rho_0^{-2}\mathbf{p} \cdot \mathbf{p}' 1_\omega)\, d\mathbf{x}\, dt = \int_\Omega \mathbf{y}_0(\mathbf{x}) \cdot \mathbf{p}'(\mathbf{x}, 0)\, d\mathbf{x} \\ \forall (\mathbf{p}', \sigma') \in \mathbf{\Phi};\ (\mathbf{p}, \sigma) \in \mathbf{\Phi}. \end{cases} \tag{6.38}$$

Once more, (6.38) can be viewed as the weak formulation of a (non-scalar) boundary-value problem for a PDE that is fourth-order in $\mathbf{x}$ and second-order in t. Indeed, arguing as in Section 6.2, we can easily deduce that $(\mathbf{p}, \sigma)$ satisfies, together with some $\pi \in \mathcal{D}'(Q_T)$, the following:

$$\begin{cases} \mathbf{L}(\rho^{-2}(\mathbf{L}^*\mathbf{p} + \nabla\sigma)) + \nabla\pi + 1_\omega \rho_0^{-2}\mathbf{p} = 0 & \text{in} \quad Q_T, \\ \nabla \cdot (\rho^{-2}(\mathbf{L}^*\mathbf{p} + \nabla\sigma)) = 0,\ \nabla \cdot \mathbf{p} = 0 & \text{in} \quad Q_T, \\ \mathbf{p} = \mathbf{0},\ \ \rho^{-2}(\mathbf{L}^*\mathbf{p} + \nabla\sigma) = \mathbf{0} & \text{on} \quad \Sigma_T, \\ \rho^{-2}(\mathbf{L}^*\mathbf{p} + \nabla\sigma)\big|_{t=0} = \mathbf{y}_0,\ \ \rho^{-2}(\mathbf{L}^*\mathbf{p} + \nabla\sigma)\big|_{t=T} = \mathbf{0} & \text{in} \quad \Omega. \end{cases} \tag{6.39}$$

By setting

$$\langle \zeta_0, (\mathbf{p}, \sigma) \rangle := \int_\Omega \mathbf{y}_0(\mathbf{x}) \cdot \mathbf{p}(\mathbf{x}, 0)\, d\mathbf{x}, \tag{6.40}$$

it is found that (6.38) can be rewritten in the form

$$m((\mathbf{p}, \sigma), ((\mathbf{p}', \sigma')) = \langle \zeta_0, (\mathbf{p}', \sigma') \rangle \quad \forall (\mathbf{p}', \sigma') \in \mathbf{\Phi};\ (\mathbf{p}, \sigma) \in \mathbf{\Phi}. \tag{6.41}$$

Thus, if $\mathbf{\Phi}_h$ denotes a finite dimensional subspace of $\mathbf{\Phi}$, a natural approximation of (6.41) is the following:

$$m((\mathbf{p}_h, \sigma_h), ((\mathbf{p}_h', \sigma_h')) = \langle \zeta_0, (\mathbf{p}_h', \sigma_h') \rangle \quad \forall (\mathbf{p}_h', \sigma_h') \in \mathbf{\Phi}_h;\ (\mathbf{p}_h, \sigma_h) \in \mathbf{\Phi}_h. \tag{6.42}$$

However, the couples $(\mathbf{p}, \sigma) \in \mathbf{\Phi}$ satisfy several properties that make it considerably difficult to construct explicitly finite dimensional spaces $\mathbf{\Phi}_h \subset \mathbf{\Phi}$. These are the following:

- As in Section 6.2, since $\rho^{-1}(\mathbf{L}^*\mathbf{p} + \nabla\sigma)$ must belong to $\mathbf{L}^2(Q_T)$ and $\rho_0^{-1}\mathbf{p}\big|_{q_T}$ must belong to $\mathbf{L}^2(q_T)$, the p_i must possess first-order time derivatives and up to second-order spatial derivatives in $L^2(Q_T)$. As before, this means that, in practice, the functions in $\mathbf{\Phi}_h$ must be C^0 in $(\mathbf{x}, t)$ and C^1 in $\mathbf{x}$.

- We now have $\nabla \cdot \mathbf{p} \equiv 0$. It is possible, but not simple at all, to give explicit expressions of zero (or approximately zero) divergence functions associated to a triangulation of Q_T with this regularity.

The second inconvenient is classical in computational fluid dynamics when one considers incompressible fluids. As in many other works, it will be overcome by introducing additional "pressure-like" multipliers and C^0 finite elements; see Section 6.3.2. On the other hand, the first difficulty will be circumvented as in Section 6.2.2, by introducing new variables and associated multipliers and eliminating all the second-order derivatives in the formulation.

In the following Sections, we will present several mixed problems connected to (6.41).

6.3.1 A first mixed formulation of (6.41)

Arguing as in the case of the heat equation and introducing the variable

$$\mathbf{z} = \mathbf{L}^*\mathbf{p} + \nabla\sigma,$$

we see that (6.41) is equivalent to:

$$\begin{cases} \mathbf{a}((\mathbf{z},\mathbf{p},\sigma),(\mathbf{z}',\mathbf{p}',\sigma')) + \mathbf{b}((\mathbf{z}',\mathbf{p}',\sigma'),\lambda) = \boldsymbol{\ell}(\mathbf{z}',\mathbf{p}',\sigma'), \\ \mathbf{b}((\mathbf{z},\mathbf{p},\sigma),\lambda') = 0, \\ \quad \forall((\mathbf{z}',\mathbf{p}',\sigma'),\lambda') \in \mathbf{W};\ ((\mathbf{z},\mathbf{p},\sigma),\lambda) \in \mathbf{W} \end{cases} \tag{6.43}$$

where

$$\mathbf{W} = \ \mathbf{X} \times \boldsymbol{\Lambda}, \quad \mathbf{X} = \ \mathbf{Z} \times \boldsymbol{\Phi}, \quad \mathbf{Z} = \mathbf{L}^2(\rho^{-1}; Q_T), \quad \boldsymbol{\Lambda} = \mathbf{L}^2(\rho; Q_T)$$

and the bilinear forms $\mathbf{a}(\cdot\,,\cdot) : \mathbf{X} \times \mathbf{X} \mapsto \mathbb{R}$ and $\mathbf{b}(\cdot\,,\cdot) : \mathbf{X} \times \mathbf{Y} \mapsto \mathbb{R}$ are given by

$$\mathbf{a}((\mathbf{z},(\mathbf{p},\sigma)),(\mathbf{z}',(\mathbf{p}',\sigma'))) := \iint_{Q_T} \left(\rho^{-2}\mathbf{z}\,\mathbf{z}' + \rho_0^{-2}\mathbf{p}\,\mathbf{p}' 1_\omega\right)\, d\mathbf{x}\, dt$$

and

$$\mathbf{b}((\mathbf{z},(\mathbf{p},\sigma)),\lambda) := \iint_{Q_T} [\mathbf{z} - (\mathbf{L}^*\mathbf{p} + \nabla\sigma)]\,\lambda\, d\mathbf{x}\, dt \tag{6.44}$$

and the linear form $\boldsymbol{\ell} : \mathbf{X} \mapsto \mathbb{R}$ is given by

$$\langle \boldsymbol{\ell}, (\mathbf{z},(\mathbf{p},\sigma))\rangle := \int_\Omega \mathbf{y}_0(\mathbf{x})\,\mathbf{p}(\mathbf{x},0)\, d\mathbf{x}.$$

6.3.2 A second mixed formulation of (6.41)

As we have said, numerical drawbacks are found when we consider finite element approximations (C^1 in space, C^0 in time) with divergence equal to zero. Accordingly, before approximating, we will reformulate (6.38) as another mixed system involving a multiplier associated to this constraint.

Let us introduce

$$\tilde{\boldsymbol{\Phi}}_0 = \left\{ (\mathbf{p},\sigma) : p_i, \sigma \in C^2(\overline{Q}_T),\ p_i = 0 \text{ on } \Sigma_T,\ \int_\Omega \sigma(\mathbf{x},t)d\mathbf{x} = 0 \ \ \forall t \in [0,T] \right\}. \tag{6.45}$$

We have the following Carleman estimates for the couples in $\tilde{\mathbf{\Phi}}_0$:

Proposition 6.4. *There exist weights ρ, ρ_0 and ρ_* and a constant C_2, only depending on Ω, ω and T, with the following property:*

$$\begin{aligned}&\iint_{Q_T} \rho_0^{-2}|\mathbf{p}|^2 d\mathbf{x}\,dt + \|\mathbf{p}(\cdot,0)\|^2_{\mathbf{V}'}\\ &\leq C_2\left(\iint_{Q_T}\rho^{-2}|\mathbf{L}^*\mathbf{p}+\nabla\sigma|^2\,d\mathbf{x}\,dt + \iint_{q_T}\rho_0^{-2}|\mathbf{p}|^2\,d\mathbf{x}\,dt + \iint_{Q_T}|\nabla\cdot\mathbf{p}|^2\,d\mathbf{x}\,dt\right)\end{aligned} \tag{6.46}$$

for all $(\mathbf{p},\sigma)\in\tilde{\mathbf{\Phi}}_0$.

Proof. The proof follows easily by splitting $(\mathbf{p},\sigma)\in\tilde{\mathbf{\Phi}}_0$ in the form

$$(\mathbf{p},\sigma) = (\widetilde{\mathbf{p}},\widetilde{\sigma}) + (\widehat{\mathbf{p}},\widehat{\sigma})$$

where $(\widehat{\mathbf{p}},\widehat{\sigma})$ solves the linear problem

$$\begin{cases} \mathbf{L}^*\widehat{\mathbf{p}} + \nabla\widehat{\sigma} = \mathbf{f} & \text{in} \quad Q_T,\\ \nabla\cdot\widehat{\mathbf{p}} = 0 & \text{in} \quad Q_T,\\ \widehat{\mathbf{p}} = \mathbf{0} & \text{on} \quad \Sigma_T,\\ \widehat{\mathbf{p}}(\cdot,T) = \mathbf{p}(\cdot,T) & \text{in} \quad \Omega \end{cases} \tag{6.47}$$

with $\mathbf{f} := \mathbf{L}^*\mathbf{p}+\nabla\sigma$ and $(\widetilde{\mathbf{p}},\widetilde{\sigma})$ solves the linear problem

$$\begin{cases} \mathbf{L}^*\widetilde{\mathbf{p}} + \nabla\widetilde{\sigma} = \mathbf{0} & \text{in} \quad Q_T,\\ \nabla\cdot\widetilde{\mathbf{p}} = \nabla\cdot\mathbf{p} & \text{in} \quad Q_T,\\ \widetilde{\mathbf{p}} = \mathbf{0} & \text{on} \quad \Sigma_T,\\ \widetilde{\mathbf{p}}(\cdot\,,T) = \mathbf{0} & \text{in} \quad \Omega. \end{cases} \tag{6.48}$$

In view of the Carleman estimates (6.32) for $(\widehat{\mathbf{p}},\widehat{\sigma})$, we have

$$\iint_{Q_T}\rho_0^{-2}|\widehat{\mathbf{p}}|^2\,d\mathbf{x}\,dt + \|\mathbf{p}(\cdot\,,0)\|^2_{\mathbf{V}} \leq C\left(\iint_{Q_T}\rho^{-2}|\mathbf{f}|^2\,d\mathbf{x}\,dt + \iint_{q_T}\rho_0^{-2}|\widehat{\mathbf{p}}|^2\,d\mathbf{x}\,dt\right).$$

On the other hand, $(\widetilde{\mathbf{p}},\widetilde{\sigma})$ solves (6.48) in the sense of transposition, that is,

$$\langle\widetilde{\mathbf{p}},\boldsymbol{\psi}\rangle_{\mathbf{L}^2(Q_T),\mathbf{L}^2(Q_T)} + \langle\widetilde{\mathbf{p}}(\cdot,0),\mathbf{u}_0\rangle_{\mathbf{V}',\mathbf{V}} = -\iint_{Q_T}(\nabla\cdot\mathbf{p})\,h\,d\mathbf{x}\,dt,$$

for all $(\boldsymbol{\psi}, \mathbf{u}_0) \in \mathbf{L}^2(Q_T) \times \mathbf{V}$, where $(\mathbf{u}, h)$ is the unique strong solution to

$$\begin{cases} \mathbf{L}\mathbf{u} + \nabla h = \boldsymbol{\psi} & \text{in} \quad Q_T, \\ \nabla \cdot \mathbf{u} = \mathbf{0} & \text{in} \quad Q_T, \\ \mathbf{u} = \mathbf{0} & \text{on} \quad \Sigma_T, \\ \mathbf{u}(\cdot\,, 0) = \mathbf{u}_0 & \text{in} \quad \Omega. \end{cases}$$

Consequently, we can argue as in [109] and deduce that

$$\|\widetilde{\mathbf{p}}\|^2_{\mathbf{L}^2(Q_T)} + \|\widetilde{\mathbf{p}}(\cdot, 0)\|^2_{\mathbf{V}'} \leq C\, \|\nabla \cdot \mathbf{p}\|^2_{L^2(Q_T)}.$$

Now, putting together the estimates for $(\widehat{\mathbf{p}}, \widehat{\sigma})$ and $(\widetilde{\mathbf{p}}, \widetilde{\sigma})$, we are led easily to (6.46). □

Let us introduce the bilinear form

$$\widetilde{m}((\mathbf{p}, \sigma), (\mathbf{p}', \sigma')) := m((\mathbf{p}, \sigma), (\mathbf{p}', \sigma')) + \iint_{Q_T} (\nabla \cdot \mathbf{p})(\nabla \cdot \mathbf{p}')\, d\mathbf{x}\, dt. \tag{6.49}$$

Again, in view of the unique continuation property of the Stokes system, $\widetilde{m}(\cdot\,, \cdot)$ is a scalar product in $\tilde{\mathbf{\Phi}}_0$.

Let $\tilde{\mathbf{\Phi}}$ be the completion of $\tilde{\mathbf{\Phi}}_0$ with respect to this scalar product. As before, $\tilde{\mathbf{\Phi}}$ is a Hilbert space, the functions $(\mathbf{p}, \sigma) \in \tilde{\mathbf{\Phi}}$ satisfy

$$\iint_{Q_T} \rho^{-2} |\mathbf{L}^*\mathbf{p} + \nabla \sigma|^2\, d\mathbf{x}\, dt + \iint_{q_T} \rho_0^{-2} |\mathbf{p}|^2\, d\mathbf{x}\, dt + \iint_{Q_T} |\nabla \cdot \mathbf{p}|^2 d\mathbf{x}\, dt < +\infty \tag{6.50}$$

and, from Proposition 6.4 and a density argument, we also have (6.46) for all $(\mathbf{p}, \sigma) \in \tilde{\mathbf{\Phi}}$.

On the other hand, any $(\mathbf{p}, \sigma) \in \tilde{\mathbf{\Phi}}$ satisfies

$$\rho_0^{-1}\mathbf{p} \in \mathbf{L}^2(Q_T), \quad \exists\, \mathbf{p}(\cdot, 0) \in \mathbf{V}'$$

and

$$\|\mathbf{p}(\cdot\,, 0)\|_{\mathbf{V}'} \leq C\, \widetilde{m}((\mathbf{p}, \sigma), (\mathbf{p}, \sigma))^{1/2} \quad \forall (\mathbf{p}, \sigma) \in \tilde{\mathbf{\Phi}}. \tag{6.51}$$

By setting

$$\langle \tilde{\ell}, (\mathbf{p}, \sigma) \rangle := \langle \mathbf{p}(\cdot, 0), \mathbf{y}_0 \rangle_{\mathbf{V}', \mathbf{V}} \tag{6.52}$$

thanks to (6.51), we have that $\tilde{\ell}$ is continuous on $\tilde{\mathbf{\Phi}}$.

Let us introduce the space

$$\widetilde{M} = \left\{ \mu \in L^2(Q_T) : \int_\Omega \mu(\mathbf{x}, t) d\mathbf{x} = 0 \ \text{a.e. in} \ (0, T) \right\}$$

and the following reformulation of (6.41):

$$\begin{cases} \widetilde{m}((\mathbf{p},\sigma),(\mathbf{p}',\sigma')) + \displaystyle\iint_{Q_T} (\nabla\cdot\mathbf{p}')\,\mu\,d\mathbf{x}\,dt = \langle\tilde{\boldsymbol{\ell}},(\mathbf{p},\sigma)\rangle, \\ \displaystyle\iint_{Q_T} (\nabla\cdot\mathbf{p})\,\mu'\,d\mathbf{x}\,dt = 0, \\ \qquad \forall((\mathbf{p}',\sigma'),\mu') \in \tilde{\mathbf{\Phi}}\times\widetilde{M};\ ((\mathbf{p},\sigma),\mu)\in\tilde{\mathbf{\Phi}}\times\widetilde{M}. \end{cases} \tag{6.53}$$

Once more, notice that the definitions of $\tilde{\mathbf{\Phi}}$ and $\widetilde{M}$ are the appropriate to keep all the terms in (6.53) meaningful.

Let us introduce the bilinear forms $\tilde{\mathbf{a}}(\cdot\,,\cdot) : \tilde{\mathbf{\Phi}}\times\tilde{\mathbf{\Phi}} \mapsto \mathbb{R}$ and $\tilde{\mathbf{b}}(\cdot\,,\cdot) : \tilde{\mathbf{\Phi}}\times\widetilde{M} \mapsto \mathbb{R}$, with

$$\tilde{\mathbf{a}}((\mathbf{p},\sigma),(\mathbf{p}',\sigma')) := m((\mathbf{p},\sigma),(\mathbf{p}',\sigma'))$$

and

$$\tilde{\mathbf{b}}((\mathbf{p},\sigma),\mu) := \iint_{Q_T} (\nabla\cdot\mathbf{p})\,\mu\,d\mathbf{x}\,dt.$$

Then, $\tilde{\mathbf{a}}(\cdot\,,\cdot)$ and $\tilde{\mathbf{b}}(\cdot\,,\cdot)$ are well-defined and continuous and (6.53) reads:

$$\begin{cases} \tilde{\mathbf{a}}((\mathbf{p},\sigma),(\mathbf{p}',\sigma')) + \tilde{\mathbf{b}}((\mathbf{p}',\sigma'),\mu) = \langle\tilde{\boldsymbol{\ell}},(\mathbf{p}',\sigma')\rangle, \\ \tilde{\mathbf{b}}((\mathbf{p},\sigma),\mu') = 0, \\ \qquad \forall((\mathbf{p}',\sigma'),\mu') \in \tilde{\mathbf{\Phi}}\times\widetilde{M};\ ((\mathbf{p},\sigma),\mu)\in\tilde{\mathbf{\Phi}}\times\widetilde{M}. \end{cases} \tag{6.54}$$

One has the following:

Proposition 6.5. *There exists exactly one solution to* (6.54)*. Furthermore,* (6.41) *and* (6.54) *are equivalent problems in the following sense:*

(i) If $((\mathbf{p},\sigma),\mu)$ solves (6.54)*, then $(\mathbf{p},\sigma)$ solves* (6.41)*.*

(ii) If $(\mathbf{p},\sigma)$ solves (6.41)*, there exists $\mu\in\widetilde{M}$ such that $((\mathbf{p},\sigma),\mu)$ solves* (6.54)*.*

Proof. Let us set

$$\widetilde{\mathcal{V}} := \{\, (\mathbf{p},\sigma)\in\tilde{\mathbf{\Phi}} : \tilde{\mathbf{b}}((\mathbf{p},\sigma),\mu) = 0 \quad \forall\mu\in\widetilde{M}\,\}.$$

We will check that

- $\tilde{\mathbf{a}}(\cdot\,,\cdot)$ is coercive in $\widetilde{\mathcal{V}}$.
- $\tilde{\mathbf{b}}(\cdot\,,\cdot)$ satisfies the usual "inf-sup" condition in $\tilde{\mathbf{\Phi}}\times\widetilde{M}$.

The proofs of these assertions are straightforward. Indeed, we have $\widetilde{\mathcal{V}} = \mathbf{\Phi}$ (the completion of $\mathbf{\Phi}_0$ with respect to $m(\cdot,\cdot)$, see (6.33)). Thus,

$$\tilde{\mathbf{a}}((\mathbf{p},\sigma),(\mathbf{p},\sigma)) = m((\mathbf{p},\sigma),(\mathbf{p},\sigma)) = \widetilde{m}((\mathbf{p},\sigma),(\mathbf{p},\sigma)) \quad \forall(\mathbf{p},\sigma)\in\widetilde{\mathcal{V}}$$

and this proves that $\tilde{\mathbf{a}}(\cdot\,,\cdot)$ is coercive in $\widetilde{\mathcal{V}}$.

On the other hand, the inf-sup condition is a consequence of the fact that, for any $\mu \in \widetilde{M}$, there exists $(\mathbf{p},\sigma) \in \tilde{\mathbf{\Phi}}$ such that

$$\tilde{\mathbf{b}}((\mathbf{p},\sigma),\mu) = \|\mu\|^2_{\widetilde{M}} \quad \text{and} \quad \|(\mathbf{p},\sigma)\|_{\tilde{\mathbf{\Phi}}} \le C\|\mu\|_{\widetilde{M}}. \tag{6.55}$$

This can be seen as follows: for any fixed $\mu \in \widetilde{M}$, let $(\mathbf{p},\sigma)$ be the solution to

$$\mathbf{L}^*\mathbf{p} + \nabla\sigma = \mathbf{0} \text{ in } Q_T, \quad \nabla\cdot\mathbf{p} = \mu \text{ in } Q_T, \quad \mathbf{p} = \mathbf{0} \text{ on } \Sigma_T, \quad \mathbf{p}(\cdot,T) = \mathbf{0} \text{ in } \Omega;$$

then $\mathbf{p}$ belongs to $\mathbf{L}^2(Q_T)$ and

$$\|\mathbf{p}\|_{\mathbf{L}^2(Q_T)} \le C\|\mu\|_{L^2(Q_T)} \tag{6.56}$$

(see [109]). Therefore, one has (6.55). □

6.3.3 A mixed reformulation of (6.54) with an additional multiplier

As in Section 6.3.1, introducing the variable

$$\mathbf{z} := \mathbf{L}^*\mathbf{p} + \nabla\sigma,$$

we observe that (6.54) is equivalent to:

$$\begin{cases} \hat{\mathbf{a}}((\mathbf{z},(\mathbf{p},\sigma)),(\mathbf{z}',(\mathbf{p}',\sigma'))) + \hat{\mathbf{b}}((\mathbf{z}',(\mathbf{p}',\sigma')),(\boldsymbol{\lambda},\mu)) = \langle\hat{\boldsymbol{\ell}},(\mathbf{z}',(\mathbf{p}',\sigma'))\rangle, \\ \hat{\mathbf{b}}((\mathbf{z},(\mathbf{p},\sigma)),(\boldsymbol{\lambda}',\mu')) = 0, \\ \qquad \forall((\mathbf{z}',(\mathbf{p}',\sigma')),(\boldsymbol{\lambda}',\mu')) \in \widehat{\mathbf{W}};\ ((\mathbf{z},(\mathbf{p},\sigma)),(\boldsymbol{\lambda},\mu)) \in \widehat{\mathbf{W}}, \end{cases} \tag{6.57}$$

where

$$\widehat{\mathbf{W}} := \widehat{\mathbf{X}} \times \widehat{\mathbf{Y}}, \quad \widehat{\mathbf{X}} := \mathbf{Z} \times \tilde{\mathbf{\Phi}}, \quad \widehat{\mathbf{Y}} := \mathbf{\Lambda} \times \widetilde{M},$$

the bilinear forms $\hat{\mathbf{a}}(\cdot\,,\cdot) : \widehat{\mathbf{X}} \times \widehat{\mathbf{X}} \mapsto \mathbb{R}$ and $\hat{\mathbf{b}}(\cdot\,,\cdot) : \widehat{\mathbf{X}} \times \widehat{\mathbf{Y}} \mapsto \mathbb{R}$ are given by

$$\hat{\mathbf{a}}((\mathbf{z},(\mathbf{p},\sigma)),(\mathbf{z}',(\mathbf{p}',\sigma'))) := \iint_{Q_T} \left(\rho^{-2}\mathbf{z}\,\mathbf{z}' + \rho_0^{-2}\mathbf{p}\,\mathbf{p}' 1_\omega\right)\, d\mathbf{x}\, dt$$

and

$$\hat{\mathbf{b}}((\mathbf{z},(\mathbf{p},\sigma)),(\boldsymbol{\lambda},\mu)) := \iint_{Q_T} [\mathbf{z} - (\mathbf{L}^*\mathbf{p} + \nabla\sigma)]\,\boldsymbol{\lambda}\, d\mathbf{x}\, dt + \iint_{Q_T} (\nabla\cdot\mathbf{p})\,\mu\, d\mathbf{x}\, dt \tag{6.58}$$

and the linear form $\hat{\boldsymbol{\ell}} : \widehat{\mathbf{X}} \mapsto \mathbb{R}$ is given by

$$\langle\hat{\boldsymbol{\ell}},(\mathbf{z},(\mathbf{p},\sigma))\rangle := \int_\Omega \mathbf{y}_0(\mathbf{x})\,\mathbf{p}(\mathbf{x},0)\, d\mathbf{x}. \tag{6.59}$$

Now, the following holds:

Proposition 6.6. *There exists exactly one solution to (6.57). Furthermore, (6.54) and (6.57) are equivalent problems in the following sense:*

(i) If $((\mathbf{z},(\mathbf{p},\sigma)),(\boldsymbol{\lambda},\mu))$ solves (6.57), then $((\mathbf{p},\sigma),\mu)$ solve (6.54).

(ii) If $((\mathbf{p},\sigma),\mu)$ solves (6.54), there exists $\boldsymbol{\lambda}\in\boldsymbol{\Lambda}$ such that $((\mathbf{z},(\mathbf{p},\sigma)),(\boldsymbol{\lambda},\mu))$, with

$$\mathbf{z} := \mathbf{L}^*\mathbf{p} + \nabla\sigma,$$

solves (6.57).

Proof. Let us introduce the space

$$\widehat{\mathcal{V}} = \{\, (\mathbf{z},(\mathbf{p},\sigma)) \in \widehat{\mathbf{X}} : \hat{\mathbf{b}}((\mathbf{z},(\mathbf{p},\sigma)),(\boldsymbol{\lambda},\mu)) = 0 \quad \forall(\boldsymbol{\lambda},\mu)\in\widehat{\mathbf{Y}}\,\}$$

and, as before, let us check that

- $\hat{\mathbf{a}}(\cdot\,,\cdot)$ is coercive in $\widehat{\mathcal{V}}$.
- $\hat{\mathbf{b}}(\cdot\,,\cdot)$ satisfies the usual “inf-sup” condition in $\widehat{\mathbf{W}}$.

Again, the proofs of these assertions are easy. Indeed, for any $(\mathbf{z},(\mathbf{p},\sigma))\in\widehat{\mathcal{V}}$, $\mathbf{z} = \mathbf{L}^*\mathbf{p}+\nabla\sigma$ and $\nabla\cdot\mathbf{p}=0$ and, therefore,

$$\begin{aligned}
\hat{\mathbf{a}}((\mathbf{z},(\mathbf{p},\sigma)),(\mathbf{z},(\mathbf{p},\sigma))) &= \iint_{Q_T} \left(\rho^{-2}|\mathbf{z}|^2 + \rho_0^{-2}|\mathbf{p}|^2\,1_\omega\right)\,d\mathbf{x}\,dt \\
&= \frac{1}{2}\|(\mathbf{z},(\mathbf{p},\sigma)\|_{\widehat{\mathbf{X}}}^2 + \frac{1}{2}\iint_{Q_T} \rho_0^{-2}|\mathbf{p}|^2 1_\omega\,d\mathbf{x}\,dt \\
&\geq \frac{1}{2}\|(\mathbf{z},(\mathbf{p},\sigma))\|_{\widehat{\mathbf{X}}}^2,
\end{aligned}$$

whence $\hat{\mathbf{a}}(\cdot\,,\cdot)$ is coercive in $\widehat{\mathcal{V}}$.

On the other hand, the inf-sup condition is a consequence of the fact that, for any $(\boldsymbol{\lambda},\mu)\in\widehat{\mathbf{Y}}$, there exists $(\mathbf{z},(\mathbf{p},\sigma))\in\widehat{\mathbf{X}}$ such that

$$\hat{\mathbf{b}}((\mathbf{z},(\mathbf{p},\sigma)),(\boldsymbol{\lambda},\mu)) = \|(\boldsymbol{\lambda},\mu)\|_{\widehat{\mathbf{Y}}}^2 \quad \text{and} \quad \|(\mathbf{z},(\mathbf{p},\sigma))\|_{\widehat{\mathbf{X}}} \leq C\|(\boldsymbol{\lambda},\mu)\|_{\widehat{\mathbf{Y}}}. \tag{6.60}$$

This time, the argument is as follows: for any fixed $(\boldsymbol{\lambda},\mu)\in\widehat{\mathbf{Y}}$, let $(\mathbf{p},\sigma)$ be the solution to

$$\mathbf{L}^*\mathbf{p}+\nabla\sigma = \mathbf{0} \text{ in } Q_T, \quad \nabla\cdot\mathbf{p} = \mu \text{ in } Q_T, \quad \mathbf{p} = 0 \text{ on } \Sigma_T, \quad \mathbf{p}(\cdot,T) = \mathbf{0} \text{ in } \Omega;$$

then $\mathbf{p}$ belongs to $\mathbf{L}^2(Q_T)$ and

$$\|\mathbf{p}\|_{\mathbf{L}^2(Q_T)} \leq C\|\mu\|_{L^2(Q_T)}. \tag{6.61}$$

Taking $\mathbf{z} = \rho^2\boldsymbol{\lambda}$, one finally has (6.60). □

6.3.4 A mixed reformulation of (6.43)

Let us introduce the spaces

$$\begin{aligned}
\mathbf{Z}^* &:= \mathbf{L}^2(\rho^{-1}; Q_T),\\
\boldsymbol{\Phi}^* &:= \Big\{ (\mathbf{p}, \sigma) : \iint_{Q_T} \left[\rho_2^{-2}|\mathbf{p}_t|^2 + \rho_1^{-2}|\nabla \mathbf{p}|^2 + \rho_0^{-2}|\mathbf{p}|^2 + \rho^{-2}|\nabla\sigma|^2\right] d\mathbf{x}\,dt < +\infty,\\
&\qquad \nabla\cdot\mathbf{p} \equiv 0,\ \mathbf{p}\big|_{\Sigma_T} = \mathbf{0} \Big\},\\
\boldsymbol{\Lambda}^* &:= \Big\{ \boldsymbol{\lambda} : \iint_{Q_T} \left[\rho_2^2|\boldsymbol{\lambda}|^2 + \rho_1^2|\nabla\boldsymbol{\lambda}|^2\right] d\mathbf{x}\,dt < +\infty,\ \boldsymbol{\lambda}\big|_{\Sigma_T} = \mathbf{0} \Big\},\\
\mathbf{X}^* &:= \mathbf{Z}^* \times \boldsymbol{\Phi}^*,\quad \mathbf{W}^* := \mathbf{X}^* \times \boldsymbol{\Lambda}^*,
\end{aligned}$$

the bilinear forms $\mathbf{a}^*(\cdot\,,\cdot) : \mathbf{X}^* \times \mathbf{X}^* \mapsto \mathbb{R}$ and $\mathbf{b}^*(\cdot\,,\cdot) : \mathbf{X}^* \times \boldsymbol{\Lambda}^* \mapsto \mathbb{R}$, with

$$\mathbf{a}^*((\mathbf{z}, (\mathbf{p}, \sigma)), (\mathbf{z}', (\mathbf{p}', \sigma'))) := \iint_{Q_T} \left(\rho^{-2}\mathbf{z}\,\mathbf{z}' + \rho_0^{-2}\mathbf{p}\,\mathbf{p}' 1_\omega\right) d\mathbf{x}\,dt$$

and

$$\mathbf{b}^*((\mathbf{z}, (\mathbf{p}, \sigma)), \boldsymbol{\lambda}) := \iint_{Q_T} \{[\mathbf{z} + \mathbf{p}_t - \nabla\sigma]\,\lambda - \nu\nabla\mathbf{p}\cdot\nabla\lambda\}\, d\mathbf{x}\,dt$$

and the linear form $\boldsymbol{\ell}^* : \mathbf{X}^* \mapsto \mathbb{R}$, with

$$\langle \boldsymbol{\ell}^*, (\mathbf{z}, (\mathbf{p}, \sigma)) \rangle := \int_\Omega \mathbf{y}_0(\mathbf{x})\,\mathbf{p}(\mathbf{x}, 0)\,d\mathbf{x}. \tag{6.62}$$

The bilinear form $\mathbf{b}^*(\cdot,\cdot)$ appears when we integrate by parts the second–order terms in $\mathbf{b}(\cdot,\cdot)$, see (6.44). Accordingly, at least formally, we can reformulate (6.43) as follows:

$$\left\{\begin{array}{l}
\mathbf{a}^*((\mathbf{z}, (\mathbf{p}, \sigma)), (\mathbf{z}', (\mathbf{p}', \sigma'))) + \mathbf{b}^*((\mathbf{z}', (\mathbf{p}', \sigma')), \boldsymbol{\lambda}') = \langle \boldsymbol{\ell}^*, (\mathbf{z}', (\mathbf{p}', \sigma')) \rangle,\\
\mathbf{b}^*((\mathbf{z}, (\mathbf{p}, \sigma)), \boldsymbol{\lambda}) = 0,\\
\qquad \forall ((\mathbf{z}', (\mathbf{p}', \sigma')), \boldsymbol{\lambda}') \in \mathbf{X}^* \times \boldsymbol{\Lambda}^*;\ ((\mathbf{z}, (\mathbf{p}, \sigma)), \boldsymbol{\lambda}) \in \mathbf{X}^* \times \boldsymbol{\Lambda}^*.
\end{array}\right. \tag{6.63}$$

This can be viewed as a new mixed formulation of (6.41). However, that these two problems are equivalent in the sense of Proposition 6.5 and 6.6 is, at present, an open question.

6.3.5 A fifth (and final) mixed formulation

Finally, let us introduce the spaces

$$\begin{aligned}
\overline{\mathbf{Z}} &:= \mathbf{L}^2(\rho^{-1}; Q_T),\\
\overline{\mathbf{\Phi}} &:= \Big\{ (\mathbf{p},\sigma) : \iint_{Q_T} \big[\rho_2^{-2}|\mathbf{p}_t|^2 + \rho_1^{-2}|\nabla\mathbf{p}|^2 + \rho_0^{-2}|\mathbf{p}|^2 + |\nabla\cdot\mathbf{p}|^2 + \rho^{-2}|\nabla\sigma|^2\big]\, d\mathbf{x}\, dt < +\infty,\\
&\qquad \mathbf{p}\big|_{\Sigma_T} = \mathbf{0} \Big\},\\
\overline{\mathbf{\Lambda}} &:= \Big\{ \boldsymbol{\lambda} : \iint_{Q_T} \big[\rho_2^2|\boldsymbol{\lambda}|^2 + \rho_1^2|\nabla\boldsymbol{\lambda}|^2\big]\, d\mathbf{x}\, dt < +\infty,\ \boldsymbol{\lambda}\big|_{\Sigma_T} = \mathbf{0} \Big\},\\
\overline{\mathbf{X}} &:= \overline{\mathbf{Z}} \times \overline{\mathbf{\Phi}},\quad \overline{\mathbf{Y}} := \overline{\mathbf{\Lambda}} \times \widetilde{M},
\end{aligned}$$

the bilinear forms $\overline{\mathbf{a}}(\cdot\,,\cdot) : \overline{\mathbf{X}} \times \overline{\mathbf{X}} \mapsto \mathbb{R}$ and $\overline{\mathbf{b}}(\cdot\,,\cdot) : \overline{\mathbf{X}} \times \overline{\mathbf{Y}} \mapsto \mathbb{R}$, with

$$\overline{\mathbf{a}}((\mathbf{z},(\mathbf{p},\sigma)),(\mathbf{z}',(\mathbf{p}',\sigma'))) := \iint_{Q_T} \left(\rho^{-2}\mathbf{z}\,\mathbf{z}' + \rho_0^{-2}\mathbf{p}\,\mathbf{p}' 1_\omega\right)\, d\mathbf{x}\, dt$$

and

$$\overline{\mathbf{b}}((\mathbf{z},(\mathbf{p},\sigma)),(\lambda,\mu)) := \iint_{Q_T} \{[\mathbf{z} + \mathbf{p}_t - \nabla\sigma]\,\lambda - \nu\nabla\mathbf{p}\cdot\nabla\lambda\}\, d\mathbf{x}\, dt + \iint_{Q_T} (\nabla\cdot\mathbf{p})\,\mu\, d\mathbf{x}\, dt$$

and the linear form $\overline{\boldsymbol{\ell}} : \overline{\mathbf{X}} \mapsto \mathbb{R}$, with

$$\langle \overline{\boldsymbol{\ell}}, (\mathbf{z},(\mathbf{p},\sigma))\rangle := \int_\Omega \mathbf{y}_0(\mathbf{x})\,\mathbf{p}(\mathbf{x},0)\, d\mathbf{x}. \tag{6.64}$$

In accordance with (6.63), it can be accepted that, at least formally, (6.57) possesses the following reformulation:

$$\begin{cases}
\overline{\mathbf{a}}((\mathbf{z},(\mathbf{p},\sigma)),(\mathbf{z}',(\mathbf{p}',\sigma'))) + \overline{\mathbf{b}}((\mathbf{z}',(\mathbf{p}',\sigma')),(\lambda,\mu)) = \langle \overline{\boldsymbol{\ell}}, (\mathbf{z}',(\mathbf{p}',\sigma'))\rangle,\\
\overline{\mathbf{b}}((\mathbf{z},(\mathbf{p},\sigma)),(\lambda',\mu')) = 0,\\
\qquad \forall((\mathbf{z}',(\mathbf{p}',\sigma')),(\boldsymbol{\lambda}',\mu')) \in \overline{\mathbf{X}} \times \overline{\mathbf{Y}};\ ((\mathbf{z},(\mathbf{p},\sigma)),(\boldsymbol{\lambda},\mu)) \in \overline{\mathbf{X}} \times \overline{\mathbf{Y}}.
\end{cases} \tag{6.65}$$

Remark 6.2. The previous mixed formulations possess several relevant properties:

- If we were able to construct finite dimensional subspaces of $\mathbf{\Phi}$, we would be led to a standard mixed approximation of (6.43). But this is not obvious: recall that, to have $(\mathbf{p},\sigma) \in \mathbf{\Phi}$, we need (among other things) $\mathbf{L}^*\mathbf{p} + \nabla\sigma \in \mathbf{L}^2(\rho^{-1}; Q_T)$ and $\nabla\cdot\mathbf{p} = 0$.

- Contrarily, it is relatively easy to construct numerically efficient finite dimensional subspaces of $\tilde{\mathbf{\Phi}}$, for instance, based on the *Bell triangle* or the *Bogner-Fox-Schmidt* *rectangle*. Consequently, we can get finite element approximations of (6.54) for which, furthermore, a convergence analysis can be performed.

- The same can be said for (6.57). In this case, the fact that the variable $\mathbf{z}$ appears explicitly can be very useful for computing and viewing the controlled state.

- The mixed formulations (6.63) and (6.65) share an advantageous characteristic: they can be approximated in a rather standard way by C^0 finite elements since, after integration by parts, no second-order spatial derivative appears. Unfortunately, up to our knowledge, it is unknown whether or not they are well posed. More precisely, the proof of the inf–sup condition is open and, moreover, the well-posedness of their associated discrete versions is not clear.

6.3.6 A numerical approximation of (6.65) (without justification)

Let us use the same notation of Section 6.2.4.

For any couple of integers $m, n \geq 1$, we will set

$$\begin{aligned}
\overline{\mathbf{Z}}_h(m,n) &:= \{\, \mathbf{z}_h \in \mathbf{C}^0(\overline{Q}_{\kappa,T}) : \mathbf{z}_h|_K \in (\mathbb{P}_{m,\mathbf{x}} \otimes \mathbb{P}_{n,t})(K) \;\; \forall K \in \mathcal{Q}_h \,\}, \\
\overline{\mathbf{V}}_h(m,n) &:= \{\, \mathbf{p}_h \in \overline{\mathbf{Z}}_h(m,n) : \mathbf{p}_h = \mathbf{0} \;\text{ on }\; \Sigma_T \,\}, \\
\overline{M}_h(m,n) &:= \{\, \sigma_h \in C^0(\overline{Q}_{\kappa,T}) : \sigma_h|_K \in (\mathbb{P}_{m,\mathbf{x}} \otimes \mathbb{P}_{n,t})(K) \;\; \forall K \in \mathcal{Q}_h \,\}.
\end{aligned}$$

Then, $\overline{\mathbf{Z}}_h(m,n)$ and $\overline{\mathbf{V}}_h(m,n)$ are finite dimensional subspaces of the Hilbert space $\mathbf{H}^1(Q_{\kappa,T})$. Moreover, $\overline{\mathbf{V}}_h(m,n) \times \overline{M}_h(m,n) \subset \overline{\mathbf{\Phi}}$, $\overline{\mathbf{V}}_h(m,n) \subset \overline{\mathbf{\Lambda}}$ and $\overline{M}_h(m,n) \subset \widetilde{M}$. Therefore, for any m, n, m', n', m'', n'', m''', n''', m'''', $n'''' \geq 1$, we can define

$$\overline{\mathbf{X}}_h := \overline{\mathbf{Z}}_h(m,n) \times \overline{\mathbf{V}}_h(m',n') \times \overline{M}_h(m'',n'') \text{ and } \overline{\mathbf{Y}}_h := \overline{\mathbf{V}}_h(m''',n''') \times \overline{M}_h(m'''',n''''),$$

that are finite dimensional subspaces of $\overline{\mathbf{X}}$ and $\overline{\mathbf{Y}}$, respectively.

The following mixed approximation of (6.65) makes sense:

$$\begin{cases}
\overline{\mathbf{a}}((\mathbf{z}_h,(\mathbf{p}_h,\sigma_h)),(\mathbf{z}'_h,(\mathbf{p}'_h,\sigma'_h))) + \overline{\mathbf{b}}((\mathbf{z}'_h,(\mathbf{p}'_h,\sigma'_h)),(\lambda_h,\mu_h)) = \langle \overline{\ell},(\mathbf{z}'_h,(\mathbf{p}'_h,\sigma'_h))\rangle, \\
\overline{\mathbf{b}}((\mathbf{z}_h,(\mathbf{p}_h,\sigma_h)),(\lambda'_h,\mu'_h)) = 0, \\
\quad \forall ((\mathbf{z}'_h,(\mathbf{p}'_h,\sigma'_h)),(\boldsymbol{\lambda}'_h,\mu'_h)) \in \overline{\mathbf{X}}_h \times \overline{\mathbf{Y}}_h;\; ((\mathbf{z}_h,(\mathbf{p}_h,\sigma_h)),(\boldsymbol{\lambda}_h,\mu_h)) \in \overline{\mathbf{X}}_h \times \overline{\mathbf{Y}}_h.
\end{cases} \tag{6.66}$$

6.3.7 A numerical experiment

This Section deals with some numerical results. We have solved (6.66) with the following data: $\Omega = (0,1) \times (0,1)$, $\omega = (0.2,0.6) \times (0.2,0.6)$, $T = 1$, $K_1 = 1$, $K_2 = 2$, $\beta_0 = \beta_0^{(0.5,0.5)}$ (as in Section 6.2); $\nu = 1$, $\mathbf{y}_0(\mathbf{x}) \equiv \nabla \times \psi(\mathbf{x})$, $\psi(x_1,x_2) \equiv M(x_1x_2)^2[(1-x_1)(1-x_2)]^2$.

Again, the computations have been performed with the software *Freefem++*, using P_2-Lagrange approximations in $(\mathbf{x},t)$ for all the variables. The linear system in (6.66) has been solved with the Arrow-Hurwicz algorithm, where we have taken $r = 0.01$ and $s = 0.1$. The convergence of this algorithm is illustrated in Table 6.2, where the first and the second relative errors are given by

$$\frac{\|(\mathbf{z}_h^{(k+1)}, \mathbf{p}_h^{(k+1)}, \sigma_h^{(k+1)}) - (\mathbf{z}_h^{(k)}, \mathbf{p}_h^{(k)}, \sigma_h^{(k)})\|_{\mathbf{L}^2(Q_T)}}{\|(\mathbf{z}_h^{(k+1)}, \mathbf{p}_h^{(k+1)}, \sigma_h^{(k+1)}))\|_{\mathbf{L}^2(Q_T)}}$$

and

$$\frac{\|(\lambda_h^{(k+1)}, \mu_h^{(k+1)}) - (\lambda_h^{(k)}, \mu_h^{(k)})\|_{\mathbf{L}^2(Q_T)}}{\|(\lambda_h^{(k+1)}, \mu_h^{(k+1)})\|_{\mathbf{L}^2(Q_T)}}.$$

The computed control and state are displayed in Fig. 6.5–6.9.

Iterate	Rel. error 1	Rel. error 2
1	0.659686	0.202439
10	0.063864	0.106203
20	0.016147	0.076083
30	0.008874	0.046212
40	0.000464	0.001397
50	0.000206	0.000762

Table 6.2: The behavior of the Arrow-Hurwicz algorithm for (6.66).

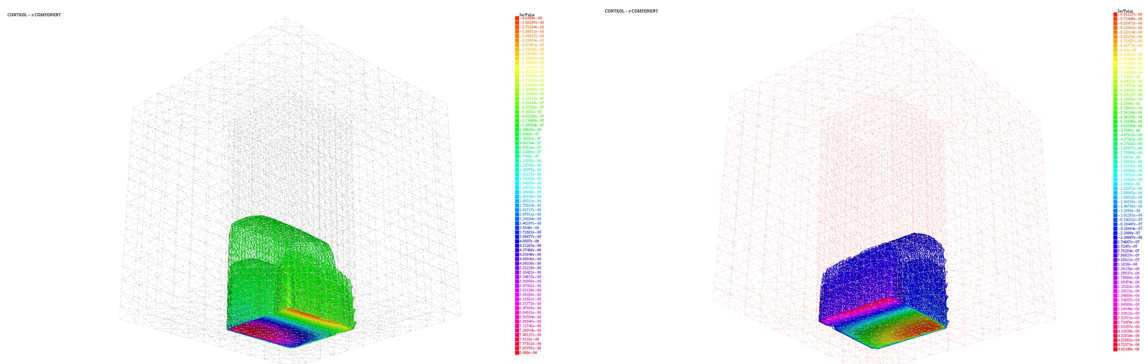

Figure 6.5: A view of the computed control: first component (**Left**) and second component (**Right**). Minimal (maximal) values of the first and second components of $\mathbf{v}_h$: -4.04×10^{-6} and -9.91×10^{-6} (resp. 8.09×10^{-6} and 4.92×10^{-6}).

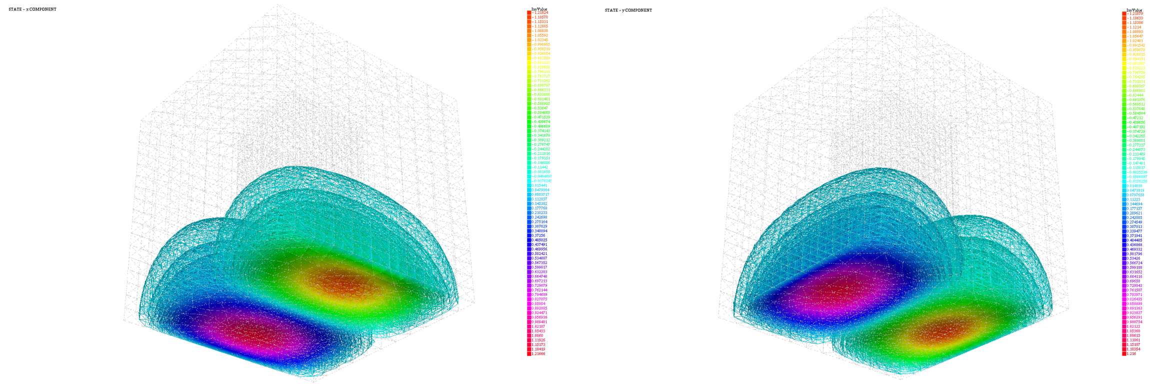

Figure 6.6: A view of the computed state: first component (**Left**) and second component (**Right**). Minimal (maximal) values of the first component of $\mathbf{y}_h$: -1.22 (resp. 1.22) and minimal (maximal) values of the second component of $\mathbf{y}_h$: -1.22 (resp. 1.22).

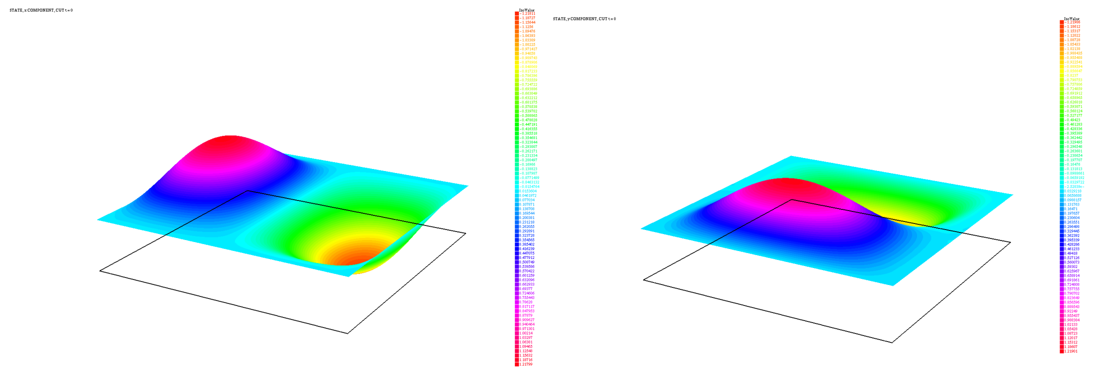

Figure 6.7: Cuts of the computed state at $t = 0$: first component (**Left**) and second component (**Right**). Minimal (maximal) values of the first and second components of $\mathbf{v}_h$: -1.21 and -1.21 (resp. 1.21 and 1.21).

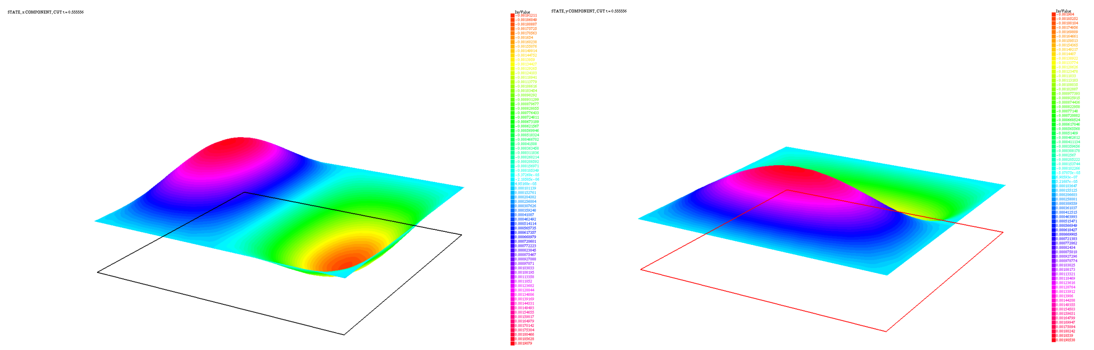

Figure 6.8: Cuts of the computed state at $t = 0.5$: first component (**Left**) and second component (**Right**). Minimal (maximal) values of the first and second components of $\mathbf{y}_h$ at $t = 0.5$: -1.91×10^{-3} and -1.9×10^{-3} (resp. 1.9×10^{-3} and 1.9×10^{-3}).

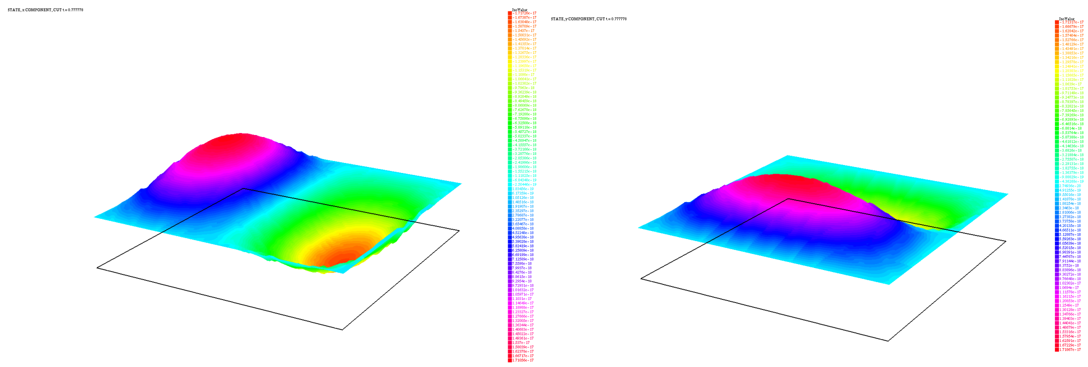

Figure 6.9: Cuts of the computed state at $t = 0.8$: first component (**Left**) and second component (**Right**). Minimal (maximal) values of the first and second components of $\mathbf{y}_h$ at $t = 0.8$: -1.71×10^{-17} and -1.71×10^{-17} (resp. 1.71×10^{-17} and 1.71×10^{-17}).

6.4 An application: numerical local exact controllability to the trajectories of the Navier-Stokes equations

In this Section, we will present a numerical method for the computation of solutions to the problem of exact controllability to the trajectories of (6.3) that is inspired by the previous ideas. This controllability property was proved in [46] under suitable regularity assumptions on the trajectories. More precisely, we have to assume that:

$$\overline{\mathbf{y}} \in L^2(0,T;D(\mathbf{A})) \cap C^0([0,T];\mathbf{V}) \cap \mathbf{L}^\infty(Q_T), \quad \overline{\mathbf{y}}_t \in L^2(0,T;\mathbf{H}), \tag{6.67}$$

where $D(\mathbf{A}) := \mathbf{H}^2(\Omega) \cap \mathbf{V}$ is the domain of the usual Stokes operator $\mathbf{A}$; see [81] for a fundamental previous result.

6.4.1 A fixed-point algorithm and a mixed formulation

First of all, let us rewrite the local exact controllability to the trajectories as a local null controllability problem. To do this, let us put $\mathbf{y} = \overline{\mathbf{y}} + \mathbf{u}$ and $\pi = \overline{\pi} + q$ and let us use (6.3). Taking into account that $(\overline{\mathbf{y}}, \overline{\pi})$ solves (6.6), we find:

$$\begin{cases} \mathbf{u}_t - \nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\overline{\mathbf{y}} + ((\overline{\mathbf{y}}+\mathbf{u})\cdot\nabla)\mathbf{u} + \nabla q = \mathbf{v}1_\omega & \text{in} \quad Q_T, \\ \nabla\cdot\mathbf{u} = 0 & \text{in} \quad Q_T, \\ \mathbf{u} = \mathbf{0} & \text{on} \quad \Sigma_T, \\ \mathbf{u}(0) = \mathbf{u}_0 := \mathbf{y}_0 - \overline{\mathbf{y}}_0 & \text{in} \quad \Omega. \end{cases} \tag{6.68}$$

This way, we have reduced our problem to a local null controllability result for the solution $(\mathbf{u}, q)$ to the nonlinear problem (6.68).

Let us suppose that $\mathbf{u}_0 \in D(\mathbf{A}^\sigma)$, with $1/2 < \sigma < 1$ ($\mathbf{A}^\sigma$ is the fractional power of the Stokes operator) and let us introduce the fixed-point mapping $F : \mathbf{W} \mapsto \mathbf{W}$, where

$$\mathbf{W} := \{ \mathbf{u} \in \mathbf{L}^\infty(Q_T) : \nabla \cdot \mathbf{u} = 0 \text{ in } Q_T,\ \mathbf{u} \cdot \mathbf{n} = 0 \text{ on } \Sigma_T \}.$$

Here, for any $\mathbf{w} \in \mathbf{W}$, $\mathbf{u} = F(\mathbf{w})$ is, together with some $\mathbf{v}$ and q, the unique solution to the extremal problem

$$\text{Minimize } \ J(\mathbf{w}; \mathbf{u}, \mathbf{v}) = \frac{1}{2} \iint_{Q_T} \rho^2 |\mathbf{u}|^2 \, d\mathbf{x}\, dt + \frac{1}{2} \iint_{q_T} \rho_0^2 |\mathbf{v}|^2 \, d\mathbf{x}\, dt \tag{6.69}$$

subject to $\mathbf{v} \in \mathbf{L}^2(q_T)$ and

$$\begin{cases} \mathbf{u}_t - \nu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla)\overline{\mathbf{y}} + ((\overline{\mathbf{y}} + \mathbf{w}) \cdot \nabla)\mathbf{u} + \nabla q = \mathbf{v} 1_\omega & \text{in } \ Q_T, \\ \nabla \cdot \mathbf{u} = 0 & \text{in } \ Q_T, \\ \mathbf{u} = \mathbf{0} & \text{on } \ \Sigma_T, \\ \mathbf{u}(0) = \mathbf{u}_0 & \text{in } \ \Omega. \end{cases} \tag{6.70}$$

It is again assumed that the weights ρ and ρ_0 satisfy (6.9).

More precisely, we have:

Theorem 6.8. *For any $\mathbf{u}_0 \in D(\mathbf{A}^\sigma)$ and $T > 0$, there exists exactly one solution to (6.69)-(6.70).*

This can be regarded as a consequence of the following *Carleman inequality* for Oseen systems (the proof can be found in [82]):

Proposition 6.7. *For all $R > 0$, the function β_0 and the associated weights ρ, ρ_0 and ρ_1 furnished by Proposition 6.1 can be chosen such that, for some C_3, only depending on Ω, ω, T and R, and for all $\mathbf{w} \in \mathbf{W}$ with $\|\mathbf{w}\|_{\mathbf{L}^\infty(Q_T)} \leq R$, one has:*

$$\begin{aligned} &\iint_{Q_T} (\rho_1^{-2} |\nabla \mathbf{p}|^2 + \rho_0^{-2} |\mathbf{p}|^2 + \rho^{-2} |\nabla \sigma|^2) \, d\mathbf{x}\, dt \\ &\leq C_3 \left(\iint_{Q_T} \rho^{-2} |\mathbf{M}^* \mathbf{p} + \nabla \sigma|^2 \, d\mathbf{x}\, dt + \iint_{q_T} \rho_0^{-2} |\mathbf{p}|^2 \, d\mathbf{x}\, dt \right) \end{aligned} \tag{6.71}$$

for all $(\mathbf{p}, \sigma) \in \mathbf{\Phi}_0$. Here, we have used the notation

$$\mathbf{M}^* \mathbf{p} = -\mathbf{p}_t - \nu \Delta \mathbf{p} - \nabla \mathbf{p}\, (\overline{\mathbf{y}} + \mathbf{w}) - \nabla \mathbf{p}^t\, \overline{\mathbf{y}}, \quad \mathbf{M} \mathbf{u} = \mathbf{u}_t - \nu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla)\overline{\mathbf{y}} + ((\overline{\mathbf{y}} + \mathbf{w}) \cdot \nabla)\mathbf{u}.$$

For any $\mathbf{w} \in \mathbf{W}$, we will denote by $m(\mathbf{w}; \cdot\,, \cdot\,)$ the following associated bilinear form on $\mathbf{\Phi}_0$:

$$m(\mathbf{w}; (\mathbf{p}, \sigma), (\mathbf{p}', \sigma')) := \iint_{Q_T} \left(\rho^{-2} (\mathbf{M}^* \mathbf{p} + \nabla \sigma) \cdot (\mathbf{M}^* \mathbf{p}' + \nabla \sigma') + 1_\omega \rho_0^{-2} \mathbf{p} \cdot \mathbf{p}' \right) \, d\mathbf{x}\, dt;$$

recall that $\mathbf{\Phi}_0$ is given in (6.31).

This bilinear form is a scalar product in $\mathbf{\Phi}_0$. Let us denote by $\mathbf{\Phi}^{\mathbf{w}}$ the corresponding completion. Then, for a good choice of ρ and ρ_0 (the same as above), the solution to (6.69) can be characterized by the identities

$$\mathbf{u} = \rho^{-2}(\mathbf{M}^* \mathbf{p_w} + \nabla \sigma_{\mathbf{w}}), \quad \mathbf{v} = -\rho_0^{-2} \mathbf{p_w}\big|_{q_T}, \tag{6.72}$$

where $(\mathbf{p_w}, \sigma_{\mathbf{w}})$ is the solution to a variational equality in the Hilbert space $\mathbf{\Phi}^{\mathbf{w}}$:

$$\begin{cases} m(\mathbf{w}; (\mathbf{p_w}, \sigma_{\mathbf{w}}), (\mathbf{p}', \sigma')) = \displaystyle\int_\Omega \mathbf{u}_0(\mathbf{x}) \cdot \mathbf{p}'(\mathbf{x}, 0)\, d\mathbf{x} \\ \forall (\mathbf{p}', \sigma') \in \mathbf{\Phi}^{\mathbf{w}};\ (\mathbf{p_w}, \sigma_{\mathbf{w}}) \in \mathbf{\Phi}^{\mathbf{w}}. \end{cases} \tag{6.73}$$

Remark 6.3. Note that, in view of (6.71), for any fixed $R > 0$, the good choice of the weights indicated in Proposition (6.7) leads to a family of norms $m(\mathbf{w}; \cdot, \cdot)^{1/2}$ that are *equivalent* as long as $\|\mathbf{w}\|_{\mathbf{L}^\infty(Q_T)} \le R$. Consequently, the associated spaces $\mathbf{\Phi}^{\mathbf{w}}$ are the same for all $\mathbf{w}$ with $\|\mathbf{w}\|_{\mathbf{L}^\infty(Q_T)} \le R$. □

In order to solve the null controllability problem for (6.68), it suffices to find a solution to the fixed-point equation

$$\mathbf{u} = F(\mathbf{u}), \quad \mathbf{u} \in \mathbf{W}. \tag{6.74}$$

Moreover, in view of the results in [72], if $\mathbf{u}_0$ is small enough, F is well defined and possesses at least one fixed-point (by the *Schauder's Theorem*).

Consequently, a natural strategy is to use the following algorithm:

ALG 2 (Fixed-point):

(i) Choose $\mathbf{u}_0 \in D(\mathbf{A}^\sigma)$.

(ii) Then, for given $n \ge 0$ and $\mathbf{u}^n \in \mathbf{W}$ compute $\mathbf{u}^{n+1} = F(\mathbf{u}^n)$, i.e. find the unique solution $(\mathbf{u}^{n+1}, \mathbf{v}^{n+1})$ to the extremal problem

$$\text{Minimize } J(\mathbf{u}^n; \mathbf{u}^{n+1}, \mathbf{v}^{n+1}) = \frac{1}{2} \iint_{Q_T} \rho^2 |\mathbf{u}^{n+1}|^2 \, d\mathbf{x}\, dt + \frac{1}{2} \iint_{q_T} \rho_0^2 |\mathbf{v}^{n+1}|^2 \, d\mathbf{x}\, dt \tag{6.75}$$

subject to $\mathbf{v}^{n+1} \in \mathbf{L}^2(q_T)$ and

$$\begin{cases} \mathbf{u}_t^{n+1} - \nu \Delta \mathbf{u}^{n+1} + (\mathbf{u}^{n+1} \cdot \nabla)\bar{\mathbf{y}} + ((\bar{\mathbf{y}} + \mathbf{u}^n) \cdot \nabla)\mathbf{u}^{n+1} + \nabla q^{n+1} = \mathbf{v}^{n+1} 1_\omega & \text{in } Q_T, \\ \nabla \cdot \mathbf{u}^{n+1} = 0 & \text{in } Q_T, \\ \mathbf{u}^{n+1} = \mathbf{0} & \text{on } \Sigma_T, \\ \mathbf{u}^{n+1}(0) = \mathbf{u}_0 & \text{in } \Omega. \end{cases} \tag{6.76}$$

This is the classical fixed-point method for F. We start from a prescribed state $\mathbf{u}^0$ and, then, we solve a null controllability problem for a linear parabolic system at each step. Thus, we produce a sequence $\{\mathbf{u}^n, \mathbf{v}^n\}$ that is expected to converge to a solution to the null controllability problem (6.68).

For the numerical solution of the problems (6.75)-(6.76), we can apply arguments similar to those in Sections 6.3.5 and 6.3.6. Thus, a suitable mixed formulation is:

$$\begin{cases} \overline{\mathbf{a}}((\mathbf{z},(\mathbf{p},\sigma)),(\mathbf{z}',(\mathbf{p}',\sigma'))) + \overline{\mathbf{b}}((\mathbf{z}',(\mathbf{p}',\sigma')),(\lambda,\mu)) = \langle \overline{\boldsymbol{\ell}}, (\mathbf{z}',(\mathbf{p}',\sigma'))\rangle, \\ \overline{\mathbf{b}}((\mathbf{z},(\mathbf{p},\sigma)),(\lambda',\mu')) = 0, \\ \quad \forall ((\mathbf{z}',(\mathbf{p}',\sigma')),(\boldsymbol{\lambda}',\mu')) \in \overline{\mathbf{X}}\times\overline{\mathbf{Y}};\ ((\mathbf{z},(\mathbf{p},\sigma)),(\boldsymbol{\lambda},\mu)) \in \overline{\mathbf{X}}\times\overline{\mathbf{Y}}. \end{cases} \tag{6.77}$$

where, the spaces $\overline{\mathbf{X}}$ and $\overline{\mathbf{Y}}$ and the forms $\overline{\mathbf{a}}(\cdot\,,\cdot)$, $\overline{\mathbf{b}}(\cdot\,,\cdot)$ and $\overline{\boldsymbol{\ell}}$ are given by

$$\begin{aligned}
&\overline{\mathbf{Z}} := \mathbf{L}^2(\rho^{-1}; Q_T),\\
&\overline{\mathbf{\Phi}} := \Big\{ (\mathbf{p},\sigma) : \iint_{Q_T} \big[\rho_2^{-2}|\mathbf{p}_t|^2 + \rho_1^{-2}|\nabla\mathbf{p}|^2 + \rho_0^{-2}|\mathbf{p}|^2 + |\nabla\cdot\mathbf{p}|^2 + \rho^{-2}|\nabla\sigma|^2\big]\, d\mathbf{x}\, dt < +\infty,\\
&\qquad \mathbf{p}\big|_{\Sigma_T} = \mathbf{0} \Big\},\\
&\overline{\mathbf{\Lambda}} := \Big\{ \boldsymbol{\lambda} : \iint_{Q_T} \big[\rho_2^2|\boldsymbol{\lambda}|^2 + \rho_1^2|\nabla\boldsymbol{\lambda}|^2\big]\, d\mathbf{x}\, dt < +\infty,\ \boldsymbol{\lambda}\big|_{\Sigma_T} = \mathbf{0} \Big\},\\
&\overline{\mathbf{X}} := \overline{\mathbf{Z}} \times \overline{\mathbf{\Phi}},\quad \overline{\mathbf{Y}} := \overline{\mathbf{\Lambda}} \times \widetilde{M},\\
&\overline{\mathbf{a}}((\mathbf{z},(\mathbf{p},\sigma)),(\mathbf{z}',(\mathbf{p}',\sigma'))) := \iint_{Q_T} \left(\rho^{-2}\mathbf{z}\,\mathbf{z}' + \rho_0^{-2}\mathbf{p}\,\mathbf{p}' 1_\omega\right) d\mathbf{x}\, dt,\\
&\overline{\mathbf{b}}((\mathbf{z},(\mathbf{p},\sigma)),(\lambda,\mu)) := \iint_{Q_T} \big\{ \big[\mathbf{z} + \mathbf{p}_t + \nabla\mathbf{p}^t\overline{\mathbf{y}} + \nabla\mathbf{p}(\overline{\mathbf{y}}+\mathbf{w}) - \nabla\sigma\big]\,\lambda - \nu\nabla\mathbf{p}\cdot\nabla\lambda \big\}\, d\mathbf{x}\, dt\\
&\qquad\qquad + \iint_{Q_T} (\nabla\cdot\mathbf{p})\,\mu\, d\mathbf{x}\, dt,\\
&\langle \overline{\ell}, (\mathbf{z},(\mathbf{p},\sigma))\rangle := \int_\Omega \mathbf{y}_0(\mathbf{x})\,\mathbf{p}(\mathbf{x},0)\, d\mathbf{x}.
\end{aligned}$$

6.4.2 Numerical experiments

In this Section, we are going to present some numerical experiments concerning the Poiseuille flow $\overline{\mathbf{y}}_P$ and the Taylor-Green vortex $\overline{\mathbf{y}}_{TG}$. In both cases, we try to solve a local exact controllability problem:

$$\overline{\mathbf{y}}(\mathbf{x},T) \equiv \overline{\mathbf{y}}_P(\mathbf{x}) \quad \text{or} \quad \overline{\mathbf{y}}(\mathbf{x},T) \equiv \overline{\mathbf{y}}_{TG}(\mathbf{x},T).$$

In the case of the Poiseuille flow, we will take the following data are fixed: $\Omega = (0,5)\times(0,1)$, $\omega = (1,2)\times(0,1)$, $T = 2$, $K_1 = 1$, $K_2 = 2$, $\beta_0 = \beta_0^{(1.5,0.5)}$, $\nu = 1$, $\overline{\mathbf{y}}_P(x_1,x_2) := (4x_2(1-x_2),0)$, $\mathbf{y}_0(\mathbf{x}) :\equiv \overline{\mathbf{y}}_p + M(\nabla\times\psi)(\mathbf{x})$ where $\psi(x_1,x_2) \equiv (x_1x_2)^2[(1-x_1)(1-x_2)]^2$ and $M = 0.1$. Again, the computations have been performed with the software *Freefem++*, using P_2-Lagrange approximations and the linear systems have been solved with the Arrow-Hurwicz algorithm, with parameters $r = 0.01$ and $s = 0.1$.

In the case of the Taylor–Green flow, we have taken the same data, except the following: $\Omega = (0, \pi) \times (0, \pi)$, $\omega = (\pi/3, 2\pi/3) \times (\pi/3, 2\pi/3)$, $T = 1$ and

$$\overline{\mathbf{y}}_{TG}(x_1, x_2, t) := (\sin(2x_1)\cos(2x_2)e^{-8t}, -\cos(2x_1)\sin(2x_2)e^{-8t}).$$

The same software and the same kind of approximation were considered.

The computational domains and the corresponding triangulations are displayed in Fig. 6.10 and 6.13. The behavior of the fixed-point iterates is depicted in Table 6.3. There, the relative error is given by

$$\frac{\|\mathbf{u}^{n+1} - \mathbf{u}^n\|_{\mathbf{L}^2(Q_T)}}{\|\mathbf{u}^{n+1}\|_{\mathbf{L}^2(Q_T)}}.$$

The computed controls and states are shown in Fig. 6.11 and 6.12 for the Poiseuille test and Fig. 6.14–6.17 for the Taylor-Green test.

Iterate	Rel. error (P)	Rel. error (TG)
1	0.499140	0.622740
10	0.039318	0.044985
20	0.010562	0.012376
30	0.003035	0.003366
40	0.000331	0.000851
50	0.000122	0.000209

Table 6.3: The behavior of ALG 2 (P: Poiseuille, TG: Taylor-Green).

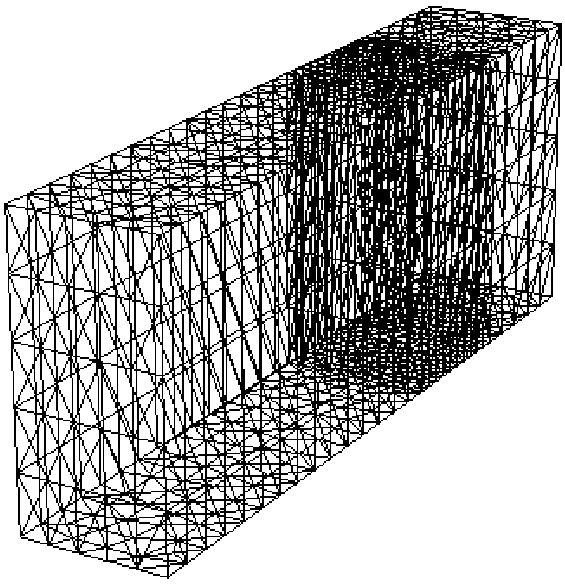

Figure 6.10: Poiseuille test – The domain and the mesh. Number of vertices: 1830. Number of elements (tetrahedra): 7830. Total number of variables: 12810.

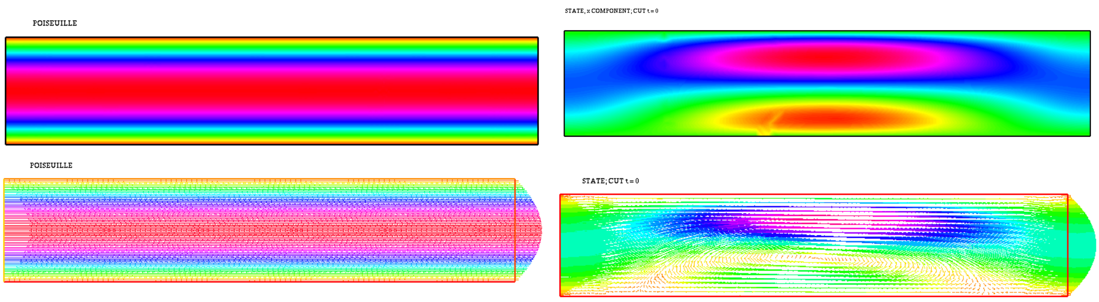

Figure 6.11: Poiseuille test – the target (**Left**) and the initial state (**Right**).

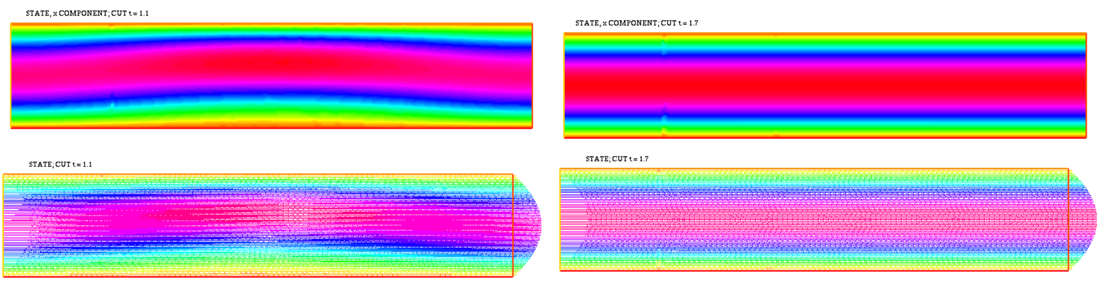

Figure 6.12: Poiseuille test – The state at $T = 1.1$ (**Left**) and the state at $T = 1.7$ ((**Right**).

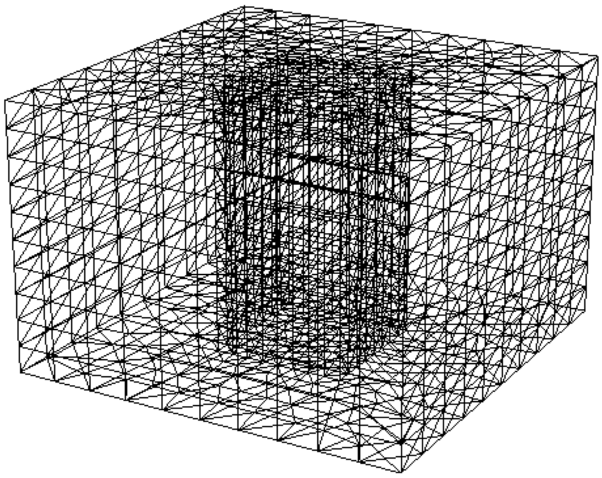

Figure 6.13: Taylor-Green test – The domain and the mesh. Number of vertices: 3146. Number of elements (tetrahedra): 15900. Total number of variables: 22022.

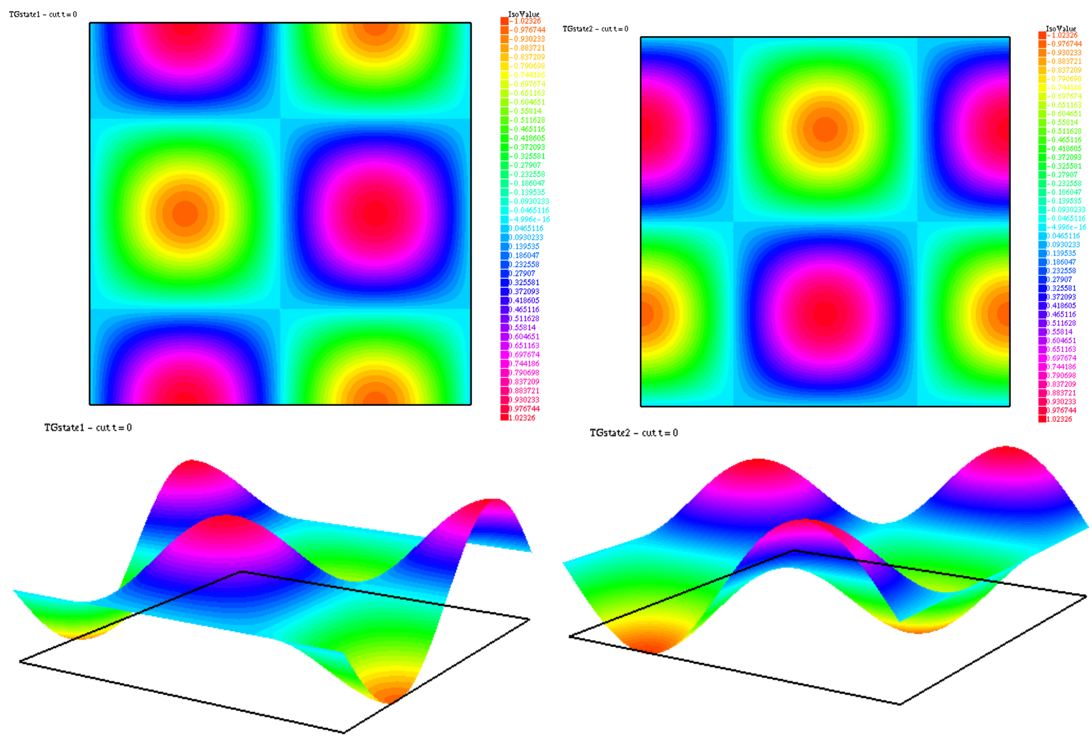

Figure 6.14: Taylor-Green test – First component of the initial datum (**Left**) and second component of the initial datum (**Right**).

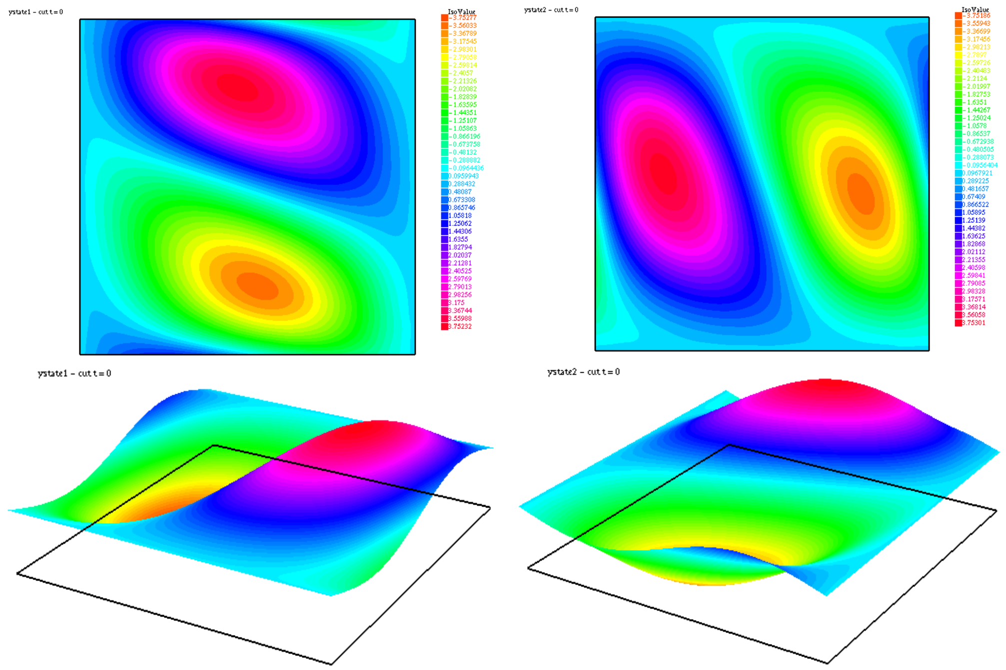

Figure 6.15: Taylor-Green test – The initial data: first component (**Left**) and second component (**Right**).

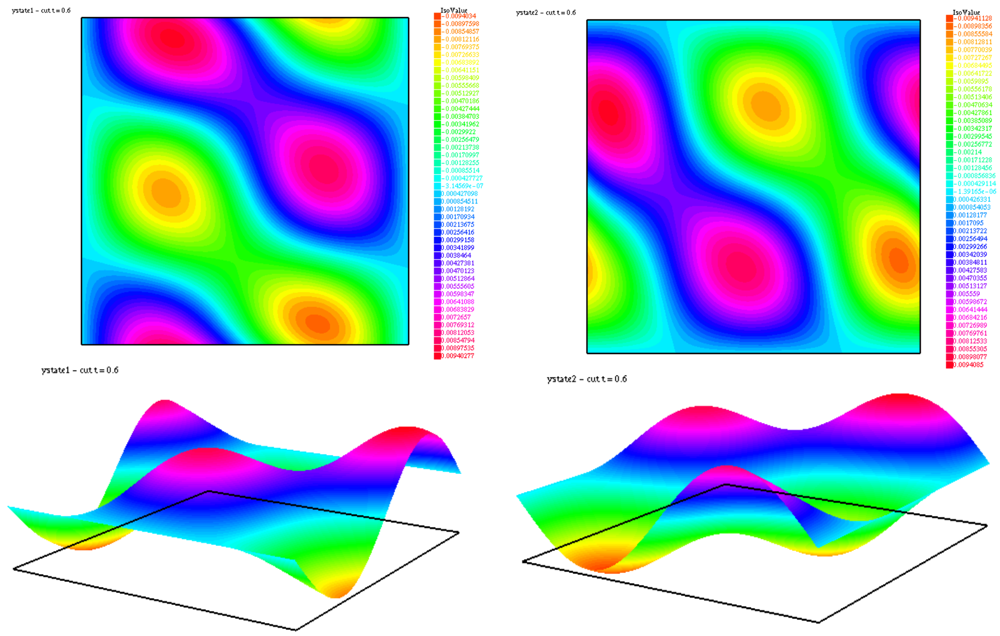

Figure 6.16: Taylor-Green test – The first component of the state (**Left**) and the second component of the state at $T = 0.6$ ((**Right**).

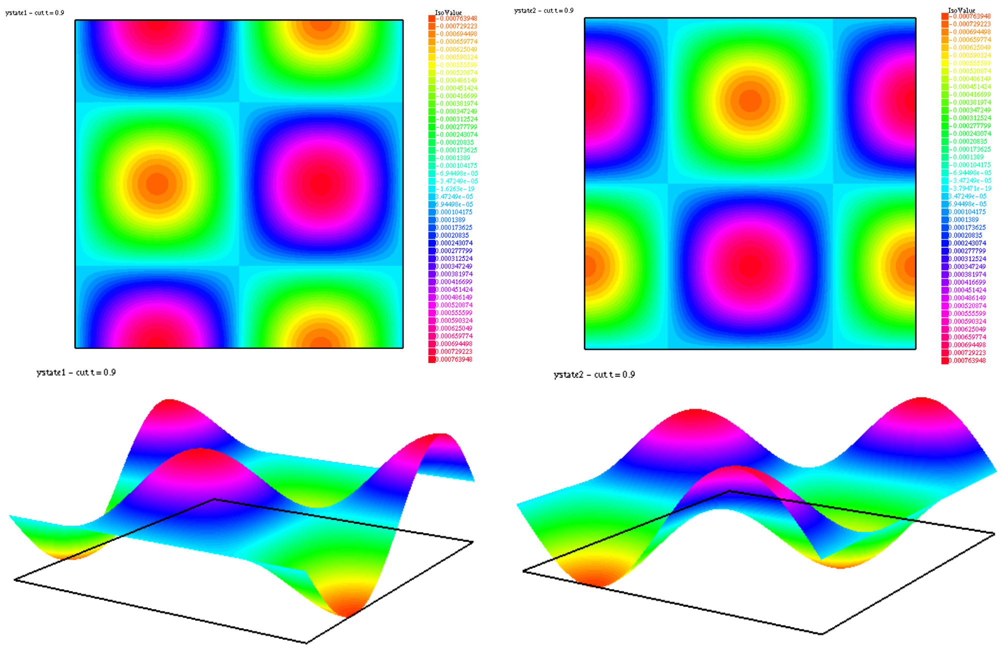

Figure 6.17: Taylor-Green test – The first component of the state (**Left**) and the second component of the state at $T = 0.9$ ((**Right**).

Chapter 7

Conclusions

Conclusions

Let us present some conclusions.

- In Chapter 2, we study the distributed null controllability of the Burgers-α system. We can use an extension argument to prove similar boundary controllability. More precisely, let us introduce the system

$$\begin{cases} y_t - y_{xx} + zy_x = 0 & \text{in} \quad (0,L)\times(0,T), \\ z - \alpha^2 z_{xx} = y & \text{in} \quad (0,L)\times(0,T), \\ z(0,\cdot) = y(0,\cdot) = 0,\ z(L,\cdot) = y(L,\cdot) = u & \text{in} \quad (0,T), \\ y(\cdot,0) = y_0 & \text{in} \quad (0,L), \end{cases} \tag{7.1}$$

  where $u = u(t)$ stands for the control and the initial datum satisfies $y_0 \in H_0^1(0,L)$.

  Let a, b and $\overline{L}$ be given, with $L < a < b < \overline{L}$. Then, let us define $\overline{y}_0 : [0,\overline{L}] \mapsto \mathbb{R}$, with $\overline{y}_0 := y_0 1_{[0,L]}$. Arguing as in Theorem 2.1, it can be proved that there exists $(\overline{y},\overline{v})$, with $\overline{v} \in L^\infty((a,b)\times(0,T))$,

$$\begin{cases} \overline{y}_t - \overline{y}_{xx} + z1_{[0,L]}\overline{y}_x = \overline{v}1_{(a,b)} & \text{in} \quad (0,\tilde{L})\times(0,T), \\ z - \alpha^2 z_{xx} = \overline{y} & \text{in} \quad (0,L)\times(0,T), \\ \overline{y}(0,\cdot) = z(0,\cdot) = \overline{y}(\tilde{L},\cdot) = 0,\ z(L,\cdot) = \overline{y}(L,\cdot) & \text{in} \quad (0,T), \\ \overline{y}(\cdot,0) = \overline{y}_0 & \text{in} \quad (0,\tilde{L}) \end{cases}$$

  and $\overline{y}(x,T) = 0$ in $(0,\tilde{L})$. Then, $y := \tilde{y}\big|_{(0,L)\times(0,T)}$, z and $u(t) := \tilde{y}(L,t)$ satisfy (7.3).

- In Chapter 3, we prove local results concerning the null controllability of the Leray-α system. It is unknown whether a general global null controllability result (not necessarily uniform in α) holds; this seems a very interesting question.

  Another interesting question is if we can get local null controllability with $N-1$ or even less scalar controls. In the distributed case, in view of the achievements in [12] and [33] for the Navier-Stokes equations, it is reasonable to expect that results similar to Theorems 3.1 and 3.3 hold with controls $\mathbf{v}$ such that $v_i \equiv 0$ for some i. On the other hand, it is completely unknown whether or not results of this kind holds in the boundary controllability case (even for the linear Stokes equations, this is not well understood).

  Roughly speaking, the results in this Chapter (and also in the previous one) show that turbulence models behave, from the viewpoint of control theory, as the original viscous Burgers or Navier-Stokes equations, at least in this case.

- In Chapter 4, we prove a global exact control result for a system of conservation laws furnished by the Euler equations coupled to a zero-diffusion heat equation through Boussinesq-like terms.

  In this result, for any initial and final data $\mathbf{y}_0$, $\mathbf{y}_T \in \mathbf{C}^{2,\alpha}$ and θ_0, $\theta_T \in C^{2,\alpha}$, we find a solution $(\mathbf{y}, p, \theta)$ with the following regularity:

  $$\mathbf{y} \in C^0([0,T]; \mathbf{C}^{1,\alpha}), \quad \theta \in C^0([0,T]; C^{1,\alpha}), \quad p \in \mathcal{D}'.$$

  and
  $$\mathbf{y}(\mathbf{x}, T) = \mathbf{y}_T(\mathbf{x}), \quad \theta(\mathbf{x}, T) = \theta_T(\mathbf{x}) \quad \text{in} \quad \Omega.$$

  Thus, we can only ensure the existence of a controlled solution that is $C^{1,\alpha}$ in space. It would be interesting to improve this result, i.e. to get a solution with the regularity
  $$\mathbf{y} \in C^0([0,T]; \mathbf{C}^{2,\alpha}), \quad \theta \in C^0([0,T]; C^{2,\alpha}), \quad p \in \mathcal{D}',$$

  but, at present, we do not know how.

  Another interesting question is whether we can get global controllability for the inviscid Boussinesq system. More precisely, is it possible to start from any initial data $\mathbf{y}_0 \in \mathbf{C}^{2,\alpha}$ and $\theta_0 \in C^{2,\alpha}$ and find a solution $(\mathbf{y}, p, \theta)$ to the following system, where $\kappa > 0$,

  $$\begin{cases} \mathbf{y}_t + (\mathbf{y} \cdot \nabla)\mathbf{y} = -\nabla p + \vec{\mathbf{k}}\theta & \text{in} \quad \Omega \times (0,T), \\ \nabla \cdot \mathbf{y} = 0 & \text{in} \quad \Omega \times (0,T), \\ \theta_t + \mathbf{y} \cdot \nabla\theta = \kappa \Delta\theta & \text{in} \quad \Omega \times (0,T), \\ \mathbf{y} \cdot \mathbf{n} = 0 & \text{on} \quad (\partial\Omega \setminus \Gamma) \times (0,T), \\ \theta = 0 & \text{on} \quad (\partial\Omega \setminus \gamma) \times (0,T), \\ \mathbf{y}(\mathbf{x},0) = \mathbf{y}_0(\mathbf{x}),\ \theta(\mathbf{x},0) = \theta_0(\mathbf{x}) & \text{in} \quad \Omega, \end{cases} \tag{7.2}$$

  that satisfies
  $$\mathbf{y}(\mathbf{x}, T) = \mathbf{0}, \quad \theta(\mathbf{x}, T) = 0 \quad \text{in} \quad \Omega?$$

  Unfortunately, we are only able to provide here a local result. Nevertheless, the main result in this Chapter shows that, for inviscid flows, the global exact controllability is preserved and not destroyed by heat effects.

- Chapters 5 and 6 contain several numerical methods that provide approximations of null controls and associated states for the heat and Stokes equations and exact to the trajectory controls for the Navier-Stokes equations.

  The approach in Chapters 5, which relies on solving directly the optimality system, may be employed to solve inverse problems where, for instance, the solution to the heat equation has to be recovered from a partial observation, typically localized on a subdomain q_T (actually, the optimality conditions associated to a least-square type functional can be expressed as a mixed formulation very closed to (5.32)). This issue will be analyzed in a future work.

The numerical methods in Chapter 6 are based on mixed formulations where only first derivatives appears. This is very convenient, since they can be approximated easily by C^0 finite elements. However, the well-posedness of those formulations seems to be a quite delicate issue and many related questions are at present open.

The results in these Chapters prove that it is possible to solve numerically many interesting controllability problems and can be viewed as a new step in the research in this area.

Conclusiones

Presentaremos a continuación algunas conclusiones.

- En el Capítulo 2, estudiamos la controlabilidad nula distribuida del sistema de Burgers-α. Podemos usar un argumento de extensión para probar resultados similares de control frontera. Más precisamente, consideremos el sistema

$$\begin{cases} y_t - y_{xx} + z y_x = 0 & \text{en} \quad (0,L)\times(0,T), \\ z - \alpha^2 z_{xx} = y & \text{en} \quad (0,L)\times(0,T), \\ z(0,\cdot) = y(0,\cdot) = 0,\ z(L,\cdot) = y(L,\cdot) = u & \text{en} \quad (0,T), \\ y(\cdot,0) = y_0 & \text{en} \quad (0,L), \end{cases} \tag{7.3}$$

  donde $u = u(t)$ es el control y el dato inicial verifica $y_0 \in H_0^1(0,L)$.

  Sean a, b y $\overline{L}$ dados, con $L < a < b < \overline{L}$. Pongamos $\overline{y}_0 : [0,\overline{L}] \mapsto \mathbb{R}$, con $\overline{y}_0 := y_0 1_{[0,L]}$. Razonando como en el Teorema 2.1, se puede probar que existe $(\overline{y},\overline{v})$, con $\overline{v} \in L^\infty((a,b)\times(0,T))$,

$$\begin{cases} \overline{y}_t - \overline{y}_{xx} + z 1_{[0,L]} \overline{y}_x = \overline{v} 1_{(a,b)} & \text{en} \quad (0,\tilde{L})\times(0,T), \\ z - \alpha^2 z_{xx} = \overline{y} & \text{en} \quad (0,L)\times(0,T), \\ \overline{y}(0,\cdot) = z(0,\cdot) = \overline{y}(\tilde{L},\cdot) = 0,\ z(L,\cdot) = \overline{y}(L,\cdot) & \text{en} \quad (0,T), \\ \overline{y}(\cdot,0) = \overline{y}_0 & \text{en} \quad (0,\tilde{L}) \end{cases}$$

  e $\overline{y}(x,T) = 0$ en $(0,\tilde{L})$. Entonces, $y := \tilde{y}\big|_{(0,L)\times(0,T)}$, z y $u(t) := \tilde{y}(L,t)$ satisfacen (7.3).

- En el Capítulo 3, probamos resultados locales de controlabilidad nula para el sistema de Leray-α. Es desconocido si se tiene un resultado general global (no necesariamente uniforme en α); esto parece una cuestión de gran interés.

  Otra cuestión interesante es si se puede conseguir la controlabilidad con $N-1$ o menos controles escalares. En el caso distribuido, a la vista de los resultados en [12] y [33] para las ecuaciones de Navier-Stokes, es razonable esperar resultados similares a los Teoremas 3.1 y 3.3 con controles $\mathbf{v}$ tales que $v_i \equiv 0$ para algún i. Por otra parte, es completamente desconocido si resultados de este tipo son ciertos o no en el caso de la controlabilidad frontera (incluso para las ecuaciones de Stokes, esta cuestión no se comprende bien).

  Hablando en términos generales, los resultados de este capítulo (y también del anterior) muestran que los modelos de turbulencia se comportan, desde el punto de vista de la teoría de control, de manera análoga a como lo hacen la ecuación viscosa de Burgers y las ecuaciones de Navier-Stokes, al menos en este caso.

- En el Capítulo 4, probamos un resultado de control exacto global para un sistema de leyes de conservación constituido por las ecuaciones de Euler acopladas con una ecuación del calor sin difusión a través de términos de tipo Boussinesq.

  En este resultado, dados los estados iniciales y finales $\mathbf{y}_0$, $\mathbf{y}_T \in \mathbf{C}^{2,\alpha}$ y θ_0, $\theta_T \in C^{2,\alpha}$, hallamos una solución $(\mathbf{y}, p, \theta)$ con la regularidad siguiente:

$$\mathbf{y} \in C^0([0,T]; \mathbf{C}^{1,\alpha}), \quad \theta \in C^0([0,T]; C^{1,\alpha}), \quad p \in \mathcal{D}',$$

$$\mathbf{y}(\mathbf{x}, T) = \mathbf{y}_T(\mathbf{x}), \quad \theta(\mathbf{x}, T) = \theta_T(\mathbf{x}) \quad \text{en} \quad \Omega.$$

  Por tanto, sólo podemos asegurar la existencia de soluciones controladas que son $C^{1,\alpha}$ en espacio. Sería interesante mejorar este resultado, i.e. conseguir una solución con la regularidad

$$\mathbf{y} \in C^0([0,T]; \mathbf{C}^{2,\alpha}), \quad \theta \in C^0([0,T]; C^{2,\alpha}), \quad p \in \mathcal{D}',$$

  pero, en el momento presente, no sabemos cómo hacer esto.

  Otra cuestión de interés es si podemos conseguir control global para el sistema de Boussinesq. Más precisamente, ¿ es posible arrancar de datos iniciales arbitrarios $\mathbf{y}_0 \in \mathbf{C}^{2,\alpha}$ y $\theta_0 \in C^{2,\alpha}$ y hallar una solución $(\mathbf{y}, p, \theta)$ del sistema que sigue, donde $\kappa > 0$,

$$\begin{cases} \mathbf{y}_t + (\mathbf{y} \cdot \nabla)\mathbf{y} = -\nabla p + \vec{\mathbf{k}}\theta & \text{en} & \Omega \times (0,T), \\ \nabla \cdot \mathbf{y} = 0 & \text{en} & \Omega \times (0,T), \\ \theta_t + \mathbf{y} \cdot \nabla\theta = \kappa\,\Delta\theta & \text{en} & \Omega \times (0,T), \\ \mathbf{y} \cdot \mathbf{n} = 0 & \text{sobre} & (\partial\Omega \setminus \Gamma) \times (0,T), \\ \theta = 0 & \text{sobre} & (\partial\Omega \setminus \gamma) \times (0,T), \\ \mathbf{y}(\mathbf{x},0) = \mathbf{y}_0(\mathbf{x}),\ \theta(\mathbf{x},0) = \theta_0(\mathbf{x}) & \text{en} & \Omega, \end{cases} \tag{7.4}$$

  que verifique

$$\mathbf{y}(\mathbf{x}, T) = \mathbf{0}, \quad \theta(\mathbf{x}, T) = 0 \quad \text{en} \quad \Omega?$$

  Desgraciadamente, sólo somos capaces de proporcionar aquí resultados locales. No obstante, el resultado principal de este capítulo muestra que, para flujos no viscosos, la controlabilidad exacta global es conservada y no destruida por los efectos térmicos.

- Los Capítulos 5 y 6 contienen varios métodos numéricos que proporcionan aproximaciones de controles nulos para las ecuaciones del calor y Stokes y controles exactos a trayectorias para las ecuaciones de Navier-Stokes.

El enfoque del Capítulo 5, que reposa sobre la resolución directa del sistema de optimalidad, puede ser utilizado también para resolver problemas inversos donde, por ejemplo, la solución de la ecuación del calor debe ser reconstruida a partir de una observación parcial, típicamente localizada en un subdominio q_T (de hecho, las condiciones de optimalidad de un funcional de tipo mínimos-cuadrados pueden ser escritas como un problema mixto muy próximo a (5.32)). Todo esto será analizado en un trabajo futuro.

Los métodos numéricos del Capítulo 6 se basan en formulaciones mixtas donde sólo aparecen derivadas de primer orden. Esto es muy conveniente, dado que estas formulaciones pueden ser aproximadas fácilmente usando elementos finitos C^0. Sin embargo, el buen planteamiento de los sistemas que surgen parece una cuestión delicada y difícil conduce a muchos problemas abiertos.

Los resultados de estos capítulos prueban que es posible resolver eficientemente muchos problems de control relevantes y pueden ser mirados como una nueva etapa en la investigación en este área.

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