BREATHERS IN FPU SYSTEMS, NEAR AND FAR FROM THE PHONON BAND

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Introduction. This work is motivated by a recent breathers existence proof in the one dimensional FPU system, given by the equations:

$$\ddot{x}_n = V'(x_{n+1} - x_n) - V'(x_n - x_{n-1}) \ , \quad n \in \mathbb{Z} \ , \tag{1}$$

where V is a smooth interaction potential satisfying $V'(0) = 0$ and $V''(0) > 0$. Using a center manifold technique[2], one can prove the existence of small amplitude breathers (SAB) with frequencies ω_b slightly above the phonon band if $B = \frac{1}{2}V''(0)V^{(4)}(0) - (V^{(3)}(0))^2 > 0$, and their non-existence for $B < 0$. Our aim is to test numerically the range of validity of this theoretical result and to explore new phenomena. For this purpose we shall fix $V(u) = u^2/2 + a\,u^3 + \frac{1}{4}\,u^4$, which yields $B = 3(1 - 12a^2)$.

We work with the difference variables $u_n = x_n - x_{n-1}$ more suitable for the use of our numerical method. We also use periodic boundary conditions $u_{n+2p}(t) = u_n(t)$ so that the maximum frequency of the linear phonons is exactly 2 as in the infinite lattice. Our computations are performed using a numerical scheme based on the anti-continuous limit and Newton method[3].

Test and range of validity. First, we have computed numerically SAB (i.e. breathers whose amplitudes go to zero when $w_b \to 2^+$) in the case when $B > 0$. We have obtained breathers with symmetries $u_n(t) = u_{-n}(t)$ (Page mode) and $u_n(t) = u_{-n-1}(t + T_b/2)$ (Sievers-Takeno mode), where $T_b = 2\pi/\omega_b$ is the breather period. The force $y_n = V'(u_n)$ is the variable used in reference [2]. In Fig.1 (left) it is shown that the maxima of the force are of order $\mu^{1/2}$ when $\mu = w_b - 2 \to 0^+$, as predicted by the theory, up to relatively large values. Thus if $B > 0$ breathers exist for any small value of energy in our FPU system (1).

Another property of these SAB is that their width diverges when $w_b \to 2^+$. More precisely the theory predicts that their spatial extend is of order $\mu^{-1/2}$, which is in accordance with our numerical observations.

Other numerical observations. For $B > 0$, we have numerically continued the SAB as ω_b goes away from the phonon band. We have found that the maxima amplitudes of the oscillations, $\sup |u_n|$, are also approximately linear functions of $\mu^{1/2}$. This is expected for small μ, since $u_n = y_n + O(y_n^2))$, but it occurs surprisingly far from the phonon band, at least until values of $\mu \approx 1$ (see fig.1, left). We have also checked that the Page mode fits very well to the NLS soliton $u_n(t) = \alpha \sqrt{\mu}\,(-1)^n \cos(\omega_b t)\,[\cosh(\beta \sqrt{\mu}\, n)]^{-1}$, even far from the top of the phonon band.

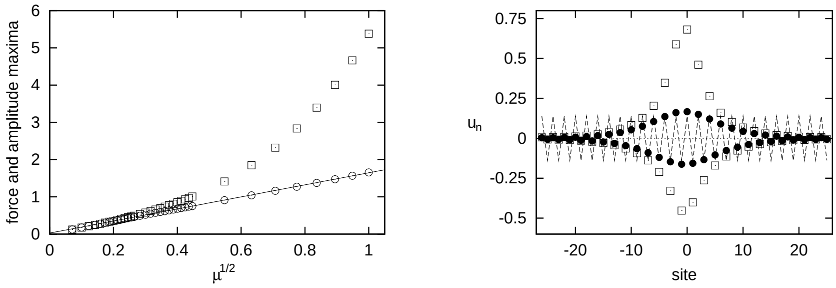

Figure 1. Left: Force (squares) and amplitudes (circles) maxima versus $\mu^{1/2}$. The cubic coefficient in V is $a = -0.1$ ($B = 2.64$). Right: Comparison between a SAB (full circles) for $a = -0.1$ and a LAB (blank squares) for $a = -1/3$ ($B = -1$) having the same frequency $w_b = 2.01$. The dashed line represents the linear phonon with frequency 2.

For $B < 0$ and V strictly convex ($\frac{1}{\sqrt{12}} < |a| < \frac{1}{\sqrt{3}}$), breathers exist near the top of the phonon band but they are large amplitude breathers[1] (LAB), i.e. their amplitudes do not go to zero when $w_b \to 2^+$. As a consequence there is an energy gap for breathers creation in these FPU systems. In figure 1 (right) we compare a SAB and a LAB having the same frequency $w_b = 2.01$. We have found LAB with the same symmetries as SAB (Page and Sievers-Takeno modes). The Page mode fits very well to an exponential profile having the form $u_n(t) = \alpha(\omega_b)\,(-1)^n \cos(\omega_b t)\,|\sigma(\omega_b)|^{|n|}$ where $\sigma(\omega_b) = 1-(\omega_b^2)/2+(\omega_b/2)(\omega_b^2-4)^{1/2} \in (-1,0)$. As (1) is formulated as a mapping in a loop space[2] and $\omega_b > 2$, the linearized operator has a purely hyperbolic spectrum and the constant $\sigma(\omega_b)$ is the closest eigenvalue to -1 (with $\sigma(2) = -1$). Consequently, for $\omega_b \approx 2$ one can ask if the iterated map admits a global center manifold containing these LAB.

References

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